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Ex 1) $\sum_{n\geq 1} \operatorname{arctg} \frac{1}{n^2+n+1}$

Rezolvare:

$$\operatorname{arctg} x - \operatorname{arctg} y = \begin{cases} \operatorname{arctg} \frac{x-y}{1+xy}, & x\cdot y > -1 \\ \pi + \operatorname{arctg} \frac{x+y}{1+xy}, & x\cdot y < -1,\ x>0 \\ -\pi + \operatorname{arctg} \frac{x+y}{1+xy}, & x\cdot y < -1,\ x<0 \end{cases}$$

$$\frac{1}{n^2+n+1} = \frac{n+1-n}{1+n(n+1)} \Rightarrow \operatorname{arctg} \frac{1}{n^2+n+1} = \operatorname{arctg} \frac{\overset{x}{\overset{\parallel}{n+1}} - \overset{y}{\overset{\parallel}{n}}}{1+n\cdot(n+1)} = \operatorname{arctg}(n+1) - \operatorname{arctg} n$$

$$\sum_{n\geq 1} [\operatorname{arctg}(n+1) - \operatorname{arctg} n] = \cancel{\operatorname{arctg} 2} - \operatorname{arctg} 1 + \cancel{\operatorname{arctg} 3} - \cancel{\operatorname{arctg} 2} + \cancel{\dots} + \operatorname{arctg}(n+1) - \cancel{\operatorname{arctg} n} + \dots$$

$$= -\operatorname{arctg} 1 + \lim_{n\to\infty} \operatorname{arctg}(n+1) = -\frac{\pi}{4} + \frac{\pi}{2} = \frac{\pi}{4}$$

$\Rightarrow$ seria este convergentă

Ex 2) $\sum_{n\geq 2} \frac{\sin n}{\ln n}$ semi convergentă

Def! S. semiconv $\begin{cases} \text{seria este convergentă} \\ \text{seria modulelor este divergentă} \end{cases}$

Rezolvare:

$$\left|\frac{\sin n}{\ln n}\right| = \frac{|\sin n|}{\ln n} \geq \frac{\sin^2 n}{\ln n} = \frac{1-\cos 2n}{2\ln n}$$

$\sin n \in [-1;1] \Rightarrow |\sin n| \in [0;1] \Rightarrow |\sin n| \geq \sin^2 n$

$\cos 2n = 1 - 2\sin^2 n \Rightarrow \sin^2 n = \frac{1-\cos 2n}{2}$

Pp (R.A). seria $\sum_{n\geq 2} \frac{1-\cos 2n}{2\ln n}$ = convergentă

Cum este $\sum_{n\geq 2} \frac{\cos 2n}{2\ln n}$?

$\left.\begin{array}{l} \frac{1}{2\ln n} \longrightarrow 0, \text{ descrescător} \\ \left|\sum_{k=2}^{n} \cos 2k\right| \leq \underbrace{1+1+\dots+1}_{n-1 \text{ ori}} = n-1 = M \end{array}\right\} \Rightarrow \sum_{n\geq 2} \frac{1}{2\ln n}\cdot \cos 2n$ convergentă (Abel-Dirichlet)

$$\Rightarrow \underbrace{\sum_{n\geq 2} \frac{1-\cos 2n}{2\ln n}}_{\text{conv}} + \underbrace{\sum_{n\geq 2} \frac{\cos 2n}{2\ln n}}_{\text{conv}} = \sum_{n\geq 2} \frac{1-\cos 2n+\cos 2n}{2\ln n} = \underbrace{\sum_{n\geq 2} \frac{1}{2\ln n}}_{\text{conv}} \not$$

Dar pt $n \geq 2 \Rightarrow \ln n \leq n \Rightarrow \frac{1}{n} \leq \frac{1}{\ln n} \Big/ \cdot \frac{1}{2} \Rightarrow \frac{1}{2n} \leq \frac{1}{2\ln n}$

Din criteriul comparației ($\sum_{n\geq 2} \frac{1}{2n} = \frac{1}{2}\cdot \sum_{n\geq 2} \frac{1}{n} \rightarrow$ seria armonică $\rightarrow$ diverg.)

$\Rightarrow \sum_{n\geq 2} \frac{1}{2\ln n}$ diverg CONTRADICȚIE *

CRIT. COMPARAȚIE:

$a_n \leq b_n$, $\sum b_n$ conv $\Rightarrow \sum a_n$ conv.

$a_n \leq b_n$, $\sum a_n$ div $\Rightarrow \sum b_n$ div

$\sum_{n\geq 2} \frac{1-\cos 2n}{2\ln n}$ diverg. $\Rightarrow \sum_{n\geq 2} \left|\frac{\sin n}{\ln n}\right|$ diverg $\Rightarrow$ seria nu este abs. converg. (1)

Verificăm $\sum_{n\geq 2} \frac{\sin n}{\ln n}$

Aplicăm Abel - Dirichlet

$\left.\begin{array}{l} \frac{1}{\ln n} \to 0, \text{ desc.} \\ \left|\sum_{k=2}^{n} \sin k\right| \leq \underbrace{1+1+\ldots+1}_{n-1 \text{ ori}} = n-1 = M \end{array}\right\} \Rightarrow \sum_{n\geq 2} \frac{1}{\ln n} \cdot \sin n$ converg. (2)

Din (1) și (2) $\Rightarrow$ seria este semiconverg.

Ex 3) $(x_n)_{n\in\mathbb{N}}$ șir $\subset (0,1)$

$x_n(1-x_{n+1}) > \frac{1}{4}$

convergența șirului + limita

Rezolvare:

$x_n(1-x_{n+1}) > \frac{1}{4} \Leftrightarrow 1-x_{n+1} > \frac{1}{4x_n} \Leftrightarrow 1-\frac{1}{4x_n} > x_{n+1} \Leftrightarrow \frac{4x_n-1}{4x_n} > x_{n+1} \Leftrightarrow$

$-x_{n+1} > -\frac{4x_n-1}{4x_n} \quad / + x_n$

$\left.\begin{array}{l} x_n - x_{n+1} > x_n - \frac{4x_n-1}{4x_n} = \frac{4x_n^2-4x_n+1}{4x_n} = \frac{(2x_n-1)^2}{4x_n} \\ (2x_n-1)^2 \geq 0 \\ 4x_n > 0 \text{ (din ip.)} \end{array}\right\} \Rightarrow x_n - x_{n+1} > 0 \Rightarrow x_n > x_{n+1}$

$\Rightarrow$ $\left.\begin{array}{l}\text{șir monoton} \\ \text{șir mărginit}\end{array}\right\} \Rightarrow$ șir convergent

$l(1-l) \geq \frac{1}{4} \Leftrightarrow l - l^2 \geq \frac{1}{4} \Leftrightarrow l^2 - l + \frac{1}{4} \leq 0$

$l^2 - 2\cdot l \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 \leq 0$

$\left.\begin{array}{l} \left(l-\frac{1}{2}\right)^2 \leq 0 \\ \left(l-\frac{1}{2}\right)^2 \geq 0 \end{array}\right\} \Rightarrow \left(l-\frac{1}{2}\right)^2 = 0 \Rightarrow l - \frac{1}{2} = 0$

$\boxed{l = \frac{1}{2}}$

Serii de puteri

Serii de puteri

Forma: $\sum_{n=0}^{\infty} a_n \cdot (x - x_0)^n$

Elemente caracteristice: x_0, a_0, a_n

Cerințe: ρ – raza de convergență

$(x_0-\rho, x_0+\rho)$ – interval de convergență

D – mulțime de convergență

$f: D \to \mathbb{R}$ funcția corespunzătoare seriei (suma seriei)

$$(x_0-\rho, x_0+\rho) \subseteq \underset{\underset{\mathbb{R}}{\cap}}{D} \subseteq [x_0-\rho, x_0+\rho]$$

Exemple de serii remarcabile

1) $\sum_{n\geq 0} (-1)^n \cdot x^n = \frac{1}{1+x} \quad \forall x \in (-1;1)$

2) $\sum_{n\geq 0} x^n = \frac{1}{1-x} \quad \forall x \in (-1;1)$

3) $\sum_{n\geq 0} \frac{x^n}{n!} = e^x \quad \forall x \in \mathbb{R}$

4) $\sum_{n\geq 0} \frac{(-1)^n \cdot x^{2n}}{(2n)!} = \cos x \quad \forall x \in \mathbb{R}$

5) $\sum_{n\geq 0} \frac{(-1) \cdot x^{2n+1}}{(2n+1)!} = \sin x \quad \forall x \in \mathbb{R}$

Ex 1) $\sum_{n=0}^{+\infty} (-1)^n \cdot (n+1) \cdot x^n$

Rezolvare:

$$\sum_{n=0}^{+\infty} \underbrace{\left[(-1)^n \cdot (n+1)\right]}_{a_n} (x-0)^n \Rightarrow \begin{array}{l} a_n = (-1)^n \cdot (n+1) \\ x_0 = 0 \\ a_0 = (-1)^0 \cdot (0+1) = 1 \end{array}$$

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{|(-1)^n \cdot (n+1)|} = \lim_{n\to\infty} \sqrt[n]{(n+1)}$$

Folosim criteriul radicalului de la șiruri; $(t_n = n+1)$

$$\lim_{n\to\infty} \frac{t_{n+1}}{t_n} = \lim_{n\to\infty} \frac{n+2}{n+1} = 1 \Rightarrow \lim_{n\to\infty} \sqrt[n]{(n+1)} = 1$$

$$\rho = \frac{1}{L} = 1$$

$(x_0-\rho, x_0+\rho) = (-1;1)$

$$\rho = \frac{1}{L} = 1$$

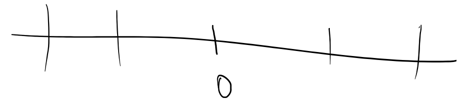
$$(x_0-\rho, x_0+\rho) = (-1;1)$$

$$(-1;1) \subseteq D \subseteq [-1;1]$$

$$\sum_{n=0}^{+\infty} (-1)^n \cdot (n+1)(-1)^n = \sum_{n=0}^{+\infty} (n+1)\cdot(-1)^{2n} = \sum_{n=0}^{+\infty} (n+1) \rightarrow \text{divergentă} \Rightarrow -1 \notin D$$

$$\lim_{n\to\infty} (n+1) = +\infty \neq 0$$

$$\sum_{n=0}^{+\infty} (-1)^n \cdot (n+1) \cdot 1^n = \sum_{n=0}^{+\infty} (n+1)(-1)^n \rightarrow \text{divergentă} \Rightarrow 1 \notin D$$



$$\Rightarrow D = (-1;1)$$

$$f : (-1;1) \rightarrow \mathbb{R}$$

$$f(x) = \sum_{n=0}^{+\infty} (-1)^n \cdot (n+1) \cdot x^n$$

D coincide cu intervalul de convergență $\Rightarrow$ f de clasă C^∞ (derivabilă și integrabilă de ∞ ori)

$$f(x_0) = a_0$$

$$\int (-1)^n \cdot (n+1) \cdot x^n dx = (-1)^n \cdot (n+1) \int x^n dx = (-1)^n \cdot \cancel{(n+1)} \cdot \frac{x^{n+1}}{\cancel{n+1}} + C$$

$$= (-1)^n \cdot x^{n+1} + C$$

$$\int f(x)\,dx = \int \sum_{n=0}^{+\infty} (-1)^n (n+1) \cdot x^n\,dx = \sum_{n=0}^{+\infty} \int (-1)^n (n+1) \cdot x^n\,dx = \sum_{n=0}^{+\infty} (-1)^n \cdot x^{n+1} + C$$

$$\sum_{n=0}^{+\infty} (-1)^n \cdot x^n = \frac{1}{1+x} \Big/ \cdot x$$

$$\sum_{n=0}^{+\infty} (-1)^n \cdot x^{n+1} = \frac{x}{x+1}$$

$$\int f(x)\,dx = \frac{x}{x+1} + C$$

$$f(x) = \left(\frac{x}{x+1} + C\right)' = \frac{x' \cdot (x+1) - x \cdot (x+1)'}{(x+1)^2} = \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2}$$

$$f : (-1;1) \rightarrow \mathbb{R} \quad f(x) = \frac{1}{(x+1)^2}$$

Observații:

(I) f de clasă C^∞ înseamnă că f = derivabilă și integrabilă de ∞ ori

Observații:

I. f de clasă C^∞ înseamnă că f = derivabilă și integrabilă de ∞ ori

II. $f(x_0) = a_0$

III. Dacă $D \neq (x_0-\rho, x_0+\rho) \Rightarrow \begin{cases} f \text{ funcție de clasă } C^\infty \text{ pe } (x_0-\rho, x_0+\rho) \\ f \text{ va fi continuă în } x_0-\rho,\ x_0+\rho \end{cases}$

↳ f se va calcula folosind integrarea/derivarea de un nr. suficient de ori pe $(x_0-\rho, x_0+\rho)$

Vom calcula $f(x_0-\rho)$ și/sau $(x_0+\rho)$

$$f(x_0-\rho) = \lim_{\substack{x \to x_0-\rho \\ x > x_0-\rho}} f(x)$$

$$f(x_0+\rho) = \lim_{\substack{x \to x_0+\rho \\ x < x_0+\rho}} f(x)$$

IV. $\rho = \frac{1}{L} \Rightarrow L = \limsup \sqrt[n]{|a_n|}$