24.02.~~feb~~ 2022

Seminar 1

Dynamical Systems

Differential equations = equations that envolve derivatives of a function, where the variables and unknowns are functions

$x(t)$ - unknown function

The general form of a n^{th} order linear diff. equation

$x^{(n)}(t) + a_{n-1}(t)\cdot x^{(n-1)}(t) + \ldots + a_1(t)\cdot x'(t) + a_0(t)\cdot x(t)$

$= f(t)$ → non-homogenous part

$x' = 0$

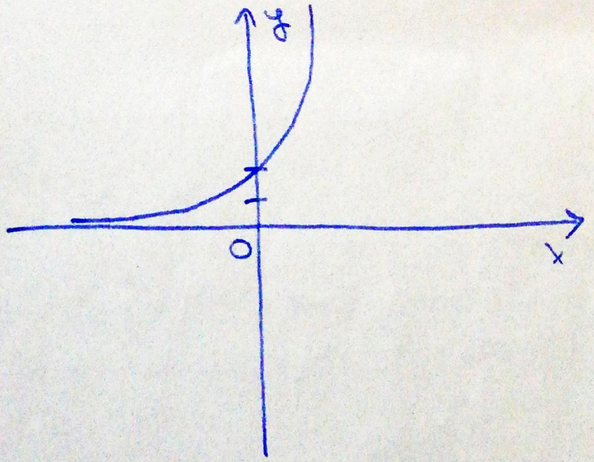
$x'(t) = 0$

$$\int_{t_0}^{t} x'(s)\,ds = \int_{t_0}^{t} 0s = 0 \Rightarrow x(s)\Big|_{t_0}^{t} = x(t) - \underbrace{x(t_0)}_{\in\mathbb{R}}^{=0} \Rightarrow x(t) = c, \quad c\in\mathbb{R}$$

1) Show that the function $\varphi:\mathbb{R}\to\mathbb{R}$, $\varphi(t) = 2e^{3t}$, $\forall t\in\mathbb{R}$ is a solution of the Initial Value Problem (IVP):
$x' = 3x$, $x(0) = 2$

$\varphi'(t) = (2e^{3t})' = 2\cdot 3\cdot e^{3t} = 6e^{3t} = 3\cdot 2\cdot e^{3t} = 3\cdot\varphi(t)$

$\varphi(0) = 2\cdot e^0 = 2\cdot 1 = 2$



$\varphi'(t) = 6e^{3t} > 0 \Rightarrow \varphi$ is incr.

$\lim_{t\to-\infty}\varphi(t) = 0$

$\lim_{t\to\infty}\varphi(t) = +\infty$, unbounded

Ox axis is a horizontal asymp

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2) Let $\eta \in \mathbb{R}^*$ be fixed.
Show that the function $\varphi: \mathbb{R} \to \mathbb{R}$, $\varphi(t) = \eta \cdot \sin t$ is a solution of the IVP.

$$x'' + x = 0 \qquad x(0) = 0,\ x'(0) = \eta$$

$\varphi(t) = \eta \sin t$

$\varphi'(t) = \eta \cdot \cos t$

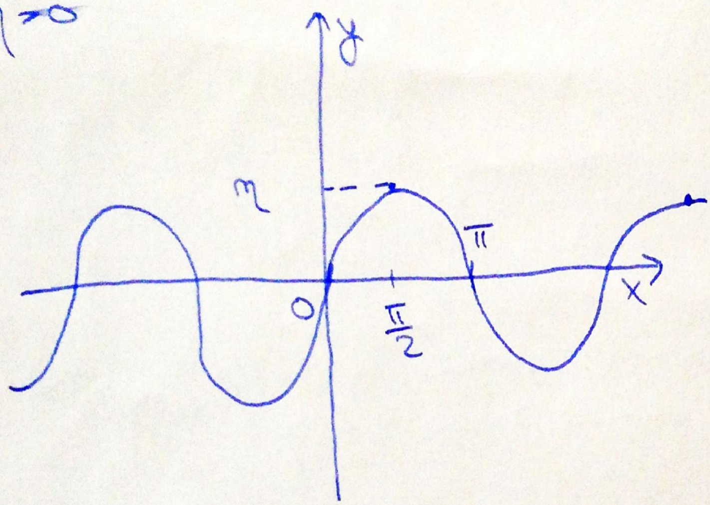
$\varphi''(t) = -\eta \cdot \sin t$

$x'' + x = -\eta \cdot \sin t + \eta \sin t = \eta(\sin t - \sin t) = 0$

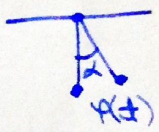
$x(0) = \varphi(0) = \sin 0 \cdot \eta = 0$

$x'(0) = \varphi'(0) = \eta \cdot \cos 0 = \eta$

$\eta > 0$



- periodic with period 2π
- bounded by η and $-\eta$, because $\eta \cdot \sin t \in [-\eta, \eta]$
- $\lim\limits_{t \to \pm\infty} \varphi(t)$ doesn't exist, oscillating around 0
- $\varphi(t) = \varphi(t + 2\pi)$



$\varphi(0) = 0$

$\varphi'(0) = \eta$

(it never stops; it doesn't have a limit)

3) Show that the function $\varphi(t) = e^{-2t} \cos t$ is a solution of the IVP:

$$x'' + 4x' + 5 = 0,\ x(0) = 1,\ x'(0) = 2$$

$\varphi'(t) = \cancel{-2e^{-2t}} - 2e^{-2t} \cos t - e^{-2t} \sin t = -\cancel{2}e^{-2t}(2\cos t + \sin t)$

$\varphi''(t) = \cancel{4e^{-2t}}\ 2e^{-2t}(2\cos t + \sin t) - e^{-2t}(-2\sin t + \cos t) =$

$\cancel{4e^{-2t} - 8e^{-2t} + 5 = -4e^{-2t} + 5}$

$= e^{-2t}(4\cos t + 2\sin t + 2\sin t - \cos t)$

$= e^{-2t}(4\sin t + 3\cos t)$

$f''(t) + 4f'(t) + 5 =$

$e^{-2t}(4\sin t + 3\cos t) + 4(-e^{-2t})(2\cos t + \sin t) + 5 =$

$= e^{-2t}(4\sin t + 3\cos t - 8\cos t - 4\sin t) + 5 =$

$= e^{-2t} \cdot (-5\cos t) + 5 = \ldots$

$f(0) = e^0 \cdot \cos 0 = 1$

$f'(0) = -2e^0 \cdot \cos 0 - e^0 \sin 0 = -2$

5) Find all constant solutions: $\Rightarrow x(t) = c,\ c \in \mathbb{R}$

a) $x' = x - x^3$

$x(t) = c,\ c \in \mathbb{R}$

$0 = x(t) - x^3(t) = x(1 - x^2) = x(1-x)(1+x)$

$\Rightarrow x(t) \in \{-1, 0, 1\}$

b) $x' = \sin x$

$x(t) = c,\ c \in \mathbb{R}$ $\cancel{\Rightarrow 0 = \cos x}$

$0 = \sin(x(t)) \Rightarrow x(t) = \arcsin 0 + k\pi,\ k \in \mathbb{R}$

$\Rightarrow x(t) = k\pi,\ k \in \mathbb{Z}$

4) $x' + x = 0$

$x' = -x$

$x(t) = e^{-t} \cdot c$

$x' + x = 0 \mid \cdot e^t > 0$

$x' \cdot e^t + x \cdot e^t = 0 \mid \Rightarrow x'(t) \cdot e^t + x(t) \cdot e^t = 0$

$\Leftrightarrow (x \cdot e^t)' = 0 \Rightarrow x \cdot e^t = c,\ c \in \mathbb{R}$

$\Downarrow$

$x(t) = c \cdot e^{-t},\ c \in \mathbb{R}$

8) Find $r \in \mathbb{R}$ s.t. $x(t) = e^{rt}$ is a solution of $x'' - 5x' + 6x = 0$.

$x'(t) = (e^{rt})' = r \cdot e^{rt}$

$x''(t) = r \cdot r \cdot e^{rt} = r^2 \cdot e^{rt}$

$\Rightarrow r^2 \cdot e^{rt} - 5r \cdot e^{rt} + 6e^{rt} =$

$\cancel{r^2 e^{rt} + e^{rt}}$

$e^{rt}(r^2 - 5r + 6) = 0$

$e^{rt} > 0,\ \forall r \in \mathbb{R}$

$r^2 - 5r + 6 = 0 \Rightarrow r_1 = 2,\ r_2 = 3$

$\Rightarrow r^2 + r - \cancel{6r + 6} = 0$

$r(r+1) - 6(r+1) = 0$

$(r-6)(r+1) = 0$

$\Rightarrow x(t) \in \{e^{2t}, e^{3t}\}$

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9 martie 2022

Seminar 2

Find the general solutions of the following differential equations.

1) $x' + 6x = 0 \xrightarrow{LHDE} r+6=0$ characteristic equation

$r+6=0 \Rightarrow r=-6$ is the solution

$-6 \in \mathbb{R} \Rightarrow e^{-6t}$ is a solution of the differential eq.

$\Rightarrow x(t) = c \cdot e^{-6t}$ is the general solution.

2) $x''=0 \longrightarrow r^2=0 \Rightarrow r=0$ is a solution with multiplicity 2

$\Rightarrow r_{1,2}=0 \longrightarrow e^{0\cdot t}, t\cdot e^{0\cdot t}$ are the solutions of the LHDE

$\Rightarrow x(t) = c_1\cdot e^0 + c_2\cdot t\cdot e^0 = c_1 + c_2 t,\ c_{1,2}\in\mathbb{R}$

If $r\in\mathbb{R}$ is a solution of the ch. eq. $\Rightarrow$ ~~e^{rt} is a solut~~

$\Rightarrow e^{rt}$ is a solution of the DE

If r has multiplicity $m\geq 1 \Rightarrow$ the solutions of the DE are $e^{rt}, t\cdot e^{rt}, \dots, t^{m-1}\cdot e^{rt}$

$\Rightarrow$ general sol. $x(t) = c_1\cdot e^{rt} + c_2\cdot te^{rt} + \dots + c_m\cdot t^{m-1} e^{rt}$

3) $x'' - 2x' - 15x = 0 \Rightarrow r^2 - 2r - 15 = 0 \to$ ch. eq.

$\Delta = 4 + 4\cdot 15 = 4\cdot 16 = (2\cdot 4)^2 = 8^2$

$r_{1,2} = \frac{2\pm 8}{2}$ — $r_1 = -3 \in \mathbb{R}$, $r_2 = 5 \in \mathbb{R}$

~~x~~ $\Rightarrow e^{-3t}, e^{5t} \Rightarrow$ sol of the DE

- gen. sol $x(t) = c_1\cdot e^{-3t} + c_2\cdot e^{5t},\ c_1, c_2\in\mathbb{R}$

4) $x^{(4)} - x = 0 \to r^4 - 1 = 0 \Leftrightarrow (r^2+1)(r^2-1)=0$

$(r^2+1)(r-1)(r+1)=0$

$\Rightarrow r_{1,2}\in\{\pm 1\} \to e^{-t}, e^{t}$ solutions

$r_{3,4}\in\{\pm i\} \to \cos t, \sin t$ (solutions), $e^{it} = \cos t + i\sin t$

$e^{-it} = \cos t - i\sin t$

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$e^{a+bi} = e^a \cdot e^{bi}$

$10^i = a+bi, \quad i^2 = -1$

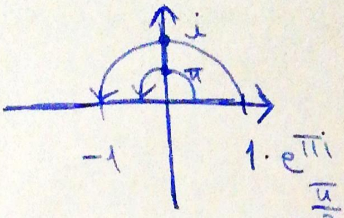
$10^{-i} = a-bi, \quad (-i)^2 = i^2 = 1$

$1 = a^2 + b^2$

$1 = \sin^2 + \cos^2$

$e^{\theta i} = \cos\theta + i\sin\theta$

$e^{\pi i} = -1$



$10^{\pi i} = e^{\ln 10 \cdot \pi \cdot i}$

$\Rightarrow$ general solution $x(t) = c_1 \cdot e^{-t} + c_2 \cdot e^{t} + c_3 \cdot \cos t + c_4 \cdot \sin t$
$c_{1,2,3,4} \in \mathbb{R}$.

5) Find the homogeneous differential equation with constant coefficient, having the following solutions with minimal order.

a) $5t \cdot e^{-3t}$ and $-3e^{5t}$ $\longrightarrow$ the ch. eq.

$\downarrow \qquad\qquad \downarrow$

$r_{1,2} = -3 \qquad r_3 = 5$

$$(r-5)(r+3)^2 = 0$$
$$(r-5)(r^2+6r+9) = 0$$
$$r^3 + 6r^2 + 9r - 5r^2 - 30r - 45 = 0$$
$$r^3 + r^2 - 21r - 45 = 0$$

$\longrightarrow x^{(3)} + x^{(2)} - 21x' - 45x = 0$

The general solution

$x(t) = c_1 \cdot t \cdot e^{-3t} + c_2 \cdot e^{5t} + c_3 \cdot e^{-3t}, \quad c_1, c_2, c_3 \in \mathbb{R}$

6) $e^{5t} \cdot \sin 3t$

$\downarrow$

$\left.\begin{array}{l} r_1 = 5+3i \\ r_2 = 5-3i \end{array}\right\} \Rightarrow$ char. eq.: $(r-r_1)(r-r_2) = 0$

$$(r-5-3i)(r-5+3i) = 0$$
$$r^2 - 10r + 34 = 0$$

$x'' - 10x' + 34x = 0$

$\Rightarrow$ general solution $x(t) = c_1 \cdot e^{5t} \cdot \sin 3t + c_2 \cdot e^{5t} \cdot \cos 3t$
$c_{1,2} \in \mathbb{R}$

Find $\lambda \in \mathbb{R}$ parameter with the property that there exist nonnull 2π - periodic solutions of $x'' + \lambda x = 0$.

- char. eq: $r^2 + \lambda = 0 \Rightarrow r^2 = -\lambda \cancel{\Rightarrow r = \pm\sqrt{\lambda}}$

Case I: $\lambda = 0 \Rightarrow r^2 = 0 \Rightarrow r_{1,2} = 0 \Rightarrow$ multiplicity 2.

- general solution $x(t) = c_1 \cdot t \cdot e^{0t} + c_2 \cdot e^{0t}$, $c_{1,2} \in \mathbb{R}$
$= c_1 t + c_2$ not 2π - periodic

Case II: $\lambda < 0 \Rightarrow r_{3,4} \in \mathbb{R} \Rightarrow c_1 \cdot e^{\sqrt{\lambda} t} + c_2 \cdot e^{-\sqrt{\lambda} t}$ not 2π periodic

Case III $\lambda > 0 \Rightarrow r_{1,2} = \pm i\sqrt{\lambda} \Rightarrow$

$\Rightarrow x(t) = c_1 \cdot \cos\sqrt{\lambda} \cdot t + c_2 \sin\sqrt{\lambda} \cdot t =$

$= x(t + \frac{2\pi}{\sqrt{\lambda}})$, sin and cos being 2π-periodic

$\cos(t + 2\pi) = \cos t$

$\cancel{\cos(\sqrt{\lambda} t + \frac{2\pi}{\sqrt{\lambda}})} = \cos\left(\cancel{\frac{\lambda t + 2\pi}{\sqrt{\lambda}}}\right)$

$\cos\left(\sqrt{\lambda}\left(t + \frac{2\pi}{\sqrt{\lambda}}\right)\right) = \cos(\sqrt{\lambda} + 2\pi) = \cos(\sqrt{\lambda}\, t)$

$\Rightarrow x(t)$ is a periodic function with period $= \frac{2\pi}{\sqrt{\lambda}}$,

so $\lambda = 1$, for $x(t)$ to be a 2π periodic

$r_1 = 3 + 2i$
$r_2 = 3 - 2i$
$\Rightarrow e^{3t} \sin 2t$, $t \cdot e^{3t} \cdot \sin 2t$
$e^{3t} \cos 2t$, $t \cdot e^{3t} \cdot \cos 2t$

Find all the solutions of the following boundary value problem (BVP): $x'' + x = 0$, $x(0) = x(\pi) = 0$.

$r^2 + 1 = 0$ - ch. eq. $\Rightarrow r^2 = -1 \Rightarrow r_{1,2} = \pm i$

$\Rightarrow x(t) = c_1 \cdot \cos t + c_2 \cdot \sin t$, $c_{12} \in \mathbb{R}$

$x(0) = c_1 \cdot \cos 0 + c_2 \cdot \sin 0 = c_1 = 0$

$x(\pi) = c_1 \cdot \cos\pi + c_2 \cdot \sin\pi = \cancel{c_1 (-1)}\ c_1 \cdot (-1) + c_2 \cdot 0 = -c_1$

$\Rightarrow x(t) = c_2 \cdot \sin t = c \cdot \sin t$, $c \in \mathbb{R}$

Seminar 3
24.03.2022

$\lambda > 0$ and let $\varphi(\cdot, \lambda)$ be the solution of the IVP.
$$x'' - 4x = e^{\lambda t}, \quad x(0) = x'(0) = 0$$

i) When $\lambda \neq 2$ then find ~~the~~ a solution of the form
$x_p(t) = a \cdot e^{\lambda t}$ for $x'' - 4x = e^{\lambda t}$. $(a = ?)$

$x_p' = a\lambda \cdot e^{\lambda t} \Rightarrow x_p'' = a\lambda^2 e^{\lambda t}$

$a \cdot \lambda^2 \cdot e^{\lambda t} - 4 \cdot a \cdot e^{\lambda t} = e^{\lambda t}$

$e^{\lambda t}(a \cdot \lambda^2 - 4a) = e^{\lambda t} \Rightarrow a\lambda^2 - 4a = 1$

$a(\lambda^2 - 4) = 1 \Rightarrow a = \frac{1}{\lambda^2 - 4}$

$\Rightarrow x_p(t) = \frac{1}{\lambda^2 - 4} \cdot e^{\lambda t}$

ii) Find a solution of the form $x_p = a \cdot t \cdot e^{2t}$ for
$x'' - 4x = e^{2t}$

$x_p' = ae^{2t} + at \cdot 2e^{2t}$

$x_p'' = 2ae^{2t} + 2a \cdot e^{2t} + at \cdot 4 \cdot e^{2t}$

$$\left.\begin{aligned} x_p'' - 4x_p &= 4ae^{2t} + 4ate^{2t} - 4 \cdot at \cdot e^{2t} = 4ae^{2t} \\ &= e^{2t} \end{aligned}\right\} \Rightarrow$$

$\Rightarrow 4a = 1 \Rightarrow a = \frac{1}{4}$

$\Rightarrow x_p(t) = \frac{1}{4} \cdot t \cdot e^{2t}$

$x'' - 4x = e^{\lambda t}$

HDE: $x'' - 4x = 0$

$r^2 - 4 = 0$

$r = \pm 2 \Rightarrow e^{2t}, e^{-2t}$

$\longrightarrow x_h(t) = c_1 \cdot e^{-2t} + c_2 \cdot e^{2t}$, $c_1, c_2 \in \mathbb{R}$.

I $\lambda \neq 2 \Rightarrow x(t) = x_h(t) + x_p(t) = c_1 \cdot e^{-2t} + c_2 \cdot e^{2t} + \frac{1}{\lambda^2 - 4} \cdot e^{\lambda t}$, $c_i \in \mathbb{R}$

II $\lambda = 2 \Rightarrow x(t) = c_1 \cdot e^{-2t} + c_2 \cdot e^{2t} + \frac{t}{4} \cdot e^{2t}$, $c_i \in \mathbb{R}$

$x'' + x = \cos(\lambda t)$

$x_p = a \cdot \cos \lambda t + b \cdot \sin \lambda t$, $\lambda \neq 1$

$x_p = t(a \cdot \cos t + b \sin t)$, $\lambda = 1$

$x' + a(t) \cdot x = f(t)$, $t \in I$

$$x' + \underbrace{\frac{1}{t}}_{a(t)} x = \frac{1}{t} \cdot e^{-2t+1}$$

$$A(t) = -\int_{t_0}^{t} a(s)\, ds$$

$A'(t) = a(t)$

$A(t_0) = 0$

$\mu(t) = e^{-A(t)}$ - integrating function

$\varphi(t) \cdot e^{A(t)} = x_p(t)$ - particular solution

$$x' + \frac{1}{t} x = \frac{1}{t} \cdot e^{-2t+1}, \quad t \in (0, \infty)$$

Method 1: $x = x_h + x_p$

x_h: $x' + \frac{1}{t} x = 0$

$$\frac{dx}{dt} + \frac{x}{t} = 0$$

$$\frac{dx}{dt} = -\frac{x}{t}$$

$$\frac{dx}{x} = -\frac{dt}{t} \Big| \int$$

$$\int \frac{dx}{x} = -\int \frac{dt}{t}$$

$\ln|x| = -\ln t + c$

$\Rightarrow |x| = e^c \cdot \frac{1}{t}$

$$\Rightarrow x = \begin{cases} \pm e^c \cdot \frac{1}{t}, & c \in \mathbb{R} \\ 0 \end{cases}$$

$\Rightarrow x_h(t) = c_1 \cdot \frac{1}{t}$, $c_1 \in \mathbb{R}$

general solution

You can guess one solution that is non-zero

$x(t) = \frac{1}{t}$ is a solution $\Rightarrow x_h(t) = c \cdot \frac{1}{t}$ general solution

$x_p = ?$ Lagrange method $\Rightarrow x_p(t) = \varphi(t) \cdot e^{A(t)} = \varphi(t) \cdot e^{-\ln t}$

$= \varphi(t) \cdot \frac{1}{t}$

$a(t) = \frac{1}{t}$

$A(t) = -\ln t$

$$\Rightarrow x_p' + \frac{1}{t} \cdot x_p = \frac{1}{t} \cdot e^{-2t+1}$$

$$p'(t) \cdot \frac{1}{t} + p(t) \cdot \left(-\frac{1}{t^2}\right) + \frac{1}{t} \cdot p(t) \cdot \frac{1}{t} = \frac{1}{t} \cdot e^{-2t+1}$$

$$\Rightarrow p'(t) = e^{-2t+1} \Rightarrow p(t) = -\frac{1}{2} e^{-2t+1} \Rightarrow$$

$$\Rightarrow x_p = -\frac{1}{2t} \cdot e^{-2t+1}$$

$$\Rightarrow x(t) = x_h(t) + x_p(t) = c \cdot \frac{1}{t} - \frac{1}{2t} \cdot e^{-2t+1}, \quad c \in \mathbb{R}$$

Seminar 4

7.04.2022

$x' + \frac{1}{t}x = \frac{1}{t}e^{-2t+1}$

$a(t) = \frac{1}{t}$, $A(t) = -\int_{t_0}^{t} a(s)ds = -\ln t$

$\mu(t) = e^{-A(t)}$ – integrating factor

$\mu(t) = e^{\ln t} = t$

$tx' + x = e^{-2t+1} \Rightarrow (xt)' = e^{-2t+1} \Rightarrow xt = -\frac{1}{2}e^{-2t+1} + c \mid \cdot \frac{1}{t}$

$\Rightarrow x(t) = \underbrace{\frac{1}{2t}\cdot e^{-2t+1}}_{x_p} + \underbrace{\frac{c}{t}}_{x_h}$, $c \in \mathbb{R}$

$x = t^r$

$x' = r \cdot t^{r-1}$

$x'' = r(r-1)t^{r-2}$

$e^{t-1}(t + \sin 2t)$

$\parallel$

$\frac{1}{e}e^{t} \to r = 1$ mult.$=2$

$r = 1 \pm 2i$

$e^t, t\cdot e^t, e^t\cdot \sin 2t, e^t \cdot \cos 2t$

1. Let $\eta \in \mathbb{R}$ be a fixed parameter.

a) Find $\varphi_\eta : I_\eta \to \mathbb{R}$ the unique solution of the IVP

$\uparrow$ open interval

$\begin{cases} x' = -x \\ x(0) = \eta \end{cases}$ (0 is a solution)

b) Study the properties of φ_η, find its image $= \gamma_\eta$, represent its graph with respect to η.

a) $x' = -x \Rightarrow \frac{dx}{dt} = -x \Rightarrow \frac{dx}{-x} = dt \Rightarrow \frac{dx}{x} = -dt \Rightarrow \int \frac{1}{x}\,dx = -\int 1\,dt$

$\Rightarrow \ln|x| = -t + C,\ C \in \mathbb{R}$

$\Rightarrow \pm x = e^{-t+c} \Rightarrow x = (\pm e^{c}) \cdot e^{-t}$, $c_1 \in \mathbb{R}$

$\Rightarrow x(t) = c_1 \cdot e^{-t}$ – general solution

$x(0) = c_1 \cdot e^{-0} = c_1 = \eta$

$\Rightarrow \varphi_\eta(t) = \eta \cdot e^{-t}$

b) $\varphi_\eta(t) = \eta \cdot e^{-t}$

I $\eta > 0 \Rightarrow \gamma_\eta = (0, +\infty)$

$\lim\limits_{t \to -\infty} (\eta \cdot e^{-t}) = +\infty$

$\lim\limits_{t \to +\infty} (\eta \cdot e^{-t}) = 0$

$\Rightarrow \varphi_\eta$ is strictly decreasing

II $\eta < 0 \Rightarrow \gamma_\eta = (-\infty, 0)$

φ_η is strictly increasing

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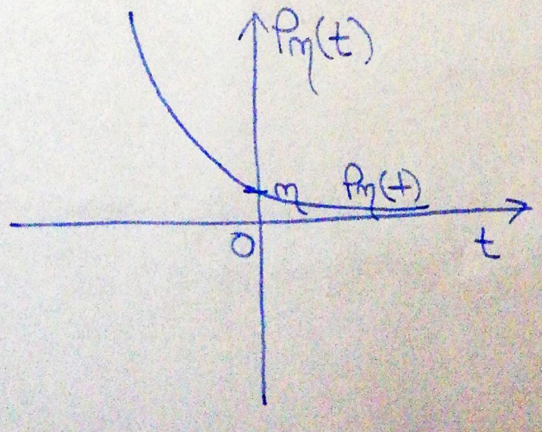
III $\eta = 0 \Rightarrow \cancel{\gamma_\eta = 0}$ $\varphi_\eta = 0 \Rightarrow \gamma_\eta = 0$

$\eta > 0$

$\varphi_\eta(0) = \eta$

$\varphi_\eta(1) = \frac{\eta}{e}$



c) Represent γ_η on a horizontal line and insert an arrow to indicate the future.

$\varphi_\eta(t)$

$$\longrightarrow \underset{\gamma_0}{\times} \longleftarrow$$ phase portrait

$x' = -x$

$\eta_* = 0 \quad \gamma_0 = \{0\} \Rightarrow 0$ equilibrium point attractor

η_* equilibrium $\Leftrightarrow \varphi_{\eta_*}(t) = \eta_*,\ \forall t \in I_{\eta_*}$

η_* attractor $\Leftrightarrow \exists\ \forall V \in \mathcal{V}(\eta_*) : \forall \eta \in V$

$$\lim_{t\to\infty} \varphi_\eta(t) = \eta_*$$

$\varphi_1(t) = e^{t}$ $\qquad\qquad \varphi_{-1}(t) = -e^{-t}$

$\lim_{t\to\infty} \varphi_1(t) = 0$

η_* repeller $\Leftrightarrow \lim_{t\to\infty} \varphi_\eta(t) = \eta_*$

$\varphi_\eta(t)$ - flow, η - initial state

2) $x' = x$

3) $x' = 1 - x^2$, constant solutions $\{-1, 1\}$

$x' = 1 - x^2 \Rightarrow \frac{dx}{dt} = 1 - x^2 \Rightarrow \frac{dx}{1-x^2} = dt$

$$\frac{dx}{(1-x)(1+x)} = \cancel{\frac{(1-x)-(1+x)}{}} \cancel{= \frac{1}{2}\left(\frac{1}{1-x}\right.} = -\frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{1+x}\right)$$

$$\frac{-2\,dx}{1-x^2} = -2\,dt \Rightarrow -2\int \frac{dx}{1-x^2} = -2\int dt$$

$$\Rightarrow -2\int dt = \int\left(\frac{1}{x-1} - \frac{1}{x+1}\right)dx \Rightarrow$$

$$\Rightarrow \ln|x-1| - \ln|x+1| = -2t + C,$$

$$\ln \frac{|x-1|}{|x+1|} = -2t + \mathcal{C}$$

$$\ln\left|\frac{x-1}{x+1}\right| = -2t + \mathcal{C}$$

$$\frac{x-1}{x+1} = \pm e^{-2t+\mathcal{C}}$$

$$\frac{x-1}{x+1} = \underbrace{\pm e^{\mathcal{C}}}_{c_1 \in \mathbb{R}^*} \cdot e^{-2t} \Rightarrow \frac{x-1}{x+1} = c_1 \cdot e^{-2t} \Rightarrow \frac{x+1}{x-1} = \frac{e^{2t}}{c_1} \Rightarrow$$

$$\Rightarrow \frac{x-1}{x-1} + \frac{2}{x-1} = \frac{e^{2t}}{c_1}$$

$$x(t) = \frac{c \cdot e^{2t} + 1}{c \cdot e^{2t} - 1}$$

$$x(0) = \eta \Rightarrow \frac{c+1}{c-1} = \eta \Rightarrow 1 - \frac{2}{c-1} = \eta \Rightarrow \frac{+2}{c-1} = \eta - 1 \Rightarrow c - 1 = \frac{2}{\eta - 1}$$

$$\Rightarrow c = \frac{2 + \eta - 1}{\eta - 1} \Rightarrow c = \frac{\eta + 1}{\eta - 1}$$

$$\varphi_\eta(t) = \frac{\frac{\eta+1}{\eta-1} \cdot e^{2t} + 1}{\frac{\eta+1}{\eta-1} \cdot e^{2t} - 1} = \frac{(\eta+1) \cdot e^{2t} + (\eta - 1)}{(\eta+1) \cdot e^{2t} - (\eta - 1)} \quad \text{flow}$$

Choose $\eta \in \{-2, -1, 0, 1, 2\}$

$\varphi_1(t) = 1 \Rightarrow \gamma_1 = \{1\} \Rightarrow 1$ is an equilibrium

$\varphi_{-1}(t) = -1 \Rightarrow \gamma_{-1} = \{-1\} \Rightarrow -1$ is an equilibrium

$$\varphi_{-2}(t) = \frac{-e^{2t} - 3}{-e^{2t} + 3} = \frac{e^{2t} + 3}{e^{2t} - 3}$$

a) Prove that $\varphi_{-2} < -1$ and $\varphi_{-2}' < 0 \quad \forall t \in (-\infty, \ln\sqrt{3})$

$$\frac{e^{2t} + 3}{e^{2t} - 3} < -1 \iff \frac{\overbrace{e^{2t} \cancel{+3} + e^{2t} \cancel{-3}}^{>0}}{e^{2t} - 3} < 0$$

$$t < \ln\sqrt{3} \Rightarrow 2t < \ln 3 \Rightarrow e^{2t} < 3 \Rightarrow e^{2t} - 3 < 0$$

$\varphi'_{-2} = 1 - \underbrace{\varphi_{-2}^2}_{>1} < 0 \Rightarrow \varphi_{-2}$ strictly decreasing on $(-\infty; \ln\sqrt{3}) = I_{-2}$

$\lim\limits_{t\to-\infty} \varphi_{-2}(t) = -1$, $\lim\limits_{t\to\ln\sqrt{3}} \varphi_{-2}(t) = -\infty$

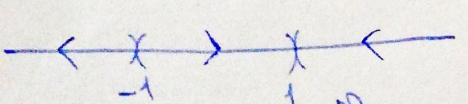
$\Rightarrow \gamma_{-2} = (-\infty, -1)$ – orbit

$\varphi_0(t) = \dfrac{e^{2t}-1}{e^{2t}+1}$ Hint: $-1 < \varphi_0(t) < 1$, $\forall t \in \mathbb{R}$

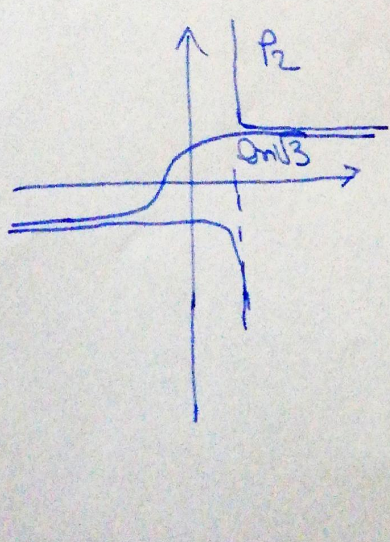
$\varphi_0'(t) = 1 - \underbrace{\varphi_0^2}_{\in[0,1)} > 0 \Rightarrow \varphi_0$ is strictly increasing

$$\left.\begin{array}{l} \lim\limits_{t\to-\infty} \dfrac{e^{2t}-1}{e^{2t}+1} = -1 \\ \lim\limits_{t\to\infty} \dfrac{e^{2t}-1}{e^{2t}+1} = 1 \end{array}\right\} \Rightarrow \gamma_0 = (-1, 1)$$

$\varphi_2(t) = \dfrac{3e^{2t}+1}{3e^{2t}-1} = \dfrac{e^{2t}+\frac{1}{3}}{e^{2t}-\frac{1}{3}} > 1$ and $\varphi_2' < 0$, $\forall t \in (-\ln\sqrt{3}, \infty)$



-1 repeller
1 attractor



$$\begin{cases} x' = x + 3y \\ y' = x - y \end{cases} \Rightarrow y = \frac{1}{3}(x' - x)$$

Reduction method

$$x' = x + 3y \Rightarrow x'' = x' + 3y'$$

$$\Rightarrow x'' = x' + 3(x - y)$$

$$x'' = x' + 3x - 3 \cdot \frac{1}{3}(x' - x)$$

$$x'' = 4x$$

$$x(t) = c_1 \cdot e^{2t} + c_2 \cdot e^{-2t}$$

Seminar 5
21.04.2022

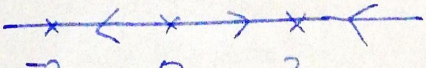
(S) 1. Represent the phase portrait of the scalar dynamical system $\dot{x}=2x-x^2$.
Study the stability of the equilibrium points using the linearization method. Find $\varphi(t,2)$, $\varphi(t,0)$ and study the properties of $\varphi(t,-2)$, $\varphi(t,1)$, $\varphi(t,3)$.
If there is an attractor, specify its basin of attraction.

$\dot{x}=2x-x^2 \Rightarrow f(x)=2x-x^2$

$f(x)=0 \Rightarrow x(2-x)=0 \Rightarrow x\in\{0,2\}$ eq. points

$f'(x)=2-2x$, $f'(0)=2>0 \to$ repeller, unstable

$f'(2)=-2<0 \to$ attractor, stable



Orbits: $(-\infty,0)$, $\{0\}$, $(0,2)$, $\{2\}$, $(2,\infty)$

$\varphi(t,2)=2$ $\quad \gamma_2=\{2\}$ $\quad \forall t\in\mathbb{R}$ $\quad I_2=\mathbb{R}$

$\varphi(t,0)=0$ $\quad \gamma_0=\{0\}$ $\quad I_0=\mathbb{R}$

$\varphi(t,-2)$ strictly decreasing

$\begin{cases}\dot{x}=2x-x^2\\ x(0)=\eta\end{cases}$

$\varphi(t,\eta)$

$\lim\limits_{t\to-\infty}\varphi(t,-2)=0$ $\quad \gamma_{-2}=(-\infty,0)$

$\varphi(t,1)$ strictly increasing

$\lim\limits_{t\to\infty}\varphi(t,1)=2$ $\quad I_1=\mathbb{R}$

$\lim\limits_{t\to-\infty}\varphi(t,1)=0$ $\quad \gamma_1=(0,2)$

$\varphi(t,3)$ strictly decreasing

$\lim\limits_{t\to}\varphi(t,3)=2$

$A_{\eta^*}=\{\eta\in\mathbb{R} \mid \lim\limits_{t\to\infty}\varphi(t,\eta)=\eta^*\}$

$A_2=(0,\infty)$ basin of attraction of 2

(a) 2. $\varphi(-,\eta)$ for any $\eta \in \mathbb{R}$

$\dot{x} = x - x^3 + 1$

$$\left.\begin{aligned} f(x) &= x - x^3 + 1 \\ f(x) &= 0 \end{aligned}\right\} \Rightarrow x - x^3 + 1 = 0$$

Case I: 3 solutions $\in \mathbb{R}$

Case II: 1 solution $\in \mathbb{R}$

$$\left.\begin{aligned} f'(x) &= 1 - 3x^2 \\ f'(x) &= 0 \end{aligned}\right\} \Rightarrow 3x^2 = 1 \Leftrightarrow x = \pm\frac{1}{\sqrt{3}}$$

x	$-\infty$		$-\frac{1}{\sqrt{3}}$		$\frac{1}{\sqrt{3}}$		$+\infty$
$f(x)$		$\searrow$	>0	$\nearrow$	>0	$\searrow$	$-\infty$
$f'(x)$	$-$	$-$	0	$+ + +$	0	$-$	$-$

$$f\left(-\frac{1}{\sqrt{3}}\right) = -\frac{1}{\sqrt{3}} + \frac{1}{3\sqrt{3}} + 1$$

$$= \cancel{\frac{-3\sqrt{3}}{\ }} = 1 - \frac{2}{3\sqrt{3}} >$$

$\lim\limits_{x \to \infty} f(x) = -\infty \Rightarrow f$ has exactly 1 sol, $\eta \in \left(\frac{1}{\sqrt{3}}, +\infty\right)$

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$-\infty \longrightarrow \ast \longleftarrow +\infty$

$\eta^* > 0$

Orbits: $(-\infty, \eta^*)$, $\{\eta^*\}$, $(\eta^*, +\infty)$

$f(\eta^*) < 0 \Rightarrow \eta^*$ is an attractor

If $\eta \in (-\infty, \eta^*) \Rightarrow \varphi(t,\eta)$ is strictly increasing

If $\eta \in (\eta^*, +\infty) \Rightarrow \varphi(t,\eta)$ is strictly decreasing

$A_{\eta^*} = (-\infty, +\infty) = \mathbb{R}$

If $\eta \in (-\infty, \eta^*)$: $\lim\limits_{\eta \to +\infty} \varphi(t,\eta) = \eta^*$

If $\eta \in (\eta^*, +\infty)$: $\lim\limits_{\eta \to -\infty} \varphi(t,\eta) = \eta^*$

$\Rightarrow A_{\eta^*} = \mathbb{R}$

$f(1) > 0$, $f(2) = < 0$

First integrals

$$\begin{cases} \dot{x} = f_1(x,y) \\ \dot{y} = f_2(x,y) \end{cases}$$

$\mathcal{U} \subseteq \mathbb{R}^2$

$H: \mathcal{U} \to \mathbb{R}$ first integral if it's not constant and

$H(\varphi(t,\eta)) = H(\eta)$, $\eta \in \mathcal{U}$, $t \in I_\eta$

$\Updownarrow$

$$f_1(x,y) \cdot \frac{\partial H}{\partial x}(x,y) + f_2(x,y) \cdot \frac{\partial H}{\partial y}(x,y) = 0, \ \forall (x,y) \in \mathcal{U}$$

Find a first integral

$$\begin{cases} \dot{x} = -2x \\ \dot{y} = 3y \end{cases}$$

$$\frac{dy}{dx} = \frac{f_2(x,y)}{f_1(x,y)}$$

If $H(x,y) = C$, $C \in \mathbb{R}$

$\Downarrow$

H is a first integral.

$$\frac{dy}{dx} = \frac{3y}{-2x} \iff \cancel{\frac{-2x}{dx} = \frac{3y}{3dy}} \quad \frac{dy}{3y} = \frac{dx}{-2x} \Big| \int \Rightarrow \frac{1}{3}\ln|y| = \frac{-1}{2}\ln|x| + C_1, \ C_1 \in \mathbb{R}$$

$\iff \ln|y| = -\frac{3}{2}\ln|x| + C$, $C \in \mathbb{R}$

$\ln|y| + \frac{3}{2}\ln|x| = C \Rightarrow H(x,y) = \ln|y| + \frac{3}{2}\ln|x|$ first integral

$\ln|y| + \frac{3}{2}\ln|x| = C \Rightarrow \ln|y \cdot \sqrt{x^3}| = C \Rightarrow y \cdot \sqrt{x^3} = e^C = K \in \mathbb{R} \Rightarrow$

$\Rightarrow H_2(x,y) = y\sqrt{x^3}$.

b) $$\begin{cases} \dot{x} = y \\ \dot{y} = -4\sin x \end{cases}$$

$$\frac{dy}{dx} = \frac{-4\sin x}{y} \iff y\,dy = -4\sin dx \ \Big| \int$$

$$\frac{y^2}{2} = 4\cos x + C$$

$$\frac{y^2}{2} - 4\cos x = C$$

$$\begin{cases} \dot{x} = x - xy \\ \dot{y} = -0.3y + 0.3xy \end{cases} \Rightarrow H(x,y) = \frac{y^2}{2} - 4\cos x \text{, the first integral}$$

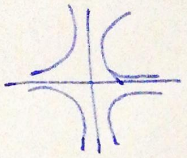
$$\frac{dy}{dx} = \frac{-0.3y + 0.3xy}{x - xy} = \frac{-0.3y(1+x)}{x(1-y)}$$

$$\frac{1-y}{y}\,dy = -0.3 \cdot \frac{1+x}{x}\,dx \Big| \int$$

$\dot{X} = AX$, $A \in M_2(\mathbb{R})$, $\det A \neq 0$, $\lambda_1, \lambda_2 \in \mathbb{C}$ eigenvalues

The eq. point is the origin $(0,0)$.

node: $\underbrace{\lambda_1 \le \lambda_2 < 0}_{\text{asymptotically stable}}$ or $\lambda_2 \ge \lambda_1 > 0$ unstable



saddle: $\lambda_1 < 0 < \lambda_2$ – unstable

center: $\lambda_{1,2} = \cancel{\pm i\sqrt{3}, \sqrt{3}}$ $\pm i\beta$, $\beta \neq 0 \Rightarrow$ stable but not asymptotically

focus: $\lambda_{1,2} = \alpha \pm i\beta$, $\alpha \neq 0$, $\beta \neq 0$

Specify the type and stability of the linear system and find the flow.

a) $\begin{cases} \dot{x} = 4x - 5y \\ \dot{y} = x - 2y \end{cases}$ $\quad A = \begin{pmatrix} 4 & -5 \\ 1 & -2 \end{pmatrix}$

$\det(A - \lambda I) = 0$

$\lambda^2 - \operatorname{tr} A\,\lambda + \det A = 0$

$\lambda^2 - 2\lambda - 3 = 0 \Rightarrow (\lambda - 3)(\lambda + 1) = 0$

$\Rightarrow \lambda_1 = 3$, $\lambda_2 = -1$ eigenvalues

$\lambda_2 = -1 < 0 < 3 = \lambda_1 \Rightarrow$ unstable saddle

$\lambda_1 = 3$

$$\begin{pmatrix} 4-3 & -5 \\ 1 & -2-3 \end{pmatrix} = \begin{pmatrix} 1 & -5 \\ 1 & -5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -5 \\ 1 & -5 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow a - 5b = 0 \Rightarrow a = 5b$

$v_1 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ eigenvector associated to λ_1

$\lambda_2 = -1$

$$\begin{pmatrix} 5 & -5 \\ 1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$ eigenvector of λ_2

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \cdot \begin{pmatrix} 5 \\ 1 \end{pmatrix} \cdot e^{3t} + c_2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot e^{-t}, \quad c_1, c_2 \in \mathbb{R}$$

$$= \begin{pmatrix} 5c_1 e^{3t} + c_2 \cdot e^{-t} \\ c_1 \cdot e^{3t} + c_2 \cdot e^{-t} \end{pmatrix}$$

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \iff \begin{cases} 5c_1 + c_2 = \eta_1 \\ c_1 + c_2 = \eta_2 \end{cases}$$

$$c_1 = \frac{\eta_1 - \eta_2}{4}$$

$$c_2 = \frac{5\eta_2 - \eta_1}{4}$$

$$P(t, \eta) = \frac{\eta_1 - \eta_2}{4} \cdot \begin{pmatrix} 5 \\ 1 \end{pmatrix} e^{3t} + \frac{5\eta_2 - \eta_1}{4} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$$