$I = \frac{\pi}{8}\ln 2$

(11) $I = \int_0^{\infty} \frac{\ln x\, dx}{x^2 + 2ax\cos\lambda + a^2} = \int_0^{\infty} \frac{\ln x\, dx}{(x + a\cos\lambda)^2 + a^2\sin^2\lambda} =$

$= \int_0^{\infty} \frac{\ln x\, dx}{(x + ae^{i\lambda})(x + ae^{-i\lambda})}$.

Аналогично пред. задаче замкнем контур и возведем ln в квадрат

$\sum_1 \operatorname{res} = \frac{1}{a\sin\lambda}\left(\left(\ln a + i(\pi-\lambda)\right)^2 - \left(\ln a + i(\pi+\lambda)\right)^2\right) =$

$= \frac{2\lambda}{a\sin\lambda}\left(-4\pi\lambda i - \ln a\right)$

$\oint = -4\pi i I - 4\pi^2 \int_0 \Rightarrow -4\pi i I = \frac{4\pi i\lambda}{a\sin\lambda}(-\ln a) \to$

$\Rightarrow I = \frac{\lambda \ln a}{a\sin\lambda}$

Задача 12

$I = \int_0^{\infty} \frac{x^{a-1}}{1+x^b}\, dx$

Чтобы был простой полюс замени $x^b = y$

$I = \frac{1}{b}\int_0^{\infty} \frac{y^{\frac{a}{b}-1}}{1+y}\, dy$

Замкнем контур как в пред. задаче

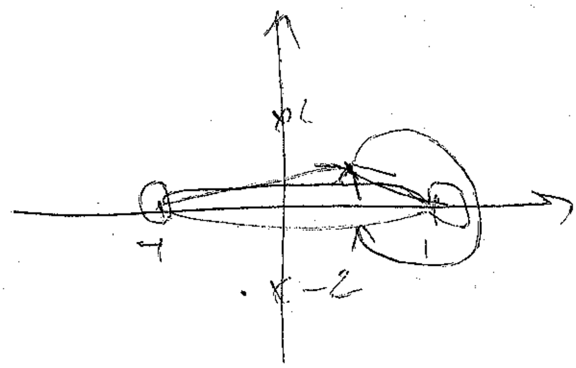
$I_- = -I e^{2\pi i\left(\frac{a}{b}-1\right)}; \quad 2\pi i \operatorname{Res}_{t=-1} = I\left(1 - e^{2\pi i\left(\frac{a}{b}-1\right)}\right)$

$\operatorname{Res}_{t=-1} = +\frac{1}{b} e^{\frac{a}{b}} \Rightarrow I = \frac{\pi}{b\sin\frac{\pi a}{b}}$

$$= -\frac{\pi}{\sin\pi\alpha}\left(-\left(\frac{\alpha^2}{2} - \frac{5\alpha}{2} + 4\right) + 2^{-\alpha+2}\right) = \frac{\pi}{\sin\pi\alpha}\left(\frac{\alpha^2}{2} - \frac{5\alpha}{2} + 4 - 2^{-\alpha+2}\right)$$

8) $$I(\alpha) = \int_{-1}^{1} \frac{(1-x)^{\alpha}(1+x)^{1-\alpha}}{x^2+1}\,dx$$

$$g(x-i0) = g(x+i0)\, e^{i\Delta \arg g(z)}$$



$$\Delta \arg g(z) = (1-\alpha)\cdot(-2\pi) + 0 = -2\pi + 2\pi\alpha$$

$$g(x-i0) = g(x+i0)\, e^{\cancel{-2\pi i} + 2\pi i\alpha} = e^{+2\pi i\alpha}$$

$$I_- = -I e^{2\pi i\alpha}. \quad \oint = I\left(1 - e^{2\pi i\alpha}\right) = \sum \operatorname{Res}_{z=\pm i,\ z=\infty} \cdot 2\pi i$$

Для $z=\pm i$ – точка простой полюс, т.е.

$$\operatorname{Res}_{z=-i} = (1-i)^{\alpha-1}(1+i)^{\alpha}$$

$$\operatorname{Res}_{z=i} = \frac{(1-i)^{\alpha+1}(1+i)^{-\alpha}}{2}$$

Для вычета на бесконечности:

$$g(\infty) = g(x)\cdot e^{-i\pi(1-\alpha)} \quad \text{тогда} \quad \operatorname{Res}_{z=\infty} = \left(\ldots + \left(\frac{1}{z}\right)^2 - \left(\frac{1}{z}\right)^4 \ldots\right.$$

$$\left(1 + (1-2\alpha)z \ldots\right) \cdot (-1)\, e^{\alpha-1}$$

$$I = \frac{2\pi i \sum \operatorname{Res}}{1 - e^{2\pi i}} = \left[\frac{e^{i\pi\alpha}}{2} + \frac{e^{-\frac{i\pi(\alpha+\alpha)}{2}} - e^{\frac{i\pi\alpha}{2}} - e^{-\frac{i\pi\alpha}{2}} - e^{i\pi\ldots}}{4}\right.$$

$$\cancel{= \frac{e^{i\pi\alpha}}{2}\left(1 - \cos\right.}\Big] \cdot \frac{-\pi}{\sin\pi\alpha} = -\left(1 - \cos\frac{\pi\alpha}{2} - \sin\frac{\pi\alpha}{2}\right)\frac{\pi}{\sin\pi\alpha}$$

б) $\Delta_{13} = 2\pi i \operatorname{Res}_{t=\frac{1}{x}} e^{i\Delta \arg} = -2\pi i e^{-2\pi i \alpha} \frac{(x-1)^\alpha}{x^{2\alpha+1}}$

③

$I_0(z) = \int_0^\infty t^z e^{-t} dt$; $I_1(z) = \frac{1}{z+1} \int_0^\infty z^{z+1} e^{-t} dt$

$I_2(z) = \frac{1}{(z+1)(z+2)} \int_0^\infty t^{z+2} e^{-t} dt$

$I_3(z) = \frac{1}{(z+1)(z+2)(z+3)} \int_0^\infty t^{z+3} e^{-t} dt$

Тогда при $z \to -3$ $\operatorname{Res}_{z=-3} I(z) = \frac{1}{2}$

Задача

$f(z) = \int_1^z \left(\frac{1}{w} + \frac{\alpha}{w^3}\right) \cos w \, dw = \int_1^z \frac{\cos w}{w} dw + \alpha \int_1^z \frac{\cos w}{w^3} dw$

Путем интегрирования по частям (дважды)

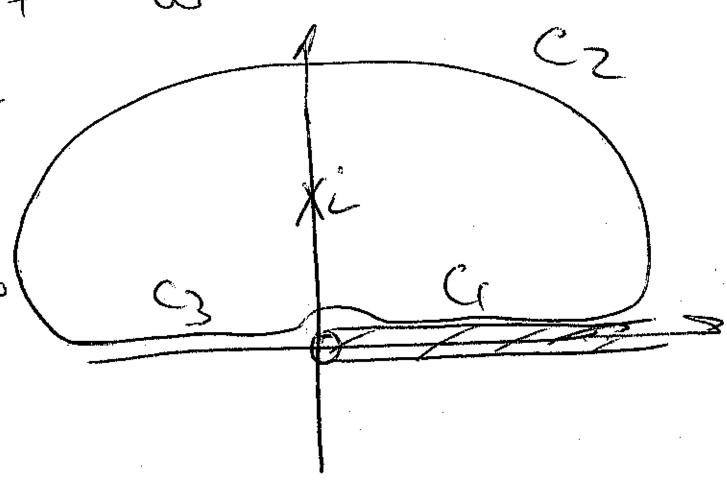
$= \int_1^z \frac{\cos w}{w} dw + \frac{\alpha}{2} \int_1^z \frac{\cos w}{w} dw - \frac{\alpha}{2} \underbrace{\left(\sin 1 - \cos 1 + \dots \right)}_{\text{const}}$.

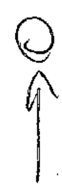
Можно предположить, что многозначность функции может быть у $\int^z \frac{\cos w}{w} dw$

то есть $1 - \frac{1}{2}\alpha = 0 \Rightarrow \alpha = 2$

Задача 6

$I = \int_0^\infty \frac{\ln x \, dx}{x^2+1} = \int_0^\infty \frac{\ln x}{(x+i)(x-i)} dx$





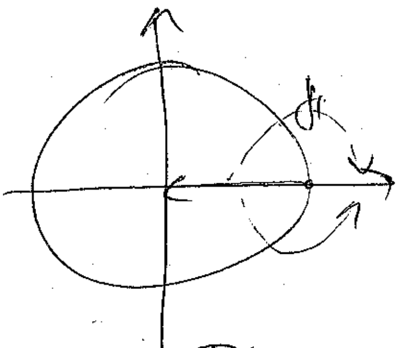
$f(z) = \ln(z^2-1)$; $\xi = \frac{1}{z}$ => $f(\xi) = \ln(\frac{1}{\xi^2} - 1) = -2\ln\xi +$
$+ \ln(1-\xi^2)$

имеет ветвлен

Задача 10

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} z^n = \sum_0^{\infty} \frac{z^n}{n+1} - \sum_0^{\infty} \frac{z^n}{n+2} =$$

$$= \frac{1}{z}\sum \frac{z^{n+1}}{n+1} - \frac{1}{z^2}\sum \frac{z^{n+2}}{n+2} = -\frac{1}{z}\ln(1-z) + \frac{1}{z^2}\ln(1-z) + \frac{1}{z}$$



т.к. $\ln f(z) = \ln|f(z)| + i\Delta_\gamma \arg f$

$$f(2)_{\gamma_1} = \frac{1}{2} - \frac{1}{2}(\ln|1| - i\pi) + \frac{1}{4}(\ln|1| - i\pi) = \frac{1}{2} + \frac{\pi i}{4}$$

$$f(2)_{\gamma_2} = \frac{1}{2} - \frac{1}{2}(\ln|1| + i\pi) + \frac{1}{4}(\ln|1| + i\pi) = \frac{1}{2} - \frac{\pi i}{4}$$

Задача 12

а) $$\frac{e^{iz}}{\cos z - 1} \approx \frac{1+iz-\frac{z^2}{2}}{1-\frac{z^2}{2}+\frac{z^4}{4!}-1} = \frac{1+iz-\frac{z^2}{2}}{\frac{z^4}{4!}-\frac{z^2}{2!}} = \frac{12}{z^2}\,\frac{1+iz-z^2}{z^2-12} =$$

$$= -\frac{1}{z^2}(1+iz-z^2)\frac{1}{1-z^2/12} = -\frac{1}{z^2}(1+iz-z^2)\left(1+\frac{z^2}{12}+\frac{z^4}{12^2}\right) =$$

$$= -\frac{1}{z^2} - \frac{i}{z} - \frac{1}{12} + \dots \quad C_{-3} = 0$$

б) $|z| \in (2\pi; 4\pi)$: Известно, что: $C_{-3} = \frac{1}{2\pi i}\oint \frac{e^{iz} z^2}{\cos z - 1} dz =$

$$= \frac{1}{2\pi i}\cdot 2\pi i \sum \operatorname{Res} g(z) = \operatorname*{Res}_{z=2\pi} + \operatorname*{Res}_{z=-2\pi}$$

Задача 9

$f(z) = z^a(z-1)^b$

$N(1;1)$: $f(z) = z(z-1)$ $z=0$

~~$z=1$~~ $z=1$

$z=\infty$, т.к. $\frac{1}{t} \to 0$ $\frac{1}{t-1}$

$N(1;1/2)$: $f(z) = z\sqrt{z-1}$ $z=1$

$z=\infty$

$N(1/2;1/3) = \sqrt{z} \cdot (z-1)^{1/3}$ $z=0$

$z=1$

$z=\infty$.

$N(2/3, 1/3) = z^{2/3}(z-1)^{1/3}$ $z=0$ $z=1$ $z=\infty$

Задача 8

$f(z) = \ln z$ $z=0$

$z=\infty$

$f(z) = \ln\left(\frac{z-1}{z+1}\right)$ $z\to\infty$ $\cancel{\ln 1} \to 0$

$f(z) = \ln(z^2-1) = \ln\left(\frac{1}{\delta^2}-1\right) = -i\pi - \infty$

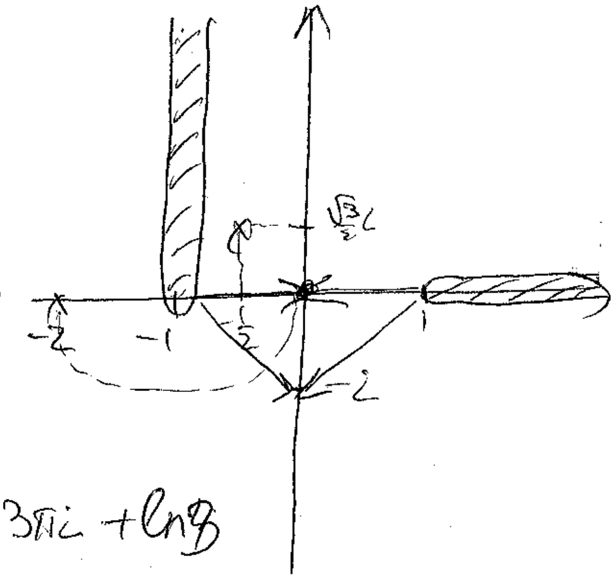
$$\underset{x\to a}{F(x)} = \cancel{\frac{|x|^{M+1}(1-x)^{1-M}}{|F(x_0)|}F(x_0)}\, \cancel{e^{i\cdot\pi M}}\, e^{-i\pi(-1-M)} = \cancel{x^{M+1+1-M}}\, e^{+i\pi(M-1)}$$

Задача 5

$$\varphi(z) = \ln(1-z)^2 = \ln(1-z)(1+z)$$

$$\varphi(0) = -2\pi i$$

$$\varphi(z) = \varphi(z_0) + \ln\left|\frac{\varphi(z)}{\varphi(z_0)}\right| + i\Delta\arg\varphi$$



для точки „-2"

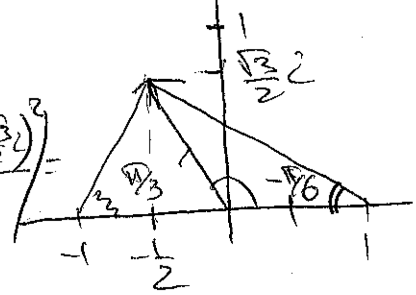
$$\Delta\arg\varphi = 0 - \pi = -\pi$$

$$\varphi(-2) = -2\pi i + \ln\left|\frac{1-4}{1}\right| - i\pi = -3\pi i + \ln 3$$

$$\varphi(-1) = -2\pi i + \ln\left|\frac{1-(i)^2}{1}\right| = -2\pi i + \ln 2$$

$$\Delta\arg\varphi = \frac{\pi}{6}$$

$$\varphi\left(\frac{-1+\sqrt{3}i}{2}\right) = -2\pi i + \frac{\pi i}{6} + \ln\left|\frac{1-\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^2}{1}\right| =$$

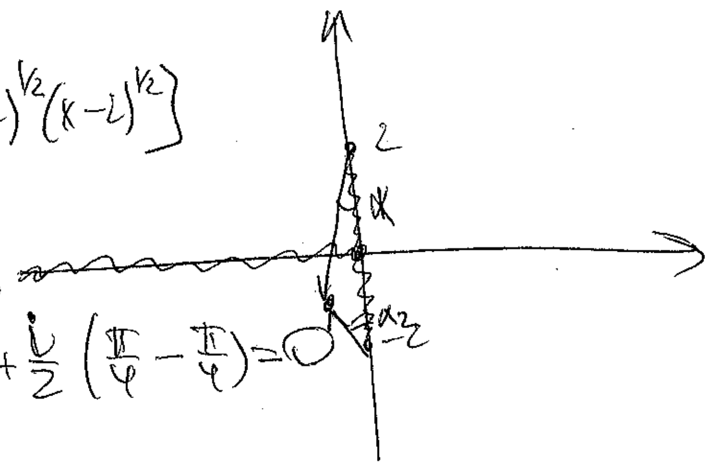


$$= -\frac{11\pi i}{6} + \frac{\ln 3}{2}$$

Задача 6

$$F(x) = \ln\left[(1+x^2)^{1/2}\right] = \ln\left[(x+i)^{1/2}(x-i)^{1/2}\right]$$

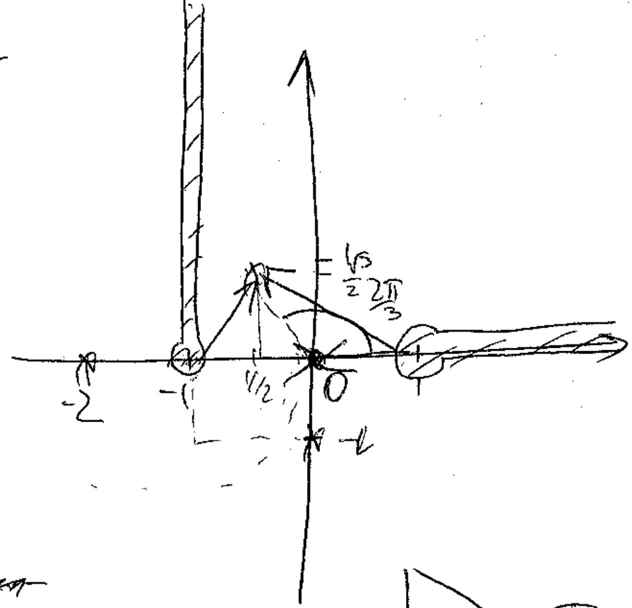
т.к. $F(1) = \frac{1}{2}\ln(2)$



$$1)\ f(i) = \cancel{\ln\left|\frac{\sqrt{1+i^2}}{\sqrt{2}}\right|} + \frac{1}{2}\ln 2 + \frac{i}{2}\left(\frac{\pi}{4} - \frac{\pi}{4}\right) = 0$$

$\varphi(i) = 0 + i\left(-\frac{3\pi}{2}\right) = -\frac{3\pi}{2}i$

Задача 5



$\varphi(z) = \ln(1 - z^2)$

$\varphi(0) = -2\pi i$

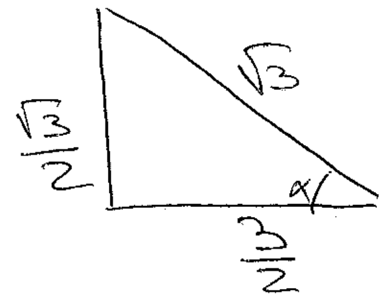
$1 - z^2 = (1-z)(1+z) \Rightarrow$

$\Rightarrow \arg(1-z^2) = \arg(1-z) + \arg(1+z)$

для $\varphi(-2)$: $\arg(1-z^2) = -\pi + 0 = -\pi$

$\varphi(-i)$: $\arg(1-z^2) = 0$

$\varphi\left(\frac{-1+\sqrt{3}i}{2}\right) = +\frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$



$\varphi(-2) = -2\pi i + \ln\left|\frac{-2}{1}\right| + i\cdot(-\pi) = -3\pi i + \ln 2$

$\varphi(-i) = -2\pi i + \ln\left|\frac{-i}{1}\right| + 0 = -2\pi i$ $\quad \ln\left|\frac{-i}{1}\right| \to 0$

$\varphi\left(-\frac{1+\sqrt{3}i}{2}\right) = -2\pi i + \ln\left|\frac{-1+\sqrt{3}i}{2}\right| + i\frac{\pi}{6} = 1 - \frac{11\pi i}{6}$ $\quad \ln\left|\frac{-1+\sqrt{3}i}{2}\right| \to 1$

Задача 6