Задание 5 ①

$$|f| = e^{r^2\cos 2\varphi} = e^{r^2(\cos^2\varphi - \sin^2\varphi)} = e^{x^2-y^2}$$

$z = x+iy$

$$\left.\begin{aligned} z^2 &= (x^2-y^2)+2ixy \\ z^{*2} &= (x^2-y^2)-2ixy \end{aligned}\right\} \Rightarrow |f|$$

$$f\cdot f^* = e^{z^2} e^{z^{*2}} = e^{z^{*2}} e^{z^2} \Rightarrow f = e^{z^2}$$

$\operatorname{Arg} f = xy$

$f = |f|\, e^{i\arg f}$

$\ln f = \ln|f| + i\arg f$. Пусть $g = \ln f$

$\operatorname{Im} g = xy$

$u = xy$

$$\frac{\partial u}{\partial x} = y = \frac{\partial v}{\partial y} \Rightarrow v = \frac{y^2}{2} + \varphi(x)$$

$v = y^2 + \varphi(x)$

$$\frac{\partial v}{\partial x} = \varphi'(x) = -\frac{\partial u}{\partial y} = -x \Rightarrow$$

$$\Rightarrow \varphi'(x) = -x \Rightarrow \varphi(x) = -\frac{x^2}{2}$$

7.0.

$u = xy$

$v = \frac{y^2}{2} - \frac{x^2}{2}$

$W = xy + i\left(\frac{y^2}{2} - \frac{x^2}{2}\right) = \cancel{xy + i\frac{y^2}{2} + \frac{i^3 x^2}{2}}$

$= xy + i\left(\frac{y^2}{2} - \frac{x^2}{2}\right) \Rightarrow f = e^{\frac{z^2}{2}}$

$z = x + iy$

$|z| = \sqrt{x^2 + y^2}$

$\cancel{z^2 = (x - iy)^2 = x^2 -}$

Задание 6

$f(x;y) = u(x;y) + i\,v(x;y)$

$u(x,y) = \varphi(x^2 - y^2)$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\qquad \frac{\partial u}{\partial x} = 2x\,\varphi_t' = \frac{\partial v}{\partial y} \Rightarrow$

$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$. $\qquad \Rightarrow \boxed{v = 2xy\varphi_t' + \varphi(x)}$

$\Delta u = 0$ $\qquad \frac{\partial u^2}{\partial x^2} = \frac{\partial v^2}{\partial y}$ $\qquad \frac{\partial v}{\partial x} = 2y\varphi_t' + \varphi'(x) = 2y\varphi_t'$

$\Delta v = 0$

$2\varphi_t' + \varphi'' \cdot 2x = 2x\varphi_t' + \varphi_t'' \cdot 2xy$.

Задача 8

а) $\int \frac{y\,dx - x\,dy}{x^2+y^2}$ $\quad x=\cos\varphi$, $y=\sin\varphi$

$= -\int d\varphi = -2\pi$

б) $x = 2+\cos\varphi$

$y = \cancel{2}+\sin\varphi$

По ф-ле Грина

$\frac{dz}{z} = \frac{dx+i\,dy}{x+iy} = \frac{(dx+i\,dy)(x-iy)}{x^2+y^2} =$

$\int_C \frac{y\,dx - x\,dy}{x^2+y^2} = \qquad = \frac{x\,dx + i\,dy}{x^2+y^2}$

$= -\int_C \frac{dz}{z}$: $\quad z = 2+e^{i\varphi} \Rightarrow dz = i e^{i\varphi} d\varphi$

$= -\int_0^{2\pi} \frac{i e^{i\varphi}}{2+e^{i\varphi}} d\varphi = -\ln(2+e^{i\varphi})\Big|_0^{2\pi}$

Задача 9

$p(n) = \frac{1}{2\pi i} \int_C dz\, z^{-1-n} \prod_{k=1}^{\infty} \frac{1}{1-z^k}$

$p(1) = \frac{1}{2\pi i} \int_C dz\, z^{-2} \frac{1}{1-z} \cdot \frac{1}{1-z^2}$

$$\frac{1}{1-z}\cdot\frac{1}{1-z^2}\cdot\frac{1}{1-z^3}=(1+z+z^2+\dots)+(1+z^2+z^4+z^5)\dots=$$

$$=(1+z^3+z^6+\dots)=1+z+2z^2+3z^3+4\dots$$

$\oint z^{-2}$ вписываем коэфф. ряда.

$$\frac{1}{z^2}+\frac{1}{z}+2=1 \quad n=1.$$

<u>Задача 5 пр.</u>

$$\partial x^2 u=\partial x^2\varphi(x^2-y^2)=\partial x\{2x\varphi'(x^2-y^2)\}=$$

$$=4x^2\varphi''(x^2-y^2)+2\varphi'(x^2-y^2)$$

$$\partial y^2 u=-2\varphi'(x^2-y^2)'+4y^2\varphi''(x^2-y^2)\Rightarrow$$

$$\Rightarrow(x^2-y^2)\varphi''(x^2-y^2)=0$$

$$t\varphi''(t)=0\Rightarrow\varphi$$

$$F(z)=C_0+C_1z^2$$

<u>Задача 4</u>

$$\frac{1+2z^2}{z^3+z^5}: \quad \text{Ряд Лорана в окрестн. } 0:$$

$$=\frac{1}{z^3}+\frac{1}{z}-z+z^3-z^5 \text{ т.к. } \frac{1}{z^3}(1+2z^2)\frac{1}{1+z^2}$$

~~Задание 12~~

$$\cancel{f(z) = \frac{1}{z(e^{z}-1)}}$$

$$\frac{1}{z^3}\left[1+2z^2\right]\left[1-z^2+z^4+\dots\right] \; -1$$

$$\to \frac{a_{-2}}{z^2} \quad a_{-2}=0$$

$$a_{-3}=1$$

$$a_{-1}=1$$

Задание 2

$$f(z)=\frac{1}{z(e^{z}-1)}$$

$$\lim_{z\to 0}\frac{1}{e^{z}-1} \to \infty$$

$$\lim_{z\to 0}\frac{z}{e^{z}-1}=1=\frac{1}{e^{z}}$$

Задание 2

$$\cancel{S=\frac{b_1}{1-q}=\frac{\varepsilon}{1-\varepsilon}}$$

$$S=\frac{\varepsilon(1-\varepsilon^{n})}{1-\varepsilon} \qquad S'=\frac{n\varepsilon^{n+1}-(n+1)\varepsilon^{n}+1}{(1-\varepsilon)^2}=$$

$$\cancel{S'=n\varepsilon^{n-1}-(\varepsilon+1)}$$

$$\varepsilon_1 \qquad \varepsilon^{n}(n(\varepsilon-1)-n+1$$

$$=$$

Задание 10

$y(1)=0$ $\qquad z=e^{i\varphi};\ \varphi\in 0;\pi$

$y'(z)=\frac{1}{2z}$

$\int\limits_C y'(z)\,dz=\int\limits_C \frac{1}{2}e^{-i\varphi}ie^{i\varphi}d\varphi=\frac{1}{2}i\pi=\frac{\pi i}{2}$ $\qquad y(1)=\frac{\pi i}{2}$

$y(-1)-\frac{\pi i}{2}$

Задача 1

$$\lim_{z\to\pi}\frac{-\pi^2 z^2+2z\sin z}{\sin z(z-\pi)(z+\pi)}=\frac{1}{2\pi}.$$

Задача 2

1) $f(z)=\frac{\sin z}{1-\operatorname{tg} z}$ $\qquad 1-\operatorname{tg} z=0$

$z=\frac{\pi}{4}+\pi n,\ n\in\mathbb{Z}.$

~~Нокрыд:~~ $\cancel{z_1=\frac{3\pi}{4}}$

2) $f(z)=\frac{e^{c(z-a)}}{e^{za}-1}$ $\qquad e^{za}=1 \Rightarrow z=2\pi a n$

Вправи 8

$$\frac{i}{x+iy}\frac{(x-iy)}{(x-iy)} = \frac{x-iy}{x^2+y^2}$$

$$\frac{-y}{x^2+y^2} = 1 \qquad -y = x^2+y^2 \Rightarrow x^2+\left(y+\frac{1}{2}\right)^2 = 1$$

Вправа 13

$|z|<1$

$$\frac{1}{z(z-1)} = \frac{1}{z} - 1 - z - z^2 - \text{геом прогресія}$$

$|z|>1$

$$\frac{1}{z(z-1)} = \frac{1}{z-1} - \frac{1}{z} = \frac{1}{z(1-\frac{1}{z})} = \frac{1}{z}\sum_0^\infty \frac{1}{z^n} =$$

$$= \sum_0^\infty \frac{1}{z^{n+1}} - \frac{1}{z}.$$

Вправа 14

$$\frac{z}{z^2+1} = \frac{z}{(z-i)(z+i)} = \frac{z}{z-i} \cdot \frac{1}{2i+z-i} =$$

$$= \frac{1}{z-i} \cdot \frac{1}{2i} \cdot \frac{1}{1+\frac{z-i}{2i}} = \frac{1}{2i}\frac{z}{z-i} + \frac{1}{4}\sum_0^\infty (-1)^k \frac{(z-i)^k}{(2i)^k}$$

Задача 3

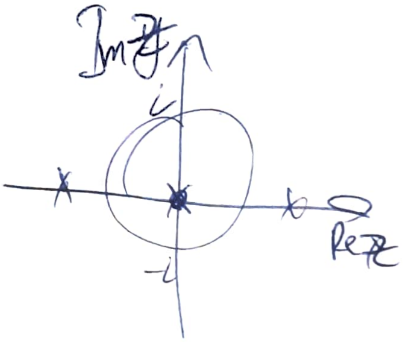
$f(z) = ze^{\frac{1}{z}} e^{-\frac{1}{z^2}}$

Рассмотрим $\lim_{z\to 0} z \sqrt[z]{e}\, e^{-1/z^2} = \lim_{z\to 0} z\cdot \exp\left(\frac{z-1}{z^2}\right) =$

$= \lim_{z\to 0} z \exp\left(\frac{1}{z^2}\right)$ ~~∞~~ – предела не сущ.

Задача 4

$\int_C \frac{ze^z}{\tan z^2}$ $\qquad \operatorname{tg} z^2 = 0 \Leftrightarrow z = \sqrt{\pi n},\ n\in\mathbb{Z}$



1 изол. особ. точка внутри контура $\Rightarrow$ $\int_C = 2\pi i \cdot \operatorname{res} f(0)$

Вблизи 0: $\quad ze^z = z + z^2 + \frac{z^3}{2} + \dots$

$\tan z^2 = z^2 + \frac{z^6}{3} + \dots$

$\frac{ze^z}{\operatorname{tg} z^2} = \frac{z + z^2 + \frac{z^3}{2}}{z^2 + \frac{z^6}{3}\dots} = \frac{1}{z} + 1 + \frac{z}{2} \dots$

$\int_C = 2\pi i$

$\int_C e^{-1/z} \sin\frac{1}{z}\, dz \qquad$ Особая точка $z = 0$.

Найдем вычет на бесконечн.: $\frac{1}{w^2} e^{-w} \sin w$

Рез. нормален: $\frac{1}{z} - 1 + \dots$

$I = +\left(\frac{1}{2\pi i}\right)^{-1}$

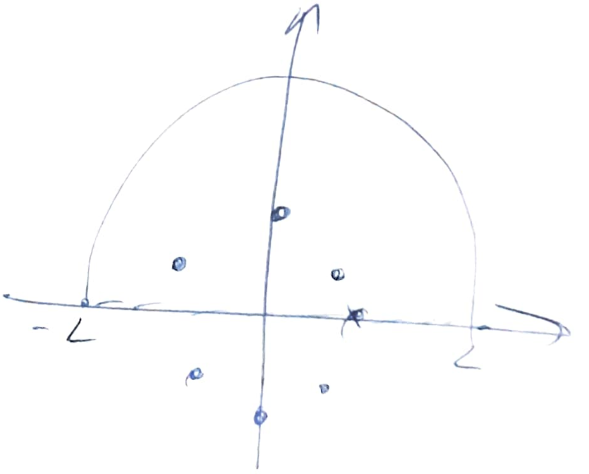
в) $\int\limits_C \frac{e^z}{z^n}$ - n пъти

$\frac{e^z}{z} = \frac{1}{z}\left(1+z+\frac{z^2}{2}+\dots\right) \Rightarrow \operatorname{Res}_{z=0} = 1$

$\frac{e^z}{z^2} = \frac{1}{z^2}(1+z)\dots \Rightarrow \operatorname{Res}_{z=0} = 1 \Rightarrow \int\limits_C = \frac{2\pi i}{(2-1)!}$

$\frac{e^z}{z^3} = \frac{1}{z^3}\left(1+z+\frac{z^2}{2}+\frac{z^3}{6}\dots\right) \Rightarrow \operatorname{Res} = \frac{1}{2} \Rightarrow$

$\Rightarrow \int\limits_C = \frac{2\pi i}{(3-1)!}$



Задача 5

1) $\int\limits_{-\infty}^{\infty} \frac{x^4}{1+x^6}\,dx \overset{*}{=}$

$1+z^6=0 \Leftrightarrow \left[\begin{array}{l} z=\pm i \\ z=\pm \sqrt{\;}i \\ z=\pm(-1)^{5/6} \end{array}\right.$

$\overset{*}{=} 2\pi i\left[\operatorname{Res}_{z_1}+\operatorname{Res}_{z_2}+\operatorname{Res}_{z_3}\right]$. Възползвай се от теорема за полюс първи порядък.

$\operatorname{Res}_{z_i} = \frac{x_i^4}{6x_i^5} = \frac{1}{6}\cdot\frac{1}{x_i} \Rightarrow I = \frac{2\pi i}{6}\left[\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right]$

$$I = \frac{\pi i}{3}\left[\frac{1}{i} + \frac{1}{\sqrt[6]{i}} + \frac{1}{(-1)^{5/6}}\right] = \frac{2\pi}{3}$$

2)

$$\int_0^{2\pi} \frac{\cos 2\theta}{\cos\theta + 2}\, d\theta = \left\{ \begin{matrix} e^{i\theta} = z \\ e^{i\theta} i\, d\theta = dz \end{matrix} \right\} =$$

$$= \left| \begin{matrix} \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) \\ \cos 2\theta = \cancel{\frac{1}{2}(z^2} \ \frac{1}{4} z^{-2} + \frac{1}{4} z^2 + \cancel{\frac{1}{2}} \cancel{- \frac{(z^2+1)^2}{4z^2}} \end{matrix} \right. +$$

$$+ \frac{1}{4} z^{-2} + \frac{1}{4} z^2 - \cancel{\frac{1}{2}} = \frac{1}{2} z^{-2} + \frac{1}{2} z^2 =$$

$$= \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right) = \frac{1}{2}\left(\frac{z^4+1}{z^2}\right) \Bigg| = \int \frac{\frac{1}{2}\left(\frac{z^4+1}{z^2}\right)}{\frac{1}{2}\left(z + \frac{1}{z}\right) + 2} =$$

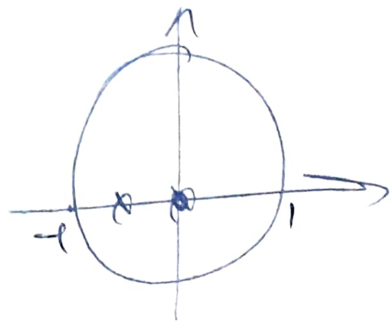
$$= \int \frac{z^4+1}{z^2\left(\frac{1}{2}\left(z + \frac{1}{z}\right) + 2\right)} = \int \frac{z^4+1}{z^3 + z + 4z^2} = \int \frac{z^4+1}{z(z^2+4z+1)} =$$

$$= \int \frac{z^4+1}{z(z+2+\sqrt{3})(z+2-\sqrt{3})} =$$

$$\operatorname{Res}_{z=0} = \frac{1}{3z^2 + 1 + 8z} = 1$$

$$\operatorname{Res}_{z=\sqrt{3}-2} = \frac{97 - 56\sqrt{3}}{2(39 - 22\sqrt{3})}$$

$\Rightarrow$



Задача 6

$$\int_C \frac{z^5\,dz}{z^6+1}$$

Из пред. задачи, видно, что 6 вычетов равны $\frac{1}{6}$. $\Rightarrow$

$\Rightarrow \sum \operatorname{Res} = 1 \Rightarrow I = 2\pi i$

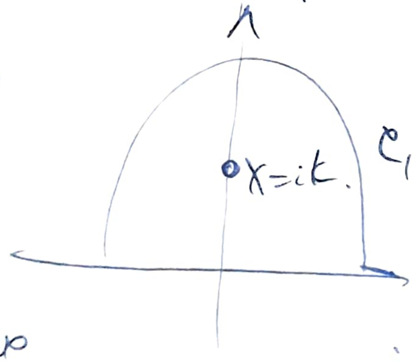
Задача

$$\int_{-\infty}^{\infty} \cancel{\frac{\sin^2 x\,dx}{x^2(x^2+1)}} \quad \cancel{\sin^2 x} \quad \frac{x \sin ax}{x^2+k^2}\,dx =$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{x \sin ax\,dx}{x^2+k^2} = \left|\sin x = \operatorname{Im} e^{ix}\right| =$$

$$= \frac{1}{2}\operatorname{Im}\int_{-\infty}^{+\infty} \frac{x e^{iax}\,dx}{x^2+k^2}$$

1) $a > 0$



$$\int_{C_1} \frac{z e^{iaz}}{z^2+k^2}\,dz \to 0$$

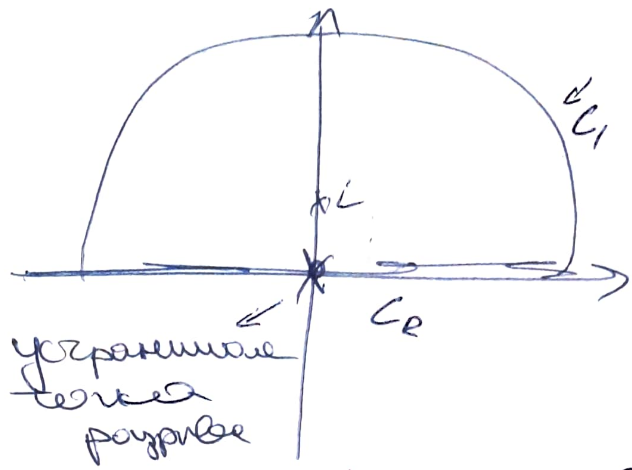
$$\int_C \frac{x e^{iax}}{x^2+k^2}\,dx = 2\pi i \frac{ik e^{iaik}}{2ik} = \pi i e^{-ak}$$

$$I = \frac{1}{2}\operatorname{Im}\int = \frac{1}{2}\pi e^{-ak}$$

В общем случае $I = \frac{\pi}{2} e^{-|a||k|} \operatorname{sign}(a)$

Задача 8

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2(x^2+1)}\,dx$$



$I_{C_1}=0$

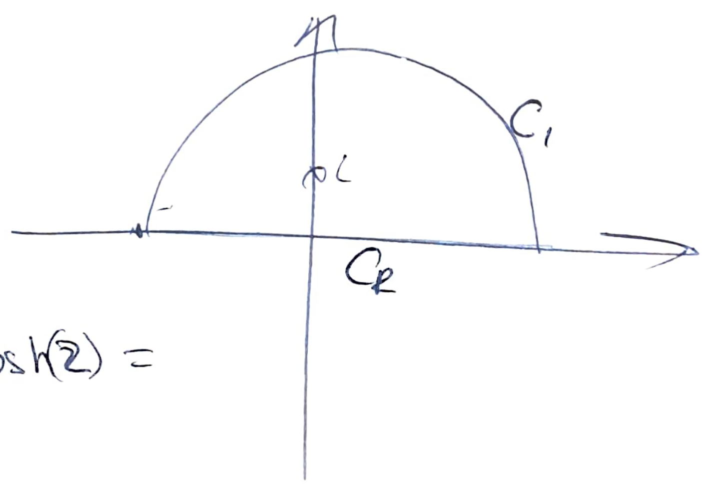
$x^2=-1 \Rightarrow x=\pm i$ - полюс $\Rightarrow$

$\Rightarrow I = \operatorname{res}_{z=i} = 2\pi i\cdot\left(-\frac{1}{2} i\right)\sinh^2(1) = -\pi\cdot\left(-\frac{1}{2}+\frac{1}{4e^2}+\frac{e^2}{4}\right) =$

$= \frac{\pi}{2} - \frac{\pi}{4e^2} \cancel{+} - \frac{\pi e^2}{4} = -\pi\sinh^2 1 = \frac{\pi}{2}\left(1+\frac{1}{e}\exp^{-2}\right)$

Задача 9

$$\int_{-\infty}^{\infty} \frac{\cos\left(x-\frac{1}{x}\right)}{1+x^2}\,dx$$



$I_{C_1}=0$

$I = \operatorname{res}_{z=i} = 2\pi i\left(-\frac{1}{2}\right) i\cosh(2) =$

$= \pi\cosh 2$

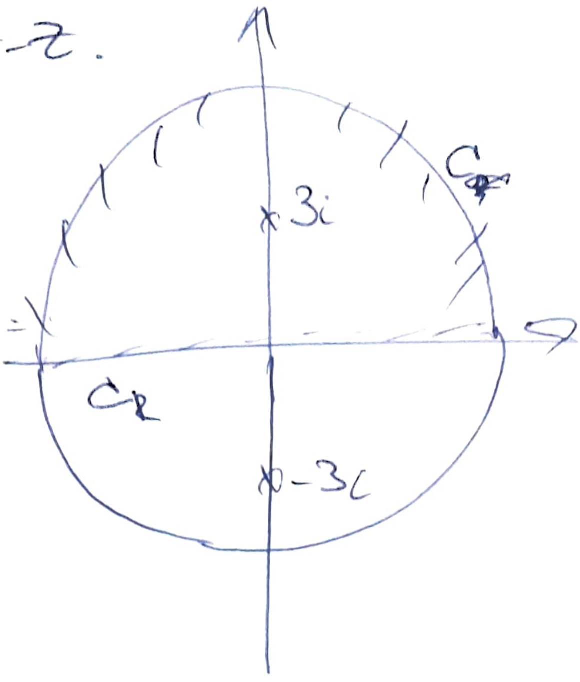
Задача 10

$\int_0^\infty \frac{x-\sin x}{x^3}\,dx = \cancel{\int_0^\infty \frac{1}{x^2} - \int\frac{\sin x}{x^3}\,dx}$

$x \to -x$

$\int \frac{-x+\sin x}{-x^3}(-dx) = -\int\frac{x-\sin x}{x^3}\,dx$

б) $\int_{-\infty}^{\infty} \frac{e^{-iz}dz}{z^2+9}$ Пусть $\xi = -z$.



$I = \cancel{\int_{C_R}}^{0} + \int_{C_R} = 2\pi i \sum \text{Res} =$

$= 2\pi i \left(-\frac{ie^3}{6e^3}\right) = \frac{\pi e^3}{3e^3}$

Задача 4

$f(z) = z^3 \cos\frac{1}{z-2}$ $\quad z = \infty$

$z \to \infty$ $\cos\frac{1}{z-2} \approx 1 - \frac{z^{-2}}{2} + \frac{z^{-4}}{24}$

$z \to \infty$ $f(z) \approx z^3 - \frac{z}{2} + \frac{1}{24z}$

$\underset{z=\infty}{\text{Res}} = -\frac{1}{24}$

Задача 5

$f(z) = \frac{1}{z^3 - z^5} = \frac{1}{z^3} + \frac{1}{z} - \frac{1}{z(z+1)} - \frac{1}{z(z-1)}$

$\sum\limits_{1, -\frac{1}{2}, \frac{1}{2}} \text{Res} = \underset{\text{ТПА}}{\text{Res}}$

Задача 13

при $z=1$ $\underset{z=1}{\mathrm{Res}} = -\sin(1)$

$$f(z) = \frac{\sin\frac{1}{z}}{1-z}$$

при $z \to \infty$. $\frac{1 - \frac{z^3}{3!}}{1-z} \Rightarrow \underset{z=\infty}{\mathrm{res}} = 0$.

$$g(z) = \exp(-\exp(\tfrac{1}{z}))$$

$z \to \infty$. $\Rightarrow g = \exp(-e) \approx \frac{1}{e} - \frac{1}{ez} + \dots$

$$\underset{z=0}{\mathrm{Res}}\, z = \underset{z=\infty}{\mathrm{Res}} \Rightarrow -\frac{1}{ez}.$$

Задача 10

$$\int \frac{x - \sin x}{x^3} = \int_{\mathbb{R}} \frac{x - \operatorname{Im} e^{ix}}{x^3} dx = \operatorname{Im} \int_{\mathbb{R}} \frac{x - e^{ix}}{x^3} dx$$

$$\int_{\mathbb{R}} \frac{x - \frac{e^{ix} - e^{-ix}}{2}}{x^3} = \int_C \frac{z - e^{iz}/2 + e^{-iz/2}}{z^3} =$$

$$= \int \frac{z}{z^3} dz - \int \frac{e^{iz/2}}{z^3} dz + \cancel{\int \frac{e^{-iz/2}}{z^3} dz} \to 0 = \frac{\pi}{2}$$

Задача 9

$$\int_{\mathbb{R}} \frac{\cos(z - \frac{1}{z})}{1+z^2} dz \quad z - \frac{1}{z} = t \Rightarrow dt = (1 + \tfrac{1}{z^2}) dz$$

$$dt = (z^2+1) dz$$