Exam July 8, 2020.

Complete solutions of selected problems.

$\boxed{1.}$ There exists a linear homogeneous differential equation with constant coefficients of order 7 that has as solutions the functions: (a) $t^2\cos 2t$, $t\sin 2t$ and e^{7t}?; (b) $t^2\cos 2t$, $t\sin 7t$ and e^t?; (c) $t^2\cos 2t$?. Justify the answers.

<u>Solution</u> We have that:

$t^2\cos 2t$ is a sol. of a LHDE with CC iff $\pm 2i$ is a root of order (multiplicity) 3 of the charac. eq.

$t\sin 2t$ is a sol. of a LHDE with CC iff $\pm 2i$ is a root of multiplicity 2 of the charact. eq.

e^{7t} is a sol. of a LHDE with CC iff 7 is a root of mult. 1 (simple root) of the charact. eq.

e^t is a sol of a LHDE with CC iff 1 is a simple root of the charact. eq.

(a) These functions are solutions of a LHDE with CC iff $\pm 2i$ is a root of multiplicity 3 and 7 is a simple root of the charact. eq. Thus the charact. eq. has at least degree $3\times 2+1=7$.

Conclusion: the answer at (a) is "YES".

(b) These functions are solutions of a LHDE with CC iff $\pm 2i$ is a root of multiplicity 3, $\pm 7i$ is a root of multiplicity 2, and 1 is a simple root of the charact. eq. Thus, the charact. eq. has at least degree $3\times 2+2\times 2+1=11$.

Conclusion: the answer at (b) is "NO".

(c) Here the charact. eq. has at least degree 6.

Conclusion: the answer at (c) is "YES".

2. (a) Does the formula $x=A_0\cos(3t-\varphi_0)$, $A_0\geq 0$, $\varphi_0\in[0,2\pi]$ describe the general solution of the differential equation $x''+9x=0$?

<u>Solution</u> First we apply the characteristic eq. method to find the general solution of $x''+9x=0$.

$r^2+9=0 \qquad r_{1,2}=\pm 3i \longmapsto \cos 3t\ ,\ \sin 3t$

$x=c_1\cos 3t+c_2\sin 3t\ ,\quad c_1, c_2\in\mathbb{R}$.

Now we work with the given formula.

$x=A_0\cos(3t-\varphi_0)=A_0\cos 3t\cos\varphi_0+A_0\sin 3t\sin\varphi_0=$

$=(A_0\cos\varphi_0)\cos 3t+(A_0\sin\varphi_0)\sin 3t$

We know that $\forall\ (c_1,c_2)\in\mathbb{R}^2\setminus\{(0,0)\}\ \exists!\ (A_0,\varphi_0)\in(0,\infty)\times[0,2\pi)$ such that $c_1=A_0\cos\varphi_0$ and $c_2=A_0\sin\varphi_0$

(A_0,φ_0 are the polar coordinates of (c_1,c_2)).

Of course, for $(c_1,c_2)=(0,0)$ we can take $A_0=0$.

Conclusion: The answer is "YES".

$\boxed{2.(b)}$ Let $A_1 \in \mathbb{R}$. Find a particular solution of $x'' + 9x = A_1 \cos 3t$, knowing that it has the form $x_p = a t \sin 3t$.

<u>Solution.</u> $x_p' = a \sin 3t + 3at \cos 3t$

$x_p'' = 3a \cos 3t + 3a \cos 3t - 9at \sin 3t$

we replace in the DE:

$$6a \cos 3t - 9at \sin 3t + 9at \sin 3t = A_1 \cos 3t, \quad \forall t \in \mathbb{R}.$$

$$6a \cos 3t = A_1 \cos 3t, \quad \forall t \in \mathbb{R}$$

$$\Leftrightarrow \quad 6a = A_1 \quad \Leftrightarrow \quad a = \frac{A_1}{6}$$

conclusion: $x_p = \frac{A_1}{6} t \sin 3t$.

$\boxed{2.(c)}$ Find the solution of the IVP $\begin{cases} x'' + 9x = A_1 \cos 3t \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$.

<u>Solution.</u>
Let $\theta(t) = \frac{A_1}{6} t \sin 3t$. We know that θ is a sol. of the D.E. (from (b)). It is easy to check that θ satisfies also the IC's. Thus, θ is a solution of this IVP. We know from the lecture that this IVP has a unique sol.

Conclusion: The unique solution of this IVP is $\theta(t) = \frac{A_1}{6} t \sin 3t$.

Describe the motion of a simple pendulum in the case that $\theta(t)$ is the measure in radians of the angle between the rod and the vertical.

<u>Solution.</u> We have that $\theta(0) = 0$ and $\theta'(0) = 0$. Thus, the critical position of the pendulum is

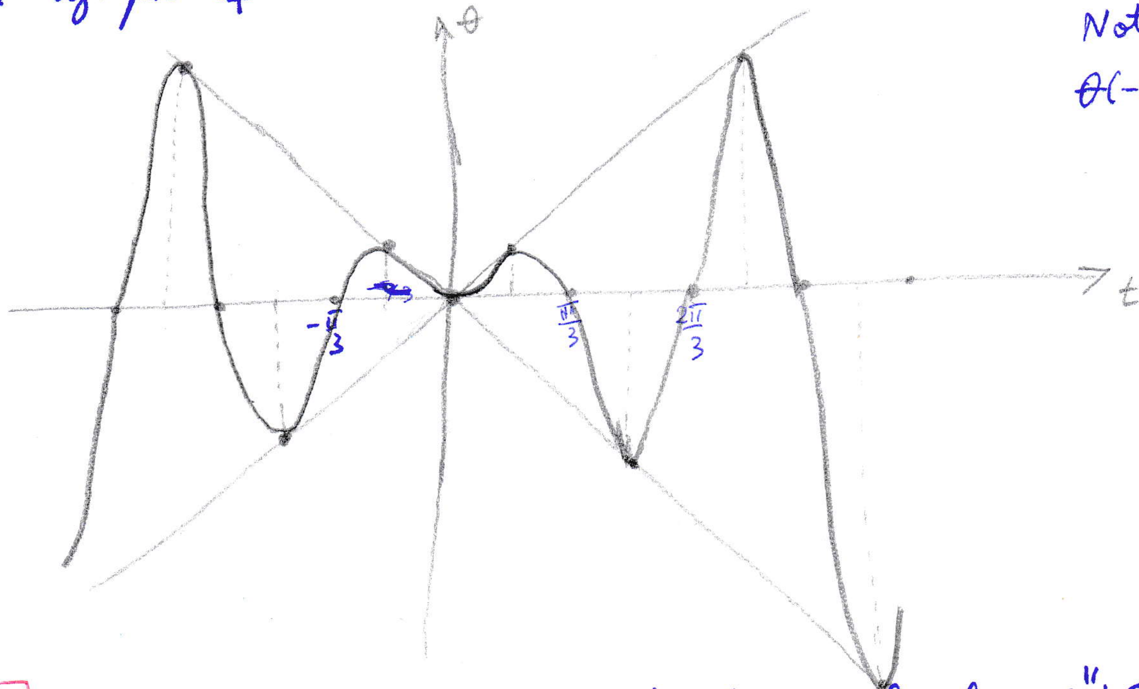
we also have that $\theta\left(\frac{k\pi}{3}\right) = 0 \quad \forall k \in \mathbb{Z}$ and θ changes sign around $\frac{k\pi}{3}$. Thus, the pendulum oscillates around the position $\theta^* = 0$.

Since $\lim_{t\to\infty} |\theta(t)| = \lim_{t\to\infty} \left|\frac{A_1}{6} t \sin 3t\right| = +\infty$

we deduce that the amplitude of the oscillations is increasing to ∞ as $t \to \infty$. Of course, practically, this is not possible. This phenomenon is called Resonance.

The graph of $\theta(t)$ looks like (fix $A_1 = 6$)

Note that $\theta(-t) = \theta(t)$ $\forall t \geq 0$



$\boxed{2.(d)}$ Let $A_2 \in \mathbb{R}$. Find a particular sol. of $x'' + 9x = A_2$.

<u>Solution</u>. $x_p = \frac{A_2}{9}$ (constant function).

$\boxed{2.(e)}$ Find a particular sol. of $x'' + 9x = A_1 \cos 3t + A_2$.

<u>Solution</u>. We know that $x_{p_1} = \frac{A_1}{6} t \sin 3t$ is a sol. of $x'' + 9x = A_1 \cos 3t$ and that $x_{p_2} = \frac{A_2}{9}$ is ~~of~~ $x'' + 9x = A_2$. We apply the Superposition Principle and deduce that

$$x_p = \frac{A_1}{6} t \sin 3t + \frac{A_2}{9}.$$

$\boxed{3.(a)}$ • Find the solution of the IVP $y' = y$, $y(0) = 1$.

Sol: $\varphi(x) = e^x$.

$\boxed{3.(b)}$ Write the Euler's numerical formula with stepsize $h = 0.01$ to approximate the solution of this IVP in the interval $[0,1]$.

Solution. $x_k = k h = k \cdot 0.01$, $k = \overline{0, 100}$

$x_0 = 0$

$y_0 = 1$

$$y_{k+1} = y_k + h y_k \iff y_{k+1} = 1.01 \cdot y_k \quad k = \overline{0, 100}.$$

$\boxed{3.(c)}$ Using (b) find a rational approximation of the Euler's constant e.

Solution. Since $x_{100} = 1$ and $\varphi(1) = e$ we have that y_{100} is the approximation given by (b) for e.

$$y_{k+1} = 1.01 \cdot y_k, \ y_0 = 1 \iff y_k = (1.01)^k, \ \forall k \geq 0$$

Thus, $y_{100} = \left(\frac{101}{100}\right)^{100}$ which is a rational number.

$\boxed{4.(a)}$ We consider the planar system

$$\begin{cases} \dot{x} = -y\sqrt{3} + x(9 - x^2 - 3y^2) \\ \dot{y} = \dfrac{x}{\sqrt{3}} + y(9 - x^2 - 3y^2). \end{cases}$$

(a) Study the type and stability of the equilibrium point $(0,0)$ using the linearization method.

<u>Solution</u>. First note that $(0,0)$ is, indeed, an equil. point.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$, $\quad f(x,y) = \begin{pmatrix} -y\sqrt{3} + 9x - x^3 - 3xy^2 \\ \dfrac{x}{\sqrt{3}} + 9y - x^2 y - 3y^3 \end{pmatrix}$.

f is a polynomial function, thus $f \in C^1(\mathbb{R}^2)$.

$$Jf(x,y) = \begin{pmatrix} 9 - 3x^2 - 3y^2 & -\sqrt{3} - 6xy \\ \dfrac{1}{\sqrt{3}} - 2xy & 9 - x^2 - 9y^2 \end{pmatrix}$$

$$Jf(0,0) = \begin{pmatrix} 9 & -\sqrt{3} \\ \dfrac{1}{\sqrt{3}} & 9 \end{pmatrix} \qquad \begin{vmatrix} 9-\lambda & -\sqrt{3} \\ \dfrac{1}{\sqrt{3}} & 9-\lambda \end{vmatrix} = 0 \iff$$

$\iff (9-\lambda)^2 + 1 = 0 \iff (9-\lambda)^2 = -1 \iff 9 - \lambda = \pm i$

$\iff \lambda_{1,2} = 9 \pm i$

$\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) = 9 \neq 0 \Rightarrow$ the eq. $(0,0)$ is hyperbolic $\Rightarrow$

$\Rightarrow$ the linearization method works

The linearized system around $(0,0)$ is $\begin{cases} \dot{x} = 9x - y\sqrt{3} \\ \dot{y} = \dfrac{x}{\sqrt{3}} + 9y \end{cases}$

which is of focus type since the eigenvalues are complex with non-zero real part. It is also a ~~fince~~ global repellor since $\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) > 0$.

Applying the linearization method, we deduce that the eq. $(0,0)$ of the given nonlinear system is a repellor.

$\boxed{4.(b)}$ Check that $\varphi(t,3,0) = (3\cos t, \sqrt{3}\sin t)$, $\forall t\in\mathbb{R}$. Represent the corresponding orbit. What shape is it?

Solution By definition, $\varphi(t,3,0)$ denotes the solution of the system that also satisfies the initial conditions

$x(0)=3$, $y(0)=0$.

Let $x(t) = 3\cos t$ and $y(t)=\sqrt{3}\sin t$. It is easy to check that, indeed, they satisfy the ~~given~~ above initial cond. Now we check that ~~they~~ ~~(x(t),~~ they satisfy the system.

$$-3\sin t = -\sqrt{3}\cdot\sqrt{3}\sin t + 3\cos t\,(9 - 9\cos^2 t - 3\cdot 3\sin^2 t)$$

$$\sqrt{3}\cos t = \frac{3}{\sqrt{3}}\cos t + \sqrt{3}\sin t\,(9 - 9\cos^2 t - 3\cdot 3\sin^2 t)$$

$\forall t\in\mathbb{R}$.

it is easy to see that this is valid.

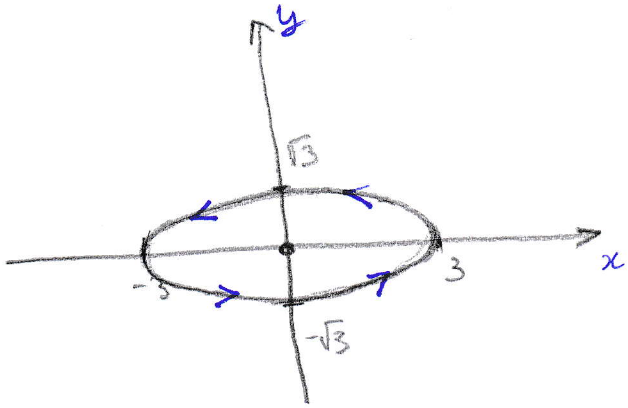
Conclusion: $\varphi(t,3,0) = (3\cos t, \sqrt{3}\sin t)$, $\forall t\in\mathbb{R}$.

By definition, the corresponding orbit is

$$\Gamma = \{ (\underbrace{3\cos t}_{x}, \underbrace{\sqrt{3}\sin t}_{y}) : t\in\mathbb{R} \}.$$

we have: $$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{\sqrt{3}}\right)^2 = 1$$

we know that this is the implicit equation of an ellipse. Thus, Γ is an ellipse. Its representation is:



$\underline{t=0}$ $\quad x(0)=3,\ y(0)=0$

$t=\frac{\pi}{2}$ $\quad x\left(\frac{\pi}{2}\right)=0,\ y\left(\frac{\pi}{2}\right)=\sqrt{3}$

$(3,0) \to (0,\sqrt{3})$

$t=0 \to t=\frac{\pi}{2}$

$\boxed{4. (e)}$ Transform the given system to the coordinates $(r,\varphi) \in [0,\infty) \times [0, 2\pi)$ related to the cartesian coordinates (x,y) by $\frac{x}{\sqrt{3}} = r\cos\varphi$, $y = r\sin\varphi$.

<u>Solution</u>. we have $\left(\frac{x}{\sqrt{3}}\right)^2 + y^2 = r^2$, $tg\,\varphi = y\cdot\frac{\sqrt{3}}{x}$

$$\Leftrightarrow \begin{cases} x^2+3y^2 = 3r^2 \\ tg\,\varphi = \sqrt{3}\cdot\frac{y}{x} \end{cases}.$$

Derivating w.r.t. t we obtain:

$$3r\dot{r} = x\dot{x} + 3y\dot{y}$$

$$\frac{\dot{\varphi}}{\cos^2\varphi} = \sqrt{3}\cdot\frac{\dot{y}x - \dot{x}y}{x^2}.$$

Then:

$$\begin{cases} 3r\dot{r} = x\left[-y\sqrt{3} + x(9-x^2-3y^2)\right] + 3y\left[\frac{x}{\sqrt{3}} + y(9-x^2-3y^2)\right] \\ \frac{\dot{\varphi}}{\cos^2\varphi} = \sqrt{3}\cdot\frac{x\left[\frac{x}{\sqrt{3}} + y(9-x^2-3y^2)\right] - y\left[-y\sqrt{3} + x(9-x^2-3y^2)\right]}{x^2} \end{cases}$$

$$\Leftrightarrow \begin{cases} 3r\dot{r} = (x^2+3y^2)(9-x^2-3y^2) \\ \frac{\dot{\varphi}}{\cos^2\varphi} = \sqrt{3}\cdot\frac{\frac{x^2}{\sqrt{3}} + y^2\sqrt{3}}{x^2} = \frac{x^2+3y^2}{x^2} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 3r\dot{r} = 3r^2(9-3r^2) \\ \frac{\dot{\varphi}}{\cos^2\varphi} = \frac{3r^2}{3r^2\cos^2\varphi} \end{cases} \Leftrightarrow \begin{cases} \dot{r} = r(9-3r^2) \\ \dot{\varphi} = 1 \end{cases}.$$

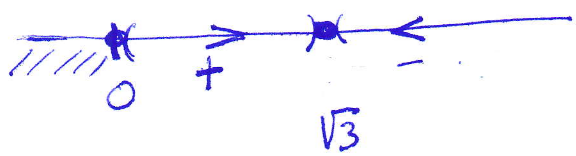
4.(d) Sketch the phase portrait of this planar system.

In the coordinates (r, φ) we obtained ~~an~~ the uncoupled system

$$\begin{cases} \dot{r} = r(9-3r^2) \\ \dot{\varphi} = 1 \end{cases}$$

We represent first the phase portrait of each equation.

$$\dot{r} = r(3+r\sqrt{3})(3-r\sqrt{3})$$



$$\dot{\varphi} = 1$$

Note that the new coordinates (r, φ) are very much like the polar coordinates, just a little bit deformed. r is like a distance to the origin, and φ is like an angle of the vector (x, y) with the $[Ox$ semi-axis. Using the geometric interpretation of polar coordinates and taking into account the particularities of the new coordinates we deduce:

- since the angle φ is strictly increasing, any orbit will rotate around the origin in the trigonometric sense.

- the equilibrium point $r^* = \sqrt{3}$ of $\dot{r} = r(9-3r^2)$ corresponds to the orbit Γ found at (b).

A point with $r \in (0, \sqrt{3})$ lies inside Γ, while a point with $r > \sqrt{3}$ lies outside Γ.

- reading the phase portrait of $\dot{r} = r(9-3r^2)$ and taking into account the ~~abo~~ above remarks, we deduce that an orbit the starts inside Γ, ~~sto~~ stays inside Γ for ~~all t~~ any time and is departing from the origin. ~~Also, an orbit that star~~ and approaching Γ as $t \to \infty$. Also, an orbit that starts outside Γ, stays outside Γ for all time and is approaching Γ as $t \to \infty$.

Thus, the phase portrait looks like