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Tutoriat 8

Integrale improprii

→ F. Leibniz-Newton:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$, f continuă pe $[a,b]$, F primitiva

→ Integrare prin părți

$$\int f(x)\cdot g'(x)\,dx = f(x)\,g(x) - \int f'(x)\cdot g(x)\,dx$$

→ Schimbarea de variabilă:

$$\int f(u(x))\cdot u'(x)\,dx = \underbrace{F(u(x))}\; + C$$

→ Integrale improprii

$$\int_a^b f(x) = \underbrace{F(b)} - \underbrace{F(a)}$$

$$\int_a^{b\pm 0} f(x)\,dx \rightsquigarrow \lim_{\substack{x\to b\\ (x>b)+\\ (x<b)-}} \int_a^x f(t)\,dt$$

$$\int_{a-0}^{b} f(x)\,dx \rightsquigarrow \lim_{\substack{x\to a\\ x<0}} \int_x^b f(t)\,dt$$

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$I \subseteq \mathbb{R}$, unde I e de forma $(a,b]$, $[a,b)$, (a,b), $(a,+\infty)$, $[a,+\infty)$, $(-\infty,a)$, $(-\infty,a]$.

Alegem $I=[a,b)$

Def: Fie $f \in R_{loc}([a,b))$ o funcție local integrabilă pe $[a,b)$, $a<b$.

a). Integrala improprie $\int_a^{b-0} f(x)\,dx$ e convergentă dacă $\exists \lim\limits_{\substack{x\to b\\ x<b}} \int_a^x f(t)\,dt \in \mathbb{R}$

b). Integrala improprie $\int_a^{b-0} f(x)\,dx$ e divergentă dacă nu e convergentă

c). Integrala impr $\int_a^{b-0} f(x)\,dx$ e abs. convergentă dacă $\int_a^{b-0} |f(x)|\,dx$ e convergentă

$\left\{ \begin{array}{l} \text{abs conv} \Rightarrow \text{convergenta} \\ \quad (\text{reciproc nu e valid}) \end{array} \right.$

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1 $$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \lim_{b\to -\infty} \int_{b}^{0} \frac{1}{1+x^2}\,dx = \lim_{b\to\infty} \int_{-b}^{0} \frac{1}{1+x^2}\,dx =$$

$$= \lim_{b\to\infty}\left(\operatorname{arctg} x\Big|_{-b}^{0}\right) = \lim_{b\to\infty}\left(\underbrace{\operatorname{arctg} 0}_{0} - \operatorname{arctg}(-b)\right) =$$

$$= \lim_{b\to\infty} \operatorname{arctg} b = \operatorname{arctg} \infty = \frac{\pi}{2}.$$

1. Crit. de comparație cu inegalități

- $f, g : [a, \infty) \to [0, \infty)$

$0 \le f(x) \le g(x) \quad \forall\, x \in [a, \infty)$.

1. $\int_a^\infty g(x)\,dx$ Convergentă $\to \int_a^\infty f(x)\,dx$ convergentă
2. $\int_a^\infty f(x)\,dx$ Divergent $\to \int_a^\infty g(x)\,dx$ Divergent

② $\int_1^\infty \frac{1}{1+x^4}\,dx$

natura?

$1 + x^4 \ge x^4 \quad \forall\, x \in [1, \infty)$

$\frac{1}{1+x^4} \le \frac{1}{x^4}$

Fie $f, g : [1, \infty) \to [0, \infty)$

$f(x) = \frac{1}{1+x^4}, \quad g(x) = \frac{1}{x^4}$

$$x^{-4} = \frac{x^{-4+1}}{-4+1} = \frac{x^{-3}}{-3}$$

$$\int_1^\infty g(x)\,dx = \int_1^\infty \frac{1}{x^4}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^4}\,dx = \lim_{b\to\infty} \frac{-1}{3x^3}\Big|_1^b$$

$$= \lim_{b\to\infty}\Big(\underbrace{\frac{-1}{3b^3}}_{\to 0} + \frac{1}{3}\Big) = \frac{1}{3} \Rightarrow \text{Convergent}$$

Conform C. Comp. cu inegalități $\Rightarrow \int_1^\infty \frac{1}{1+x^4}\,dx$ Convergentă

Criteriu de comparație cu limită

$f, g \in R_{loc}([a,b))$ pozitive pt care $\exists \lim\limits_{\substack{x \to b \\ x < b}} \frac{f(x)}{g(x)} = l \in \overline{\mathbb{R}}$

a) $l \in (0, \infty)$, atunci $\int_a^{b-0} f(x)\,dx$, $\int_a^{b-0} g(x)\,dx$ au ac natură

b) $l = 0$ și $\int_a^{b-0} g(x)\,dx$ e conv $\Rightarrow \int_a^{b-0} f(x)\,dx$ e conv

c) $l = \infty$ și $\int_a^{b-0} g(x)\,dx$ e div $\Rightarrow \int_a^{b-0} f(x)$ e div

Corolar $\rightarrow$ $I = \int_a^{+\infty} f(x)\,dx$, $a > 0$, f cont și $\exists p$ a.î. $\exists \lim\limits_{x \to \infty} x^p \cdot f(x) = l \in \mathbb{R}$

$$\Rightarrow \int_a^{\infty} f(x)\,dx = \begin{cases} \text{conv}, & p > 1 \\ \text{div}, & p \in (0,1] \end{cases}$$

③ $\int_1^{\infty} \boxed{\frac{x+1}{x^4+1}}\,dx$

Obs că $\exists \lim\limits_{x \to \infty} x^3 \cdot \frac{x+1}{x^4+1} = 1 \in \mathbb{R} \Rightarrow p = 3 > 1$

$\Rightarrow \int_1^{\infty} \frac{x+1}{x^4+1}\,dx$ e conv

Funcțiile BETA și GAMMA

* Funcția $B : (0,+\infty) \times (0,+\infty) \to \mathbb{R}$ definită prin

$$B(p,q) = \int_{0+0}^{1-0} x^{p-1}(1-x)^{q-1}\,dx, \ \forall p,q \in (0,\infty)$$

se num. funcția BETA a lui Euler

* Funcția $\Gamma : (0,\infty) \to \mathbb{R}$, definită prin

$$\Gamma(p) = \int_{0+0}^{+\infty} x^{p-1} \cdot e^{-x}\,dx, \ \forall p \in (0,\infty)$$

se num funcția GAMMA a lui Euler.

Proprietățile funcției Γ

a) $\Gamma(1) = 1$

b) $\Gamma(p+1) = p \cdot \Gamma(p), \ \forall p > 0$

c) $\Gamma(n+1) = n!, \ \forall n \in \mathbb{N}$

d) $\Gamma(p) \cdot \Gamma(1-p) = \dfrac{\pi}{\sin(p \cdot \pi)}, \ \forall p \in (0,1)$

e) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Proprietățile funcției B

a) $B(p,q) = B(q,p), \ \forall p,q \in (0,\infty)$

b) $B(p+1,q) = \dfrac{p}{p+q} \cdot B(p,q), \ \forall p,q \in (0,\infty)$

$B(p,q+1) = \dfrac{q}{p+q} \cdot B(p,q), \ \forall p,q \in (0,\infty)$

! c) $B(p,q) = 2 \cdot \displaystyle\int_{0+0}^{\frac{\pi}{2}-0} \sin^{2p-1}x \cdot \cos^{2q-1}x\,dx, \ \forall p,q \in (0,\infty)$

Legătura dintre B și Γ

a) $B(p,q) = \dfrac{\Gamma(p) \cdot \Gamma(q)}{\Gamma(p+q)}, \ \forall p,q \in (0,\infty)$

b) $B(p,1-p) = \Gamma(p) \cdot \Gamma(1-p) = \dfrac{\pi}{\sin p\pi}, \ \forall p \in (0,1)$

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① $\int_0^\infty \sqrt{x}\, e^{-x^3}\, dx$ $\qquad\qquad x^{p-1}\cdot e^{-x}$

!sch. de variabilă (vrem e^{-t})

$x^3 = t \Rightarrow x = t^{1/3}$

$3x^2 dx = dt \Rightarrow dx = \frac{1}{3x^2} dt$

$x = 0 \Rightarrow t = x^3 = 0$

$x \to \infty \Rightarrow t = x^3 \to \infty$

$$\int_0^\infty t^{1/6} \cdot e^{-t} \cdot \frac{1}{3\cdot t^{2/3}} dt = \frac{1}{3}\int_0^\infty t^{\frac{1}{6}-\frac{2}{3}} \cdot e^{-t} dt$$

$$= \frac{1}{3}\int_0^\infty t^{\overbrace{-1/2}^{p-1}} e^{-t} dt = \frac{1}{3}\int_0^\infty t^{\frac{1}{2}-1} \cdot e^{-t} dt$$

$$= \frac{1}{3}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{3}$$

② $I = \int_0^{\pi/2} (\sin t)^{\overbrace{\frac{5}{2}}^{2p-1}} (\cos t)^{\overbrace{\frac{3}{2}}^{2q-1}} dt$ $\qquad 2\int_0^{\pi/2} \sin^{2p-1}x \cdot \cos^{2q-1}x\, dx$

$2p - 1 = \frac{5}{2} \Rightarrow p = \frac{7}{4}$

$2q - 1 = \frac{3}{2} \Rightarrow q = \frac{5}{4}$

$$\int_0^{\pi/2} (\sin t)^{2\cdot\frac{7}{4}-1} (\cos t)^{2\cdot\frac{5}{4}-1} dt = \frac{1}{2}\cdot 2\int_0^{\pi/2} (\sin t)^{2\cdot\frac{7}{4}-1} (\cos t)^{2\cdot\frac{5}{4}-1} dt$$

$$= \frac{1}{2}\cdot B\left(\frac{7}{4}, \frac{5}{4}\right) = \frac{1}{2}\cdot\frac{\Gamma\left(\frac{7}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma(3)}$$

$$B(p,q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$$

$$\Gamma(3) = 2! = 2$$

$$\Gamma\left(\frac{7}{4}\right) = \frac{3}{4}\Gamma\left(\frac{3}{4}\right)$$

$$\Gamma\left(\frac{5}{4}\right) = \frac{1}{4}\Gamma\left(\frac{1}{4}\right) \Rightarrow$$

$\Gamma(n+1) = n!$, $n \in \mathbb{N}$

$\Gamma(p+1) = p \cdot \Gamma(p)$, $\forall p \in (0, \infty)$

$\Gamma(p)\Gamma(1-p) = \dfrac{\pi}{\sin(p \cdot \pi)}$, $p \in (0, 1)$

$$\rightarrow \Gamma\left(\frac{7}{4}\right) \cdot \Gamma\left(\frac{5}{4}\right) = \frac{3}{4} \cdot \frac{1}{4} \cdot \Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) = \frac{3}{16} \cdot \frac{\pi}{\sin\left(\frac{\pi}{4}\right)} = \frac{3\sqrt{2}\pi}{16}$$

$$I = \frac{1}{2} \cdot \frac{\Gamma\left(\frac{7}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma(3)} = \frac{1}{2} \cdot \frac{\frac{3\sqrt{2}\pi}{16}}{2} = \frac{3\sqrt{2}\pi}{64}$$

(3) $\displaystyle\int_0^\infty x^3 e^{-3x}\, dx$ $\quad \rightarrow \Gamma(p) : \displaystyle\int_0^{+\infty} x^{p-1} \cdot e^{-x}\, dx$

$\Gamma(n+1) = n!$, $n \in \mathbb{N}$

Sch. de variabilă (vrem e^{-t})

$3x = t \Rightarrow x = \frac{1}{3} t$

$3\, dx = dt \Rightarrow dx = \frac{1}{3} dt$

$x = 0 \Rightarrow t = 3 \cdot x = 0$

$x \to \infty \Rightarrow t = 3 \cdot x \to \infty$

$$\int_0^\infty x^3 e^{-3x} dx = \int_0^\infty \left(\frac{t}{3}\right)^3 e^{-t} \cdot \frac{1}{3} dt = \frac{1}{3^4}\int_0^\infty t^{\overset{4-1}{3}} e^{-t} dt$$

$$= \frac{1}{81} \cdot \Gamma(4) = \frac{1}{81} \cdot 3! = \frac{2}{27}$$