Prove *Central Limit Theorem* using *Moment Generating Function*

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- MGF & k-th moment

- $\mu_k = E[\mathrm{X}^k]$
- $M_X(t) = E(e^{tX})$ -> MGF of r.v. X
- If Z = X + Y, with X independent of Y, then $M_z(t) = M_X(t) * M_Y(t)$

Theorem: $X_1, X_2, \dots, X_n$ are i.i.d., with mean μ, variance σ^2, we can have

$$\lim_{n\to\infty} \frac{X_1+X_2+\dots+X_n}{n} \sim N(\mu, \frac{\sigma^2}{n})$$

- Let $S_n = X_1 + X_2 + \dots + X_n$, so $\bar{X} = \frac{S_n}{n}$
- $M_{S_n}(\mathrm{t}) = E(e^{t(X_1+X_2+\dots+X_n)}) = [M_X(t)]^{\mathrm{n}}$
- $M_{\bar{X}}(t) = E(e^{\frac{t(X_1+X_2+\dots+X_n)}{n}}) = E(e^{\frac{t}{n}(X_1+X_2+\dots+X_n)}) =$

$$M_{S_n}\left(\frac{\mathrm{t}}{\mathrm{n}}\right) = \left[M_X\left(\frac{t}{n}\right)\right]^{\mathrm{n}}$$

- Standardize $\bar{X}$: $Y = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} = \frac{\sqrt{n}}{\sigma}\bar{X} - \frac{\mu\sqrt{n}}{\sigma}$
- $M_Y(t) = E(e^{tY}) = E\left(e^{t\frac{\sqrt{n}}{\sigma}\bar{X}}e^{-t\frac{\mu\sqrt{n}}{\sigma}}\right) = e^{-t\frac{\mu\sqrt{n}}{\sigma}}\mathrm{M}_{\bar{X}}(\frac{\sqrt{n}}{\sigma}t) =$

  $e^{-t\frac{\mu\sqrt{n}}{\sigma}}\left[M_X\left(\frac{\sqrt{n}}{\sigma n}t\right)\right]^{\mathrm{n}} = e^{-t\frac{\mu\sqrt{n}}{\sigma}}\left[M_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^{\mathrm{n}}$
- Now we just need to prove the **MGF of Y** has the form of a stadard normal random variable as n goes to infinity!

$$M_Y(t) = e^{-t\frac{\mu\sqrt{n}}{\sigma}}\left[M_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^{\mathrm{n}}$$

- $\ln M_Y(t) = \ln\left(e^{-t\frac{\mu\sqrt{n}}{\sigma}}\right) + \ln\left[M_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^{\mathrm{n}} = -t\,\frac{\mu\sqrt{n}}{\sigma} + n\ln M_X\left(\frac{t}{\sigma\sqrt{n}}\right)$
- Let $p = \frac{t}{\sqrt{n}\sigma}, then\ \sqrt{n} = \frac{t}{p\sigma}, if\ n \to \infty, p \to 0$
- $\ln M_Y(t) = -\frac{\mu t^2}{p\sigma^2} + \frac{t^2}{p^2\sigma^2}\ln M_X(p) = \frac{t^2}{\sigma^2}(-\frac{\mu}{p} + \frac{1}{p^2}\ln M_X(p)) =$

$\frac{t^2}{\sigma^2}\left(\frac{\ln M_X(p)-\mu p}{p^2}\right) \to we\ need\ to\ use$ ***L'Hôpital's rule***

$We\ let\ p \to 0$

- $\lim_{p\to 0} \frac{t^2}{\sigma^2}\left(\frac{\ln M_X(p)-\mu p}{p^2}\right) = \lim_{p\to 0} \frac{t^2}{\sigma^2}\left(\frac{\frac{1}{M_X(p)}\frac{dM_X(p)}{dp}-\mu}{2p}\right)$ → ***L'Hôpital's rule*** $again$

- $= \lim_{p\to 0} \frac{t^2}{\sigma^2}\left(\frac{\frac{M'_X(p)}{M_X(p)}-\mu}{2p}\right) = \lim_{p\to 0} \frac{t^2}{\sigma^2}\left(\frac{\frac{M''_X(p) * M_X(p) - M'_X(p) * M'_X(p)}{\left(M_X(p)\right)^2}}{2}\right)$

- $M_X(0) = 1,\ M'_X(0) = E[X] = \mu,\ M''_X(0) = E[X^2]$

- $= \frac{t^2}{\sigma^2}\ \frac{\left(E\left[X^2\right] - E^2[X]\right)}{2} = \frac{t^2}{2}$

- $\ln M_Y(t) \to \frac{t^2}{2}$
- $so\ M_Y(t)\ \to e^{\frac{t^2}{2}}, as\ n \to\ \infty$

➢ $Proof\ finished$

$Thanks\ for\ listening$