Rezolvări:

1) $\lim_{n\to\infty} \sqrt{n+2}-\sqrt{n} = \lim_{n\to\infty} \frac{\cancel{n}+2-\cancel{n}}{\sqrt{n+2}+\sqrt{n}} = \frac{2}{\infty} = 0$

$$\sqrt{a}-\sqrt{b} = \frac{a-b}{\sqrt{a}+\sqrt{b}}$$

2) $\lim_{n\to\infty} \sqrt[3]{n^3+n^2} - \sqrt[3]{n^3-n^2} =$

$$= \lim_{n\to\infty} \frac{\cancel{n^3}+n^2-\cancel{n^3}+n^2}{\sqrt[3]{(n^3+n^2)^2}+\sqrt[3]{n^6-n^4}+\sqrt[3]{(n^3-n^2)^2}} =$$

$$= \lim_{n\to\infty} \frac{2n^2}{n^2\left(\sqrt[3]{\left(1+\frac{1}{n}\right)^2}+\sqrt[3]{1-\frac{1}{n^2}}+\sqrt[3]{\left(1-\frac{1}{n}\right)^2}\right)} = \frac{2}{3}$$

$$\sqrt[3]{a} - \sqrt[3]{b} = \frac{a-b}{\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2}}$$

3) $\lim_{n\to\infty} \sqrt[3]{6n^3+1} - \sqrt{6n^2+2} =$

$$= \lim_{n\to\infty} n\left(\sqrt[3]{6+\frac{1}{n^3}}-\sqrt{6+\frac{2}{n^2}}\right) = \infty\,\underbrace{\left(\sqrt[3]{6}-\sqrt{6}\right)}_{<0} = -\infty$$

4) Fie $(a_n)_{n\geq 1}$ cu $a_n > 0$

Dacă $l = \lim\limits_{n\to\infty} \dfrac{a_n}{a_{n+1}}$ $> 1 \Rightarrow a_n \to \infty$

$< 1 \Rightarrow a_n \to 0$

$$\frac{a_{n+1}}{a_n} = \frac{\cancel{a^{n+1}}^{\,a} \cdot \cancel{(n+1)!}^{\,n+1}}{(n+1)^{\cancel{n+1}}} \cdot \frac{n^n}{\cancel{a^n} \cdot \cancel{n!}} = a \cdot \left(\frac{n}{n+1}\right)^n =$$

$$= \frac{a}{\left(1+\frac{1}{n}\right)^n} \to \frac{a}{e}$$

Dacă $a > e \Rightarrow \dfrac{a^n n!}{n^n} \to \infty$

Dacă $a < e \Rightarrow \dfrac{a^n \cdot n!}{n^n} \to 0$.

5) $\lim\limits_{n\to\infty} \left(1+\sqrt{n+1}-\sqrt{n}\right)^{2\sqrt{n}} \overset{1^\infty}{=} \lim\limits_{n\to\infty}\left(1+\dfrac{n+1-n}{\sqrt{n+1}+\sqrt{n}}\right)^{2\sqrt{n}}$

$= \left[\left(1+\dfrac{1}{\sqrt{n+1}+\sqrt{n}}\right)^{\sqrt{n+1}+\sqrt{n}}\right]^{\frac{2\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}} =$

$= e^{\lim\limits_{n\to\infty}\frac{2\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}} = e$

6) $\lim\limits_{n\to\infty} \sqrt[n]{a^n+b^n+c^n}$

Presupunem $a \le b \le c$

$\lim\limits_{n\to\infty} \sqrt[n]{a^n+b^n+c^n} = \lim\limits_{n\to\infty} \sqrt[n]{\left(\frac{a}{c}\right)^n+\left(\frac{b}{c}\right)^n+1}\cdot c$

$= c \qquad \left(\left(\frac{a}{c}\right)^n, \left(\frac{b}{c}\right)^n \to \begin{matrix}0\\ \text{sau}\\ 1\end{matrix} \right)$

7) $\lim\limits_{n\to\infty}\left(\dfrac{a^{\frac{1}{n}}+b^{\frac{1}{n}}}{2}\right)^n \overset{1^\infty}{=} \lim\limits_{n\to\infty}\left(1+\dfrac{a^{\frac{1}{n}}+b^{\frac{1}{n}}-2}{2}\right)^n =$

$= \lim\limits_{n\to\infty}\left[\left(1+\dfrac{a^{\frac{1}{n}}+b^{\frac{1}{n}}-2}{2}\right)^{\frac{2}{a^{\frac{1}{n}}+b^{\frac{1}{n}}-2}}\right]^{\frac{n\left(a^{\frac{1}{n}}+b^{\frac{1}{n}}-2\right)}{2}}$

$= e^{\lim\limits_{n\to\infty}\frac{n\left(a^{\frac{1}{n}}-1\right)+n\left(b^{\frac{1}{n}}-1\right)}{2}} = e^{\frac{\ln a+\ln b}{2}} = \sqrt{ab}.$

$\lim\limits_{n\to\infty} n\left(a^{\frac{1}{n}}-1\right) = \lim\limits_{n\to\infty}\dfrac{a^{\frac{1}{n}}-1}{\frac{1}{n}} = \ln a.$

8) $\lim\limits_{n\to\infty} \sqrt[n]{n!} = \lim\limits_{n\to\infty} \sqrt[n]{x_n} = \lim\limits_{n\to\infty} \dfrac{x_{n+1}}{x_n} =$

$= \lim\limits_{n\to\infty} \dfrac{(n+1)\cancel{!}}{\cancel{n!}} = \infty$

9) $\lim\limits_{n\to\infty} \dfrac{\sqrt[n]{n!}}{n} = \lim\limits_{n\to\infty} \sqrt[n]{\dfrac{n!}{n^n}} =$

$= \lim\limits_{n\to\infty} \dfrac{(n+1)\cancel{!}}{(n+1)^{n+1}} \cdot \dfrac{n^n}{\cancel{n!}} = \lim\limits_{n\to\infty} \left(\dfrac{n}{n+1}\right)^n = \dfrac{1}{e}$

10) $\lim\limits_{n\to\infty} \sqrt[n]{C_{2n}^n} = \lim\limits_{n\to\infty} \dfrac{C_{2n+2}^{n+1}}{C_{2n}^n} =$

$= \lim\limits_{n\to\infty} \dfrac{\dfrac{((n+1)!)^2}{(2n+2)!}}{\dfrac{(n!)^2}{(2n)!}} = \lim\limits_{n\to\infty} \left(\dfrac{(n+1)\cancel{!}}{\cancel{n!}}\right)^2 \cdot \dfrac{(2n)!}{(2n+2)!} =$

$= \lim\limits_{n\to\infty} \dfrac{(n+1)^2}{(2n+1)(2n+2)} = \dfrac{1}{4}$

11) $\lim\limits_{n\to\infty} n\left(\sqrt[n]{n}-1\right) =$

$$= \lim_{n\to\infty} \frac{e^{\frac{1}{n}\ln n}-1}{\frac{1}{n}\ln n} \cdot \frac{1}{\cancel{n}}\ln n \cdot \cancel{n} = \infty$$

12) $$\lim_{n\to\infty} \frac{1+\frac{1}{2}+\dots+\frac{1}{n}}{\ln n} \overset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty} \frac{a_n}{b_n} \overset{C.S}{=} \lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} =$$

$$= \lim_{n\to\infty} \frac{\cancel{1}+\cancel{\frac{1}{2}}+\dots+\cancel{\frac{1}{n}}+\frac{1}{n+1}-\cancel{1}-\cancel{\frac{1}{2}}-\dots-\cancel{\frac{1}{n}}}{\ln n+1 - \ln n} =$$

$$= \lim_{n\to\infty} \frac{1}{n+1 \ln\left(1+\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{1}{\underbrace{\ln\left(1+\frac{1}{n}\right)^{n+1}}_{\to e}} = 1$$

13) $$\lim_{n\to\infty} \frac{1+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{n}}}{\sqrt{n}} =$$

$$= \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+1}}}{\sqrt{n+1}-\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}\,\frac{1}{\sqrt{n+1}+\sqrt{n}}} = 2$$

15) $$\lim_{n\to\infty} \frac{1^p+2^p+\ldots+n^p}{n^{p+1}} =$$

$$= \lim_{n\to\infty} \frac{\cancel{1^p+2^p+\ldots+n^p}+(n+1)^p - \cancel{1^p-2^p-\ldots-n^p}}{(n+1)^{p+1}-n^p} =$$

$$= \lim_{n\to\infty} \frac{(n+1)^p}{(n+1)^{p+1}-n^{p+1}} =$$

$$= \lim_{n\to\infty} \frac{\cancel{n^p}\left(1+\frac{1}{n}\right)^p}{\cancel{n^{p+1}}\left(\left(1+\frac{1}{n}\right)^{p+1}-1\right)} =$$

$$= \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^p}{\frac{\left(1+\frac{1}{n}\right)^{p+1}-1}{\frac{1}{n}}} = \frac{1}{p+1}$$

$$\lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \lim_{x\to 0} \frac{e^{\alpha\ln(1+x)}-1}{\alpha\ln(x+1)} \cdot \frac{\alpha\ln(x+1)}{x} =$$

$$= \alpha.$$

16) $$\lim_{n\to\infty} \frac{10^n+n^2+1}{6^{n+1}+3} \cdot \frac{21^n+1}{14^n+7} =$$

$$= \lim_{n\to\infty} \frac{\left(1+\frac{n^2}{10^n}+\frac{1}{10^n}\right)\overset{2^n}{\cancel{10^n}}}{\left(6+\frac{3}{6^n}\right)\underset{3^n}{\cancel{6^n}}} \cdot \frac{\overset{3^n}{\cancel{21^n}}\left(1+\frac{1}{21^n}\right)}{\underset{2^n}{\cancel{14^n}}\left(1+\frac{7}{14^n}\right)} =$$

$$= \frac{1}{6}$$

20) $$\lim_{n\to\infty} \frac{n!}{n\ln n} = \lim_{n\to\infty} \frac{\ln(n+1)! - \ln n!}{(n+1)\ln(n+1) - n\ln n} =$$

$$= \lim_{n\to\infty} \frac{\ln \frac{(n+1)!}{n!}}{\ln(n+1) + n(\ln(n+1) - \ln n)} =$$

$$= \lim_{n\to\infty} \frac{\ln(n+1)}{\ln(n+1) + \underbrace{\ln\left(1+\frac{1}{n}\right)^n}_{\to 1 = \ln e}} = 1$$

17) $\lim\limits_{n\to\infty} n\sqrt{n}\left(\sqrt{n+1}+\sqrt{n-1}-2\sqrt{n}\right) =$

$= \lim\limits_{n\to\infty} n\sqrt{n}\left(\sqrt{n+1}-\sqrt{n}+\sqrt{n-1}-\sqrt{n}\right)$

$= \lim\limits_{n\to\infty} n\sqrt{n}\left(\dfrac{n+1-n}{\sqrt{n+1}+\sqrt{n}} + \dfrac{n-1-n}{\sqrt{n-1}+\sqrt{n}}\right) =$

$= \lim\limits_{n\to\infty} n\sqrt{n}\left(\dfrac{1}{\sqrt{n+1}+\sqrt{n}} - \dfrac{1}{\sqrt{n}+\sqrt{n-1}}\right)$

$= \lim\limits_{n\to\infty} n\sqrt{n}\,\dfrac{\cancel{\sqrt{n}}+\sqrt{n-1}-\cancel{\sqrt{n}}-\sqrt{n+1}}{\left(\sqrt{n+1}+\sqrt{n}\right)\left(\sqrt{n}+\sqrt{n-1}\right)} =$

$= \lim\limits_{n\to\infty} \dfrac{\cancel{n}\sqrt{n}\,\left(n-1-n-1\right)}{\cancel{n}\left(\sqrt{1+\frac{1}{n}}+1\right)\left(1+\sqrt{1-\frac{1}{n}}\right)\left(\sqrt{n}+\sqrt{n+1}\right)} =$

$= \lim\limits_{n\to\infty} \dfrac{-2\sqrt{n}}{4\left(\sqrt{n}+\sqrt{n+1}\right.} = -\dfrac{1}{4}.$

18) $\lim\limits_{n\to\infty} \sqrt[n+1]{(n+1)!} - \sqrt[n]{n!} := l$

$$\sqrt[n]{n!} = e^{\frac{1}{n}\ln n!} \qquad a_n := \frac{1}{n}\ln n!$$

$$l = \lim_{n\to\infty} e^{a_{n+1}} - e^{a_n} = \lim_{n\to\infty} e^{a_n}\cdot\left(e^{a_{n+1}-a_n} - 1\right)$$

$$= \lim_{n\to\infty} \underbrace{\frac{\sqrt[n]{n!}}{n}}_{\to \frac{1}{e}} \cdot n \cdot \overbrace{\frac{e^{a_{n+1}-a_n} - 1}{a_{n+1}-a_n}}^{\to 1} \cdot \underbrace{(a_{n+1} - a_n)}_{\to 0}$$

$$a_{n+1} - a_n = \frac{1}{n+1}\ln(n+1)! - \frac{1}{n}\ln n! =$$

$$= \frac{\ln((n+1)!)^n - \ln(n!)^{n+1}}{n(n+1)} =$$

$$= \frac{\ln \frac{\cancel{(n!)^n}\cdot(n+1)^n}{\cancel{(n!)^n}\cdot n!}}{n(n+1)} = \frac{1}{n+1}\overbrace{\ln \frac{n+1}{\sqrt[n]{n!}}}^{\to \ln 1 =} \to 0$$

$$l = \lim_{n\to\infty} \frac{1}{e} \cdot n \cdot \frac{1}{n+1} \ln \frac{n+1}{\sqrt[n]{n!}} =$$

$$= \frac{1}{e}$$

1) $x_2 = a x_1 + b = a(a x_0 + b) + b =$

$= a^2 x_0 + a \cancel{x_0} + b$

inducție

$x_n = a^n x_0 + a^{n-1} b + a^{n-2} b + \dots + b$

$= a^n x_0 + b \cdot \dfrac{1 - a^n}{1 - a} \to \dfrac{b}{1-a}.$

2) $x_0 \in (0,1) \Rightarrow x_n \in (0,1)$

$x_{n+1} = x_n(1 - x_n) \in (0,1)$

$x_{n+1} - x_n = -x_n^2 \leq 0 \Rightarrow x_n \downarrow$

$\Rightarrow x_n \to l \Rightarrow$

$l = \lim\limits_{n\to\infty} x_n - x_n^2 = l - l^2 \Rightarrow l = 0$

$\lim\limits_{n\to\infty} n x_n \overset{\infty \cdot 0}{=} \lim\limits_{n\to\infty} \dfrac{n}{\frac{1}{x_n}} \underset{\frac{1}{x_n} \nearrow \infty}{\overset{C.S}{=}} \lim\limits_{n\to\infty} \dfrac{n+1-n}{\frac{1}{x_{n+1}} - \frac{1}{x_n}} =$

$= \lim\limits_{n\to\infty} \dfrac{x_{n+1} x_n}{x_n - x_{n+1}} = \lim\limits_{n\to\infty} \dfrac{x_n(1-x_n) \cdot x_n}{x_n^2} = 1.$

4) $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \geq 2 \cdot \frac{1}{2}\sqrt{\cancel{x_n} \cdot \frac{a}{\cancel{x_n}}} \geq \sqrt{a} > 0.$

$x_{n+1} - x_n$, $\frac{x_{n+1}}{x_n}$ pentru monotonie

$x_{n+1} - x_n = \frac{1}{2}\left(\frac{a}{x_n} - x_n\right) = \frac{1}{2}\frac{(a - x_n^2)}{x_n} \leq 0$

$\Rightarrow x_n \downarrow \; l \geq \sqrt{a}$. $\Rightarrow x_n$ este mărginit

$\Rightarrow l = \frac{1}{2}\left(l + \frac{a}{l}\right) \Rightarrow l^2 = a$ $\quad l = \sqrt{a}$

$\cancel{l = -\sqrt{a}}$

5) $x_{n+1} = \sqrt{2 + x_n} \geq 0 \Rightarrow$ $\quad x_{n+1} = \sqrt{2 + x_n} \geq \sqrt{2} \;\; \forall n \geq 1$

$f(x) = \sqrt{2 + x}$ $\quad f : (0, \infty) \to (0, \infty)$ $\quad f \nearrow$.

$x_{n+1} - x_n = f(x_n) - f(x_{n-1})$

Dacă $x_1 > x_0 \Rightarrow$ $\quad x_{n+1} > x_n$ $\quad x_n \nearrow$

Dacă $x_1 = x_0 \Rightarrow$ $\quad x_{n+1} = x_n$ $\quad$ constant

Dacă $x_1 < x_0 \Rightarrow$ $\quad x_{n+1} < x_n$ $\quad x_n \downarrow$.

$x_1 - x_0 = \sqrt{2 + x_0} - x_0 > 0 \Leftrightarrow \sqrt{2 + x_0} > x_0 \Leftrightarrow$

$2 + x_0 > x_0^2 \Leftrightarrow 0 > x_0^2 - x_0 - 2 = (x_0 - 2)(x_0 + 1) \Leftrightarrow$

$x_0 \in (-1, 2)$

$x_0 < 2 \xRightarrow{\text{inducție}} x_n < 2 \Rightarrow l = 2$.

$x_0 < 2 \Rightarrow x_n \nearrow \; l \leq \sqrt{5}$

$x_0 = 2 \Rightarrow x_n = 2 \Rightarrow l = 2$

$x_n > 2 \Rightarrow x_n \downarrow \; l \geq \sqrt{2} \Rightarrow l = 2$

6 a)

$$x_n = \frac{n^2}{2n^2+1} \qquad a = \frac{1}{2}$$

$$|x_n - a| < \varepsilon \iff \left| \frac{n^2}{2n^2+1} - \frac{1}{2} \right| < \varepsilon \iff$$

$$\frac{1}{4n^2+2} < \varepsilon \iff \frac{1}{\varepsilon} - 2 < 4n^2 \iff$$

$$n^2 > \frac{1}{4\varepsilon} - \frac{1}{2} \implies n_\varepsilon = \left[\sqrt{\frac{1}{4\varepsilon} - \frac{1}{2}}\right] + 1$$

b) $$\left|\sqrt{n+1} - \sqrt{n} - 0\right| < \varepsilon \iff \frac{1}{\sqrt{n+1}+\sqrt{n}} < \varepsilon \implies$$

$$\frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{2\sqrt{n+1}} < \varepsilon \implies \frac{1}{2\varepsilon} < \sqrt{n+1} \iff$$

$$n \geq \left(\frac{1}{2\varepsilon}\right)^2 - 1 \qquad n_\varepsilon = \left[\left(\frac{1}{2\varepsilon}\right)^2\right]$$

7a)

$x_n \to a \Rightarrow |x_n| \leq M$

$\forall \varepsilon > 0 \ \exists n_\varepsilon$ a.î. $\forall n \geq n_\varepsilon \Rightarrow |x_n - a| < \varepsilon$

$n > n_\varepsilon$

$$\left| \frac{x_1 + \dots + x_n}{n} - a \right| = \left| \frac{x_1 - a + x_2 - a + \dots + x_n - a}{n} \right|$$

$$\leq \frac{|x_1 - a|}{n} + \frac{|x_2 - a|}{n} + \dots + \frac{|x_{n_\varepsilon} - a|}{n_\varepsilon} + \dots + \frac{|x_n - a|}{n}$$

$$\leq \sum_{k=1}^{n_\varepsilon} \frac{\overbrace{|x_k - a|}^{\leq 2M}}{n} + \frac{n - n_\varepsilon}{n} \overbrace{|x_n - a|}^{\leq \varepsilon} \leq$$

$$\leq \underbrace{\frac{n_\varepsilon}{n} \cdot 2M}_{\substack{\downarrow n\to\infty \\ 0}} + \varepsilon .$$

Alegem $n > n'_\varepsilon \geq n_\varepsilon$ a.î. $\frac{n_\varepsilon}{n} \cdot 2M < \varepsilon$

$\Rightarrow n > n'_\varepsilon \Rightarrow \left| \frac{x_1 + \dots + x_n}{n} - a \right| \leq 2\varepsilon$