COURSES 4 and 5

The ring of square matrices over a field

Let K be a set and $m, n \in \mathbb{N}^*$. A mapping

$$A : \{1, \dots, m\} \times \{1, \dots, n\} \to K$$

is called $m \times n$ **matrix** over K. When $m = n$, we call A a **square matrix of size** n. For each $i = 1, \dots, m$ and $j = 1, \dots, n$ we denote $A(i, j)$ by $a_{ij}(\in K)$ and we represent A as a rectangular array with m rows and n columns in which the image of each pair (i, j) is written in the i'th row and the j'th column

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

We also denote this array by

$$A = (a_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$$

or, simpler, $A = (a_{ij})$. We denote the set of all $m \times n$ matrices over K by $M_{m,n}(K)$ and, when $m = n$, by $M_n(K)$.

Let $(K, +, \cdot)$ be a field. Then $+$ from K determines an operation $+$ on $M_{m,n}(K)$ defined as follows: if $A = (a_{ij})$ and $B = (b_{ij})$ are two $m \times n$ matrices, then

$$A + B = (a_{ij} + b_{ij}).$$

One can easily check that this operation is associative, commutative, it has an identity element which is the matrix $O_{m,n}$ consisting only of 0 (called **the** $m \times n$ **zero matrix**) and each matrix $A = (a_{ij})$ from $M_{m,n}(K)$ has an opposite (the matrix $-A = (-a_{ij})$). Therefore,

Theorem 1. $(M_{m,n}(K), +)$ is an Abelian group.

The scalar multiplication of a matrix $A = (a_{ij}) \in M_{m,n}(K)$ and a scalar $\alpha \in K$ is defined by

$$\alpha A = (\alpha a_{ij}).$$

One can easily check that:

i) $\alpha(A + B) = \alpha A + \alpha B,\ \forall \alpha \in K,\ \forall A, B \in M_{m,n}(K)$;
ii) $(\alpha + \beta)A = \alpha A + \beta A,\ \forall \alpha, \beta \in K,\ \forall A \in M_{m,n}(K)$;
iii) $(\alpha\beta)A = \alpha(\beta A),\ \forall \alpha, \beta \in K,\ \forall A \in M_{m,n}(K)$;
iv) $1 \cdot A = A,\ \forall A \in M_{m,n}(K)$.

The matrix multiplication is defined as follows: if $A = (a_{ij}) \in M_{m,n}(K)$ and $B = (b_{ij}) \in M_{n,p}(K)$, then

$$AB = (c_{ij}) \in M_{m,p},\ \text{cu } c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj},\ (i, j) \in \{1, \dots, m\} \times \{1, \dots, p\}.$$

For $n \in \mathbb{N}^*$ we consider the $n \times n$ square matrix

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

If $m, n, p, q \in \mathbb{N}^*$, then:

1) $(AB)C = A(BC)$, for any matrices $A \in M_{m,n}(K),\ B \in M_{n,p}(K),\ C \in M_{p,q}(K)$;

2) $I_m A = A = AI_n,\ \forall A \in M_{m,n}(K)$;

3) $A(B + C) = AB + AC$ for any matrices $A \in M_{m,n}(K),\ B, C \in M_{n,p}(K)$;

3') $(B + C)D = BD + CD$, for any matrices $B, C \in M_{n,p}(K),\ D \in M_{p,q}(K)$;

4) $\alpha(AB) = (\alpha A)B = A(\alpha B),\ \forall \alpha \in K,\ \forall A_{m,n}(K),\ \forall B \in M_{n,p}(K)$.

If we work with $n \times n$ square matrices the matrix multiplication becomes a binary (internal) operation $\cdot$ on $M_n(K)$, and the equalities 1)–3') show that $\cdot$ is associative, I_n is a multiplicative identity element (called **the identity matrix** of size n) and $\cdot$ is distributive with respect to $+$. Hence,

Theorem 2. $(M_n(K), +, \cdot)$ is a unitary ring called **the ring of the square matrices of size** n **over** K.

Remarks 3. a) If $n \geq 2$ then $M_n(K)$ is not commutative and it has zero divisors. If $a, b \in K^*$, the non-zero matrices

$$\begin{pmatrix} a & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \dots & b \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

can be used to prove this.

b) Using the properties of the addition, multiplication and scalar multiplication, one can easily prove that

$$f : K \to M_n(K),\ f(a) = \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & a & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & a \end{pmatrix} = aI_n$$

is a unitary injective ring homomorphism.

The transpose of an $m \times n$ matrix $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = (a_{ij})$ is the $n \times m$ matrix

$${}^tA = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \dots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix} = (a_{ji}).$$

The way the transpose acts with respect to the matrix addition, matrix multiplication and scalar multiplication is given bellow:

${}^t(A+B) = {}^tA + {}^tB,\ \forall A, B \in M_{m,n}(K);$

${}^t(AB) = {}^tB \cdot {}^tA,\ \forall A \in M_{m,n}(K),\ \forall B \in M_{n,p}(K);$

${}^t(\alpha A) = \alpha \cdot {}^tA,\ \forall A \in M_{m,n}(K).$

Let K be a field. The set of the units of $M_n(K)$ is

$$GL_n(K) = \{A \in M_n(K) \mid \exists B \in M_n(K) : AB = BA = I_n\}.$$

The set $GL_n(K)$ is closed in $(M_n(K), \cdot)$ and $(GL_n(K), \cdot)$ is a group called **the general linear group of degree** n over K. We know from high school that if K is one of the number fields ($\mathbb{Q}$, $\mathbb{R}$ sau $\mathbb{C}$) then $A \in M_n(K)$ is invertible if and only if $\det A \neq 0$. Thus,

$$GL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid \det A \neq 0\},$$

and analogously we can rewrite $GL_n(\mathbb{R})$ and $GL_n(\mathbb{Q})$. We will see next that this recipe works for any matrix ring $M_n(K)$ with K field. This is why our next course topic will be **the determinant of a square matrix over a field** K.

Determinants

Let $(K, +, \cdot)$ be a field, $n \in \mathbb{N}^*$ and

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \in M_n(K).$$

Definition 4. The determinant of (the squre matrix) A is

$$\det A = \sum_{\sigma \in S_n} \varepsilon(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} (\in K).$$

The map $M_n(K) \to K$, $A \mapsto \det A$ is also called **determinant**.

Remark 5. None of the products from the above definition contains 2 elements from the same row or the same column.

We also denote the determinant of A by $\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$.

Examples 6. a) $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$

b) $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$

Lemma 7. The determinant of A and the determinant of the transpose matrix ${}^tA$ are equal.

Proof. $\square$

Remark 8. Any property which refers to the rows of the determinant of a certain matrix A can also be written for the columns of A and any property valid for the columns of $\det A$ is also valid for its rows.

Proposition 9. If $n \in \mathbb{N}^*$ and $i \in \{1, \ldots, n\}$ then

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,n} \\ a_{i1} + a'_{i1} & a_{i2} + a'_{i2} & \cdots & a_{in} + a'_{in} \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,n} \\ a'_{i1} & a'_{i2} & \cdots & a'_{in} \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

This property can be generalized and restated for columns (homework).

Proof. $\square$

For the next part of the section, we consider a field $(K, +, \cdot)$, $n \in \mathbb{N},\ n \geq 2$ and $A = (a_{ij}) \in M_n(K)$.

Proposition 10. If the matrix B results from A by multiplying each element of a row (column) of A by $\alpha \in K$ then $\det B = \alpha \det A$.

Proof. $\square$

Proposition 11. If all the elements of a row (column) of A are 0, then $\det A = 0$.

Proof. $\square$

Proposition 12. If B results from A by switching two rows (columns) of A then $\det B = -\det A$.

Proof. $\square$

Proposition 13. If A has two equal rows (columns) then $\det A = 0$.

Proof. $\square$

Let us denote by $r_1, r_2, \cdots, r_n$ the rows and by $c_1, c_2, \cdots, c_n$ the columns of A. We say that **the rows (columns)** i **and** j ($i, j \in \{1, \ldots, n\}$, $i \neq j$) **are proportional** if there exists $\alpha \in K$ such that all the elements of a row (column) are the elements of the other one multiplied by α. We write, correspondingly, $l_i = \alpha l_j$ or $l_j = \alpha l_i$ or $c_i = \alpha c_j$ or $c_j = \alpha c_i$.

Corollary 14. If A has two proportional rows (columns) then $\det A = 0$.

Definition 15. We say that the i'th row of the matrix A **is a linear combination of (all) the other rows** ($i \in \{1, \ldots, n\}$) if there exists $\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n \in K$ such that

$$a_{ij} = \alpha_1 a_{1j} + \cdots + \alpha_{i-1} a_{i-1,j} + \alpha_{i+1} a_{i+1,j} + \cdots + \alpha_n a_{nj},\ \forall j \in \{1, \ldots, n\}.$$

We write

$$l_i = \alpha_1 l_1 + \cdots + \alpha_{i-1} l_{i-1} + \alpha_{i+1} l_{i+1} + \cdots + \alpha_n l_n.$$

An analogous definition can be given for columns (homework).

The property from the previous corollary can be generalized as follows:

Corollary 16. If a row (column) of A is a linear combination of the other rows (columns) then $\det A = 0$.

Corollary 17. If the matrix B results from A by adding the i'th row (column) multiplied by $\alpha \in K$ to the j'th one ($i \neq j$) then $\det B = \det A$.

Definition 18. Let $A = (a_{ij}) \in M_n(K)$, $n \geq 2$ and $i, j \in \{1, \ldots, n\}$. Let $A_{ij} \in M_{n-1}(K)$ be the matrix resulted from A by eliminating the i'th row and the j'th column (i.e. the row and the column of a_{ij}). The determinant

$$d_{ij} = \det A_{ij}$$

is called **the minor of** a_{ij} and

$$\alpha_{ij} = (-1)^{i+j} d_{ij}$$

is called **the cofactor of** a_{ij}.

Then:

Theorem 19. (the cofactor expansion of $\det A$ along the i'th row)

$$\det(A) = a_{i1}\alpha_{i1} + a_{i2}\alpha_{i2} + \cdots + a_{in}\alpha_{in},\ \forall i \in \{1, \ldots, n\}.$$

Proof. Let us denote

$$S_i = a_{i1}\alpha_{i1} + a_{i2}\alpha_{i2} + \cdots + a_{in}\alpha_{in}. \tag{$*$}$$

(A) For $i = 1$, we have $S_1 = a_{11}\alpha_{11} + a_{12}\alpha_{12} + \cdots + a_{1n}\alpha_{1n}$. Let us consider the term $a_{11}\alpha_{11} = a_{11}d_{11}$. We notice that d_{11} is the sum of all the products of the form

$$a_{2k_2} a_{3k_3} \cdots a_{nk_n} \text{ cu } \{k_2, \ldots, k_n\} = \{2, \ldots, n\},$$

and each term has the sign $(-1)^{Inv\,\tau}$ where $\tau = \begin{pmatrix} 2 & 3 & \ldots & n \\ k_2 & k_3 & \ldots & k_n \end{pmatrix}$. Each term of S_1 which contains a_{11} comes from $a_{11}\alpha_{11}$. Therefore, these terms are the products

$$(-1)^{Inv\,\tau} a_{11} a_{2k_2} a_{3k_3} \cdots a_{nk_n}.$$

On the other side, the terms of $\det A$ which contain a_{11} are (all) the products

$$a_{11} a_{2k_2} a_{3k_3} \cdots a_{nk_n} \text{ cu } \{k_2, \ldots, k_n\} = \{2, \ldots, n\},$$

and the sign of each such term is $(-1)^{Inv\,\sigma}$ with $\sigma = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 1 & k_2 & k_3 & \dots & k_n \end{pmatrix}$.

Since $1 < k_2, \dots, 1 < k_n$, we have $Inv\,\sigma = Inv\,\tau$ thus the terms which contain a_{11} are the same in S_1 and $\det A$ (and when we say the same we refer, of course, to the fact that they have the same signs in the both sums).

(B) Let us consider the general case. Let $i, j \in \{1, \dots, n\}$, and let us take the term

$$a_{ij}\alpha_{ij} = (-1)^{i+j} a_{ij} d_{ij}$$

from $(*)$. This term provides us with all the (products which are) terms of S_i which contain a_{ij}. On the other side, let us rewrite $\det A$ in the following way: by successively permuting adjacent rows, we bringa_{ij} on the first row, then, by permuting adjacent columns, we bring it in the position $(1,1)$. Let us denote by D the resulted determinant. Since we applied i row switches and j column switches this way, we have

$$\det A = (-1)^{i+j} D.$$

Based on this equality, all the terms of $\det A$ containing a_{ij} result from D as in (A). As we already saw the element which lays in the $(1,1)$ position of D is a_{ij}; from the way D occured, one deduces that its minor d_{ij}, and its cofactor is $(-1)^{1+1} d_{ij} = d_{ij}$. Therefore, the terms which contain a_{ij} are the same as in

$$(-1)^{i+j} a_{ij} d_{ij} = a_{ij}\alpha_{ij},$$

hence they are exactly the terms of S_i which contain a_{ij}.

We also notice that S_i has n terms and each such term is a sum of $(n-1)!$ products of elements of A (each one considered with the corresponding sign), thus S_i has $(n-1)!n = n!$ terms which are exactly the terms of $\det A$. This remark completes proof. $\square$

We also have:

Teorema 19'. (the cofactor expansion of $\det(A)$ **along the** j**'th column)**

$$\det A = a_{1j}\alpha_{1j} + a_{2j}\alpha_{2j} + \dots + a_{nj}\alpha_{nj}, \ \forall j \in \{1, \dots, n\}.$$

Corollary 20. If $i, k \in \{1, \dots, n\}$, $i \neq k$, then

$$a_{i1}\alpha_{k1} + a_{i2}\alpha_{k2} + \dots + a_{in}\alpha_{kn} = 0.$$

Also, if $j, k \in \{1, \dots, n\}$, $j \neq k$ then

$$a_{1j}\alpha_{1k} + a_{2j}\alpha_{2k} + \dots + a_{nj}\alpha_{nk} = 0.$$

Corollary 21. If $d = \det A \neq 0$ then A is a unit of the ring $M_n(K)$ and

$$A^{-1} = d^{-1} \cdot A^*,$$

where A^* is the matrix

$$A^* = {}^t(\alpha_{ij}) = \begin{pmatrix} \alpha_{11} & \alpha_{21} & \dots & \alpha_{n1} \\ \alpha_{12} & \alpha_{22} & \dots & \alpha_{n2} \\ \vdots & \vdots & & \vdots \\ \alpha_{1n} & \alpha_{2n} & \dots & \alpha_{nn} \end{pmatrix}$$

(called **the adjugate of** A).

Remark 22. We will see later that the converse of the previous statement is also valid, i.e. *if* A *is invertible then* $\det A \neq 0$.

Corollary 23. (Cramer) Let us consider the following system with n equations with n unknowns

$$(S) \qquad \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}, \ a_{ij}, b_i \in K \ (i,j \in \{1,\dots,n\}).$$

Denote by d the determinant $d = \det A$ of $A = (a_{ij}) \in M_n(K)$ and by d_j the determinant of the matrix resulted from A by replacing the j'th column by

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

If $d \neq 0$ then (S) has a unique solution. This solution is given by the equalities

$$x_i = d_i \cdot d^{-1}, \ i = 1, \dots, n.$$

The rank of a matrix

Let $m, n \in \mathbb{N}^*$, $A = (a_{ij}) \in M_{m,n}(K)$.

Definition 24. Let $i_1, \dots, i_k, j_1, \dots, j_l \in \mathbb{N}^*$ cu $1 \leq i_1 < \cdots < i_k \leq m$ and $1 \leq j_1 < \cdots < j_l \leq n$. A matrix

$$\begin{pmatrix} a_{i_1j_1} & a_{i_1j_2} & \dots & a_{i_1j_k} \\ a_{i_2j_1} & a_{i_2j_2} & \dots & a_{i_2j_k} \\ \vdots & \vdots & & \vdots \\ a_{i_lj_1} & a_{i_lj_2} & \dots & a_{i_lj_k} \end{pmatrix}$$

formed by taking the elementels of A which are situated at the intersections of the rows $i_1, \dots, i_k$ with the columns $j_1, \dots, j_l$ is called $k \times l$ **submatrix of** A. The determinant of a $k \times k$ submatrix is called **minor of** A **of order** k.

Definition 25. Let $A \in M_{m,n}(K)$. If A is not the zero matrix, i.e. $A \neq O_{m,n}$, we say that **the rank of the matrix** A is r, and we write $\operatorname{rank} A = r$, if A has a non-zero minor of order r all the minors of A of order greater than r (if they exist) are 0. By definition, $\operatorname{rank} O_{m,n} = 0$.

Remark 26. a) $\operatorname{rank} A \leq \min\{m, n\}$.
b) If $A \in M_n(K)$ then $\operatorname{rank} A = n$ dif and only if $\det A \neq 0$.
c) $\operatorname{rank} A = \operatorname{rank} {}^tA$.

For the following part of this section, we take $m, n \in \mathbb{N}^*$, $A = (a_{ij}) \in M_{m,n}(K)$ and $A \neq O_{m,n}$.

Finding the rank of A by definition involves, most of the time, a large number of computations (of minors). The next theorem is a first step for reducing the number of these computations.

Theorem 27. $\operatorname{rank} A = r$ if and only if A has a non-zero minor of order r and all $r+1$-size minors of A (if they exist) are 0.

Proof. □

Theorem 28. The rank of the matrix A is the maximum number of columns (rows) we can choose from the columns (rows) of A such that none of them is a linear combination of the others.

Proof. Suppose that the rank of A is r. Then A has a non-zero minor of order r. For simpler notations, we consider that

$$d = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1r} \\ a_{21} & a_{22} & \dots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{r1} & a_{r2} & \dots & a_{rr} \end{vmatrix} \neq 0$$

and any $r+1$-size minor is zero. (The proof of the general case works in the same way, only the notations are more complicated.) Therefore the determinant

$$D_{ij} = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1r} & a_{1j} \\ a_{21} & a_{22} & \dots & a_{2r} & a_{2j} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{r1} & a_{r2} & \dots & a_{rr} & a_{rj} \\ a_{i1} & a_{i2} & \dots & a_{ir} & a_{ij} \end{vmatrix}$$

of size $r+1$ resulted by adding to d the i'th row and j'th column of A ($1 \le i \le m$, $r < j \le n$) is zero, i.e. $D_{ij} = 0$. Notice that if $1 \le i \le r$ then D_{ij} has two equal rows, and if $r < i \le m$ and $r < j \le n$ then D_{ij} is a $r+1$-size minor of A resulted by adding to d the row i and the column j. Expanding D_{ij} along the row $r+1$, we get

$$a_{i1}d_1 + a_{i2}d_2 + \cdots + a_{ir}d_r + a_{ij}d = 0$$

where the cofactors $d_1, d_2, \dots, d_r$ do not depend on the added row i. It follows that

$$a_{ij} = -d^{-1}d_1a_{i1} - d^{-1}d_2a_{i2} - \cdots - d^{-1}d_ra_{ir}$$

for all $i = 1, 2, \dots, m$ and $j = r+1, \dots, n$ thus

$$c_j = \alpha_1c_1 + \alpha_2c_2 + \cdots + \alpha_rc_r \text{ for all } j = r+1, \dots, n,$$

where $\alpha_k = -d^{-1}d_k$, $1 \le k \le r$, i.e. c_j is a linear combination of $c_1, c_2, \dots, c_r$.

This way we proved that the maximum number of columns we can choose from the columns of A such that none of them is a linear combination of the others is at most r. If this number is strictly smaller than r, then one of $c_1, \dots, c_r$ will be a linear combination of the others and $d = 0$, which is not possible.

Thus the maximum number of columns we can choose from the columns of A such that none of them is a linear combination of the others is exactly r and the proof is now complete. □

Corollary 29. $\operatorname{rank} A = r$ if and only if A has a non-zero minor d of order r and all the other rows (columns) of A are linear combinations of the the rows (columns) of A whose elements are the entries of d.

Corollary 30. $\operatorname{rank} A = r$ if and only if there exists a non-zero minor d of A of order r and all the $r+1$-size minors of A resulted by adding one of remained rows and columns to d are 0 (if they exist, of course).

Corollary 31. If $m, n, p \in \mathbb{N}^*$, $A = (a_{ij}) \in M_{m,n}(K)$ and $B = (b_{ij}) \in M_{n,p}(K)$ (K field) then $\operatorname{rank}(AB) \leq \min\{\operatorname{rank} A,\ \operatorname{rank} B\}$.

If one of the given matrices is zero, the property is obvious. So, let us consider both our matrices non-zero and let us suppose that $\min\{\operatorname{rank} A,\ \operatorname{rank} B\} = \operatorname{rank} B = r \in \mathbb{N}^*$ and that a non-zero minor of B of size r can be extracted from the columns $j_1, \ldots, j_r$ with $1 \leq j_1 < \cdots < j_r \leq p$. (For the other case, one can rephrase the statement for the transposes of our matrices, then one can use the same reasoning to find the expected result.) The columns of AB are

$$A \begin{pmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{pmatrix},\ A \begin{pmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2} \end{pmatrix}, \ldots,\ A \begin{pmatrix} b_{1p} \\ b_{2p} \\ \vdots \\ b_{np} \end{pmatrix}.$$

From corollary 29 we deduce that for any $k \in \{1, \ldots, p\} \setminus \{j_1, \ldots, j_r\}$, there exist $\alpha_{1k}, \ldots, \alpha_{rk} \in K$ such that

$$\begin{pmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{nk} \end{pmatrix} = \alpha_{1k} \begin{pmatrix} b_{1j_1} \\ b_{2j_1} \\ \vdots \\ b_{nj_1} \end{pmatrix} + \alpha_{2k} \begin{pmatrix} b_{1j_2} \\ b_{2j_2} \\ \vdots \\ b_{nj_2} \end{pmatrix} + \cdots + \alpha_{rk} \begin{pmatrix} b_{1j_r} \\ b_{2j_r} \\ \vdots \\ b_{nj_r} \end{pmatrix}.$$

Hence

$$A \begin{pmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{nk} \end{pmatrix} = \alpha_{1k} \cdot A \begin{pmatrix} b_{1j_1} \\ b_{2j_1} \\ \vdots \\ b_{nj_1} \end{pmatrix} + \alpha_{2k} A \cdot \begin{pmatrix} b_{1j_2} \\ b_{2j_2} \\ \vdots \\ b_{nj_2} \end{pmatrix} + \cdots + \alpha_{rk} \cdot A \begin{pmatrix} b_{1j_r} \\ b_{2j_r} \\ \vdots \\ b_{nj_r} \end{pmatrix},$$

which means that in AB all the columns $k \in \{1, \ldots, p\} \setminus \{j_1, \ldots, j_r\}$ are linear combinations of the columns $j_1, \ldots, j_r$. Thus the rank of the matrix AB is at most r.

Corollary 32. Let $n \in \mathbb{N}^*$ and K be a field. A matrix $A \in M_n(K)$ is invertible (i.e. a unit in $(M_n(K), +, \cdot)$) if and only if $\det A \neq 0$.

An algorithm for finding the rank of a matrix:
Corollary 30 shows that for a matrix $A \neq O_{m,n}$, $\operatorname{rank} A$ can be determined in the following way: we start with a non-zero minor d of A and we compute all the minors of A obtained by adding d one of the remained rows and one of the remained columns until we find a non-zero minor, minor which will be the subject of a similar approach. In finitely many steps, we will find a non-zero minor of order r of A for which all the $r+1$-size minors resulted by adding it one of remained rows and columns are zero. Thus $r = \operatorname{rank} A$.