Seminar 12

1. Studiați posibilitatea aplicării Teoremei de permutare a limitei cu integrala pentru limitele de mai jos și apoi calculați-le:

a) $\lim\limits_{n\to\infty} \int_0^1 \frac{x \sin(nx)}{n^2+nx^2+x}\,dx.$

<u>Soluție.</u> Fie $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = \frac{x \sin(nx)}{n^2+nx^2+x}$ $\forall n \in \mathbb{N}^*$.

f_n continuă $\forall n \in \mathbb{N}^* \Rightarrow f_n$ integrabilă Riemann $\forall n \in \mathbb{N}^*$.

<u>Convergența simplă</u>

Fie $x \in [0,1]$.

$$0 \le |f_n(x)| = \frac{|x \sin(nx)|}{n^2+nx^2+x} \le \frac{1}{n^2} \quad \forall n \in \mathbb{N}^* \Rightarrow$$

$$0$$

$$\Rightarrow \lim_{n\to\infty} |f_n(x)| = 0 \Rightarrow \lim_{n\to\infty} f_n(x) = 0 \Rightarrow f_n \xrightarrow[n\to\infty]{\Delta} f,$$

unde $f : [0,1] \to \mathbb{R}$, $f(x) = 0$.

Convergența uniformă

$$\sup_{x\in[0,1]} |f_n(x)-f(x)| = \sup_{x\in[0,1]} \left|\frac{x\sin(nx)}{n^2+nx^2+x} - 0\right| =$$

$$= \sup_{x\in[0,1]} \frac{|x\sin(nx)|}{n^2+nx^2+x} \le \frac{1}{n^2} \xrightarrow[n\to\infty]{} 0 \Rightarrow$$

$$\Rightarrow f_n \xrightarrow[n\to\infty]{u} f.$$

Deci putem aplica Teorema de permutare a limitei cu integrala:

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 f(x)\,dx = \int_0^1 0\,dx = 0. \quad \square$$

b) $\lim\limits_{n\to\infty} \int_0^1 nx(1-x^2)^n\,dx.$

Soluție. Fie $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = nx(1-x^2)^n$ $\forall n\in\mathbb{N}^*$.

f_n continuă $\forall n\in\mathbb{N}^* \Rightarrow f_n$ integrabilă Riemann $\forall n\in\mathbb{N}^*$.

Convergența simplă

Fie $x\in[0,1]$.

Dacă $x=0$, atunci $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} f_n(0) = 0$.

Dacă $x=1$, atunci $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} f_n(1) = 0$.

Presupunem că $x \in (0,1)$.

$$\lim_{n\to\infty} \frac{f_{n+1}(x)}{f_n(x)} = \lim_{n\to\infty} \frac{(n+1)\cancel{x}(1-x^2)^{\cancel{n+1}}}{n\cancel{x}\cancel{(1-x^2)^n}} = 1-x^2 < 1.$$

Conform Criteriului raportului pentru șiruri cu termeni strict pozitivi rezultă că $\lim_{n\to\infty} f_n(x) = 0$.

Prin urmare am arătat că $f_n \xrightarrow[n\to\infty]{s} f$, unde

$f : [0,1] \to \mathbb{R}$, $f(x) = 0$.

<u>Convergența uniformă</u>

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} |nx(1-x^2)^n - 0| =$$

$$= \sup_{x\in[0,1]} nx(1-x^2)^n \underset{x=\frac{1}{n}}{\geq} \cancel{n}\cdot\frac{1}{\cancel{n}}\cdot\left(1-\frac{1}{n^2}\right)^n = \left(1-\frac{1}{n^2}\right)^n.$$

Deoarece $\left(1-\frac{1}{n^2}\right)^n > 0 \ \forall n \in \mathbb{N}^* \setminus \{1\}$ și $\lim_{n\to\infty}\left(1-\frac{1}{n^2}\right)^n =$

$= e^0 = 1 \neq 0$ rezultă că $f_n \not\xrightarrow[n\to\infty]{u} f$.

Deci nu putem aplica Teorema de permutare a limitei cu integrala.

$$\int_0^1 nx(1-x^2)^n\,dx = -\frac{1}{2}\int_0^1 (-2nx)(1-x^2)^n\,dx =$$

$$= -\frac{n}{2}\int_0^1 (1-x^2)'(1-x^2)^n dx = -\frac{n}{2}\cdot\frac{(1-x^2)^{n+1}}{n+1}\Big|_0^1 =$$

$$= -\frac{n}{2(n+1)}\cdot(0-1) = \frac{n}{2(n+1)} \xrightarrow[n\to\infty]{} \frac{1}{2}\left(\neq\int_0^1 f(x)dx\right).$$

$\square$

c) $\lim\limits_{n\to\infty}\int_0^1 \ln(1+x^n)\,dx.$

<u>Soluție.</u> Fie $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = \ln(1+x^n)$ $\forall n \in \mathbb{N}^*$.

f_n continuă $\forall n\in\mathbb{N}^* \Rightarrow f_n$ integrabilă Riemann $\forall n\in\mathbb{N}^*$.

<u>Convergența simplă</u>

Fie $x\in[0,1]$.

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\ln(1+x^n) = \begin{cases} 0 \,;\ x\in[0,1) \\ \ln 2\,;\ x=1 \end{cases} \Rightarrow$$

$$\Rightarrow f_n \xrightarrow[n\to\infty]{s} f, \text{ unde } f:[0,1]\to\mathbb{R},\ f(x)=\begin{cases} 0\,;\ x\in[0,1) \\ \ln 2\,;\ x=1. \end{cases}$$

<u>Convergența uniformă</u>

$\left.\begin{array}{l} f_n \text{ continuă } \forall n\in\mathbb{N}^* \\ f \text{ nu e continuă (în 1)} \end{array}\right\} \Rightarrow f_n \underset{n\to\infty}{\overset{u}{\not\longrightarrow}} f.$

Deci nu putem aplica Teorema de permutare a limitei cu integrala.

$$0 \le \ln(1+x^n) \le x^n \quad \forall x \in [0,1],\ \forall n \in \mathbb{N}^* \Rightarrow$$

$$\Rightarrow 0 \le \int_0^1 \ln(1+x^n)\,dx \le \int_0^1 x^n\,dx \Rightarrow 0 \le \int_0^1 \ln(1+x^n)\,dx \le \frac{1}{n+1}.$$

$$0$$

Deci $\lim\limits_{n\to\infty} \int_0^1 \ln(1+x^n)\,dx = 0 \left(= \int_0^1 f(x)\,dx\right)$. $\square$

2. Fie $f:[a,b] \to [0,\infty)$ o funcție continuă a.î. $\int_a^b f(x)\,dx = 0$. Arătați că $f(x)=0\ \forall x \in [a,b]$.

<u>Soluție</u>. Presupunem prin absurd că există $x_0 \in [a,b]$ a.î. $f(x_0)>0$. Deoarece f este continuă în x_0, există un interval nedegenerat $[c,d]$ a.î. $x_0 \in [c,d] \subset [a,b]$ și $f(x) > \frac{f(x_0)}{2} > 0\ \forall x \in [c,d]$.

Avem $0 = \int_a^b f(x)\,dx \ge \int_c^d f(x)\,dx \ge \int_c^d \frac{f(x_0)}{2}\,dx =$

$= \frac{f(x_0)}{2}(d-c) > 0$, contradicție.

Așadar $f(x)=0\ \forall x \in [a,b]$. $\square$

3. Fie $f:[a,b]\to\mathbb{R}$ o funcție continuă a.î.
$\int_a^b x^n f(x)dx = 0 \ \forall\, n\in\mathbb{N}$. Arătați că $f\equiv 0$.

<u>Soluție.</u> Fie $P(x)= a_0+a_1x+\ldots+a_nx^n$.

$$\int_a^b P(x)\,f(x)dx = a_0\int_a^b f(x)dx + a_1\int_a^b x f(x)dx+\ldots+$$

$$+a_n\int_a^b x^n f(x)dx = 0.$$

Deci $\int_a^b P(x)f(x)dx=0$ pentru orice funcție polinomială P.

f continuă $\xRightarrow[\text{Bernstein}]{\text{Teorema}}$ $\exists\,(P_n)_n$ un șir de funcții polinomiale ($P_n:[a,b]\to\mathbb{R}\ \forall\, n\in\mathbb{N}$) a.î. $P_n \xrightarrow[n\to\infty]{u} f$.

Avem $\int_a^b P_n(x)\,f(x)dx = 0 \ \forall\, n\in\mathbb{N}$.

Arătăm că $P_n f \xrightarrow[n\to\infty]{u} f^2$.

$$\sup_{x\in[a,b]}\left|P_n(x)f(x)-f^2(x)\right| = \sup_{x\in[a,b]}\Big(|f(x)|\,|P_n(x)-f(x)|\Big).$$

$\left.\begin{array}{l} f \text{ continuă} \\ [a,b] \text{ mulțime compactă}\end{array}\right\} \Rightarrow f$ mărginită și își atinge marginile $\Rightarrow \exists\, M>0$ a.î. $|f(x)|\le M \ \forall\, x\in[a,b]$.

Deci $\sup\limits_{x\in[a,b]} |P_n(x)f(x)-f^2(x)| \le M\sup\limits_{x\in[a,b]}|P_n(x)-f(x)| \xrightarrow[n\to\infty]{}$

$\xrightarrow[n\to\infty]{} M\cdot 0=0$ (deoarece $P_n \xrightarrow[n\to\infty]{u} f$).

Așadar $P_n f \xrightarrow[n\to\infty]{u} f^2$.

P_n, f continue $\Rightarrow P_n, f$ integrabile Riemann $\Rightarrow$

$\Rightarrow P_n\cdot f$ integrabilă Riemann $\forall\, n\in\mathbb{N}$.

Conform Teoremei de permutare a limitei cu integrala avem $\lim\limits_{n\to\infty} \underbrace{\int_a^b P_n(x)f(x)\,dx}_{\parallel \atop 0} = \int_a^b f^2(x)\,dx$.

Deci $\int_a^b f^2(x)\,dx = 0$.

Deoarece f e continuă (deci f^2 e continuă) rezultă că $f^2\equiv 0$, i.e. $f\equiv 0$. $\square$

4. Determinați:

a) $\int_0^\infty \frac{1}{1+x^2}\,dx$.

<u>Soluție.</u> $\int_0^\infty \frac{1}{1+x^2}\,dx = \lim\limits_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,dx = \lim\limits_{b\to\infty} \operatorname{arctg} x\Big|_0^b =$

$$= \lim_{b\to\infty}(\operatorname{arctg} b - \operatorname{arctg} 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}. \quad \Box$$

b) $\int_{-\infty}^{0} e^{x}\,dx$.

<u>Soluție.</u> $\int_{-\infty}^{0} e^{x}\,dx = \lim_{a\to-\infty}\int_{a}^{0} e^{x}\,dx = \lim_{a\to-\infty} e^{x}\Big|_{a}^{0} =$

$$= \lim_{a\to-\infty}(1 - e^{a}) = 1. \quad \Box$$

c) $\int_{0}^{1} \frac{1}{\sqrt{1-x^2}}\,dx$.

<u>Soluție.</u> $\int_{0}^{1} \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{b\uparrow 1}\int_{0}^{b} \frac{1}{\sqrt{1-x^2}}\,dx =$

$$= \lim_{b\uparrow 1} \arcsin x\Big|_{0}^{b} = \lim_{b\uparrow 1}(\arcsin b - \arcsin 0) =$$

$$= \arcsin 1 - \arcsin 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}. \quad \Box$$

d) $\int_{0}^{\frac{1}{2}} \frac{1}{x\ln^2 x}\,dx$.

<u>Soluție.</u> $\int_{0}^{\frac{1}{2}} \frac{1}{x\ln^2 x}\,dx \underset{\uparrow}{=} \int_{-\infty}^{\ln\frac{1}{2}} \frac{1}{t^2}\,dt =$

s.v. $\ln x = t$

$\frac{1}{x}dx = dt$

$x \searrow 0 \Rightarrow t \to -\infty$

$x = \frac{1}{2} \Rightarrow t = \ln\frac{1}{2}$

$$= \lim_{a \to -\infty} \int_a^{\ln \frac{1}{2}} t^{-2} \, dt = \lim_{a \to -\infty} \frac{t^{-1}}{-1} \Big|_a^{\ln \frac{1}{2}} =$$

$$= \lim_{a \to -\infty} -\frac{1}{t} \Big|_a^{\ln \frac{1}{2}} = \lim_{a \to -\infty} -\left(\frac{1}{\ln \frac{1}{2}} - \frac{1}{a} \right) =$$

$$= -\frac{1}{\ln \frac{1}{2}} = \frac{1}{\ln 2}. \quad \square$$

e) $\displaystyle\int_{-\infty}^{\infty} \frac{x}{1+x^4} \, dx$.

<u>Soluție.</u> $\displaystyle\int_{-\infty}^{0} \frac{x}{1+x^4} \, dx = \lim_{a \to -\infty} \int_a^0 \frac{x}{1+x^4} \, dx =$

$$= \lim_{a \to -\infty} \frac{1}{2} \int_a^0 \frac{(x^2)'}{1+(x^2)^2} \, dx = \lim_{a \to -\infty} \frac{1}{2} \operatorname{arctg} x^2 \Big|_a^0 =$$

$$= \lim_{a \to -\infty} \frac{1}{2} \left(\operatorname{arctg} 0^2 - \operatorname{arctg} a^2 \right) = -\frac{1}{2} \cdot \frac{\pi}{2} = -\frac{\pi}{4}.$$

$$\int_0^{\infty} \frac{x}{1+x^4} \, dx = \lim_{b \to \infty} \int_0^b \frac{x}{1+x^4} \, dx = \lim_{b \to \infty} \frac{1}{2} \int_0^b \frac{(x^2)'}{1+(x^2)^2} \, dx =$$

$$= \lim_{b \to \infty} \frac{1}{2} \operatorname{arctg} x^2 \Big|_0^b = \lim_{b \to \infty} \frac{1}{2} \left(\operatorname{arctg} b^2 - \operatorname{arctg} 0^2 \right) =$$

$$= \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}.$$

$$\int_{-\infty}^{\infty} \frac{x}{1+x^4}\,dx = \int_{-\infty}^{0} \frac{x}{1+x^4}\,dx + \int_{0}^{\infty} \frac{x}{1+x^4}\,dx = -\frac{\pi}{4} + \frac{\pi}{4} = 0. \square$$

f) $\int_0^1 \frac{1}{x^5}\,dx.$

<u>Soluție.</u> $\int_0^1 \frac{1}{x^5}\,dx = \lim_{a \downarrow 0} \int_a^1 x^{-5}\,dx =$

$$= \lim_{a \downarrow 0} \frac{x^{-4}}{-4}\Big|_a^1 = \lim_{a \downarrow 0} -\frac{1}{4}\left(\frac{1}{1^4} - \frac{1}{a^4}\right) = -\frac{1}{4} + \infty = \infty. \square$$

5. Determinați $\Gamma(1)$.

<u>Soluție.</u> $\Gamma(1) = \int_0^{\infty} t^{1-1} e^{-t}\,dt = \int_0^{\infty} e^{-t}\,dt =$

$$= \lim_{b \to \infty} \int_0^b e^{-t}\,dt = \lim_{b \to \infty}\left(-e^{-t}\Big|_0^b\right) = \lim_{b \to \infty}\left(-e^{-b} + e^0\right) =$$

$= 1. \square$

6. Folosind eventual funcțiile Γ și B, determinați:

a) $\int_0^{\infty} e^{-x^2}\,dx.$

<u>Soluție.</u> $\int_0^{\infty} e^{-x^2}\,dx = \int_0^{\infty} e^{-t} \frac{1}{2\sqrt{t}}\,dt =$

S.V. $x^2 = t \Leftrightarrow x = \sqrt{t}$

$2x\,dx = dt \Leftrightarrow dx = \frac{1}{2\sqrt{t}}\,dt$

$x = 0 \Rightarrow t = 0$

$x \to \infty \Rightarrow t \to \infty$

$$= \frac{1}{2}\int_0^{\infty} t^{-\frac{1}{2}} e^{-t}\,dt = \frac{1}{2}\int_0^{\infty} t^{\frac{1}{2}-1} e^{-t}\,dt = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) =$$

$$= \frac{\sqrt{\pi}}{2}. \quad \square$$

b) $\int_{-\infty}^{\infty} e^{-x^2}\,dx$.

<u>Soluție</u>. Rezolvați-l voi!

c) $\int_0^{\infty} x^6 e^{-2x}\,dx$.

<u>Soluție</u>. $\int_0^{\infty} x^6 e^{-2x}\,dx \underset{\uparrow}{=} \int_0^{\infty} \frac{1}{2^6}\cdot t^6 e^{-t}\cdot \frac{1}{2}\,dt =$

S.V. $2x = t \Leftrightarrow x = \frac{1}{2}t$

$2\,dx = dt \Leftrightarrow dx = \frac{1}{2}\,dt$

$x = 0 \Rightarrow t = 0$

$x \to \infty \Rightarrow t \to \infty$

$$= \frac{1}{2^7}\int_0^{\infty} t^6 e^{-t}\,dt = \frac{1}{2^7}\int_0^{\infty} t^{7-1} e^{-t}\,dt = \frac{1}{2^7}\Gamma(7) = \frac{6!}{2^7}. \quad \square$$

d) $\int_0^{\infty} \sqrt{x}\, e^{-x^3}\,dx$.

<u>Soluție</u>. $\int_0^{\infty} \sqrt{x}\, e^{-x^3}\,dx \underset{\uparrow}{=} \int_0^{\infty} t^{\frac{1}{6}} e^{-t}\, \frac{1}{3} t^{-\frac{2}{3}}\,dt =$

S.V. $x^3 = t \Leftrightarrow x = t^{\frac{1}{3}}$

$3x^2\,dx = dt \Leftrightarrow dx = \frac{1}{3} t^{-\frac{2}{3}}\,dt$

$x = 0 \Rightarrow t = 0$

$x \to \infty \Rightarrow t \to \infty$

$$= \frac{1}{3}\int_0^\infty t^{-\frac{3}{6}} e^{-t}\,dt = \frac{1}{3}\int_0^\infty t^{-\frac{1}{2}} e^{-t}\,dt =$$

$$= \frac{1}{3}\int_0^\infty t^{\frac{1}{2}-1} e^{-t}\,dt = \frac{1}{3}\,\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{3}. \quad \square$$

e) $\displaystyle\int_0^2 \frac{x^2}{\sqrt{2-x}}\,dx.$

Soluție. $\displaystyle\int_0^2 \frac{x^2}{\sqrt{2-x}}\,dx = \int_0^2 \frac{x^2}{\sqrt{2\left(1-\frac{x}{2}\right)}}\,dx =$

$$= \frac{1}{\sqrt{2}}\int_0^2 \frac{x^2}{\sqrt{1-\frac{x}{2}}}\,dx = \frac{1}{\sqrt{2}}\int_0^2 x^2\left(1-\frac{x}{2}\right)^{-\frac{1}{2}} dx =$$

$$= \frac{1}{\sqrt{2}}\int_0^1 4t^2(1-t)^{-\frac{1}{2}}\,2\,dt = \frac{8}{\sqrt{2}}\int_0^1 t^2(1-t)^{-\frac{1}{2}}\,dt =$$

S.V. $\frac{x}{2} = t \Leftrightarrow x = 2t$

$\frac{1}{2}dx = dt \Leftrightarrow dx = 2\,dt$

$x = 0 \Rightarrow t = 0$

$x \nearrow 2 \Rightarrow t \nearrow 1$

$$= \frac{8}{\sqrt{2}}\int_0^1 t^{3-1}(1-t)^{\frac{1}{2}-1}\,dt = \frac{8}{\sqrt{2}}\,B\left(3, \frac{1}{2}\right).$$

$$B\left(3,\frac{1}{2}\right)=\frac{\Gamma(3)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(3+\frac{1}{2}\right)}=\frac{2!\sqrt{\pi}}{\Gamma\left(3+\frac{1}{2}\right)}=\frac{2\sqrt{\pi}}{\Gamma\left(3+\frac{1}{2}\right)}.$$

$$\Gamma\left(3+\frac{1}{2}\right)=\Gamma\left(1+2+\frac{1}{2}\right)=\left(2+\frac{1}{2}\right)\Gamma\left(2+\frac{1}{2}\right)=\frac{5}{2}\Gamma\left(1+1+\frac{1}{2}\right)=$$

$$=\frac{5}{2}\cdot\left(1+\frac{1}{2}\right)\Gamma\left(1+\frac{1}{2}\right)=\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{15}{8}\sqrt{\pi}.$$

$$B\left(3,\frac{1}{2}\right)=\frac{2\cancel{\sqrt{\pi}}}{\frac{15}{8}\cancel{\sqrt{\pi}}}=2\cdot\frac{8}{15}=\frac{16}{15}.$$

$$\int_0^2\frac{x^2}{\sqrt{2-x}}\,dx=\frac{8}{\sqrt{2}}\cdot\frac{16}{15}=\frac{8\cdot 16}{15\sqrt{2}}=\frac{8\cdot\overset{8}{\cancel{16}}\cdot\sqrt{2}}{15\cdot\cancel{2}}=\frac{64\sqrt{2}}{15}.\ \square$$

f) $\displaystyle\int_0^{\frac{\pi}{2}}(\sin t)^{\frac{5}{2}}(\cos t)^{\frac{3}{2}}\,dt.$

<u>Soluție.</u> $\displaystyle B(x,y)=2\int_0^{\frac{\pi}{2}}(\sin t)^{2x-1}(\cos t)^{2y-1}\,dt.$

$$2x-1=\frac{5}{2}\Leftrightarrow x=\frac{7}{4}.$$

$$2y-1=\frac{3}{2}\Leftrightarrow y=\frac{5}{4}.$$

$$\int_0^{\frac{\pi}{2}}(\sin t)^{\frac{5}{2}}(\cos t)^{\frac{3}{2}}\,dt=\frac{1}{2}\cdot 2\int_0^{\frac{\pi}{2}}(\sin t)^{2\cdot\frac{7}{4}-1}(\cos t)^{2\cdot\frac{5}{4}-1}\,dt=$$

$$=\frac{1}{2}B\left(\frac{7}{4},\frac{5}{4}\right).$$

$$B\left(\frac{7}{4},\frac{5}{4}\right)=\frac{\Gamma\left(\frac{7}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{7}{4}+\frac{5}{4}\right)}=\frac{\Gamma\left(\frac{7}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma(3)}=\frac{\Gamma\left(\frac{7}{4}\right)\Gamma\left(\frac{5}{4}\right)}{2!}=$$

$$= \frac{\Gamma\left(\frac{7}{4}\right)\Gamma\left(\frac{5}{4}\right)}{2}.$$

$$\Gamma\left(\frac{7}{4}\right) = \Gamma\left(1+\frac{3}{4}\right) = \frac{3}{4}\Gamma\left(\frac{3}{4}\right).$$

$$\Gamma\left(\frac{5}{4}\right) = \Gamma\left(1+\frac{1}{4}\right) = \frac{1}{4}\Gamma\left(\frac{1}{4}\right).$$

$$\Gamma\left(\frac{7}{4}\right)\Gamma\left(\frac{5}{4}\right) = \frac{3}{4}\Gamma\left(\frac{3}{4}\right)\frac{1}{4}\Gamma\left(\frac{1}{4}\right) = \frac{3}{16}\Gamma\left(1-\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right) =$$

$$\underset{\uparrow}{=} \frac{3}{16}\frac{\pi}{\sin\left(\pi\cdot\frac{1}{4}\right)} = \frac{3}{16}\cdot\frac{\pi}{\frac{\sqrt{2}}{2}} = \frac{3}{16}\cdot\pi\cdot\frac{\overset{\sqrt{2}}{\cancel{2}}}{\cancel{\sqrt{2}}} =$$

$$\Gamma(1-x)\Gamma(x) = \frac{\pi}{\sin \pi x}$$

$$= \frac{3\pi\sqrt{2}}{16}.$$

$$B\left(\frac{7}{4},\frac{5}{4}\right) = \frac{\frac{3\pi\sqrt{2}}{16}}{2} = \frac{3\pi\sqrt{2}}{32}.$$

$$\int_0^{\frac{\pi}{2}} (\sin t)^{\frac{5}{2}} (\cos t)^{\frac{3}{2}}\, dt = \frac{1}{2}\cdot\frac{3\pi\sqrt{2}}{32} = \frac{3\pi\sqrt{2}}{64}. \quad \square$$