Artificial Intelligence

6. Elements of Game Theory (II)

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Elements of Game Theory (II)

1. Strategic games
    1.1. Mixed Nash equilibrium
    1.2. Pareto optimality
2. Two-player cooperative games
3. *N*-player cooperative games
    3.1. Game representation in characteristic form
    3.2. The core
    3.3. The Shapley value
4. Conclusions

Elements of Game Theory (II)

The welfare game

		Pauper	
		Try to work	*Be idle*
Government	*Aid*	3, 2	-1, 3
	No aid	-1, 1	0, 0

- The government wishes to aid a pauper (a poor man) if he searches for work but not otherwise
- The pauper searches for work only if he cannot depend on government aid

The welfare game

		Pauper	
		Try to work	*Be idle*
Government	*Aid*	3, 2 →	-1, 3
	No aid	-1, 1	0, 0

(Aid, Try to work) is not NE: Pauper prefers Be idle

The welfare game

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(Aid, Try to work) is not NE: Pauper prefers Be idle
(Aid, Be idle) is not NE: Govt prefers No aid

The welfare game

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(Aid, Try to work) is not NE: Pauper prefers Be idle
(Aid, Be idle) is not NE: Govt prefers No aid
(No aid, Be idle) is not NE: Pauper prefers Try to work

The welfare game

		Pauper: Try to work	Pauper: Be idle
Government	Aid	3, 2	-1, 3
	No aid	-1, 1	0, 0

(Aid, Try to work) is not NE: Pauper prefers Be idle
(Aid, Be idle) is not NE: Govt prefers No aid
(No aid, Be idle) is not NE: Pauper prefers Try to work
(No aid, Try to work) is not NE: Govt prefers Aid

No pure Nash equilibrium

Pure and mixed strategies

- Pure strategy
  - Player i chooses strategy s_{ij} from set S_i
- Mixed strategy
  - Player i chooses strategy s_{ij} with probability p_{ij}
    - $p_{ij} \geq 0, \sum_j p_{ij} = 1$
- Every pure strategy is also a mixed strategy
- A finite game always has a pure or mixed Nash equilibrium
  - There is always a Nash equilibrium in a mixed strategy

Mixed strategies

- Payoff in mixed strategies is the expected payoff
  - Let payoff with strategy s_1 be 1 and s_2 be 4
  - Mixed strategy (0.3, 0.7) gives the expected payoff $0.3 \cdot 1 + 0.7 \cdot 4 = 3.1$
  - It means a sure payoff of 3.1 is equivalent to a gamble where the payoffs are 1 and 4, with probabilities 0.3 and 0.7, respectively

- *Would you be willing to pay 3 units to participate in this game?*
- *How about playing 100 times?*

Mixed strategies: interpretation

- Games where multiple strategies can be simultaneously employed
  - Betting on more than one horse
- Multiple instances of the same game
  - War scenario: q_{ij}% of pilots use strategy s_{ij}
- Same game repeated infinitely
- For a single game: the probability distribution is the opponent's estimation of the player's decision

The oddment method

- A simple method for computing mixed Nash equilibria
- Not applicable if the game has a pure Nash equilibrium

Strategies for Pauper

		Pauper *Try to work*	**Pauper** *Be idle*
Government	*Aid*	3, 2	-1, 3
Government	*No aid*	-1, 1	0, 0

$$\begin{bmatrix} 3 & -1 \\ -1 & 0 \end{bmatrix}$$

$$3-(-1)=4 \quad -1-0=-1$$

$$|-1|=1 \quad |4|=4$$

$$\frac{1}{1+4}=0.2 \quad \frac{4}{1+4}=0.8$$

Strategies for Government

		Pauper	
		Try to work	*Be idle*
Government	*Aid*	3, 2	-1, 3
	No aid	-1, 1	0, 0

$$\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$$

$$2 - 3 = -1 \quad |1| = 1 \quad \frac{1}{1+1} = 0.5$$

$$1 - 0 = 1 \quad |-1| = 1 \quad \frac{1}{1+1} = 0.5$$

Mixed Nash equilibrium for the welfare game

		Pauper: *Try to work*	Pauper: *Be idle*
Government	*Aid*	3, 2	-1, 3
	No aid	-1, 1	0, 0

- If the Government chooses a probability of 0.5 for Aid, the Pauper will not be able to take advantage of the government's choice by choosing either Work or Idle
  - Pauper's payoff (Work) = $0.5 \cdot 2 + (1 - 0.5) \cdot 1 = 1.5$
  - Pauper's payoff (Idle) = $0.5 \cdot 3 + (1 - 0.5) \cdot 0 = 1.5$

Mixed Nash equilibrium for the welfare game

		Pauper: *Try to work*	Pauper: *Be idle*
Government	*Aid*	3, 2	-1, 3
	No aid	-1, 1	0, 0

- If the Pauper chooses to work with probability 0.2, then the Government will be just indifferent between Aid and No aid
  - Govt's payoff (Aid) = $0.2 \cdot 3 + (1 - 0.2) \cdot (-1) = -0.2$
  - Govt's payoff (No aid) = $0.2 \cdot (-1) + (1 - 0.2) \cdot 0 = -0.2$

Mixed Nash equilibrium for the welfare game

		Pauper: *Try to work*	Pauper: *Be idle*
Government	*Aid*	3, 2	-1, 3
	No aid	-1, 1	0, 0

- For the probabilities 0.5 and 0.2, both the Government and the Pauper have equal expected payoffs for both actions, which allows a Nash equilibrium

Method based on equations

		Pauper: Try to work	Pauper: Be idle
Government	*Aid*	3, 2	-1, 3
	No aid	-1, 1	0, 0

- Computing the strategy for Pauper
  - $3 \cdot x + (-1) \cdot (1 - x) = (-1) \cdot x + 0 \cdot (1 - x)$
  - $\Rightarrow x = 0.2,\ 1 - x = 0.8$
- Computing the strategy for Government
  - $2 \cdot y + 1 \cdot (1 - y) = 3 \cdot y + 0 \cdot (1 - y)$
  - $\Rightarrow y = 0.5,\ 1 - y = 0.5$

Stability

- If a player leaves the equilibrium strategy, the opponent can take advantage to gain more that he/she would at equilibrium

A game with an infinite number of mixed Nash equilibria

		Colin	
		Action C	*Action D*
Rose	*Action A*	(3, 1)	(4, 0)
	Action B	(3, -2)	(2, -5)

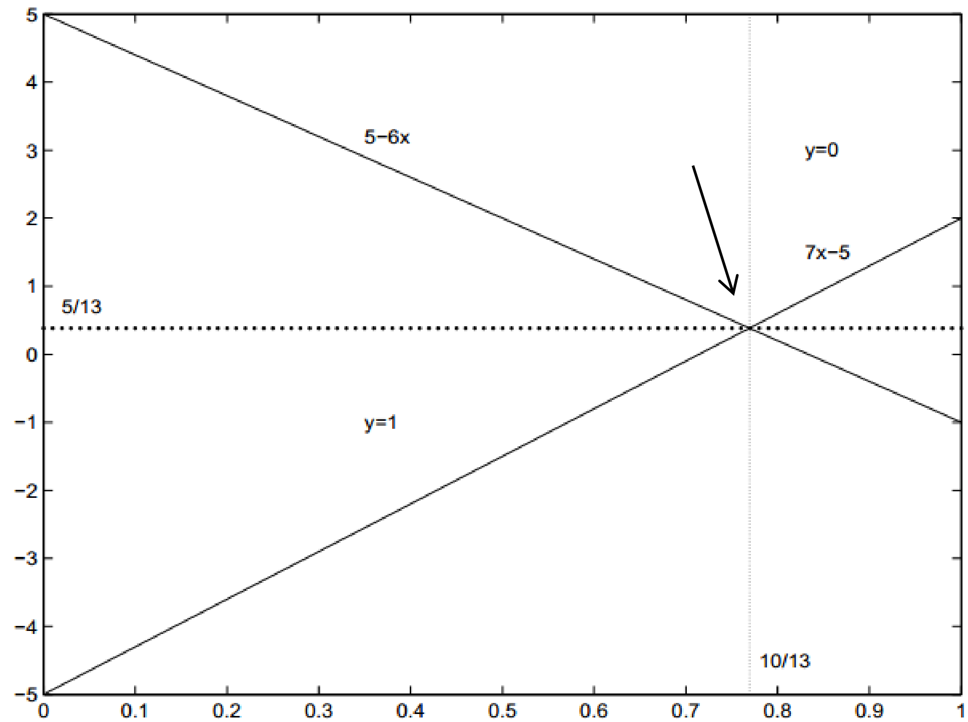
- Action *D* is dominated for Colin
- If Colin always chooses *C*, Rose is indifferent between choosing either *A* or *B*
- The equilibrium is: $(x, y) = (1, p)$, with $p \in [0, 1]$
  - $(x, 1-x)$ are Colin's probabilities for *C*, *D*
  - $(y, 1-y)$ are Rose's probabilities for *A*, *B*

Graphic perspective

For game: $\begin{bmatrix} (-1,4) & (5,0) \\ (2,-10) & (-5,5) \end{bmatrix}$

$$R = \begin{bmatrix} -1 & 5 \\ 2 & -5 \end{bmatrix} \qquad C = \begin{bmatrix} 4 & 0 \\ -10 & 5 \end{bmatrix}$$

$$[y, 1-y] \cdot R \cdot \begin{bmatrix} x \\ 1-x \end{bmatrix} = [y, 1-y] \cdot \begin{bmatrix} -1 & 5 \\ 2 & -5 \end{bmatrix} \cdot \begin{bmatrix} x \\ 1-x \end{bmatrix}$$
$$= [y, 1-y] \begin{bmatrix} 5-6x \\ 7x-5 \end{bmatrix}$$



(x, 1-x) are Colin's probabilities for *C*, *D*
(y, 1-y) are Rose's probabilities for *A*, *B*

$$\mathbf{x}_R = (10/13, 3/13) \qquad v_R = 5/13$$

Nonlinear payoffs

Moves A	Moves B	Moves C	Payoff A	Payoff B	Payoff C	Percentage of Time
1	1	1	0	0	0	x^3
1	1	2	0	0	2	$x^2(1-x)$
1	2	1	0	2	0	$x^2(1-x)$
1	2	2	1	0	0	$x(1-x)^2$
2	1	1	2	0	0	$x^2(1-x)$
2	1	2	0	1	0	$x(1-x)^2$
2	2	1	0	0	1	$x(1-x)^2$
2	2	2	0	0	0	$(1-x)^3$

All agents play 1 with probability *x* and 2 with probability 1 – *x*

Expected payoff for *A*

$$P_A(x) = x(1-x)^2 + 2x^2(1-x) = x - x^3$$

The differential solution

$$0 = \frac{dP_A(x)}{dx} = \frac{d(x - x^3)}{dx} = 1 - 3x^2 \quad \Longrightarrow x = \frac{1}{\sqrt{3}}$$

$$P_A\left(\frac{1}{\sqrt{3}}\right) = x - x^3\Big|_{x_* = \frac{1}{\sqrt{3}}} = \frac{2\sqrt{3}}{9} = 0.385$$

- The differential solution is the maximum that can be obtained, but it is not a Nash equilibrium
- Therefore, if some players use this strategy, others may gain more by using the Nash equilibrium solution

Exploiting the differential solution

- Let us assume that *B* and *C* use the $x^* = 1/\sqrt{3}$ strategy
- *A* can exploit this by choosing $y = 0$

Moves A	B	C	Payoff A	B	C	Percentage of Time
1	1	1	0	0	0	yx_*^2
1	1	2	0	0	2	$yx_*(1-x_*)$
1	2	1	0	2	0	$yx_*(1-x_*)$
1	2	2	1	0	0	$y(1-x_*)^2$
2	1	1	2	0	0	$(1-y)x_*^2$
2	1	2	0	1	0	$(1-y)x_*(1-x_*)$
2	2	1	0	0	1	$(1-y)x_*(1-x_*)$
2	2	2	0	0	0	$(1-y)(1-x_*)^2$

$$P_A(x_*) = y(1-x_*)^2 + 2(1-y)x_*^2 = y(1-2x_*-x_*^2) + 2x_*^2$$

$$P_A\left(\frac{1}{\sqrt{3}}\right) = y\underbrace{\left(1 - 2\frac{1}{\sqrt{3}} - \frac{1}{3}\right)}_{<0} + \frac{2}{3}$$

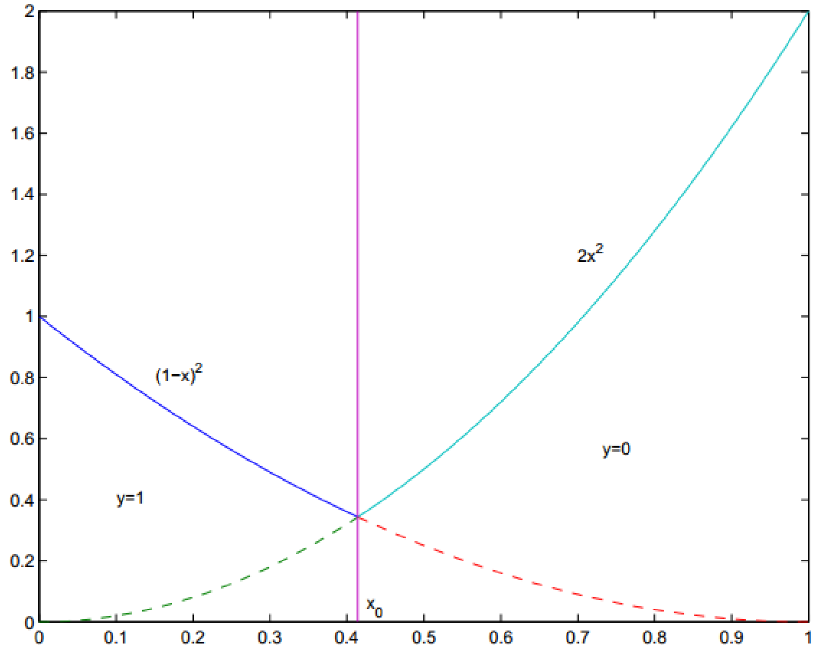
$$P_A\left(\frac{1}{\sqrt{3}}\right) = \frac{2}{3} = 0.66 \quad > 0.385$$

Graphic perspective

$$P_A(x,y) = y(1-x)^2 + 2(1-y)x^2 = y(1-2x-x^2) + 2x^2$$

$$P_A(x,0) = 0 \cdot (1-2x-x^2) + 2x^2 = 2x^2$$

$$P_A(x,1) = 1 \cdot (1-2x-x^2) + 2x^2 = 1-2x+x^2 = (1-x)^2$$

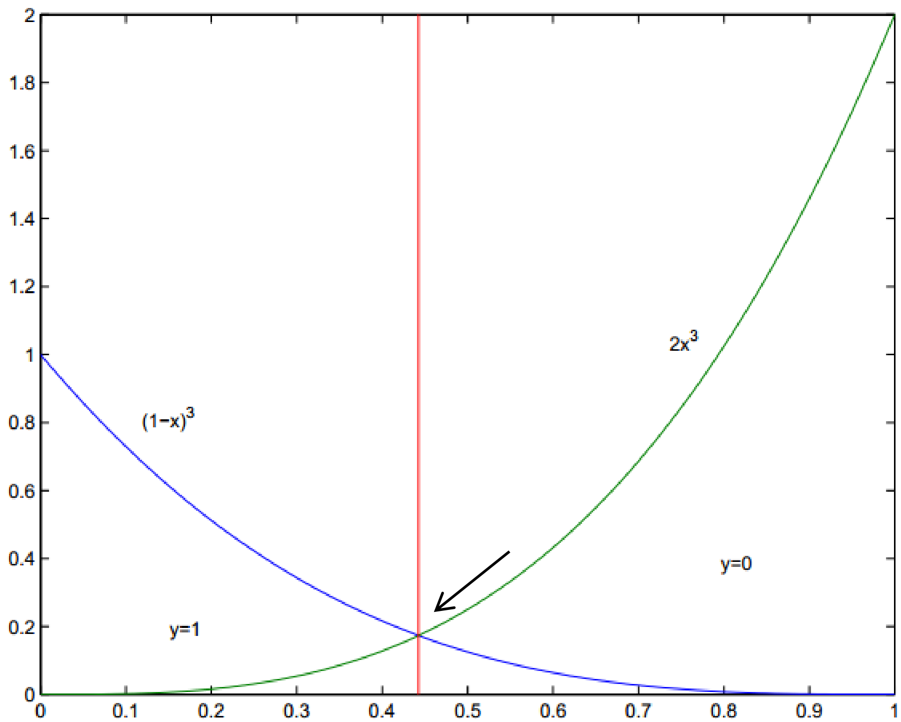


$$x = x_0 \approx 0.41$$

$$P_A(x,y) = 2x_0^2 \approx 0.343$$

A more complex example

$$P_A(x, y) = y(1-x)^3 + (1-y)(2x^3)$$



- Regardless of the shape of the functions, the Nash equilibrium point represents the abscissa of the minimum of the upper region; the ordinate represents the value of the game (the payoff)
- For more complex equations, the intersection can be determined with numerical methods

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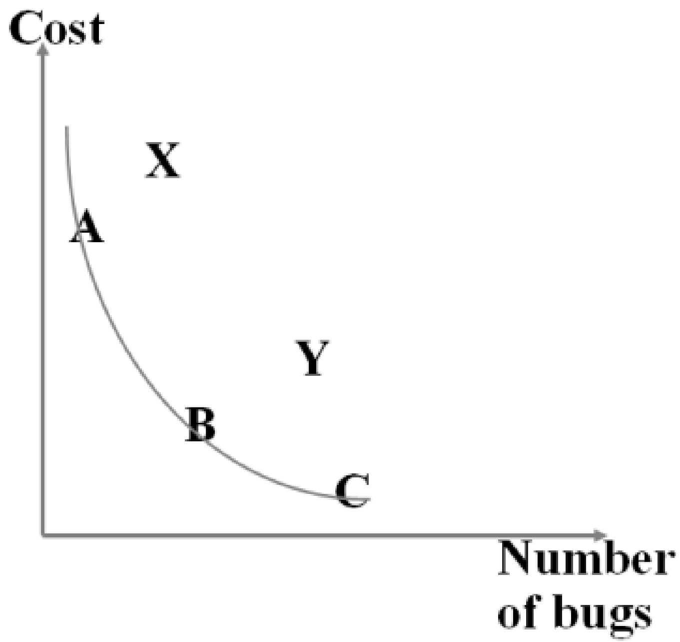
Pareto optimality

- An outcome is said to be Pareto optimal if:
  - it is better or the same as another outcome from all points of view and
  - it is strictly better from at least one point of view
- An outcome O_1 dominates another outcome O_2 if and only if:
  - O_1 is not inferior to O_2 with respect to all items: $\forall i, O_1(i) \geq O_2(i)$
  - O_1 is strictly superior to O_2 with respect to at least one item: $\exists i, O_1(i) > O_2(i)$
- The non-dominated outcomes are Pareto optimal

Example

- Minimize the cost and number of defects ("bugs") found in a software product



Solutions *A*, *B*, *C* are non-dominated
Solution *X* is dominated by *A*
Solution *Y* is dominated by *B*

Pareto optimal states

- In a Pareto optimal state, the players do not have the motivation to deviate in a coalition
- Example: the prisoner's dilemma
  - Both agents are better off together if they both deny
  - Except for the pure Nash equilibrium, all other strategy profiles are non-dominated

		Agent 2: Denies	Agent 2: Confesses
Agent 1	*Denies*	-1, -1	-5, 0
	Confesses	0, -5	-3, -3

Interpretation

- Pareto optimality means a better situation for at least one agent without harming any other agent
- Pareto optimality doesn't mean "equity"
  - E.g. dividing a pie between 3 people A, B, C
  - A gets 70%, B gets 30%, C gets nothing
  - This state is still a Pareto optimal equilibrium, because in order to give C something, A or B would have to lose
- However, it implies that all the resources are allocated
  - A state where A gets 50%, B gets 30%, C gets nothing is not Pareto optimal
  - C can get 20% without harming A or B

Applications of Pareto optimality

- Optimization problems
  - Computer network traffic
  - Task scheduling
  - Production planning
  - Component design
  - Chemical reaction processes
- Economics
  - Market efficiency analysis
  - Improving the tax system

Elements of Game Theory (II)

Cooperation

- In previous games, the agents were rational and selfish
- By cooperating, the agents can obtain a greater payoff
  - The outcome in which the sum of the payoffs is the maximum
- The problem is dividing the additional payoff
- The correct solution stands in the two agents' bargaining position
  - Bargaining position ≠ ability to negotiate

Example 1

		Peugeot F	Peugeot M
Renault	F	$(-10, -40)$	$(40, 10)$
	M	$(10, 40)$	$(-40, -10)$

- If there is no cooperation, the game has:
  - Pure Nash equilibriums: (10, 40) and (40, 10)
  - Mixed Nash equilibrium: Renault (0.5, 0.5) and Peugeot (0.8, 0.2), with payoff 0 for both companies

		Peugeot	
		F	M
Renault	F	$(-10, -40)$	$(40, 10)$
	M	$(10, 40)$	$(-40, -10)$

Cooperation

- The sum matrix of the game reflects the total payoff that can be obtained through cooperation

$$R + P = \begin{bmatrix} -50 & 50 \\ 50 & -50 \end{bmatrix}$$

- The threat matrix is used to describe the negotiation power of the agents

$$R - P = \begin{bmatrix} 30 & 30 \\ -30 & -30 \end{bmatrix}$$

Interpretation

		Peugeot	
		F	M
Renault	F	$(-10, -40)$	$(40, 10)$
	M	$(10, 40)$	$(-40, -10)$

$$R - P = \begin{bmatrix} 30 & 30 \\ -30 & -30 \end{bmatrix}$$

- The first line only has positive values and, thus, *R* has a stronger bargaining position (no matter what *P* chooses, *R* can gain a greater payoff)
- The threat differential is the value of the game for the threat matrix which is 30 in this case

		Peugeot	
		F	M
Renault	F	$(-10, -40)$	$(40, 10)$
	M	$(10, 40)$	$(-40, -10)$

The solution

- The solution for a two-agent cooperative game:
  - The total payoff is the maximum value of the sum matrix
  - The payoff difference between the two agents is the threat differential
- For the previous example:
  - The total payoff = 50
  - The difference between payoffs = 30
  - $R + P = 50$ and $R - P = 30$
  - Therefore, R gains 40 and P gains 10

$$R + P = \begin{bmatrix} -50 & 50 \\ 50 & -50 \end{bmatrix}$$

$$R - P = \begin{bmatrix} 30 & 30 \\ -30 & -30 \end{bmatrix}$$

- The played strategies do not matter, as long as the total payoff is obtained and the division method is observed
  - For the strategies (*R:F / P:M*), the agents' payoffs are those given directly by the result of the game
  - For the strategies (*R:M / P:F*), *P* has to pay 30 units to *R*

Example 2

$$R = \begin{bmatrix} -1 & 2 \\ 1 & -4 \end{bmatrix} \qquad C = \begin{bmatrix} -4 & 1 \\ 4 & -1 \end{bmatrix}$$

- The total maximum payoff is 5
- The threat matrix is:

$$R - C = \begin{bmatrix} 3 & \boxed{1} \\ -3 & -3 \end{bmatrix}$$

- The threat differential is 1, because the strategy combination (1, 2) represents a saddle point
- The solution of the game: *R* gains 3, *C* gains 2
  - The agents will play (*R*:2 / *C*:1) and Colin will pay Rose 2 units

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Definitions

- Let $\{ P_1, P_2, ..., P_n \}$ be a set of agents
- The grand coalition is the entire set of agents: $G = \{ P_1, \ldots, P_n \}$
- A coalition represents any non-empty subset of G
- Each coalition tries to maximize its payoff
- The characteristic function v records the maximum payoff for each coalition (the value of the coalition)
- Superadditive game: $v(S \cup T) \geq v(S) + v(T)$, where S and T are coalitions with no common agents

Imputation

- An imputation is the set of payoffs (x_1, x_2, ..., x_n) that satisfies the following conditions:
  - The sum of payoffs is equal to the payoff of the grand coalition
  - Each agent gains a payoff at least as good as the one it would gain when not cooperating

$$\sum_{i=1}^{n} x_i = \nu(\mathcal{G}),$$

$$x_i \geq \nu(\{P_i\}) \quad \text{for all } i.$$

- An imputation is an efficient, individually rational allocation

Example

- Let's consider a game with 3 agents: P_1, P_2, P_3
- Each agent can choose heads (H) or tails (T)
- If two agents choose the same value and the third one chooses differently, the latter will pay 1 to each other agent. Thus, all agents receive 0 in total
- $v(\{P_1, P_2, P_3\}) = 0$ (together, no amount can be gained)
- We assume that they form the coalition $S = \{P_2, P_3\}$
- The counter-coalition would be $S^c = \{P_1\}$
- Thus, we have a zero-sum game with the game matrix:

		$\mathcal{S}$			
		HH	HT	TH	TT
$\mathcal{S}^c$	H	0	1	1	−2
	T	−2	1	1	0

Example

S wants to maximize its payoff

- The 2nd and 3rd columns are dominated (0 < 1 and –2 < 1)

		$\mathcal{S}$			
		HH	$\cancel{\text{HT}}$	$\cancel{\text{TH}}$	TT
$\mathcal{S}^c$	H	0	$\cancel{1}$	$\cancel{1}$	-2
	T	-2	$\cancel{1}$	$\cancel{1}$	0

- The value of the game is –1 (mixed equilibrium)
- $v(\{P_1\}) = -1$ (what P_1 expects to gain)
- $v(\{P_2, P_3\}) = 1$ (what the *S* coalition expects to gain)
- Due to the symmetry of the game, the characteristic function is:

$$\nu(\mathcal{S}) = \begin{cases} -1, & \text{if } \mathcal{S} = \{P_1\}, \{P_2\}, \text{or } \{P_3\}, \\ +1, & \text{if } \mathcal{S} = \{P_1, P_2\}, \{P_2, P_3\}, \text{or } \{P_3, P_1\}, \\ 0, & \text{if } \mathcal{S} = \{P_1, P_2, P_3\}. \end{cases}$$

- Imputations: $x_i \geq -1$, $i = 1, 2, 3$; $x_1 + x_2 + x_3 = 0$

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The core

- The core of an n-agent game is the set of non-dominated imputations
- The core of a game with the characteristic function v is the collection of all imputations $\mathbf{x} = (x_1, x_2, ..., x_n)$ such that for any coalition $S = \{P_{i1}, P_{i2},\dots, P_{im}\}$ we have: $x_{i1} + x_{i2} + \dots + x_{im} \geq v(S)$
- Any imputation in the core can be viewed as a solution to the game
- The core is stable
- If an imputation is not in the core, then there is at least one coalition whose members do not receive the maximum payoff that they would receive otherwise. These agents prefer another imputation

Example 1: non-empty core

- 3 students wish to buy a book that costs 110
- For 2 or 3 books bought together, there is a discount of 10, and 20 respectively per copy
- The values of the coalitions express the amount of saved money

$$\nu(\{P_1\}) = \nu(\{P_2\}) = \nu(\{P_3\}) = 0,$$

$$\nu(\{P_1, P_2\}) = \nu(\{P_1, P_3\}) = \nu(\{P_2, P_3\}) = 20, \; \nu(\{P_1, P_2, P_3\}) = 60$$

Example 1: non-empty core

$$\nu(\{P_1\}) = \nu(\{P_2\}) = \nu(\{P_3\}) = 0,$$

$$\nu(\{P_1, P_2\}) = \nu(\{P_1, P_3\}) = \nu(\{P_2, P_3\}) = 20, \;\; \nu(\{P_1, P_2, P_3\}) = 60$$

- Let **x** = (x_1, x_2, x_3) be an imputation in the core, thus:

$$x_1 \geq \nu(\{P_1\}) = 0, \;\; x_2 \geq \nu(\{P_2\}) = 0, \;\; x_3 \geq \nu(\{P_3\}) = 0,$$

$$x_1 + x_2 \geq \nu(\{P_1, P_2\}) = 20, \;\; x_1 + x_3 \geq \nu(\{P_1, P_3\}) = 20,$$

$$x_2 + x_3 \geq \nu(\{P_2, P_3\}) = 20, \text{ and } x_1 + x_2 + x_3 = \nu(\{P_1, P_2, P_3\}) = 60.$$

Hence $0 \leq x_3 = 60 - (x_1 + x_2) \leq 60 - 20 \leq 40$. Similarly we get $0 \leq x_1, x_2 \leq 40$. Thus the core consists of all vectors $\mathbf{x} = (x_1, x_2, x_3)$ such that $0 \leq x_1, x_2, x_3 \leq 40$ with $x_1 + x_2 + x_3 = 60$. In particular, vectors like $(20, 20, 20)$ and $(0, 20, 40)$ are in the core.

Example 2: empty core

$$\nu(\mathcal{S}) = \begin{cases} -1, & \text{if } \mathcal{S} = \{P_1\}, \{P_2\}, \text{or } \{P_3\}, \\ +1, & \text{if } \mathcal{S} = \{P_1, P_2\}, \{P_2, P_3\}, \text{or } \{P_3, P_1\}, \\ 0, & \text{if } \mathcal{S} = \{P_1, P_2, P_3\}. \end{cases}$$

- As a zero-sum game, it has an empty core (the game is unstable)
- Let **x** = (x_1, x_2, x_3) be an imputation in the core

$$\begin{aligned} x_1 + x_2 &\geq v(\{1,2\}) = 1 \\ x_1 + x_3 &\geq v(\{1,3\}) = 1 \\ x_2 + x_3 &\geq v(\{2,3\}) = 1 \\ 2 \cdot (x_1 + x_2 + x_3) &\geq 3 \\ x_1 + x_2 + x_3 &= 0 \end{aligned}$$

- We have a contradiction: **x** cannot be an imputation and, thus, the core is empty (there is no imputation such that each agent be content)

Example 3

- A seller *S* wants to sell a horse. As far as *S* is concerned, if the horse is not sold, it has no value
- A farmer *F* and a butcher *B* want to buy the horse
- For *F*, the horse values 1000
- For *B*, the horse values 500

$$\nu(\{S\}) = \nu(\{F\}) = \nu(\{B\}) = \nu(\{F, B\}) = 0,$$

$$\nu(\{S, B\}) = 500 \quad \text{and} \quad \nu(\{S, F\}) = \nu(\{S, F, B\}) = 1000.$$

Example 3

$$\nu(\{S\}) = \nu(\{F\}) = \nu(\{B\}) = \nu(\{F, B\}) = 0,$$

$$\nu(\{S, B\}) = 500 \quad \text{and} \quad \nu(\{S, F\}) = \nu(\{S, F, B\}) = 1000.$$

If $\mathbf{x} = (x_S, x_F, x_B)$ is an imputation, then

$$x_S + x_F + x_B = 1000. \tag{7.5}$$

If $\mathbf{x}$ is in the core, we must have

$$x_S + x_F \geq \nu(\{S, F\}) = 1000.$$

Hence

$$x_B = 1000 - (x_S + x_F) \leq 1000 - 1000 \leq 0.$$

Since $x_B \geq \nu(\{B\}) = 0$, we conclude that $x_B = 0$. Then by (7.5), $x_S + x_F = 1000$.

we also have $x_S + x_B \geq \nu(\{S, B\}) = 500$.

Since $x_B = 0$, it follows that $x_S \geq 500$. On the other hand, $x_F \geq \nu(\{F\}) = 0$, and so $x_S = 1000 - x_F \leq 1000$. We conclude that an imputation in the core must be of the form

$$\mathbf{x} = (x_S, 1000 - x_S, 0) \quad \text{with } 500 \leq x_S \leq 1000.$$

Example 3

If $\mathbf{x} = (x_S, x_F, x_B)$ is an imputation, then

$$x_S + x_F + x_B = 1000. \tag{7.5}$$

If $\mathbf{x}$ is in the core, we must have

$$x_S + x_F \geq \nu(\{S, F\}) = 1000.$$

B does not gain anything, however, his presence is important for the bargaining position of seller *S*

Hence

$$x_B = 1000 - (x_S + x_F) \leq 1000 - 1000 \leq 0.$$

Since $x_B \geq \nu(\{B\}) = 0$, we conclude that $x_B = 0$. Then by (7.5), $x_S + x_F = 1000$.

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Since $x_B = 0$, it follows that $x_S \geq 500$. On the other hand, $x_F \geq \nu(\{F\}) = 0$, and so $x_S = 1000 - x_F \leq 1000$. We conclude that an imputation in the core must be of the form

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Graphical perspective of the core

$$v(1) = v(2) = v(3) = 0$$

$$v(12) = \tfrac{1}{4} \qquad v(13) = \tfrac{1}{2} \qquad v(23) = \tfrac{3}{4}$$

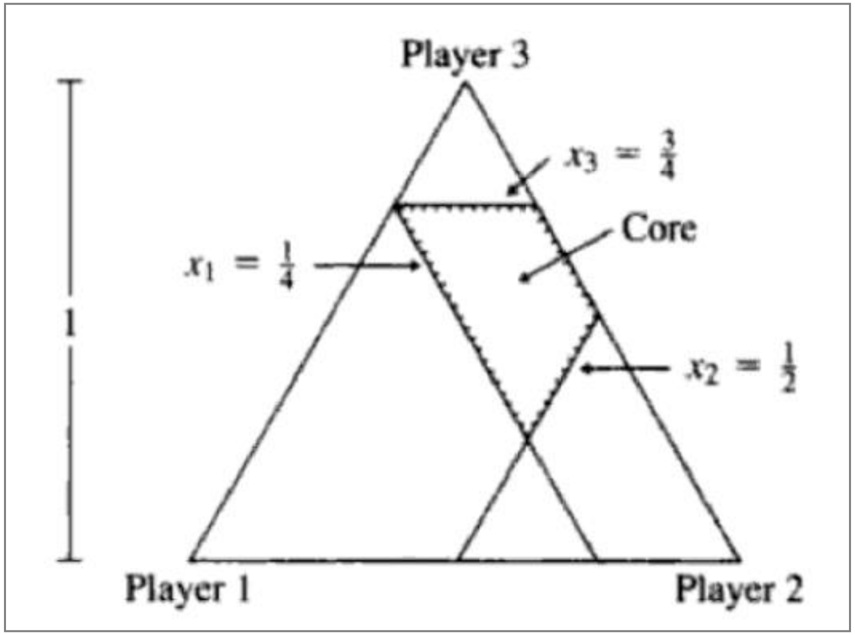
$$v(123) = 1$$

$$x_1 + x_2 \geqslant v(12) = \tfrac{1}{4}$$
$$x_1 + x_3 \geqslant v(13) = \tfrac{1}{2}$$
$$x_2 + x_3 \geqslant v(23) = \tfrac{3}{4}$$

$x_1 + x_2 \geqslant \frac{1}{4}$ if and only if $x_3 \leqslant \frac{3}{4}$

$x_1 + x_3 \geqslant \frac{1}{2}$ if and only if $x_2 \leqslant \frac{1}{2}$

$x_2 + x_3 \geqslant \frac{3}{4}$ if and only if $x_1 \leqslant \frac{1}{4}$



Elements of Game Theory (II)

The Shapley value

- The core offers a set of solutions for a game
  - Some games do not have a core
  - There is no means of evaluating the "fairness" of the imputations in the core
- The basic idea of the Shapley value:
  - Each agent must receive a payoff in accordance with its marginal contribution to the possible coalitions
- For *n* agents, there are *n*! orderings in which an agent can join other agents
  - The Shapley value represents the average for all possible orderings

Example 1

- Let's consider a two-agent game and the following characteristic form: $v(\{\ \}) = 0$, $v(\{1\}) = 1$, $v(\{2\}) = 3$, $v(\{1, 2\}) = 6$
- There are 2! possible permutations: (1, 2) and (2, 1)
- The Shapley values:

$$\begin{aligned}
\phi(1) &= \frac{1}{2} \cdot (v(1) - v() + v(21) - v(2)) \\
&= \frac{1}{2} \cdot (1 - 0 + 6 - 3) = 2 \\
\phi(2) &= \frac{1}{2} \cdot (v(12) - v(1) + v(2) - v()) \\
&= \frac{1}{2} \cdot (6 - 1 + 3 - 0) = 4
\end{aligned}$$

Example 2

$$v(A) = v(B) = v(C) = 0$$

$$v(AB) = 2 \qquad v(AC) = 4 \qquad v(BC) = 6$$

$$v(ABC) = 7,$$

Value added by

Order	A	B	C
ABC	0	2	5
ACB	0	3	4
BAC	2	0	5
$\boxed{\text{BCA}}$	1	0	6
CAB	4	3	0
CBA	1	6	0
	8	14	20

B: $v(B) - v(\phi) = 0 - 0 = 0$

C: $v(BC) - v(B) = 6 - 0 = 6$

A: $v(ABC) - v(BC) = 7 - 6 = 1.$

$$\varphi = \tfrac{1}{6}(8, 14, 20) = (1\tfrac{1}{3}, 2\tfrac{1}{3}, 3\tfrac{1}{3}).$$

The Shapley value: definition

Definition 4.4 (Shapley Value). *Let $B(\pi, i)$ be the set of agents in the agent ordering π which appear before agent i. The* Shapley value *for agent i given A agents is given by*

$$\phi(A, i) = \frac{1}{A!} \sum_{\pi \in \Pi_A} v(B(\pi, i) \cup i) - v(B(\pi, i)),$$

where Π_A is the set of all possible orderings of the set A. Another way to express the same formula is

$$\phi(A, i) = \sum_{S \subseteq A} \frac{(|A| - |S|)!\,(|S| - i)!}{|A|!} [v(S) - v(S - \{i\})].$$

Example 3

- The same expression can be rewritten with a different notation:
  - $|A| = n$
  - $|S| = s$

$$\varphi_i = \frac{1}{n!} \sum_{i \in S} (s-1)!(n-s)! \, [v(S) - v(S-i)] \qquad (s = \text{the size of } S)$$

Example 3

$$v(A) = v(B) = v(C) = v(D) = 0$$
$$v(AB) = 50 \qquad v(CD) = 70 \qquad v(AC) = 30$$
$$v(BD) = 90 \qquad v(AD) = 30 \qquad v(BC) = 90$$
$$v(ABC) = v(ABD) = v(ACD) = v(BCD) = v(ABCD) = 120.$$

Coalition S	$(s-1)!(n-s)!$	$v(S) - v(S-i)$	Product
A	$1 \times 6 = 6$	$0 - 0 = 0$	0
AB	$1 \times 2 = 2$	$50 - 0 = 50$	100
AC	$1 \times 2 = 2$	$30 - 0 = 30$	60
AD	$1 \times 2 = 2$	$30 - 0 = 30$	60
ABC	$2 \times 1 = 2$	$120 - 90 = 30$	60
ABD	$2 \times 1 = 2$	$120 - 90 = 30$	60
ACD	$2 \times 1 = 2$	$120 - 70 = 50$	100
ABCD	$6 \times 1 = 6$	$120 - 120 = 0$	0
			440

$$\varphi_A = \tfrac{1}{24} 440 = 18\tfrac{1}{3}.$$

	Value added by			
Order	A	B	C	D
ABCD	0	50	70	0
ABDC	0	50	0	70
ACBD	0	90	30	0
ACDB	0	0	30	90
ADBC	0	90	0	30
ADCB	0	0	90	30
BACD	50	0	70	0
BADC	50	0	0	70
BCAD	30	0	90	0
BCDA	0	0	90	30
BDAC	30	0	0	90
BDCA	0	0	30	90
CABD	30	90	0	0
CADB	30	0	0	90
CBAD	30	90	0	0
CBDA	0	90	0	30
CDAB	50	0	0	70
CDBA	0	50	0	70
DABC	30	90	0	0
DACB	30	0	90	0
DBAC	30	90	0	0
DBCA	0	90	30	0
DCAB	50	0	70	0
DCBA	0	50	70	0
	440	920	760	760

Properties

- The Shapley value always exists, it is unique and it is always feasible (the sum of the agents' payoffs is maximum)
- It may not belong to the core, even if the core of the game is non-empty. In this case, the Shapley value is unstable:
  - For example, the game with $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$, $v(\{1, 2\}) = v(\{1, 3\}) = 1$, $v(\{2, 3\}) = 0$, $v(\{1, 2, 3\}) = 1$ has a core with only one imputation: (1, 0, 0), but its Shapley value is: (2/3, 1/6, 1/6)

Properties

- It may involve a considerable computational effort
  - It can be used for a small number of agents
  - However, there are approximate computational methods (e.g. considering a large number of random coalitions)
- In a real multiagent system, even computing the values of sub-coalitions can be a very complex task

Convex games

- A game is convex if its characteristic function is supermodular:

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T), \forall\, S \subseteq T \subseteq N \setminus \{i\}, \forall\, i \in N$$

- The motivation to join a coalition (the possible payoff) increases as the coalition grows
- Any convex game is superadditive
- The core of a convex game is always non-empty, and the Shapley value belongs to the core and is at its center of gravity

Conclusions

- A finite strategic game always has at least one pure or mixed Nash equilibrium
- An outcome is Pareto optimal if it is better or the same as another outcome from all points of view and strictly better from at least one point of view. Non-dominated outcomes are Pareto optimal
- In a Nash equilibrium, agents do not have the motivation to deviate individually. In a Pareto optimal state, the agents do not have the motivation to deviate in a coalition
- Through cooperation, agents can obtain a higher payoff. The issue is the division of the additional payoff obtained
- The core of an *n*-player game is the set of non-dominated imputations. The core is stable
- The basic idea of the Shapley value is that each agent should receive a payoff corresponding to its marginal contribution to the possible coalitions