16.12.2020

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calculați:

Ex 1 a) $\sum_{n\geq 1} \frac{1}{(6n+5)\cdot(6n+11)}$

$$x_n = \frac{1}{(6n+5)(6n+11)} = \frac{1}{6} \cdot \frac{6}{(6n+5)(6n+11)} =$$

$$= \frac{1}{6} \cdot \frac{6n+11-(6n+5)}{(6n+5)(6n+11)} = \frac{1}{6}\left(\frac{1}{6n+5} - \frac{1}{6n+11}\right)$$

$$s_n = \sum_{k=1}^{n} x_k = x_1 + x_2 + x_3 + \dots + x_n =$$

$$= \frac{1}{6}\left(\frac{1}{11} - \cancel{\frac{1}{17}} + \cancel{\frac{1}{17}} - \cancel{\frac{1}{23}} + \cancel{\frac{1}{23}} - \cancel{\frac{1}{29}} + \dots + \cancel{\frac{1}{6n+5}} - \frac{1}{6n+11}\right)$$

$$= \frac{1}{6}\cdot\frac{1}{11} - \frac{1}{6}\cdot\frac{1}{6n+11} = \frac{1}{66} - \frac{1}{36n+66}$$

$$\lim_{n\to\infty} S_n = \frac{1}{66} - \lim_{n\to\infty} \frac{1}{36n+66} = \frac{1}{66}$$

$\sum_{n\geq 1} x_n$ converg. și are suma $\frac{1}{66}$

b) $\sum_{n\geq 0} \left(\frac{4}{11}\right)^n$

$$x_n = \left(\frac{4}{11}\right)^n$$

$$S_n = x_0 + x_1 + \dots + x_n = 1 + \frac{4}{11} + \left(\frac{4}{11}\right)^2 + \dots + \left(\frac{4}{11}\right)^n = 1 \cdot \frac{\left(\frac{4}{11}\right)^{n+1} - 1}{\frac{4}{11} - 1}$$

$$= \frac{1 - \left(\frac{4}{11}\right)^{n+1}}{1 - \frac{4}{11}} = \frac{11}{7}\left(1 - \left(\frac{4}{11}\right)^{n+1}\right)$$

$\lim_{n\to\infty} S_n = \frac{11}{7} \Rightarrow$ seria conv. și cu suma $\frac{11}{7}$

Suma termenilor unei p. geom: $S = b_1 \cdot \frac{q^n - 1}{q - 1}$

$b_1$ – I termen

$q$ – rația

$n$ – nr. de t. din sumă

Ex 2 Să se det. p. de extrem local ai

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$

$f(x,y) = x^4 + 4x^2y^2 + y^4$

Rezolvare

I $f$ continuă pe $\mathbb{R}^2 \Rightarrow$ nu are p. de discont.

II $\frac{\partial f}{\partial x}(x,y) = (x^4 + 4x^2y^2 + y^4)'_x = 4x^3 + 8x\cdot y^2 \quad \forall (x,y) \in \mathbb{R}^2$

$\frac{\partial f}{\partial y}(x,y) = (x^4 + 4x^2y^2 + y^4)'_y = 4y^3 + 8yx^2 \quad \forall (x,y) \in \mathbb{R}^2$

$\mathbb{R}^2$ mulțime deschisă
$\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ f. continue $\Big\} \Rightarrow f$ diferențiabilă pe $\mathbb{R}^2 \Rightarrow f$ nu are p. unde nu e dif

III $(\mathbb{R}^2) \begin{cases} \frac{\partial f}{\partial x}(x,y) = 0 \\ \frac{\partial f}{\partial y}(x,y) = 0 \end{cases} \Rightarrow \begin{cases} 4x^3 + 8xy^2 = 0 \\ 4y^3 + 8yx^2 = 0 \end{cases} \Rightarrow \begin{cases} x^3 + 2xy^2 = 0 \\ y^3 + 2yx^2 = 0 \end{cases} \Rightarrow \begin{cases} x(x^2 + 2y^2) = 0 \\ y(y^2 + 2x^2) = 0 \end{cases}$

$x = 0 \Rightarrow y^3 = 0 \Rightarrow y = 0 \longrightarrow (0,0)$
$x^2 + 2y^2 = 0 \Rightarrow x = y = 0 \longrightarrow (0,0)$
$\Rightarrow C = \{(0,0)\}$

IV (1) $\frac{\partial^2 f}{\partial x \partial x}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x,y) = (4x^3 + 8xy^2)'_x = 12x^2 + 8y^2 \quad \forall (x,y) \in \mathbb{R}^2$

(3) $\frac{\partial^2 f}{\partial y \partial x}(x,y) = (4x^3 + 8xy^2)'_y = 16xy \quad \forall (x,y) \in \mathbb{R}^2$

(2) $\frac{\partial^2 f}{\partial x \partial y}(x,y) = (4y^3 + 8yx^2)'_x = 16xy \quad \forall (x,y) \in \mathbb{R}^2$

(4) $\frac{\partial^2 f}{\partial y \partial y} = (4y^3 + 8yx^2)'_y = 12y^2 + 8x^2 \quad \forall (x,y) \in \mathbb{R}^2$

$\mathbb{R}^2$ mulțime deschisă
(1), (2), (3), (4) f. cont. pe $\mathbb{R}^2$ $\Big\} f$ diferenț. de 2 ori pe $\mathbb{R}^2$

V $H_f(0,0) = \begin{pmatrix} (1) & (2) \\ (3) & (4) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Delta_1 = a_{11} = 0$
$\Delta_2 = \det H_f = 0$
nu ne putem pronunța

VI Aplicăm def.
$f(x,y) - f(0,0) = x^4 + 4x^2y^2 + y^4 - 0 - 0 - 0 = x^4 + 4x^2y^2 + y^4 \geq 0 \quad \forall (x,y) \in \mathbb{R}^2$
$(0,0) =$ pct de minim local.

Ex 3 $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \quad f(x,y) = (x^2 - y^2, xy \cdot e^{xy})$.
Studiați diferențiala.

Rezolvare:

$f_1, f_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}$
$f_1(x,y) = x^2 - y^2 \qquad f_2(x,y) = xy \cdot e^{xy}$

$\frac{\partial f_1}{\partial x}(x,y) = (x^2 - y^2)'_x = 2x \quad \forall (x,y) \in \mathbb{R}^2$
$\frac{\partial f_1}{\partial y}(x,y) = (x^2 - y^2)'_y = -2y$
$\mathbb{R}^2$ mulțime deschisă $\Big\} \Rightarrow f_1$ difer. pe $\mathbb{R}^2$ (1)

$\mathbb{R}^2$ mulțime deschisă

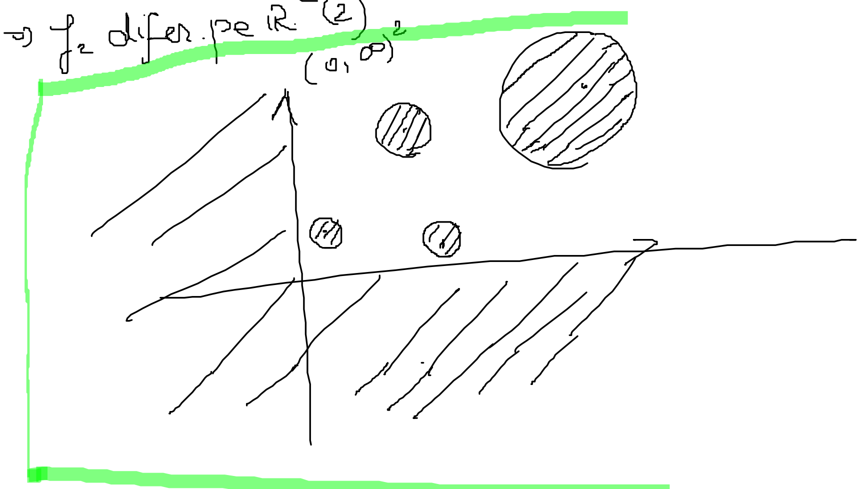
$\left.\begin{array}{l}\mathbb{R}^2 \text{ mulțime deschisă} \\ \frac{\partial f_1}{\partial x}, \frac{\partial f_1}{\partial y} \text{ continue pe } \mathbb{R}^2\end{array}\right\} \Rightarrow f_1$ difer. pe $\mathbb{R}^2$ (1)

$$\frac{\partial f_2}{\partial x}(x,y) = (xy\cdot e^{xy})'_x = (xy)'_x\cdot e^{xy} + xy\cdot(e^{xy})'_x = y\cdot e^{xy} + xy\cdot y\cdot e^{xy} = y\cdot e^{xy}(1+xy)$$

$\forall (x,y)\in\mathbb{R}^2$

$$\frac{\partial f_2}{\partial y}(x,y) = (xy\cdot e^{xy})'_y = (xy)'_y\cdot e^{xy} + xy\cdot(e^{xy})'_y = x e^{xy}(1+xy)$$

$\left.\begin{array}{l}\mathbb{R}^2 \text{ mulțime deschisă} \\ \frac{\partial f_2}{\partial x}, \frac{\partial f_2}{\partial y} \text{ f. continue pe } \mathbb{R}_2\end{array}\right\} \Rightarrow f_2$ difer. pe $\mathbb{R}^2$ (2)




(1),(2) $\Rightarrow f$ dif. pe $\mathbb{R}^2$

$f'(x,y): \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$f'(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(x,y) & \frac{\partial f_1}{\partial y}(x,y) \\ \frac{\partial f_2}{\partial x}(x,y) & \frac{\partial f_2}{\partial y}(x,y) \end{pmatrix}$$

$$f'(x,y) = \begin{pmatrix} 2x & -2y \\ y\cdot e^{xy}(1+xy) & x\cdot e^{xy}(1+xy) \end{pmatrix}$$

$$f'(x,y)(a,b) = \left[\begin{pmatrix} 2x & -2y \\ y\cdot e^{xy}(1+xy) & x\cdot e^{xy}(1+xy) \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix}\right]^t =$$

$$= \begin{pmatrix} 2xa - 2yb \\ ay\cdot e^{xy}(1+xy) + bxe^{xy}(1+xy) \end{pmatrix}^t = \left(2xa - 2yb,\ ay e^{xy}(1+xy) + bxe^{xy}(1+xy)\right)$$

$\forall (a,b)\in\mathbb{R}^2$

Ex 4. $f:\mathbb{R}^3 \longrightarrow \mathbb{R}$ $f(x,y,z) = \sqrt{x^2+y^2+z^2}$

a) Studiați difer. funcție $f$.

b) Calculați derivata funcției $f$ după direcția vectorului $(1,2,1)$ în punctul $X_0 = (1,1,2)$

Rezolvare:

a) $$\frac{\partial f}{\partial x}(x,y,z) = \left(\sqrt{x^2+y^2+z^2}\right)'_x = \frac{1}{2\sqrt{x^2+y^2+z^2}}\cdot(x^2+y^2+z^2)'_x = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2+z^2}} = \frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial f}{\partial y}(x,y,z) = \left(\sqrt{x^2+y^2+z^2}\right)'_y = \frac{y}{\sqrt{x^2+y^2+z^2}}$$

$\forall (x,y,z)\in\mathbb{R}^3\setminus\{(0,0,0)\}$

$$\frac{\partial f}{\partial z}(x,y,z) = \left(\sqrt{x^2+y^2+z^2}\right)'_z \quad \frac{z}{\ldots}$$

$$\frac{\partial f}{\partial z}(x,y,z) = \left(\sqrt{x^2+y^2+z^2}\right)'_z = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$\left.\begin{array}{l} \mathbb{R}^3 \setminus \{(0,0,0)\} \text{ m. deschisă} \\ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \text{ continue în } \mathbb{R}^3 \setminus \{(0,0,0)\} \end{array}\right\} \Rightarrow f \text{ diferen. pe } \mathbb{R}^3 \setminus \{(0,0,0)\}$

$$\lim_{t\to 0} \frac{f(x_0+t\cdot e_1)-f(x_0)}{t} = \lim_{t\to 0} \frac{f((0,0,0)+t\cdot(1,0,0))-f(0,0,0)}{t} =$$

$$= \lim_{t\to 0} \frac{f(t,0,0)}{t} = \lim_{t\to 0} \frac{\sqrt{t^2+0+0}}{t} = \lim_{t\to 0} \frac{|t|}{t}$$

$$\left.\begin{array}{l} \lim\limits_{\substack{t\to 0\\ t<0}} \frac{|t|}{t} = \lim\limits_{\substack{t\to 0\\ t<0}} \frac{-t}{t} = -1 \\ \lim\limits_{\substack{t\to 0\\ t>0}} \frac{|t|}{t} = \lim\limits_{\substack{t\to 0\\ t>0}} \frac{t}{t} = 1 \end{array}\right\} \Rightarrow \nexists \frac{\partial f}{\partial x}(0,0,0)$$

Analog $\nexists \frac{\partial f}{\partial y}, \nexists \frac{\partial f}{\partial z}$

$\Rightarrow f$ nu e dif. în $(0,0,0)$

b) $$f'(x,y,z) = \left(\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$v = (1,2,1) \quad x_0 = (1,1,2)$

$$\frac{\partial f}{\partial v}(x_0) = f'(x_0)(v)$$

$f'(x,y,z) : \mathbb{R}^3 \longrightarrow \mathbb{R}$

$$f'(x,y,z)(a,b,c) = \left[\left(\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}\right)\cdot\begin{pmatrix} a \\ b \\ c\end{pmatrix}\right]^t$$

$$= \frac{x}{\sqrt{\dots}}\cdot a + \frac{y}{\sqrt{\dots}}\cdot b + \frac{z}{\sqrt{\dots}}\cdot c$$

$$f'(1,1,2)(1,2,1) = \frac{1}{\sqrt{1+1+4}}\cdot 1 + \frac{1}{\sqrt{1+1+4}}\cdot 2 + \frac{2}{\sqrt{1+1+4}}\,1$$

$$= \frac{1}{\sqrt{6}} + \frac{2}{\sqrt{6}} + \frac{2}{\sqrt{6}} = \frac{5}{\sqrt{6}}$$