Ex. 1: Rezolvați în S_3 sistemul de ecuații:

$$\begin{cases} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \cdot x = y \\ x \cdot \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \cdot y . \end{cases}$$

Rez:

$$x \cdot \overbrace{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}}^{\sigma} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \cdot x$$

$\Rightarrow x \cdot \sigma = \sigma \cdot x$, $\sigma = (1\ 2\ 3)$.

$\Rightarrow x \in \{e, \sigma, \sigma^2\}$

$xH = Hx$, $H = \langle \sigma \rangle$.

$\sigma H = H \sigma$, $H \subseteq G$.

$x \in H \Rightarrow x^{-1} \in H$.

$x \cdot | x^{-1} \sigma = \sigma x^{-1} | \cdot x \Rightarrow \sigma x = x \sigma$ OK.

$x, y \in H \Rightarrow xy \in H$.

$(xy)\sigma \stackrel{?}{=} \sigma (xy)$ $\quad ((xy)\sigma = x(y\sigma) = (x\sigma)y = \sigma xy)$.

$\Rightarrow H \leq G$.

$S_3 = \{\underline{\underline{e}}, (1\ 2), (1\ 3), (2\ 3), \underline{\underline{\sigma, \sigma^2}}\}$.

Ex. 2: Rezolvați ecuația:

a. $x^2 = (1\ 2)(3\ 4\ 5\ 6)$, $x \in S_6$.

b. $x^2 = (1\ 5)(3\ 4)(2\ 6\ 8)(7\ 9\ 10)$, $x \in S_{10}$.

Rez:

a. $\sigma = (1\ 2)(3\ 4\ 5\ 6)$

$\varepsilon(\sigma) = 1$ $\qquad$ $ord(\sigma) = 4 = ord(x^2)$

$\varepsilon(x^2) = 1$ $\qquad$ $ord(x) = m.$

$ord(x^2) = \frac{m}{(m,2)} = 4 \Rightarrow 4 | m.$

$(m,2) = 2.$

$\Rightarrow \frac{m}{(m,2)} = 4 \Rightarrow \frac{m}{2} = 4 \Rightarrow m = 8.$

În S_6 nu avem elem. de ordin 8.

$\Rightarrow$ ecuația nu are sol.

$x^2 = \tau \cdot \sigma$, τ, σ cicli de lungime pară disj.

b. $\sigma = (1\ 5)(3\ 4)(2\ 6\ 8)(7\ 9\ 10) \in S_{10}.$

$\varepsilon(\sigma) = 1$ $\qquad$ $ord(\sigma) = 6 = ord(x^2).$

$\varepsilon(x^2) = 1$ $\qquad$ $ord(x) = m.$

$ord(x^2) = \frac{m}{(m,2)} = 6 \Rightarrow 6 | m \Rightarrow (m,2) = 2$

$ord(x) = m$, $ord(x^k) = \frac{m}{(m,k)}$

$\Rightarrow \frac{m}{2} = 6 \Rightarrow m = 12$

Căutăm elem. de ordin 12 din S_{10}.

Vrem să-l scriem pe 10 ca sumă de nr. nat. nenule ≥ 2 (*) cu cmmmc $= 12$.

$10 = 4+3+3 = 6+4$

(*) Dacă am avea $x(a) = a$, pt. un $a \in \{1, .., 10\}$, atunci $x^2(a) = a \Rightarrow \sigma(a) = a$

I. $10 = 4+3+3$

x = produs de 4 ciclu, 2 3-cicli

$x = yzt$, y = 4-ciclu, z, t 3-cicli disjuncți

$x^2 = (1\ 5)(3\ 4)(2\ 6\ 8)(7\ 9\ 10)$

$x = yzt \Rightarrow x^2 = y^2 z^2 t^2$

$z^2 = (2\ 6\ 8) \Rightarrow z = (8\ 6\ 2) = (2\ 8\ 6)$

$t^2 = (7\ 9\ 10) \Rightarrow t = (10\ 9\ 7) = (7\ 10\ 9)$

$z^3 = t^3 = e$

$y^2 = (1\ 5)(3\ 4)$

$y = (1\ \_\ 5\ \_)$, $y \in \{(1\ 3\ 5\ 4), (1\ 4\ 5\ 3)\}$.

II. $10 = 6+4$.

$x = yz$, y = 6-ciclu, z = 4-ciclu.

$x^2 = (1\ 5)(3\ 4)(2\ 6\ 8)(7\ 9\ 10) = y^2 z^2$.

$ord(y) = 6 \Rightarrow ord(y^2) = \frac{6}{2} = 3$

$ord(z) = 4 \Rightarrow ord(z^2) = 2$.

$y^2 = (2\ 6\ 8)(7\ 9\ 10)$

$\boxed{z^2 = (1\ 5)(3\ 4)}$, $z \in \{(1\ 3\ 5\ 4), (1\ 4\ 5\ 3)\}$.

$y^2 = (2\ 6\ 8)(7\ 9\ 10)$.

$y = (2\ \_\ 6\ \_\ 8\ \_)$

$y \in \{(2\ 7\ 6\ 9\ 8\ 10), (2\ 9\ 6\ 10\ 8\ 7),$
$(2\ 10\ 6\ 7\ 8\ 9)\}$

Obs: Ecuația $x^2 = \sigma \cdot \tau$, σ, τ sunt m-cicli disjuncți are m soluții.

$\sigma = (a_1 \ \dots \ a_m)$, $\tau = (b_1 \ \dots \ b_m)$

$x = (a_1 \ b_i \ a_2 \ b_{i+1} \ \dots \ a_m \ b_{i+m-1})$, $i = \overline{1,m}$

unde indicii $i, i+1, \dots, i+m-1$ se iau modulo m.

Inele. Ideale

Def: A inel comutativ, $I \subseteq A$. I este ideal dacă:

- $(I,+) \leq (A,+)$.
- $ax \in I$, $\forall a \in A$, $x \in I$

Ex. 3: Fie A un inel comutativ, $I, J \trianglelefteq A$. Atunci:

a. $I + J = \{x + y \mid x \in I, y \in J\} \trianglelefteq A$

b. $I \cap J \trianglelefteq A$

c. $I \cdot J = \{x_1 y_1 + \dots + x_m y_m \mid x_i \in I, y_i \in J, m \in \mathbb{N}^*\} \trianglelefteq A$.

Obs:

- $I \cup J$ nu este neapărat ideal.

  $x \in I, y \in J \not\Rightarrow x - y \in I \cup J$.

  $I + J$ este cel mai mic ideal ce conține $I \cup J$.

- $H = \{xy \mid x \in I, y \in J\}$ nu este neapărat ideal

  $x_1, x_2 \in I$, $y_1, y_2 \in J \not\Rightarrow x_1 y_1 - x_2 y_2 \in H$.

Rez:

c. $I \cdot J \trianglelefteq A$.

$(I \cdot J, +) \leq (A, +)$, $z_1, z_2, z \in I \cdot J$

$\begin{cases} z_1 + z_2 \in I \cdot J \\ -z \in I \cdot J \end{cases}$ Ex $\iff$ $z_1 - z_2 \in I \cdot J$.

$z_1 = x_{a_1} y_{a_1} + \dots + x_{a_m} y_{a_m}$

$z_2 = x_{b_1} y_{b_1} + \dots + x_{b_m} y_{b_m}$

$z_1 - z_2 = \dots \in I \cdot J$

$$-\underbrace{x_{b_1}}_{\in I}\underbrace{y_{b_1}}_{\in J}$$

$az \in I \cdot J$, $a \in A$.

$a(x_1 y_1 + \dots + x_m y_m) = \underbrace{(a x_1)}_{\in I}\underbrace{y_1}_{J} + \dots + \underbrace{(a x_m)}_{\in I}\underbrace{y_m}_{\in J} \in I \cdot J$.

$x_i \in I \Rightarrow a x_i \in I \quad \forall a \in A$ deoarece $I \trianglelefteq A$.

Ex. 4: Det. idealele lui $\mathbb{Z}$. Pentru $I, J \trianglelefteq \mathbb{Z}$ det. $I+J$, $I \cap J$, $I \cdot J$.

Rez: $I \trianglelefteq \mathbb{Z} \Rightarrow \begin{cases} (I,+) \leq (\mathbb{Z},+) \Rightarrow I = m\mathbb{Z} \text{ pt. } m \in \mathbb{Z}. \\ a \cdot x \in I, \forall a \in \mathbb{Z}, x \in I \end{cases}$

Idealele lui $\mathbb{Z}$ sunt de forma $m\mathbb{Z}$.

- $I + J$: $I = a\mathbb{Z}$, $J = b\mathbb{Z}$ $\qquad I \subseteq I+J$, $J \subseteq I+J$

Ex $a\mathbb{Z} + b\mathbb{Z} = d\mathbb{Z}$, $d = (a,b)$

Th: $(a,b) = d \Rightarrow \exists\, m, n \in \mathbb{Z}$ a.î. $d = am + bn$.

- $I \cap J$: $I \cap J \subseteq I, J$

$a\mathbb{Z} \cap b\mathbb{Z} = m\mathbb{Z}$, $m = [a,b]$.

- $I \cdot J$:

$a\mathbb{Z} \cdot b\mathbb{Z} = ab\mathbb{Z}$.

Obs:

1. Dacă $(a,b) = 1 \Rightarrow a\mathbb{Z} + b\mathbb{Z} = \mathbb{Z}$, $a\mathbb{Z} \cap b\mathbb{Z} = a\mathbb{Z} \cdot b\mathbb{Z}$.
2. În general, avem $I \cdot J \subseteq I \cap J$.
3. $a\mathbb{Z} \subseteq b\mathbb{Z} \Rightarrow b \mid a$

$4\mathbb{Z} \subseteq 2\mathbb{Z}$

Ex. 5: Fie A, B inele comutative. Arătați că idealele inelului $A\times B$ (cu „+" și „·" pe componente) sunt de forma $I\times J$ cu $I \trianglelefteq A$, $J \trianglelefteq B$.

Rez:

Fie $K \trianglelefteq A\times B$, $K = \{(x,y) \mid x\in A, y\in B$ + alte propr.$\}$.

$\Rightarrow \begin{cases} (K,+) \trianglelefteq (A\times B,+) \\ (a,b)(x,y)\in K, \ \forall a\in A, b\in B, (x,y)\in K \end{cases}$

$\begin{cases} (x_1,y_1)-(x_2,y_2) = (x_1-x_2, \underline{y_1-y_2})\in K. \\ (ax, \underline{by})\in K \end{cases}$

$A_1 = \{x\in A \mid \exists y\in B, (x,y)\in K\}.$

$B_1 = \{y\in B \mid \exists x\in A, (x,y)\in K\}.$

$x_1, x_2 \in A_1 \Rightarrow x_1-x_2\in A_1 \Rightarrow (A_1,+)\leq(A,+)$

$x\in A_1 \Rightarrow ax\in A_1, \forall a\in A$

$\Rightarrow A_1 \trianglelefteq A.$

Analog se arată că $B_1 \trianglelefteq B$.

Ex. 6: $G=(\mathbb{Z}_{12},+)$, $H=\{\hat{0},\hat{3},\hat{6},\hat{9}\}$.

Calculați $\mathbb{Z}_{12}/H$.

Lagrange: $|G| = |H|\cdot|G:H|$

$12 = 4\cdot|G:H| \Rightarrow |G:H| = 3$

$\Rightarrow |G/H| = 3$

Obs: G/H grup $\Rightarrow G/H \cong \mathbb{Z}_3$.

$G = \{\underline{\hat{0}}, \hat{1}, \hat{2}, \underline{\hat{3}}, \hat{4}, \hat{5}, \underline{\hat{6}}, \hat{7}, \hat{8}, \underline{\hat{9}}, \hat{10}, \hat{11}\}$

xH

$\hat{0}+H=H$, $x+H=H$, $x\in H$.

$\hat{1}+H=\{\hat{1},\hat{4},\hat{7},\hat{10}\}$

$\hat{2}+H=\{\hat{2},\hat{5},\hat{8},\hat{11}\}$. $\Rightarrow G/H=\{\hat{0},\hat{1},\hat{2}\}$.

Obs: $xH=yH$, $\forall\, y\in xH$.

Ex 7: $G=S_4$, $H=\{e,(1\,2\,3)^{(4)},(1\,3\,2)^{(4)}\}$

$(G/H)_{s,d}=?$

$|G|=|H|\cdot|G:H|$

$4!=3\cdot|G:H|$ $\Rightarrow |G:H|=8$.

$\Rightarrow |G/H|=8$.

$\hat{e}=H$.

$(1\,2)H=\{(1\,2),(2\,3),(1\,3)\}$.

$(1\,4)H=\{(1\,4),(4\,1\,2\,3),(4\,1\,3\,2)\}$.

$(2\,4)H=\{(2\,4),(4\,2\,3\,1),(4\,2\,1\,3)\}$.

$(3\,4)H=\{(3\,4),(4\,3\,1\,2),(4\,3\,2\,1)\}$.

$(1\,2\,4)H=\{(1\,2\,4),(1\,4)(2\,3),(1\,3\,4)\}$.

$(1\,4\,3)H=\{(1\,4\,3),(1\,2)(3\,4),(2\,4\,3)\}$.

$(2\,3\,4)H=\{(2\,3\,4),(1\,3)(2\,4),(1\,4\,2)\}$.

$G/H=\{\hat{e},\widehat{(1\,2)},\widehat{(1\,4)},\widehat{(2\,4)},\widehat{(3\,4)},$
$\widehat{(1\,2\,4)},\widehat{(1\,4\,3)},\widehat{(2\,3\,4)}\}$.

$A=\{a_1,\dots,a_m\}$, $\langle A\rangle=\{a_1^{b_1}\dots a_m^{b_m} \mid b_i\in\mathbb{Z}\}$.

comutativ

$0\le b_i\le ord(a_i)<\infty$