Intro to Math and Python

Lecture 1

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Today

1. Course overview
2. Linear Algebra
    - Core objects
    - Vector spaces
3. A bit of Analytic Geometry
    - Orthogonal projections
    - Hyperplanes
    - Normals

based on slides by Evgenia Korneva

About this course

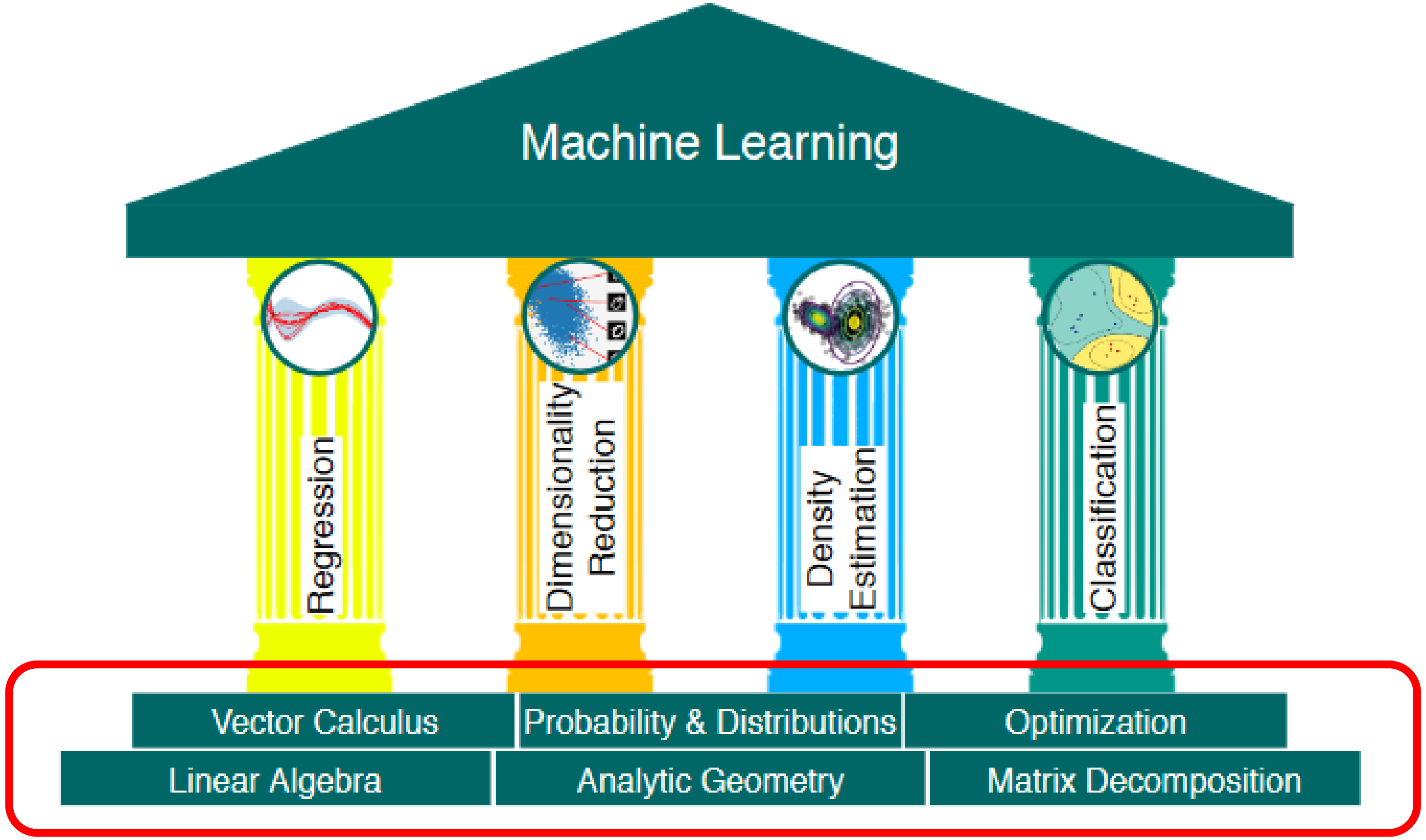

Image source: Mathematics for Machine Learning, p. 14
(https://mml-book.github.io/book/mml-book.pdf)

About this course

This week:

1. Linear algebra
2. Calculus
3. Probability theory

Prerequisites:

- basic knowledge of math;
- some Python.

About me

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- PhD researcher
- Lecturer
- Artec3D DS Team

Linear Algebra: the Basics

Linear Algebra: Core Objects

- $\alpha \in \mathbb{R}$ - a scalar *Example:* -2

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Linear Algebra: Core Objects

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- $x \in \mathbb{R}^n$ - a **vector** with n entries

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Example: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{R}^3, \quad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \in \mathbb{R}^5$

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- $A \in \mathbb{R}^{m \times n}$ - a **matrix** with m rows and n columns

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Example: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{2\times 2}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \in \mathbb{R}^{2\times 3}$

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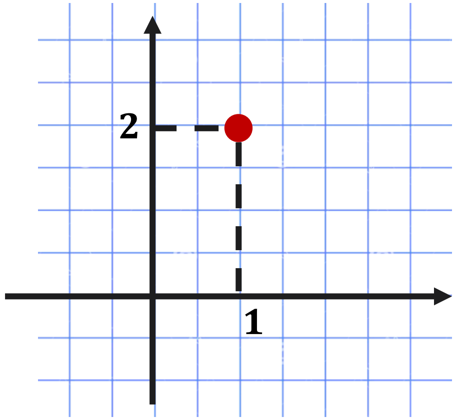

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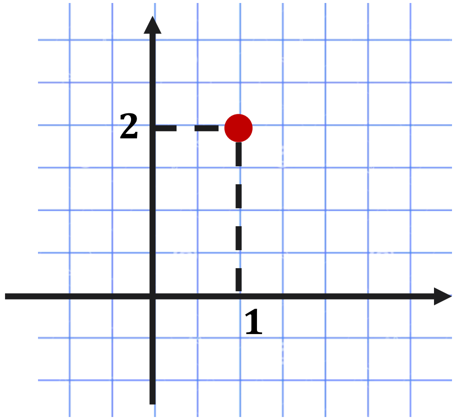
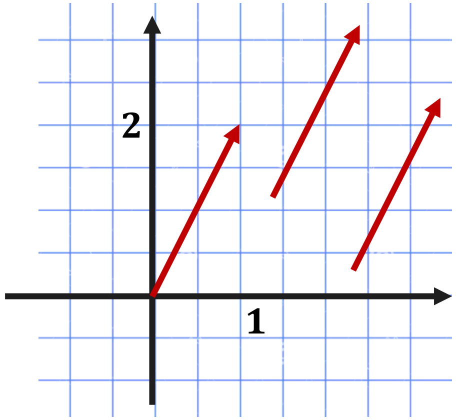


- Direction + length



Vector Spaces

Vector Space: Definition

- A real-valued **vector space** $(V, +, \cdot)$ is a set of vectors V with two operations

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that satisfy the following properties (axioms):

	Property	Meaning
1.	**Associativity** of addition	$x + (y + z) = (x + y) + z$
2.	**Commutativity** of addition	$x + y = y + x$
3.	**Identity element** of addition	$\exists 0 \in V\colon\ \forall x \in V \quad 0 + x = x$
4.	**Identity element** of scalar multiplication	$\forall x \in V \quad 1 \cdot x = x$
5.	**Inverse element** of addition	$\forall x \in V\ \exists -x \in V\colon\ x + (-x) = 0$
6.	**Compatibility** of scalar multiplication	$\alpha(\beta x) = (\alpha\beta)x$
7.	**Distributivity**	$(\alpha + \beta)x = \alpha x + \beta x$
8.		$\alpha(x + y) = \alpha x + \alpha y$

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Let’s define vector operations!

Operations with Vectors

1. Sum of two vectors:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n, \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n, \qquad x + y = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_2 \end{bmatrix} \in \mathbb{R}^n$$

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Operations with Vectors: Example

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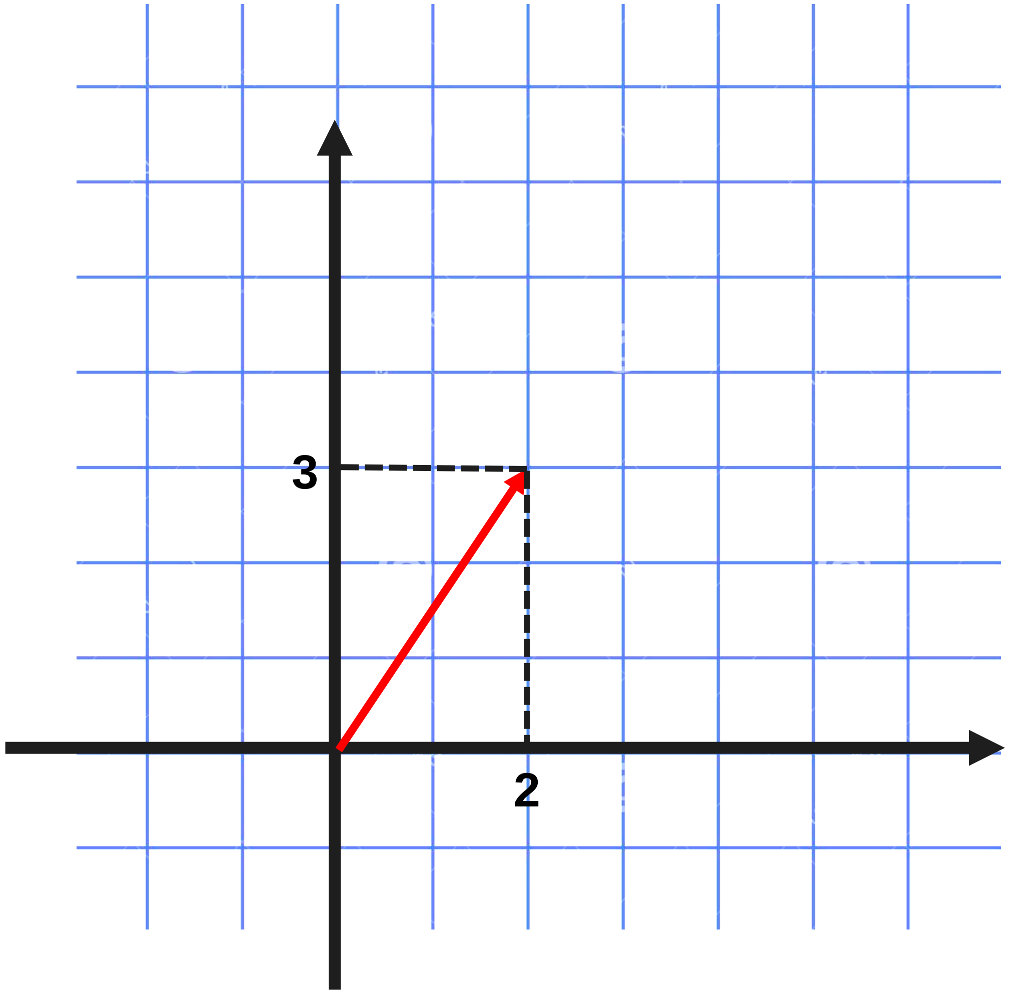
Sum:

$$x + y = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

Vector Operations: Geometrical Interpretation

Vectors: Geometrical Interpretation

$\vec{a} = [2, 3]$



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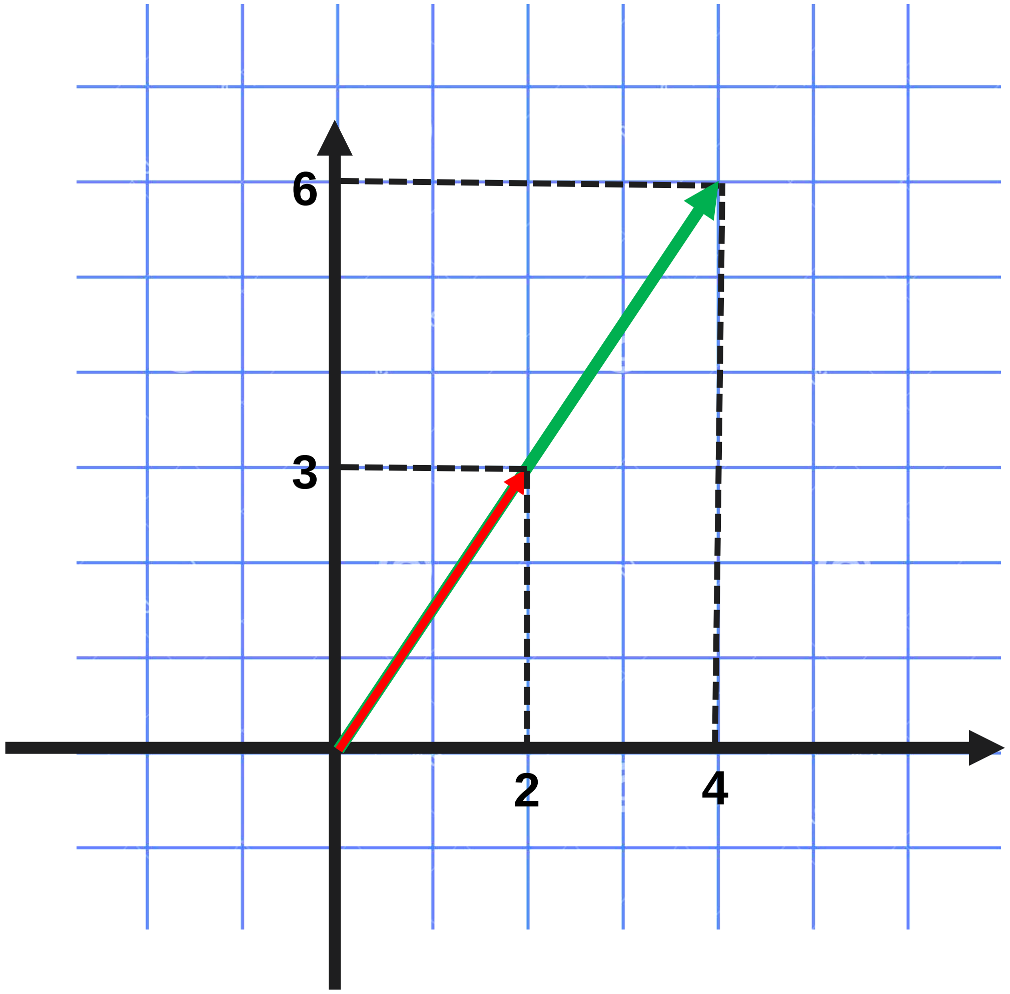
$$2\vec{a} = [4, 6]$$



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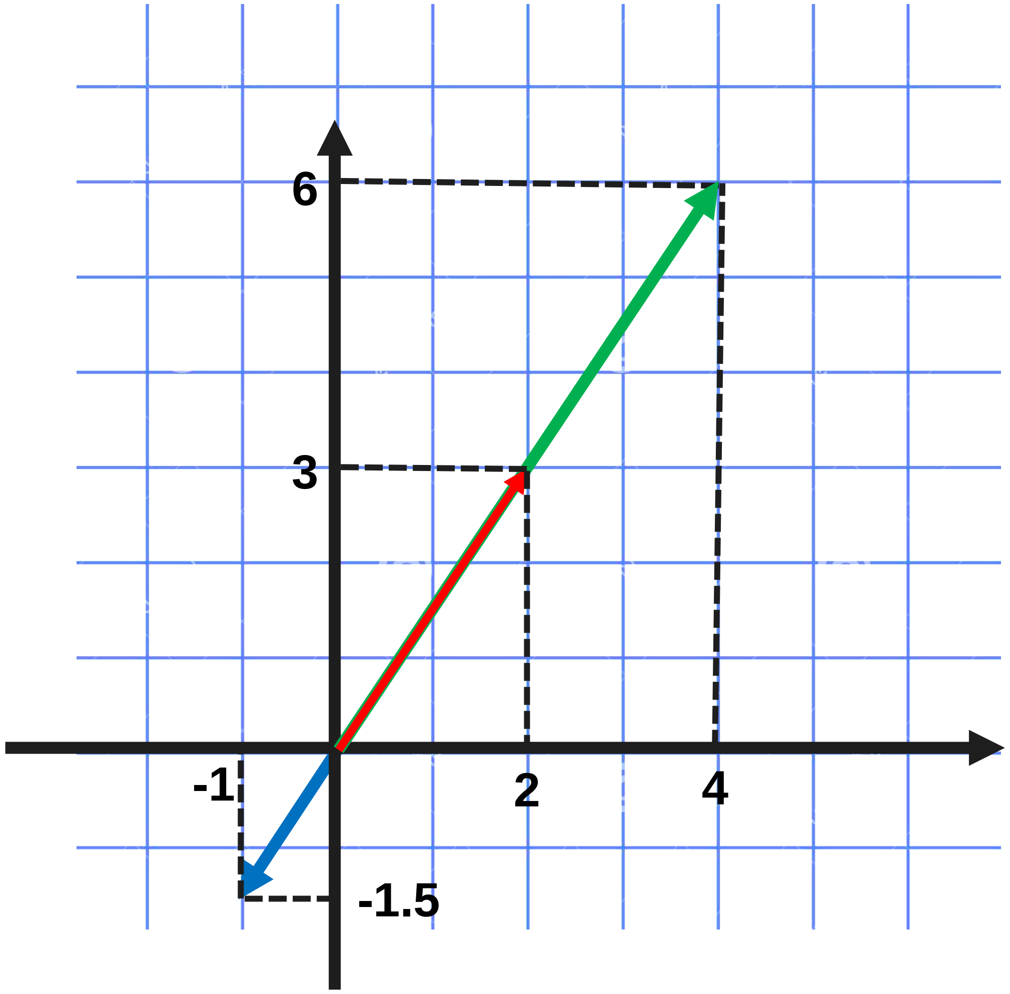

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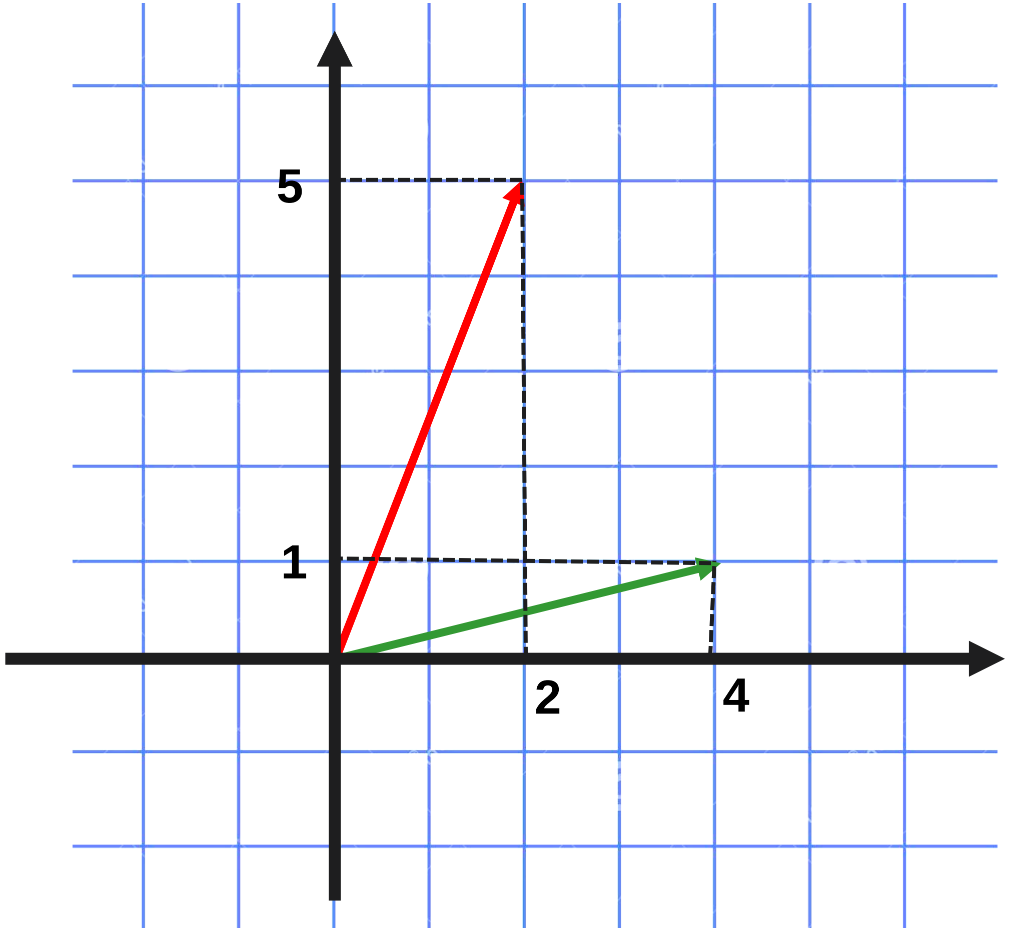

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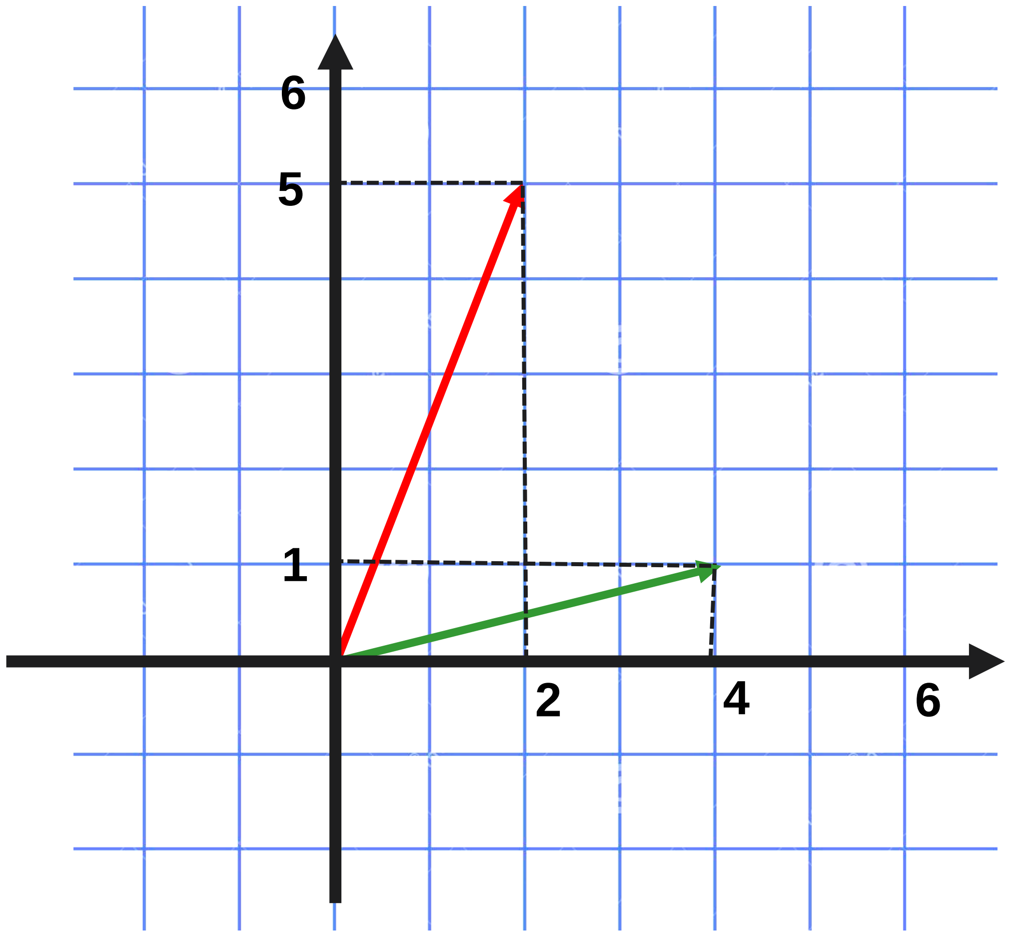

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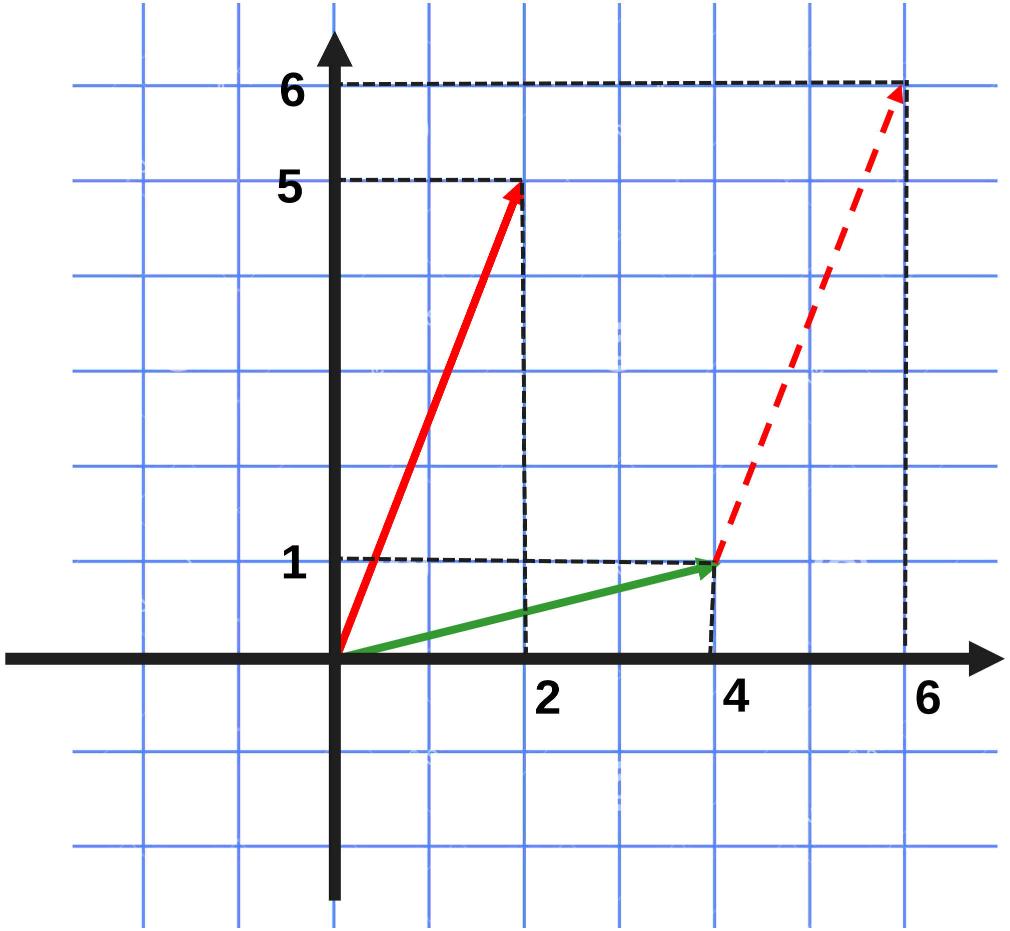

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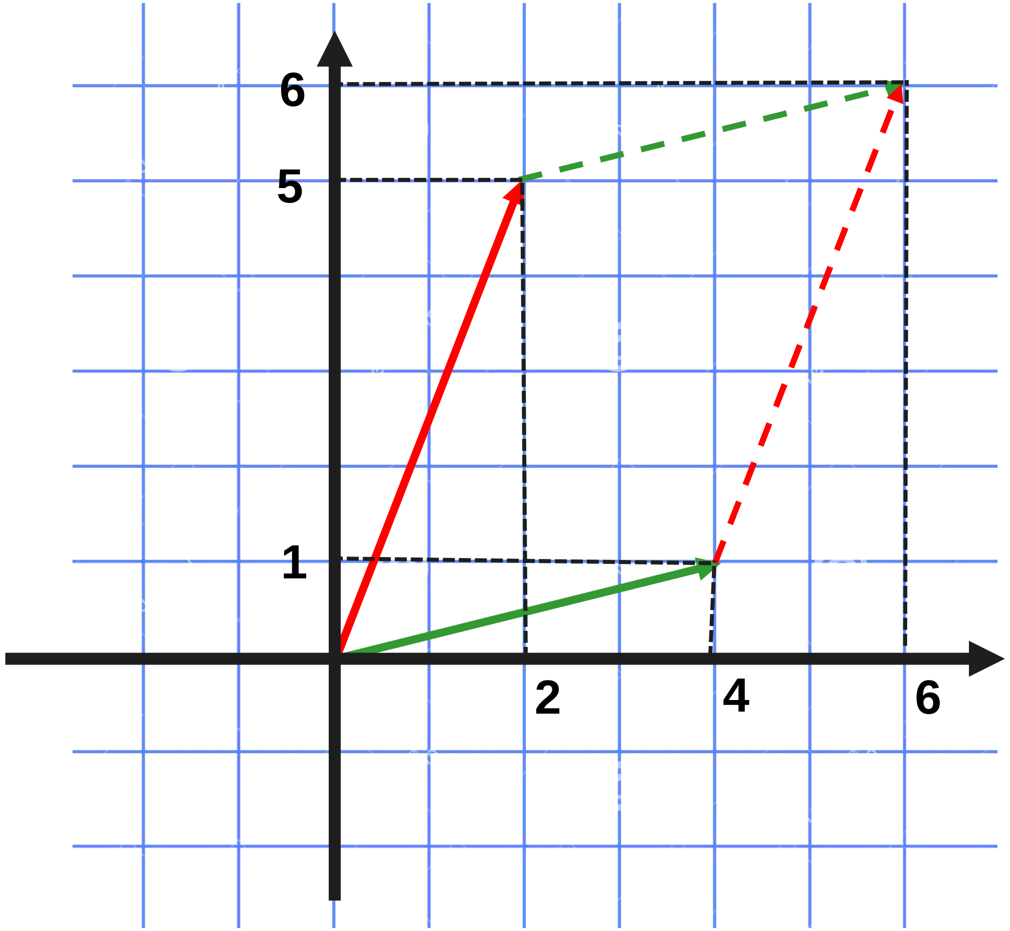

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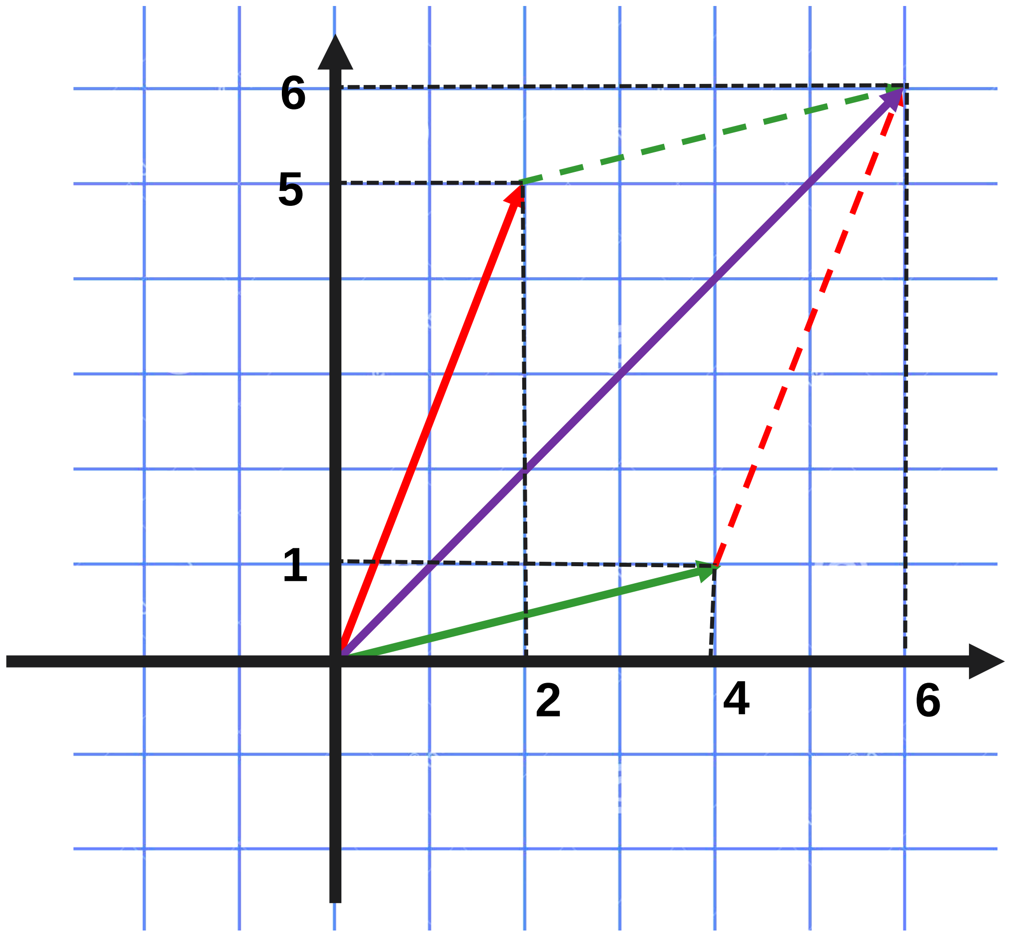

Back to Vector Spaces

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satisfy axioms (1) – (8)
(check it yourself)

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Vector Spaces

$(\mathbb{R}^n, +, \cdot), n \in \mathbb{N}$ - a vector space with operations

1. vector addition:

$$x + y = (x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$$

2. multiplication by a scalar:

$$\alpha x = \alpha(x_1, x_2, \dots, x_n) = (\alpha x_1, \alpha x_2, \dots, \alpha x_n)$$

Inner Product

Inner Product

- Inner product is a function $\langle \cdot , \cdot \rangle \colon V \times V \to \mathbb{R}$ that satisfies the following properties:
  - *Symmetric:* $\forall x, y \in V \ \ \langle x, y \rangle = \langle y, x \rangle$
  - *Positive definite:* $\forall x \in V \setminus \{0\} \ \ \langle x, x \rangle > 0$ and $\langle x, 0 \rangle = 0$.

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- Example:

$$x = [1, 2, 3, 4], \qquad y = [-1, 0, 1, 2]$$

$$(x, y) = 1 \cdot (-1) + 2 \cdot 0 + 3 \cdot 1 + 4 \cdot 2 = -1 + 0 + 3 + 8 = 10$$

Euclidian Vector Space

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- *Note: there're inner products different from dot product.*

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Norms

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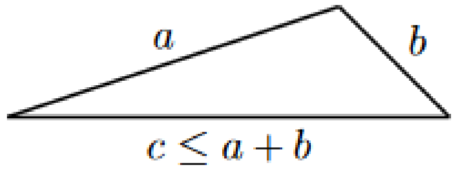
  - *Triangle inequality:* $\forall x, y \in \mathbb{V} \ \|x + y\| \leq \|x\| + \|y\|$



Examples of Norms

Manhattan Norm

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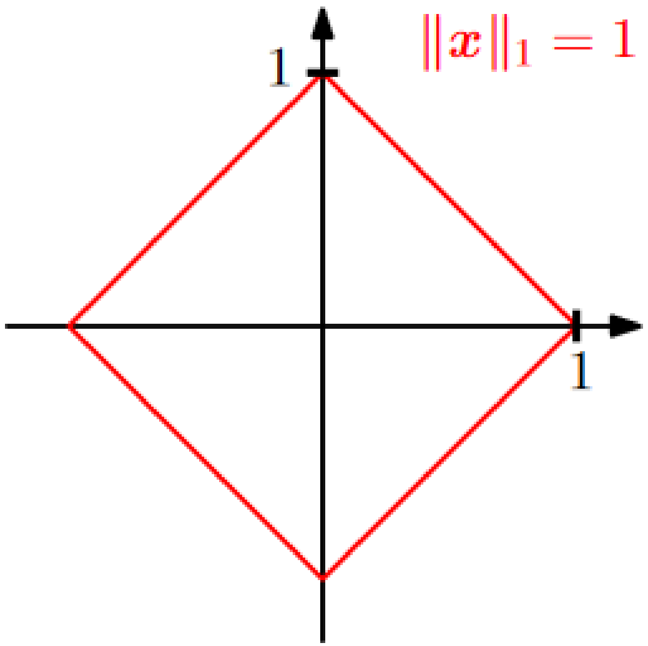

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- Examples:

$$\|[1,2,3]\|_2 = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

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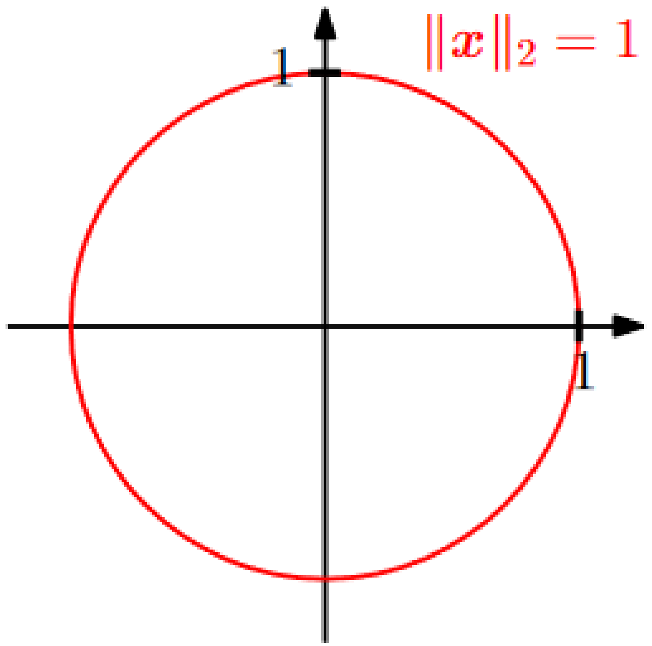

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  - ℓ_1 - Manhattan norm $\|\cdot\|_1$;
  - ℓ_2 - Euclidian norm $\|\cdot\|$ (default);
  - ℓ_∞: $\| x \|_\infty = \max_i |x_i|$

    Example: $\|[1,2,3]\|_\infty = 3$, $\|[1,0]\|_\infty = 1$, $\|[-1,0.5]\|_\infty = 1$.

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- (!) Not every norm is induced by an inner product.
  Example: Manhattan norm.

Cauchy-Schwarz Inequality

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$$|(x, y)| \leq \|x\|_2 \cdot \|y\|_2$$

Distance between Vectors

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- For dot product and Euclidian norm, we get *Euclidian distance*:

$$d(x, y) = \|x - y\|_2 = \sqrt{(x - y, x - y)} =$$

$$= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}.$$

Angles and Orthogonality

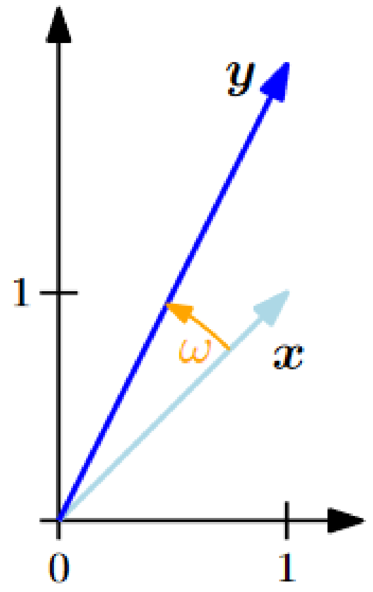
Angle between Two Vectors

- Inner product also captures the geometry of vector space by defining the angle between two vectors.

- Remember Cauchy-Schwarz inequality:

$$|(x, y)| \leq \|x\| \cdot \|y\|$$

$$-1 \leq \frac{(x, y)}{\|x\| \cdot \|y\|} \leq 1$$



ω: $\cos\omega = \frac{(x,y)}{\|x\| \cdot \|y\|}$ - angle between x and y.

Angle between Two Vectors: Example

- What is the angle ω between $x = [5, 0]$ and $y = [1, 1]$?

$$\omega = \arccos \frac{(x, y)}{\|x\| \|y\|} = \arccos \frac{5 \cdot 1 + 0 \cdot 1}{\sqrt{5^2 + 0^2} \cdot \sqrt{1^2 + 1^2}} = \arccos \frac{5}{5\sqrt{2}} = \arccos \frac{\sqrt{2}}{4} = \frac{\pi}{4}.$$

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- What is the angle ω between $x = \left[\sqrt{3}, 0,1\right]$ and $y = [1, 0, 0]$?

$$\omega = \arccos \frac{(x, y)}{\|x\| \|y\|} = \arccos \frac{\sqrt{3}}{\sqrt{3 + 1} \cdot \sqrt{1}} = \arccos \frac{\sqrt{3}}{2} = \frac{\pi}{6}.$$

Angle between Two Vectors: Example

- What is the angle ω between $x = [5, 0]$ and $y = [1, 1]$?

$$\omega = \arccos \frac{(x, y)}{\|x\|\|y\|} = \arccos \frac{5 \cdot 1 + 0 \cdot 1}{\sqrt{5^2 + 0^2} \cdot \sqrt{1^2 + 1^2}} = \arccos \frac{5}{5\sqrt{2}} = \arccos \frac{\sqrt{2}}{4} = \frac{\pi}{4}.$$

- What is the angle ω between $x = \left[\sqrt{3}, 0,1\right]$ and $y = [1, 0, 0]$?

$$\omega = \arccos \frac{(x, y)}{\|x\|\|y\|} = \arccos \frac{\sqrt{3}}{\sqrt{3+1} \cdot \sqrt{1}} = \arccos \frac{\sqrt{3}}{2} = \frac{\pi}{6}.$$

- What is the angle ω between $x = [1, 0, 0, 0, 1]$ and $y = [0, 1, 1, 0, 0]$?

$$\omega = \arccos \frac{(x, y)}{\|x\|\|y\|} = \arccos \frac{0}{\sqrt{2} \cdot \sqrt{2}} = \arccos 0 = \frac{\pi}{2}.$$

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- x and y are orthogonal if and only if $\langle x, y \rangle = 0$.

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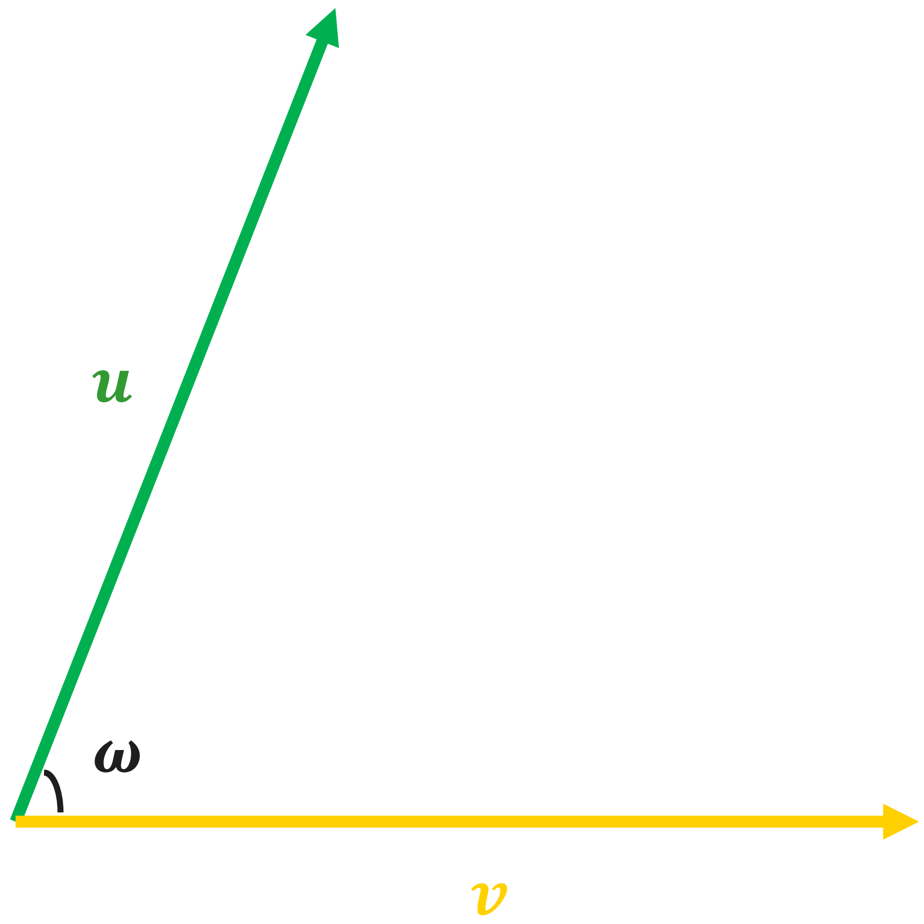
$$x = [1, 0], \qquad y = [0,1], \qquad (x, y) = 0, \qquad \|x\| = \|y\| = 1 \quad \rightarrow$$

x and y are *orthonormal*.

Orthogonal Projection

- Suppose we have two vectors u and v.



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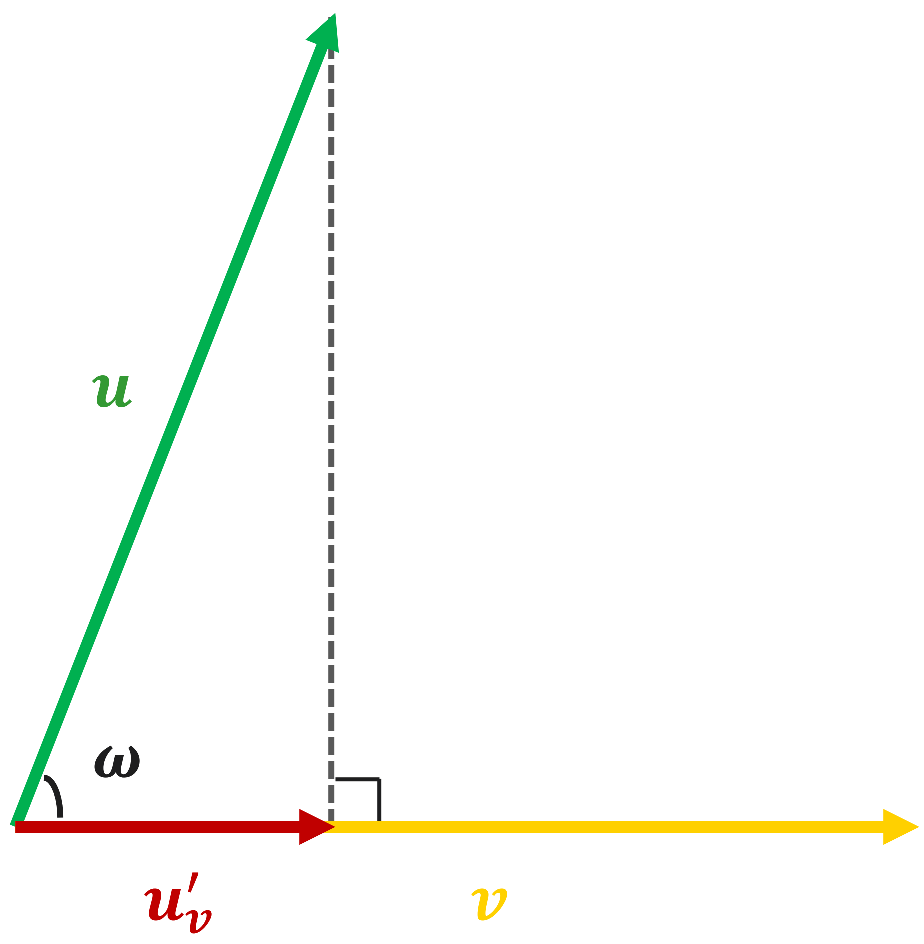

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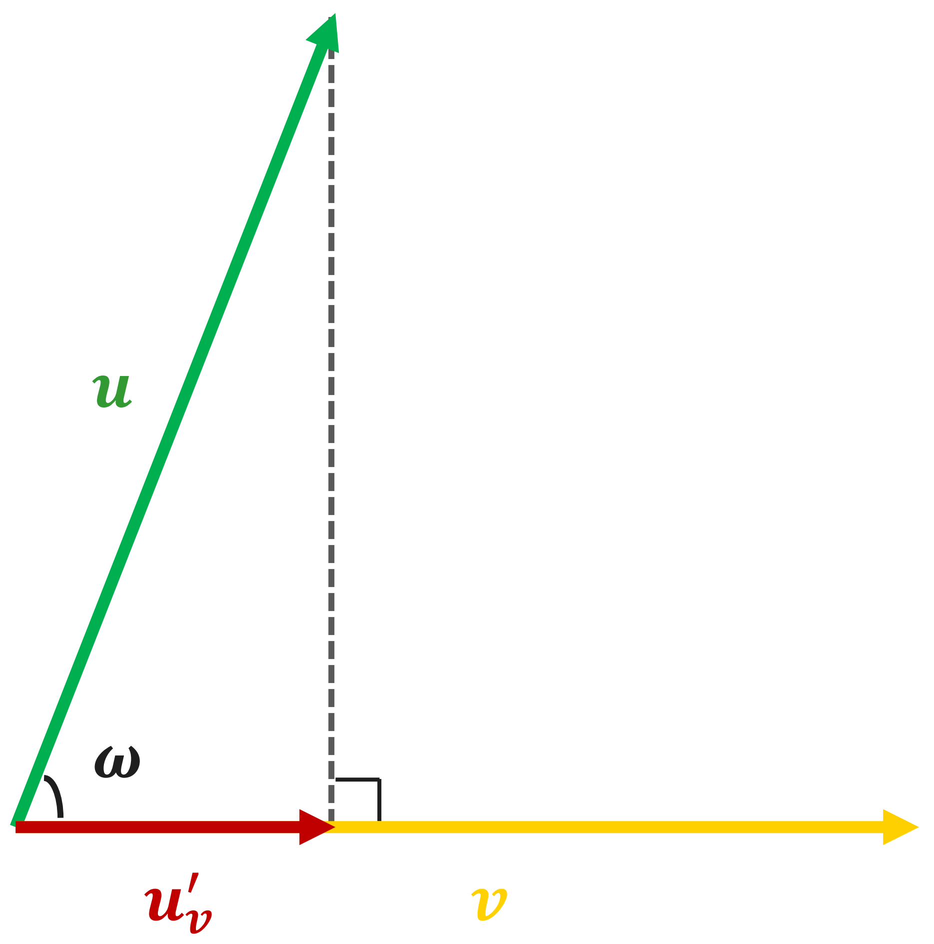

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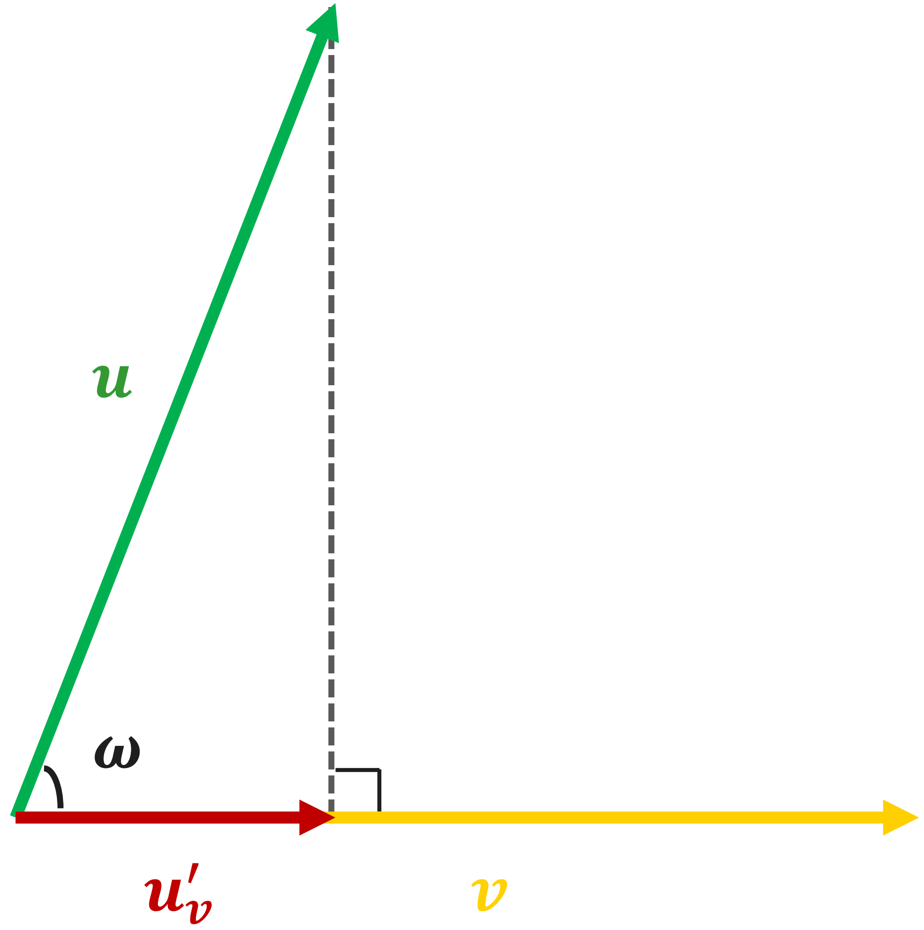
If $\mathbf{0} \leq \boldsymbol{\omega} \leq \mathbf{90}$

$$(u, v) = \|u\|\|v\| \cos \boldsymbol{\omega} = \|u\|\|v\| \frac{\|\boldsymbol{u}'_{\boldsymbol{v}}\|}{\|\boldsymbol{u}\|} =$$

$$= \|u'_v\|\|v\|$$



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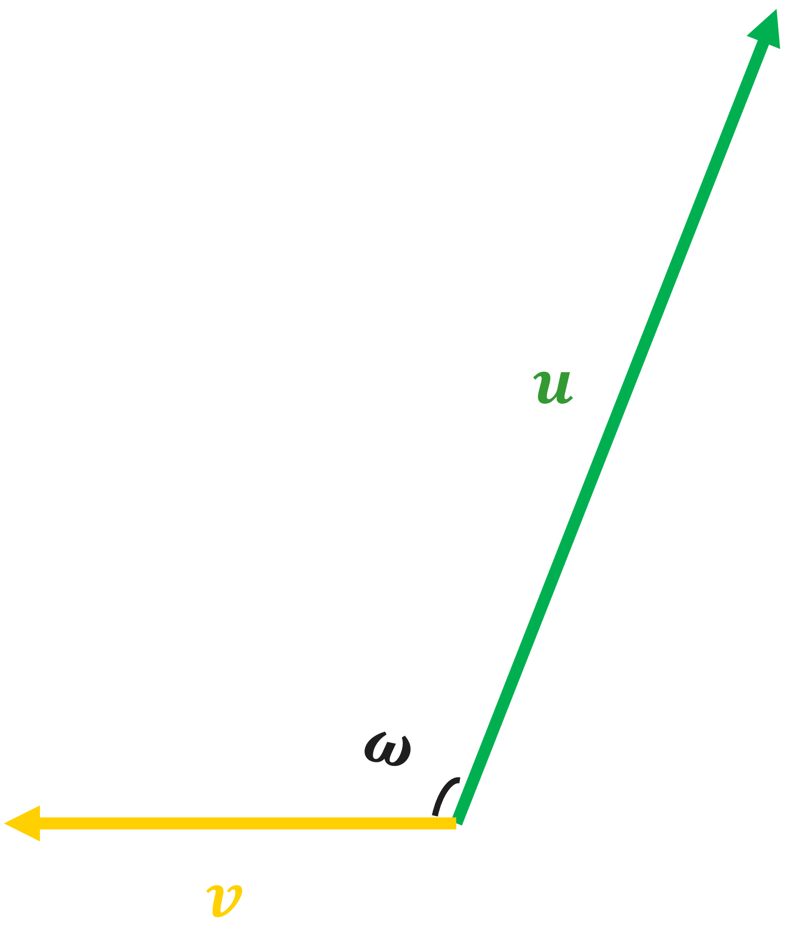

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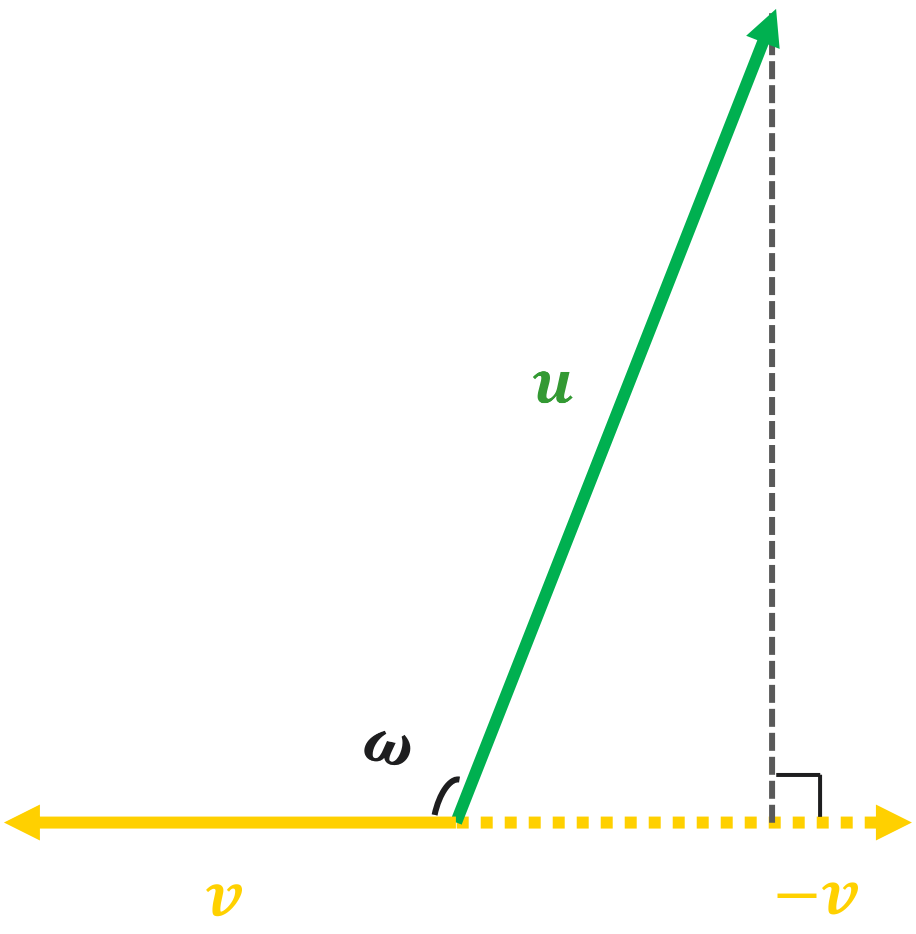

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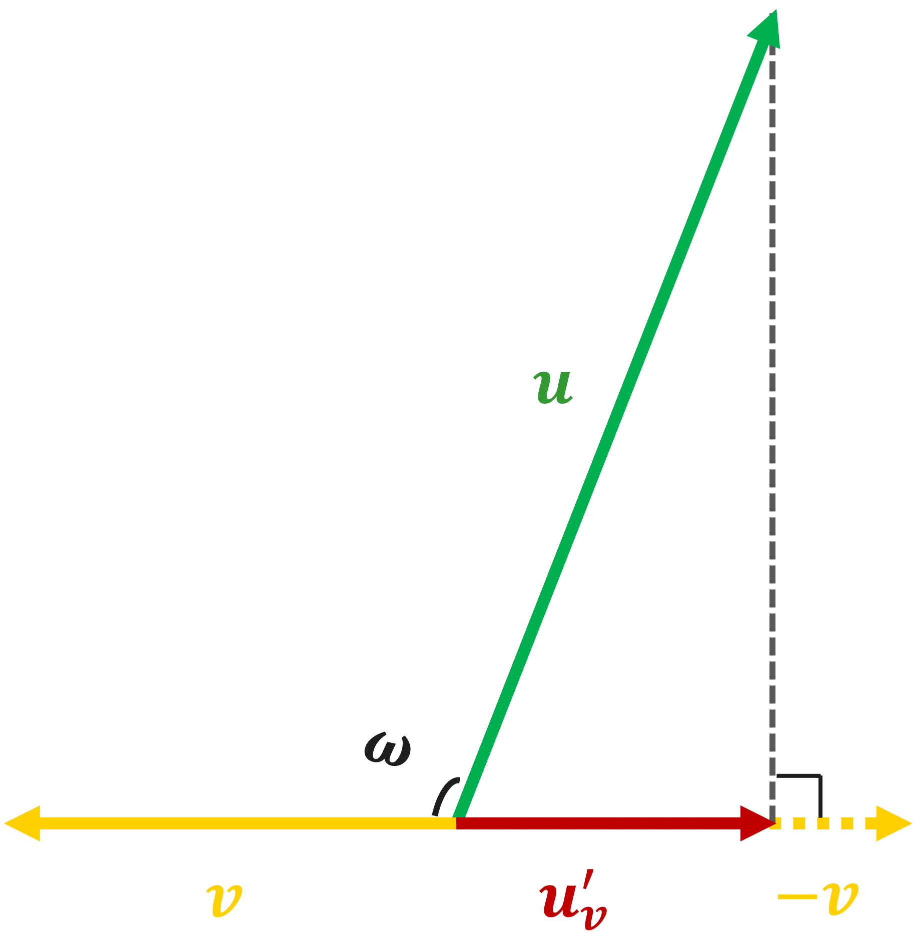

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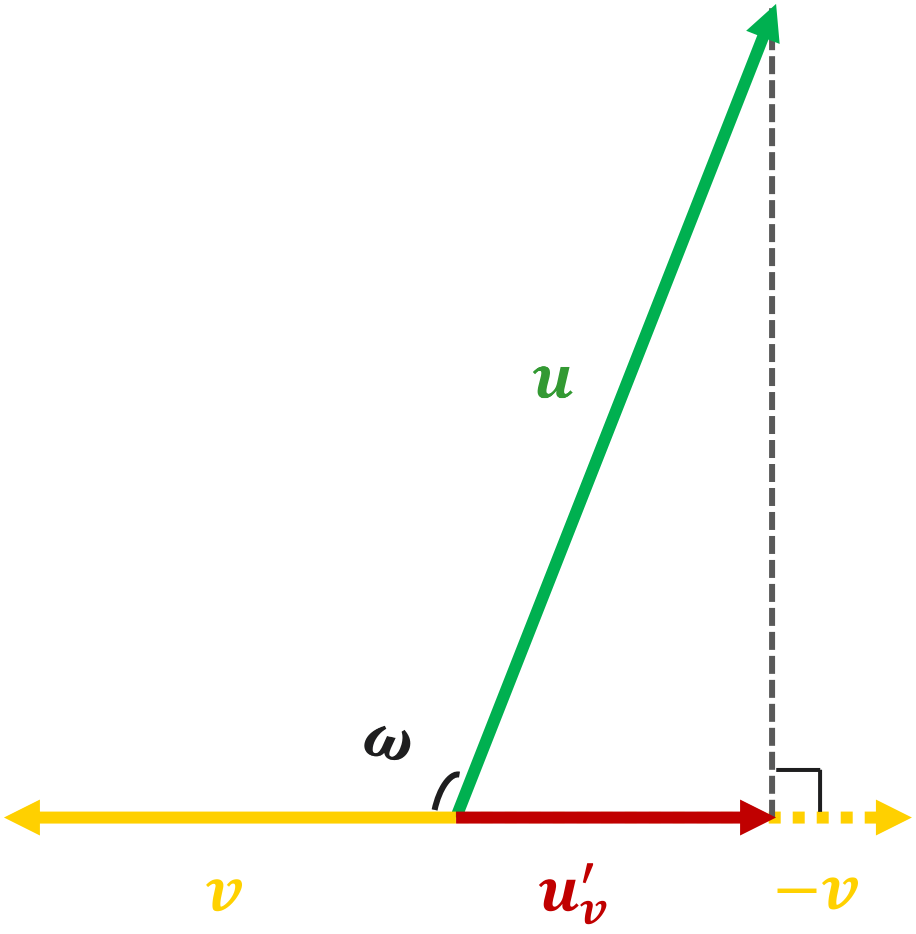
$$-(u, v) = \|u\| \|v\| \cos \omega = \|u\| \|v\| \frac{\|u_v'\|}{\|u\|} =$$

$$= \|u_v'\| \|v\|$$



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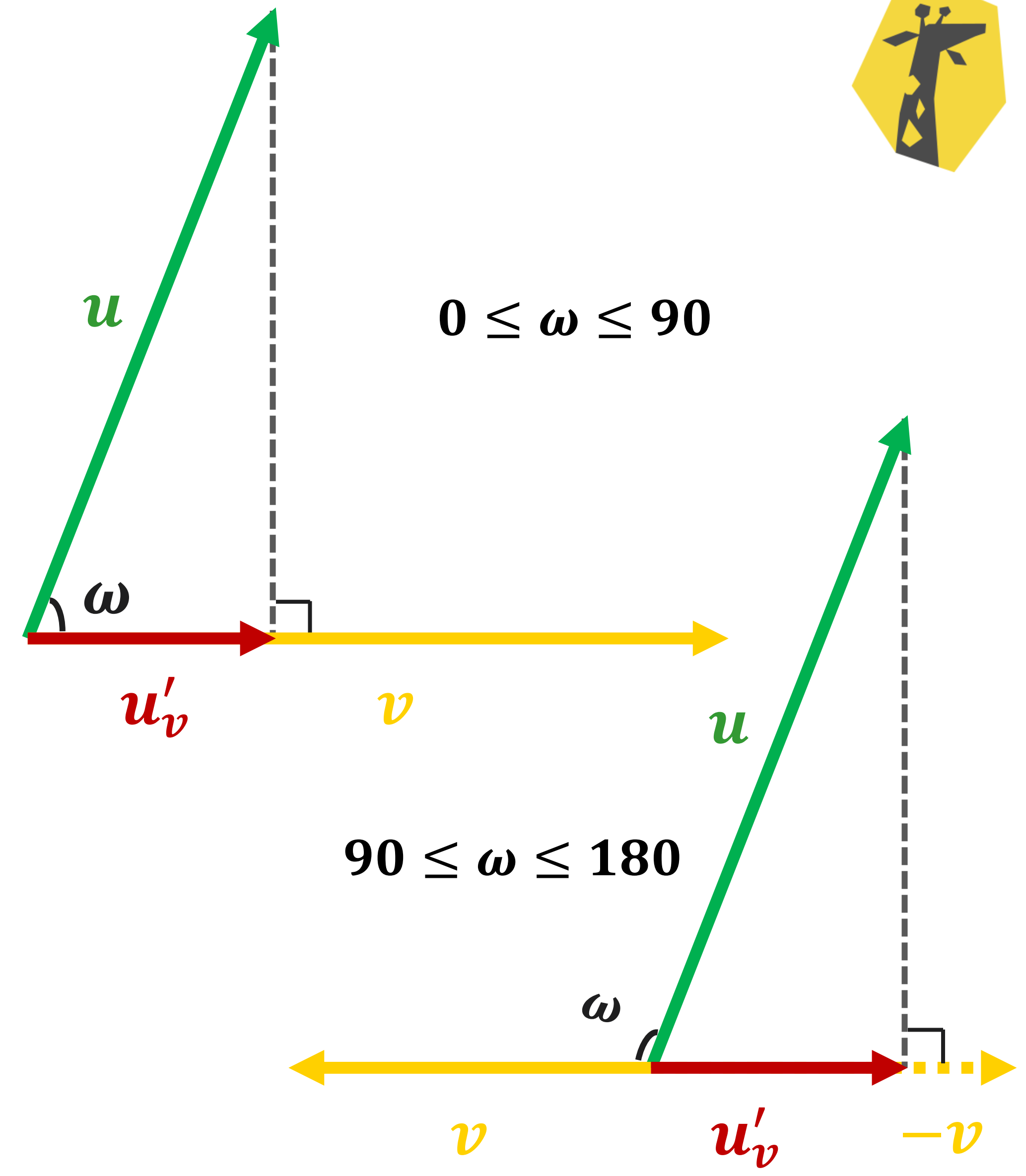
$$|(\boldsymbol{u}, \boldsymbol{v})| = \|\boldsymbol{u}'_v\| \|\boldsymbol{v}\| \leftrightarrow \|\boldsymbol{u}'_v\| = \frac{|(\boldsymbol{u}, \boldsymbol{v})|}{\|\boldsymbol{v}\|}$$



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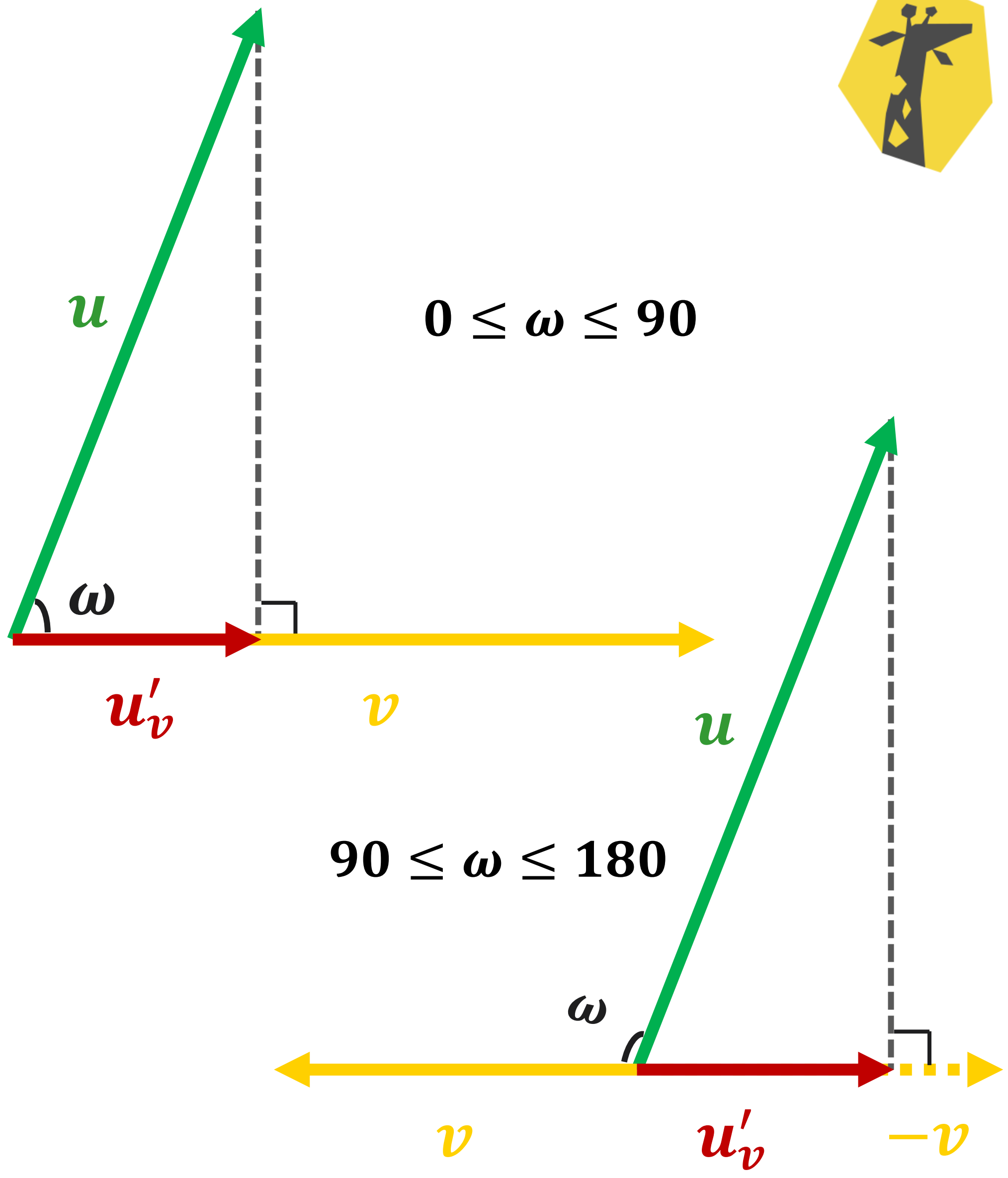
$$u'_v = \frac{(u, v)}{(v, v)}v.$$



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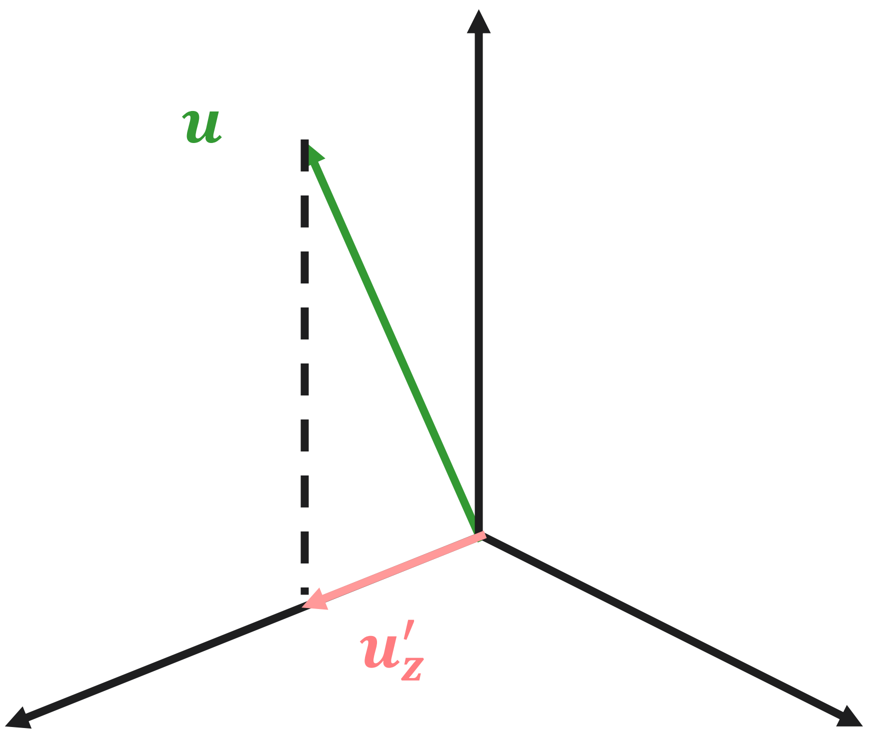

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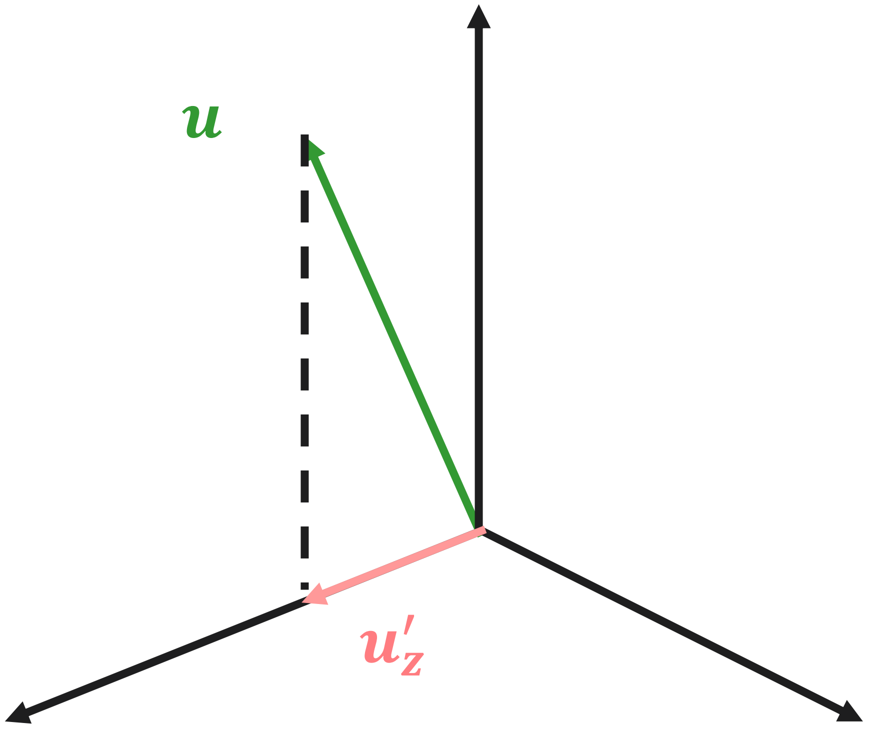
(Projection on the axis = drop other coordinates)



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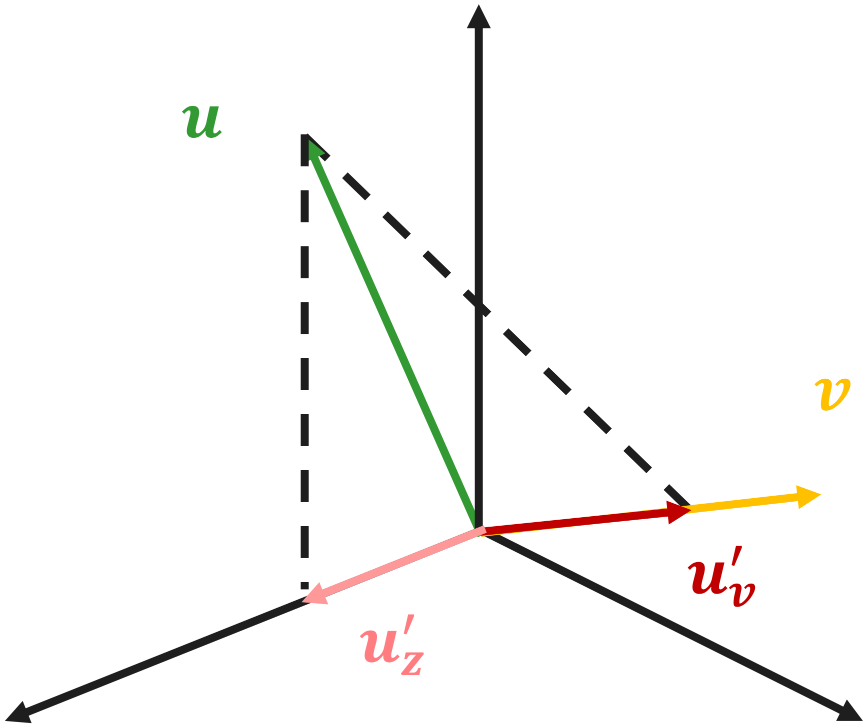

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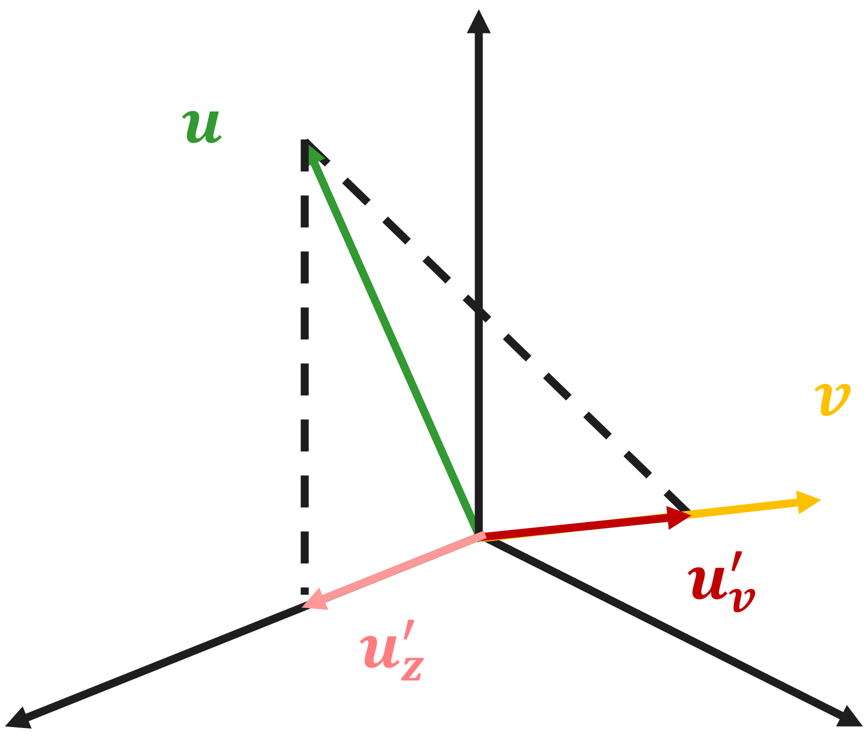
$$u'_v = \frac{(u, v)}{(v, v)} v = \frac{4 + 3 + 6}{16 + 1 + 9} v = \frac{1}{2} v = [2, 0.5, 1.5].$$



Hyperplanes

- A hyperplane is described by equation

$$w_1x_1 + w_2x_2 + \dots + w_nx_n + b = 0$$

where at least one $w_i \neq 0$.

- A more compact notation:

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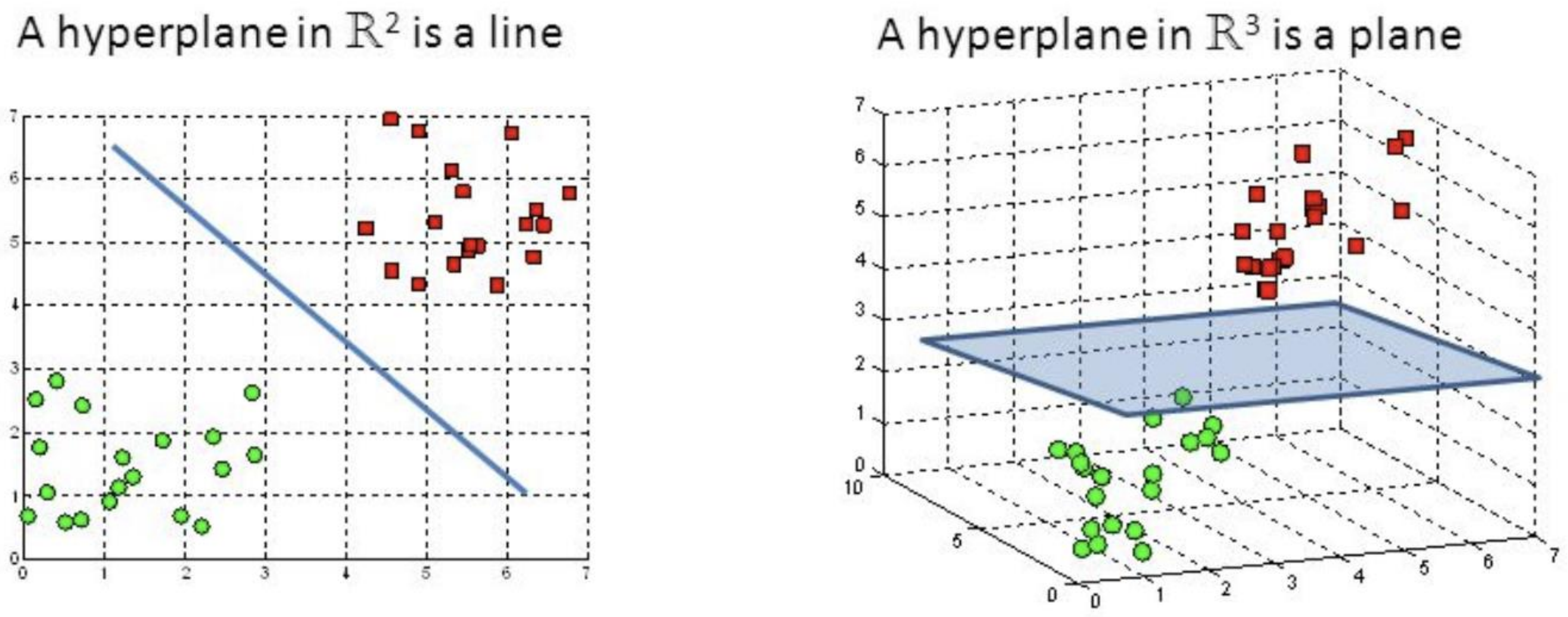

Image source: https://deepai.org/machine-learning-glossary-and-terms/hyperplane

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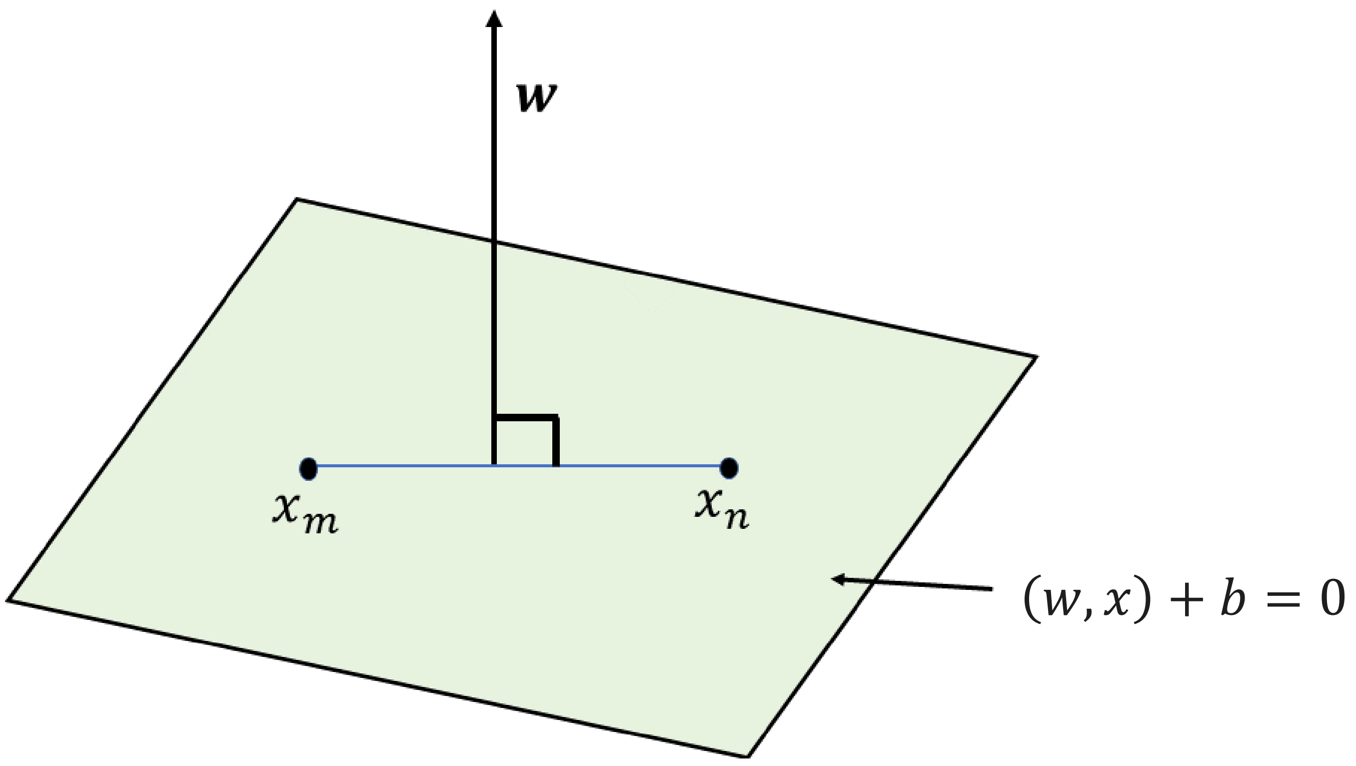
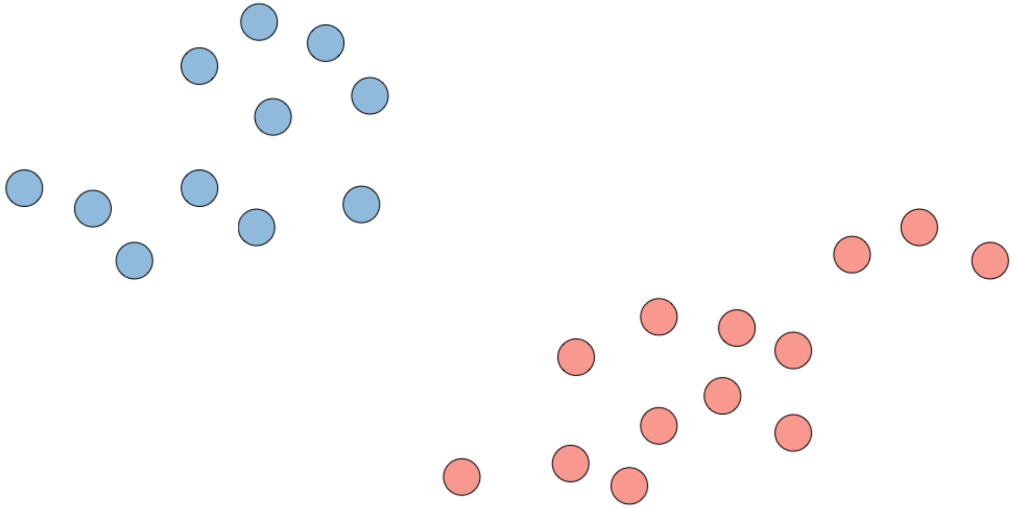
- w is a *normal vector* to this hyperplane: it's orthogonal to every vector on it.



ML Example: Linear Classifier

- Objects = 2D vectors
- Binary classification: classes +1 and -1.



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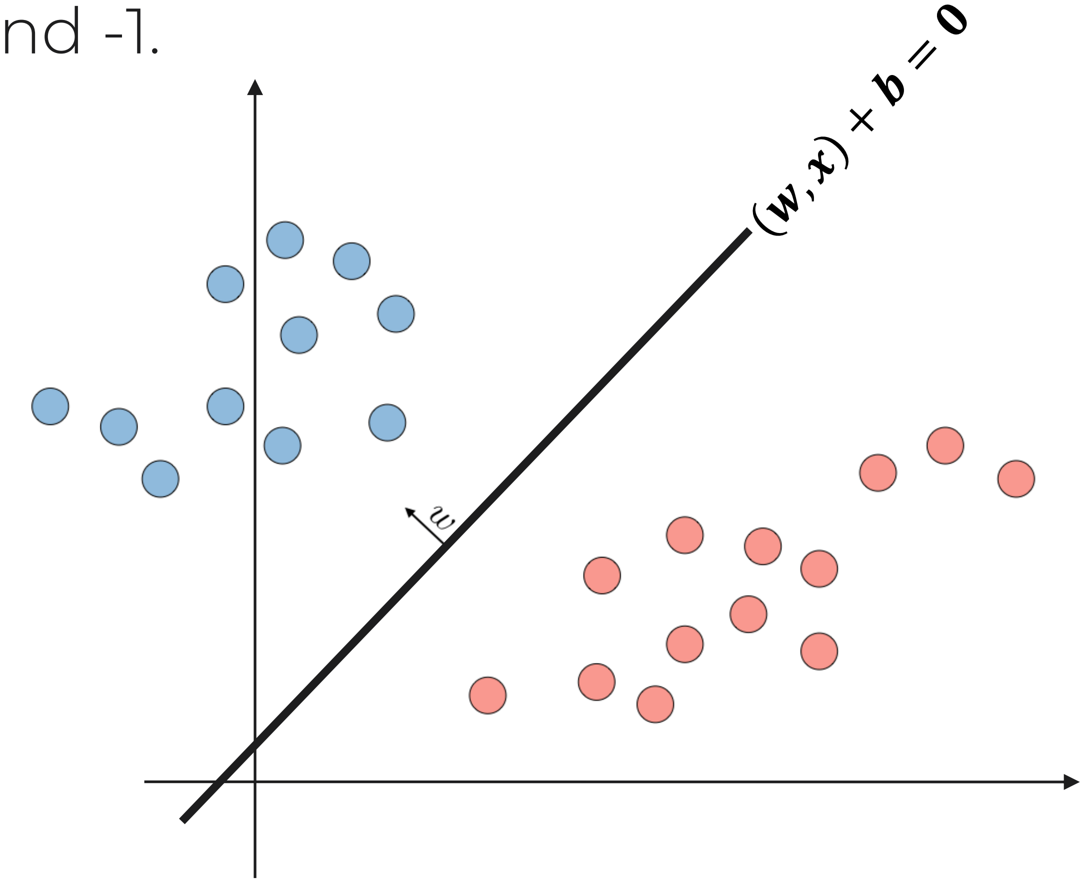

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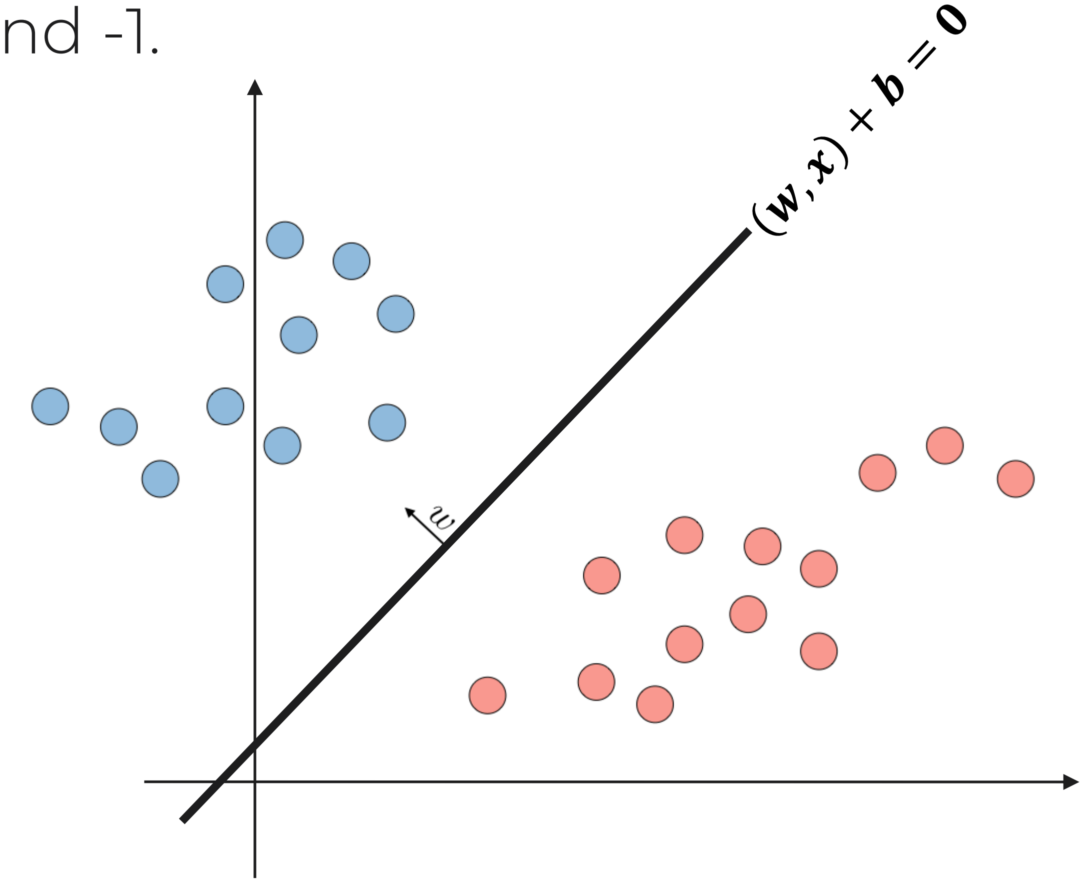

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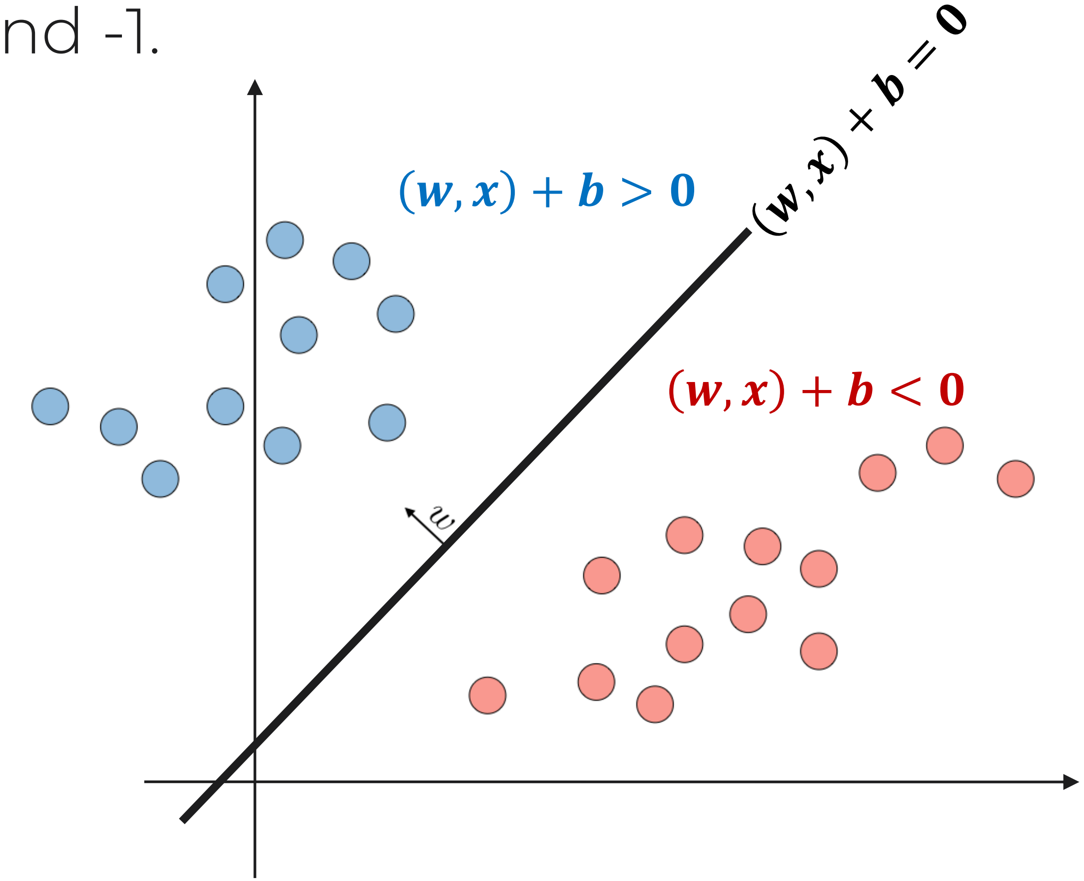
- How can we formalize this?
  - objects "above": $(w, x) + b > 0$
  - objects "below": $(w, x) + b < 0$



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$v = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k = \sum_{i=1}^{k} \lambda_i x_i \in V$ –
a *linear combination* of $x_1, x_2, \dots, x_k$.

Linear Combinations: Examples

- In $(\mathbb{R}^2, +, \ \cdot)$, consider vectors $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

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  $u = 2e_1 - e_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is a linear combination of $\boldsymbol{e_1}$ and $\boldsymbol{e_2}$.

Linear independence

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$$\leftrightarrow$$

- A set of vectors $x_1, x_2, \dots, x_k$ is linearly dependent if and only if (at least) one of the vectors is a linear combination of the others

$$x_i = \alpha_1 x_1 + \cdots + \alpha_k x_k$$

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  (Or: you cannot represent $\boldsymbol{e}_1$ as $\lambda \boldsymbol{e}_2$ or vice versa).

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Basis

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- $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix},\ \dots,\ e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ is a basis for $\mathbb{R}^n$.

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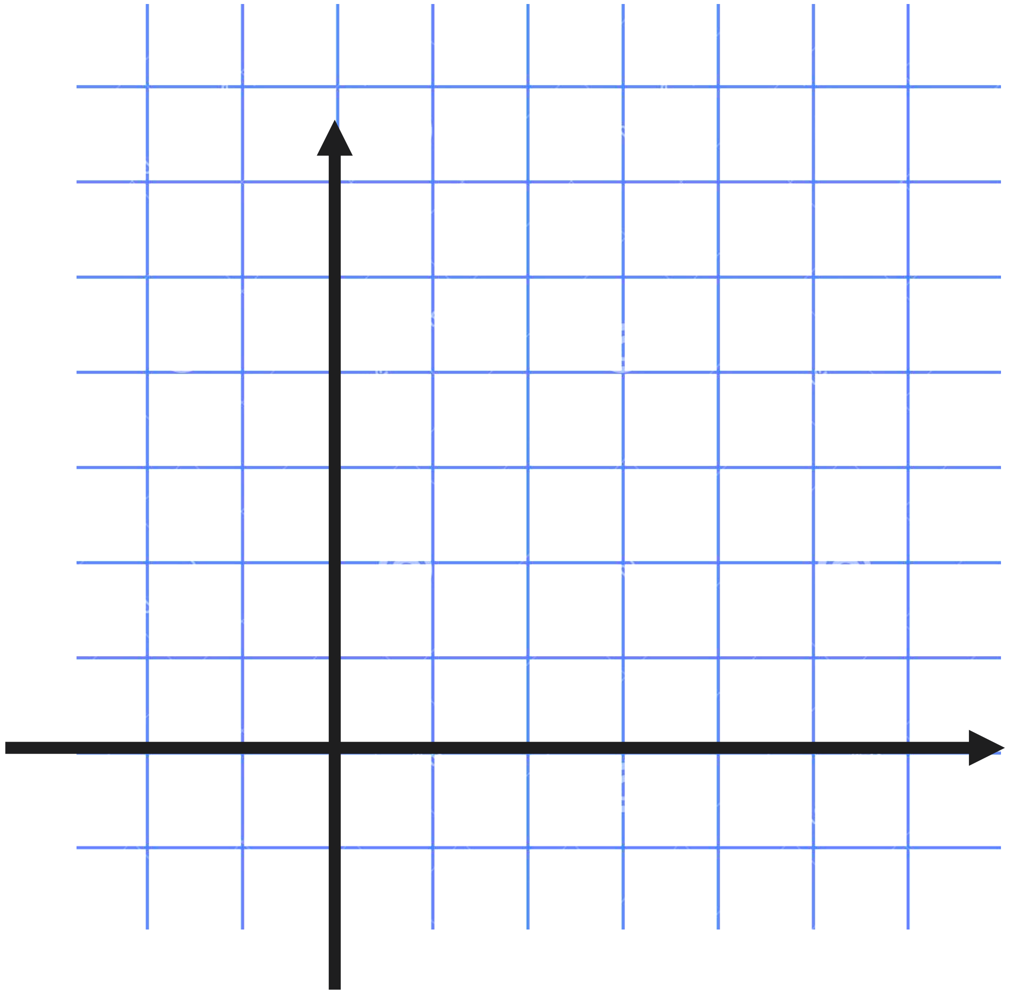
Coordinates: Example

- Consider $\mathbb{R}^2$.



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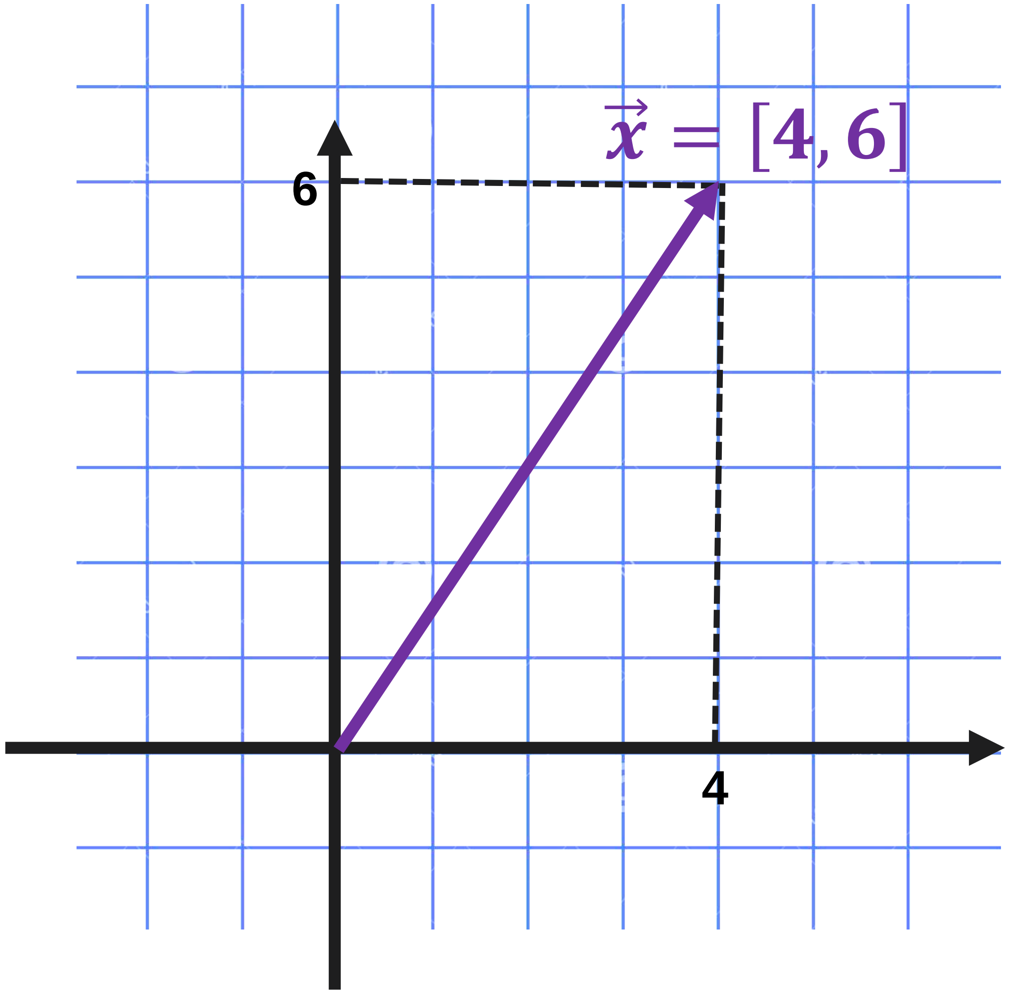

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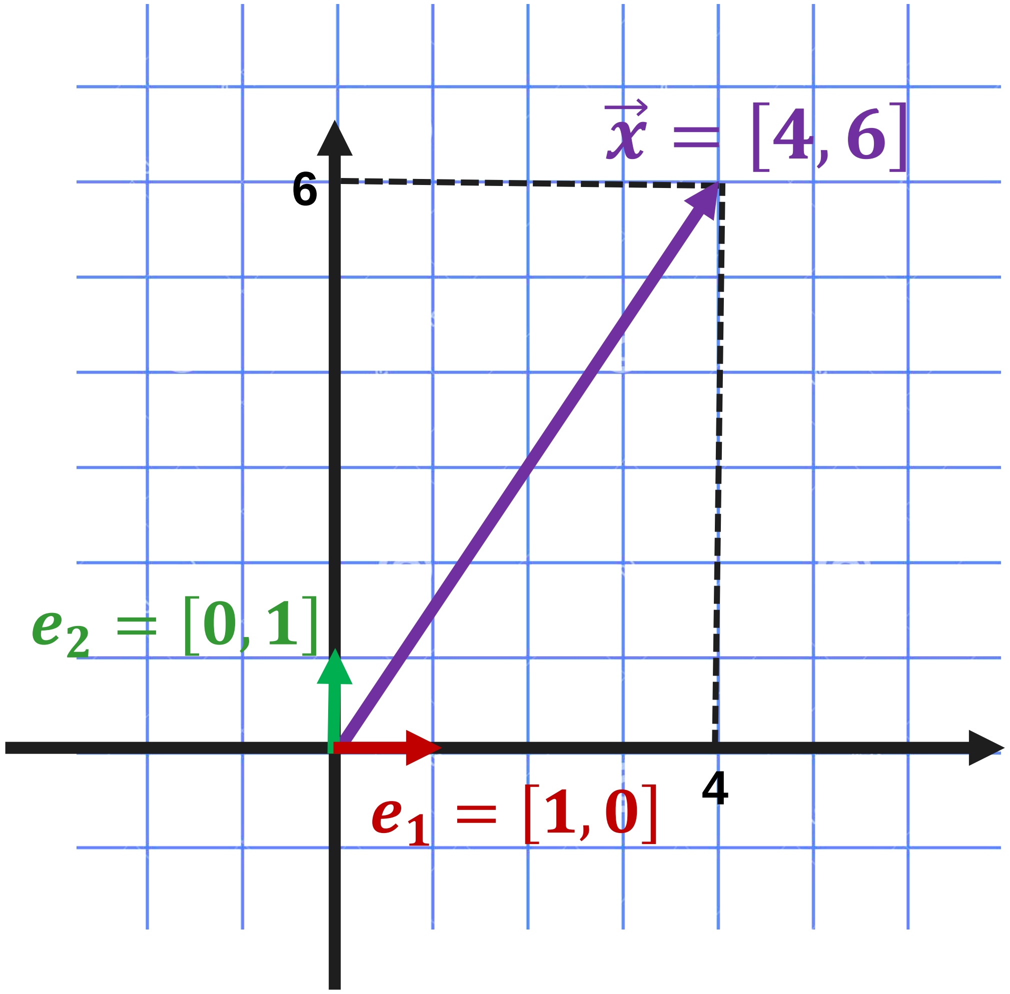

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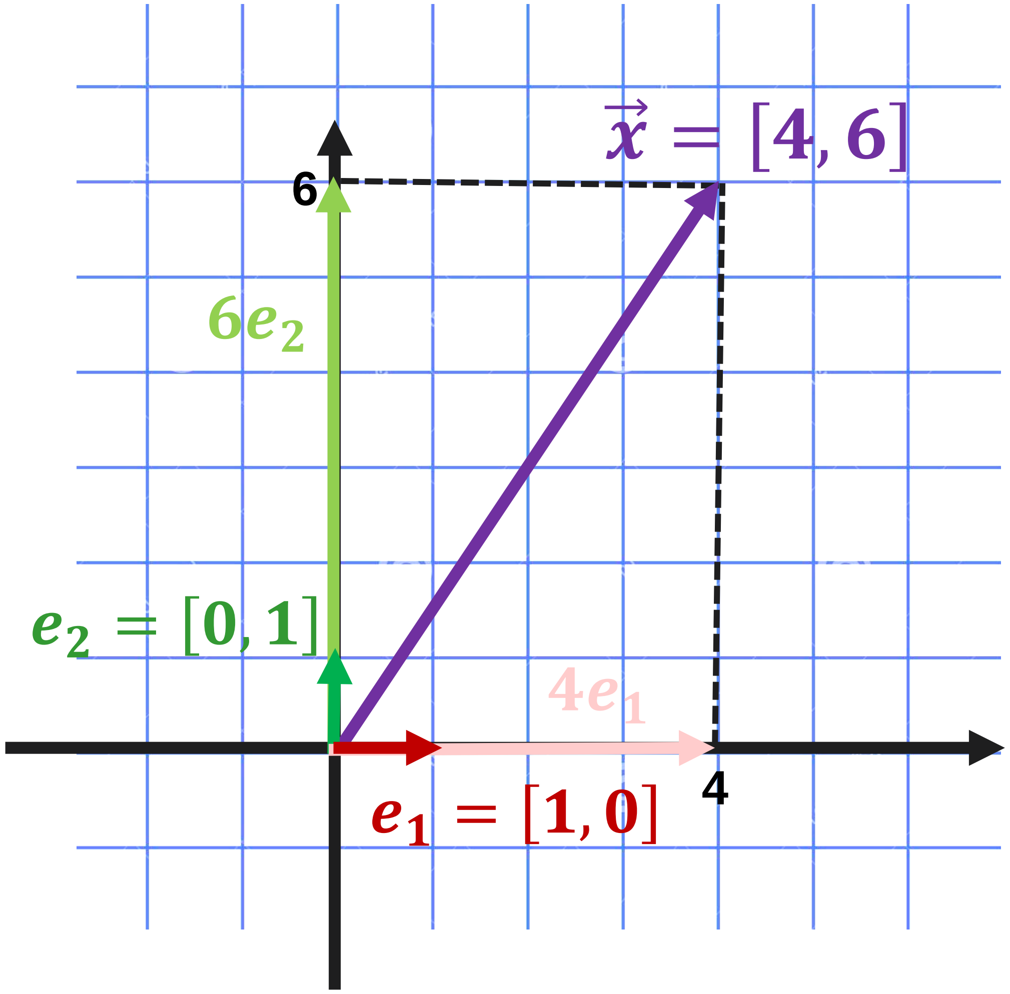
$$\boldsymbol{x} = 4\boldsymbol{e_1} + 6\boldsymbol{e_2}$$



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- If, additionally, $\forall i \; \|e_i\| = 1$, then $e_1, \dots, e_n$ is an *orthonormal basis*.

Example:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \;\; e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \;\; \dots, \;\; e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$ is an orthonormal basis of $\mathbb{R}^n$.

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Change of Basis

Basis

- A vector space has more than one basis.

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    - $b = \left\{ b_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, b_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$ - yet another one.
- Different basis = different coordinates.
  How exactly do they change?

Coordinate Change: Example

- Consider $\mathbb{R}^2$ with canonical basis

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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- $x_{old} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

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$$e'_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad e'_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

- $x_{old} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

- What are the coordinates in the new basis?

$$x_{new} =?$$

Coordinate Change: General Case

- Consider a vector space V with basis $e_1, e_2, \dots e_n$.

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- New basis: $e'_1, e'_2, \dots e'_n$.

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  $x_1, x_2, \dots, x_n$ - coordinates in basis $e_1, e_2, \dots e_n$.

- New basis: $e'_1, e'_2, \dots e'_n$.

- What are the coordinates of x in this new basis?

$$x'_1, x'_2, \dots, x'_n = ?$$

Coordinate Change

- Old basis: $e_1, e_2, \dots e_n$
  New basis: $e'_1, e'_2, \dots e'_n$
- $x_{old} = [x_1, x_2, \dots, x_n], \qquad x_{new} = [x'_1, x'_2, \dots, x'_n] =?$
- Coordinates of the new basis in the old one:

Coordinate Change

- Old basis: $e_1, e_2, \dots e_n$
  New basis: $e'_1, e'_2, \dots e'_n$

- $x_{old} = [x_1, x_2, \dots, x_n], \qquad x_{new} = [x'_1, x'_2, \dots, x'_n] = ?$

- Coordinates of the new basis in the old one:

$$
\begin{gathered}
e'_1 = \boldsymbol{\alpha_{11}} e_1 + \boldsymbol{\alpha_{21}} e_2 + \cdots + \boldsymbol{\alpha_{n1}} e_n \\
e'_2 = \boldsymbol{\alpha_{12}} e_1 + \boldsymbol{\alpha_{22}} e_2 + \cdots + \boldsymbol{\alpha_{n2}} e_n \\
\vdots \\
e'_i = \boldsymbol{\alpha_{1i}} e_1 + \boldsymbol{\alpha_{2i}} e_2 + \cdots + \boldsymbol{\alpha_{ni}} e_n \\
\vdots \\
e'_n = \boldsymbol{\alpha_{1n}} e_1 + \boldsymbol{\alpha_{2n}} e_2 + \cdots + \boldsymbol{\alpha_{nn}} e_n
\end{gathered}
$$

Coordinate Change

$$x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n = x'_1 \boldsymbol{e'_1} + x'_2 \boldsymbol{e'_2} + \cdots x'_n \boldsymbol{e'_n} =$$

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Remember: $e'_i = \boldsymbol{\alpha_{1i}} e_1 + \boldsymbol{\alpha_{2i}} e_2 + \cdots + \boldsymbol{\alpha_{ni}} e_n$

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Coordinate Change

$$x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n = x'_1 \boldsymbol{e'_1} + x'_2 \boldsymbol{e'_2} + \cdots x'_n \boldsymbol{e'_n} =$$

Remember: $e'_i = \boldsymbol{\alpha_{1i}} e_1 + \boldsymbol{\alpha_{2i}} e_2 + \cdots + \boldsymbol{\alpha_{ni}} e_n$

$$= x'_1 \cdot (\boldsymbol{\alpha_{11} e_1} + \boldsymbol{\alpha_{21} e_2} + \cdots + \boldsymbol{\alpha_{n1} e_n}) + \cdots + x'_i \cdot (\boldsymbol{\alpha_{1i} e_1} + \boldsymbol{\alpha_{2i} e_2} + \cdots + \boldsymbol{\alpha_{ni} e_n}) + \cdots$$
$$+ x'_n (\boldsymbol{\alpha_{1n} e_1} + \boldsymbol{\alpha_{2n} e_2} + \cdots + \boldsymbol{\alpha_{nn} e_n}) =$$

Coordinate Change

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$e_1, \dots, e_n$ linearly independent -> coefficients in front of them should be the same on the both sides of the equality:

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Coordinate Change

$$x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n = x'_1 \boldsymbol{e'_1} + x'_2 \boldsymbol{e'_2} + \cdots x'_n \boldsymbol{e'_n} =$$

Remember: $e'_i = \boldsymbol{\alpha_{1i}} e_1 + \boldsymbol{\alpha_{2i}} e_2 + \cdots + \boldsymbol{\alpha_{ni}} e_n$

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$$\boldsymbol{x_{old}} \quad \boxed{\begin{matrix} x_1 \\ x_2 \\ \\ x_n \end{matrix}} \begin{matrix} = x'_1 \alpha_{11} + \cdots + x'_i \alpha_{1i} + \cdots + x'_n \alpha_{1n} \\ = x'_1 \alpha_{21} + \cdots + x'_i \alpha_{2i} + \cdots + x'_n \alpha_{2n} \\ \vdots \\ = x'_1 \alpha_{n1} + \cdots + x'_i \alpha_{ni} + \cdots + x'_n \alpha_{nn} \end{matrix}$$

Coordinate Change

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$\boldsymbol{x_{new}}$

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Coordinate Change

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$$\boldsymbol{e'_i}$$

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$$\boldsymbol{x_{new}}$$

Coordinate Change: Example

- Consider $\mathbb{R}^2$ with basis $e_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $e_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
- New basis: $e'_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $e'_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$
- $x_{old} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $x_{new} = \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix}$

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- $x_{old} = \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}2\\-1\end{bmatrix}$, $x_{new} = \begin{bmatrix}x'_1\\x'_2\end{bmatrix}$

$\boldsymbol{e'_i}$

$$\boldsymbol{x_{old}} \quad \begin{aligned} x_1 &= x'_1\alpha_{11} + \cdots + x'_i\alpha_{1i} + \cdots + x'_n\alpha_{1n} \\ x_2 &= x'_1\alpha_{21} + \cdots + x'_i\alpha_{2i} + \cdots + x'_n\alpha_{2n} \\ &\vdots \\ x_n &= x'_1\alpha_{n1} + \cdots + x'_i\alpha_{ni} + \cdots + x'_n\alpha_{nn} \end{aligned}$$

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$\boldsymbol{x_{old}}$

$\boldsymbol{x_{new}}$

$$\begin{aligned} 2 &= 2x'_1 - 1x'_2 \\ -1 &= 1x'_1 - 1x'_2 \end{aligned}$$

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- $x_{old} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix},\quad x_{new} = \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix}$

e'_i

$$x_{old} \quad \begin{aligned} x_1 &= x'_1\alpha_{11} + \cdots + x'_i\alpha_{1i} + \cdots + x'_n\alpha_{1n} \\ x_2 &= x'_1\alpha_{21} + \cdots + x'_i\alpha_{2i} + \cdots + x'_n\alpha_{2n} \\ &\vdots \\ x_n &= x'_1\alpha_{n1} + \cdots + x'_i\alpha_{ni} + \cdots + x'_n\alpha_{nn} \end{aligned}$$

x_{new}

$$\begin{aligned} 2 &= 2x'_1 - 1x'_2 \\ -1 &= 1x'_1 - 1x'_2 \end{aligned} \iff \begin{aligned} x'_1 &= 3 \\ x'_2 &= 4 \end{aligned}$$

Coordinate Change: Example

- Consider $\mathbb{R}^2$ with basis $e_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $e_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
- New basis: $e'_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $e'_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$
- $x_{old} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $x_{new} = \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix}$

e'_i

x_{old}

$$\begin{aligned} x_1 &= x'_1\alpha_{11} + \cdots + x'_i\alpha_{1i} + \cdots + x'_n\alpha_{1n} \\ x_2 &= x'_1\alpha_{21} + \cdots + x'_i\alpha_{2i} + \cdots + x'_n\alpha_{2n} \\ &\vdots \\ x_n &= x'_1\alpha_{n1} + \cdots + x'_i\alpha_{ni} + \cdots + x'_n\alpha_{nn} \end{aligned}$$

x_{new}

$$\begin{matrix} 2 = 2x'_1 - 1x'_2 \\ -1 = 1x'_1 - 1x'_2 \end{matrix} \Leftrightarrow \begin{matrix} x'_1 = 3 \\ x'_2 = 4 \end{matrix} \Leftrightarrow x_{new} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Coordinate Change

- Going from one basis to the other:

e'_i

$$
\boldsymbol{x_{old}} \quad
\begin{aligned}
x_1 &= x'_1\alpha_{11} + \cdots + x'_i\alpha_{1i} + \cdots + x'_n\alpha_{1n} \\
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&\;\;\vdots \\
x_n &= x'_1\alpha_{n1} + \cdots + x'_i\alpha_{ni} + \cdots + x'_n\alpha_{nn}
\end{aligned}
$$

$\boldsymbol{x_{new}}$

Coordinate Change

- Going from one basis to the other:

$$e'_i$$

$$x_{old} \quad \begin{array}{l} x_1 = x'_1\alpha_{11} + \cdots + x'_i\alpha_{1i} + \cdots + x'_n\alpha_{1n} \\ x_2 = x'_1\alpha_{21} + \cdots + x'_i\alpha_{2i} + \cdots + x'_n\alpha_{2n} \\ \vdots \\ x_n = x'_1\alpha_{n1} + \cdots + x'_i\alpha_{ni} + \cdots + x'_n\alpha_{nn} \end{array}$$

$$x_{new}$$

- There is a more compact way of writing this down using matrices.

Matrices

A Matrix

- $A \in \mathbb{R}^{m \times n}$ - a **matrix** with m rows and n columns:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

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A Matrix

- $A \in \mathbb{R}^{m \times n}$ - a **matrix** with $\boldsymbol{m}$ rows and $\boldsymbol{n}$ columns:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- *Examples:*

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{2 \times 2}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \in \mathbb{R}^{2 \times 3}$$

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Special Matrices

- Diagonal matrix: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $\left(a_{ii} \neq 0,\ a_{ij} = 0\ \forall i \neq j\right)$

Special Matrices

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- Identity matrix: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\left(a_{ii} = 1,\ a_{ij} = 0\ \forall i \neq j\right)$

Special Matrices

- Diagonal matrix: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $(a_{ii} \neq 0,\ a_{ij} = 0\ \forall i \neq j)$

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- Symmetric matrix: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $(a_{ij} = a_{ji})$

Special Matrices

- Diagonal matrix: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $\left(a_{ii} \neq 0,\ a_{ij} = 0\ \forall i \neq j\right)$
- Identity matrix: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\left(a_{ii} = 1,\ a_{ij} = 0\ \forall i \neq j\right)$
- Symmetric matrix: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $\left(a_{ij} = a_{ji}\right)$
- Triangular matrix: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$ $\left(a_{ij} = 0\ \forall i > j\ or\ \ \forall i < j\right)$

Vectors vs Matrices

- An n-dimensional vector can be considered a $n \times 1$ matrix:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^{n \times 1}$$

Operations with Matrices

Transpose of a Matrix

- Consider a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Transpose of a Matrix

- Consider a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- Transpose = writing columns as rows:

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}, \qquad \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}^T = [x_1, \dots, x_n]$$

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Transpose of a Matrix: Example

- $\begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 5 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 2 & 5 \end{bmatrix}$

Transpose of a Matrix: Example

- $\begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 5 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 2 & 5 \end{bmatrix}$

- Transposing a symmetrical matrix = no changes:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}^{T} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

Multiplying by a Scalar

- We can multiply matrix by a scalar:

$$\lambda A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda a_{m1} & \lambda a_{m2} & \cdots & \lambda a_{mn} \end{bmatrix}$$

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- Example:

$$5 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

Sum of Two Matrices

- We can sum up matrices of the same size:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix},$$

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}$$

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- Example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 4 \end{bmatrix}$$

Matrices Also Form a Vector Space!

- $(\mathbb{R}^{m\times n}, +, \ \cdot)$ - a vector space.
  "Vectors" = matrices.

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- You can check yourself that the necessary axioms hold.

Matrix Multiplication

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- Example $\mathbb{R}^{2\times 2}$: $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$

Matrix Multiplication: Example

$$\begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 3 & 6 & 4 \\ 2 & 5 & 8 \\ 7 & 1 & 9 \end{bmatrix} =$$

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$$= \begin{bmatrix} 16 & 7 & 26 \\ 43 & 22 & 61 \end{bmatrix}$$

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To sum up

- Vectors
  - Vector spaces
  - Inner products
  - Lengths
  - Distances
  - Angles
- Analytic Geometry
  - Projections
  - Hyperplanes
  - Normal vector
- Vector spaces
  - Linear (in)dependence
  - Basis
- Matrices
  - Matrix operations

Next Time

- Calculus recap
- Probability theory recap