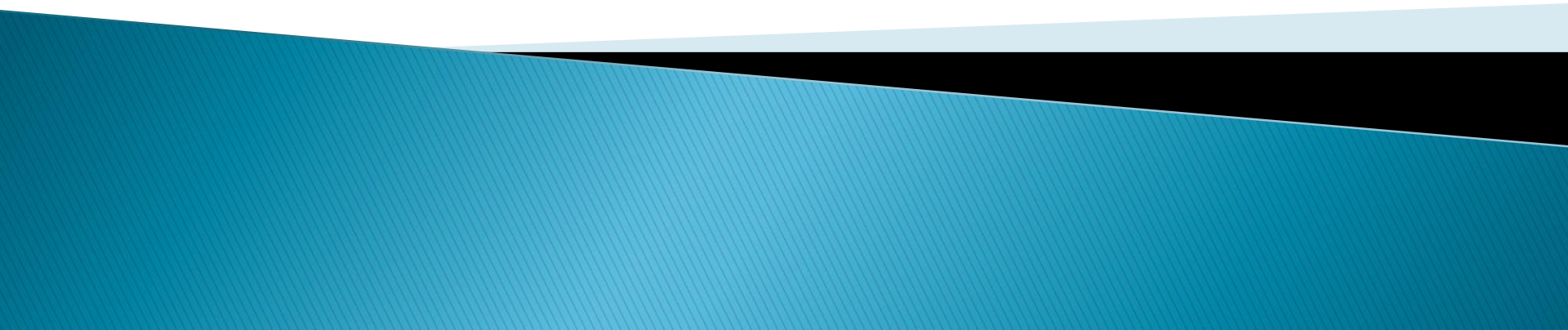


Metoda Divide et Impera

desparte și stăpânește



Metoda Divide et Impera

- ▶ Constă în
 - **împărțirea** repetată a unei probleme de dimensiuni mari în mai multe subprobleme **de același tip**

Metoda Divide et Impera

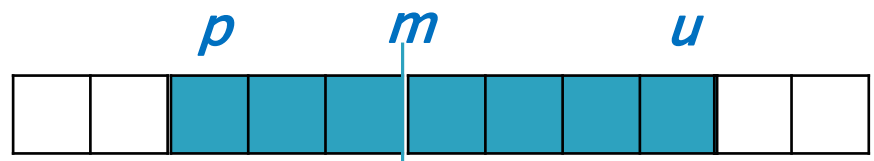
- ▶ Constă în
 - **împărțirea** repetată a unei probleme de dimensiuni mari în mai multe subprobleme **de același tip**
 - **rezolvarea** acestor subprobleme (în același mod sau prin rezolvare directă, dacă au dimensiune mică)

Metoda Divide et Impera

► Constă în

- **împărțirea** repetată a unei probleme de dimensiuni mari în mai multe subprobleme **de același tip**
- **rezolvarea** acestor subprobleme (în același mod sau prin rezolvare directă, dacă au dimensiune mică)
- **combinarea rezultatelor** obținute pentru a determina rezultatul corespunzător problemei inițiale.

Schema posibilă



– pentru $a[p..u]$

```
function DivImp(p, u)
```

```
    if  $u - p < \epsilon$ 
```

```
         $r \leftarrow \text{RezolvaDirect}(p, u)$ 
```

```
    else
```

```
         $m \leftarrow \text{Pozitie}(p, u)$  – de obicei mijlocul
```

```
         $r1 \leftarrow \text{DivImp}(p, m)$ 
```

```
         $r2 \leftarrow \text{DivImp}(m+1, u)$ 
```

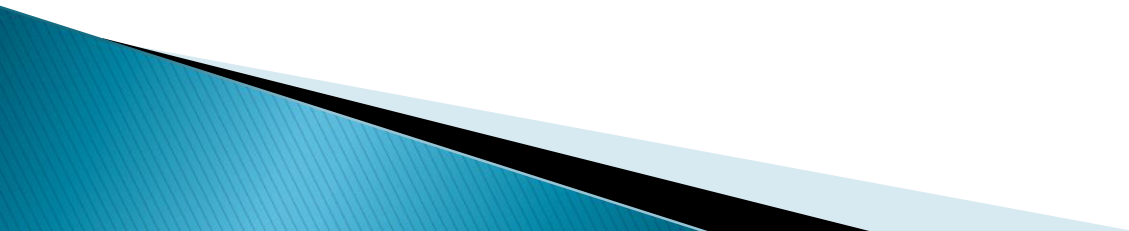
```
         $r \leftarrow \text{Combina}(r1, r2)$ 
```

```
    return r
```

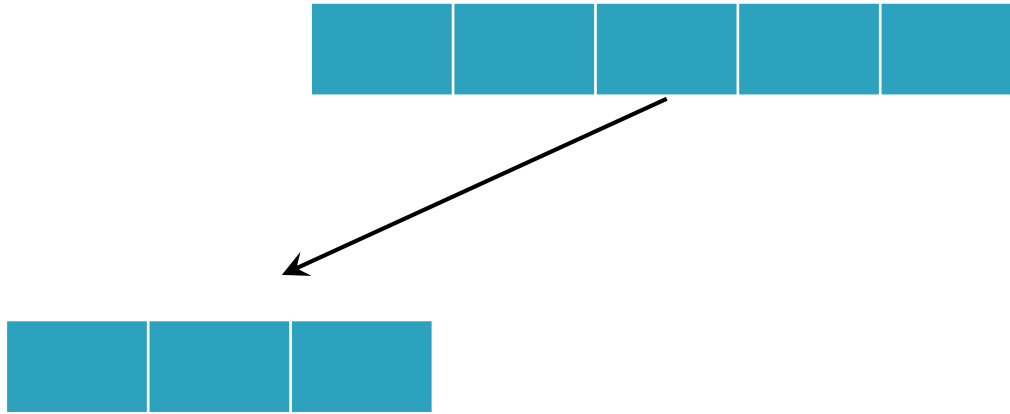
```
end
```

► **Apel:** $\text{DivImp}(1, n)$

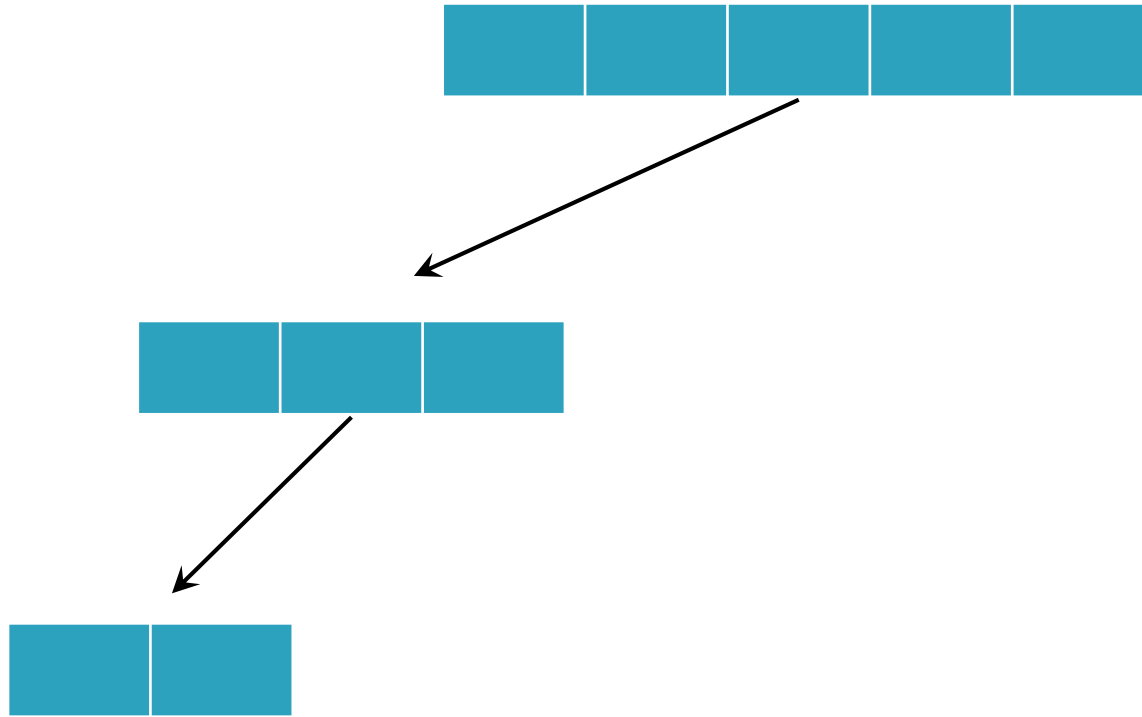
Metoda Divide et Impera



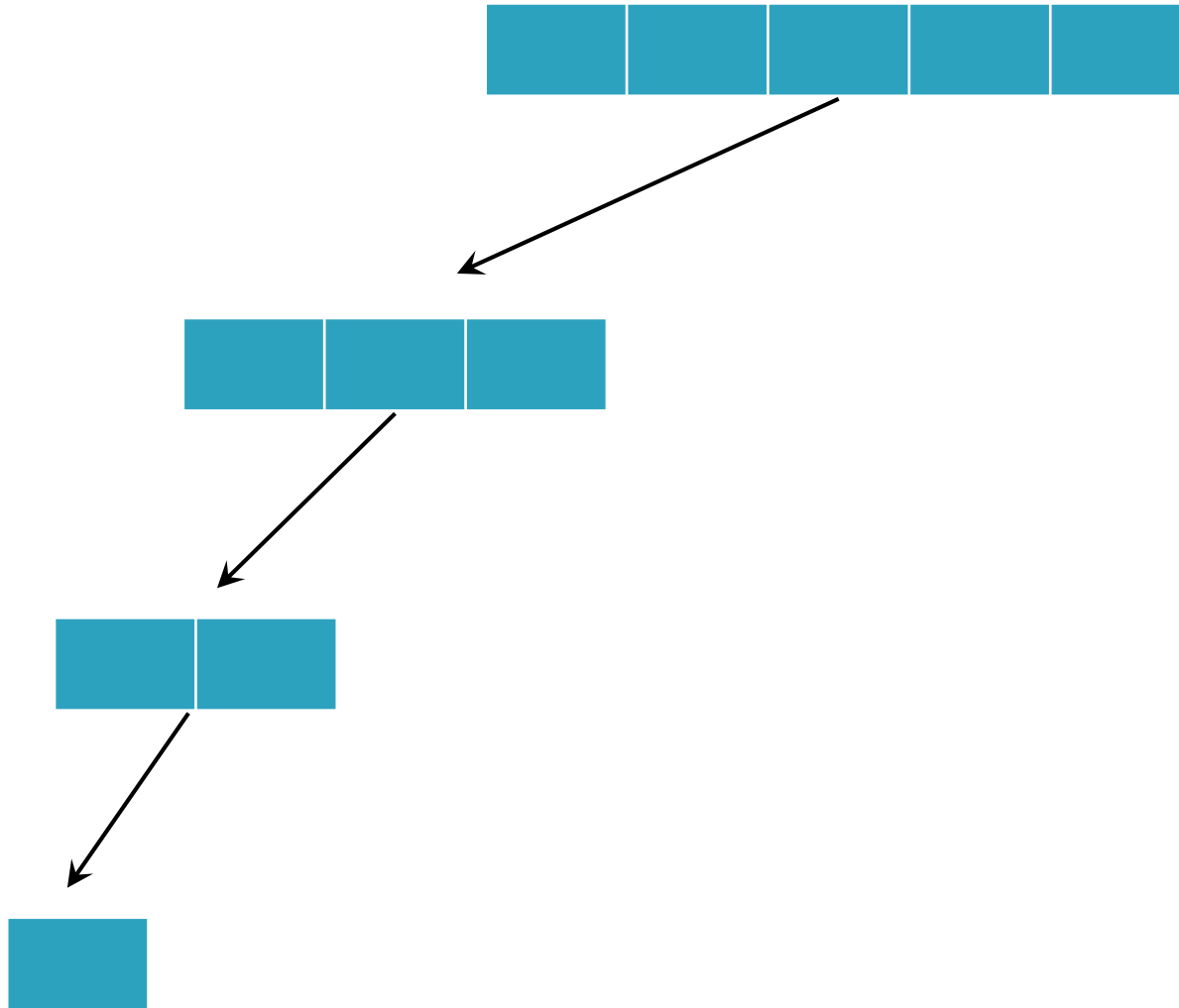
Metoda Divide et Impera



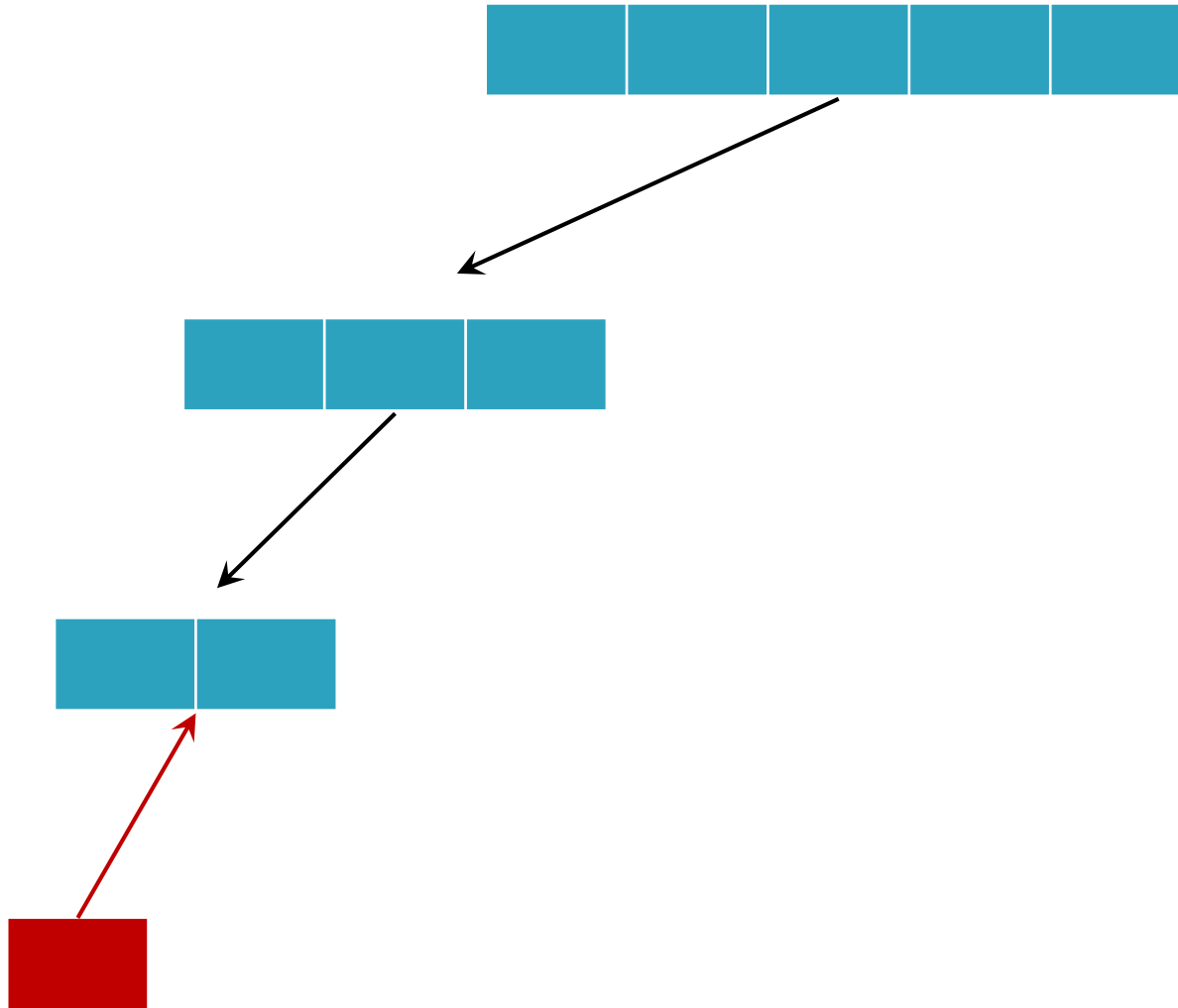
Metoda Divide et Impera



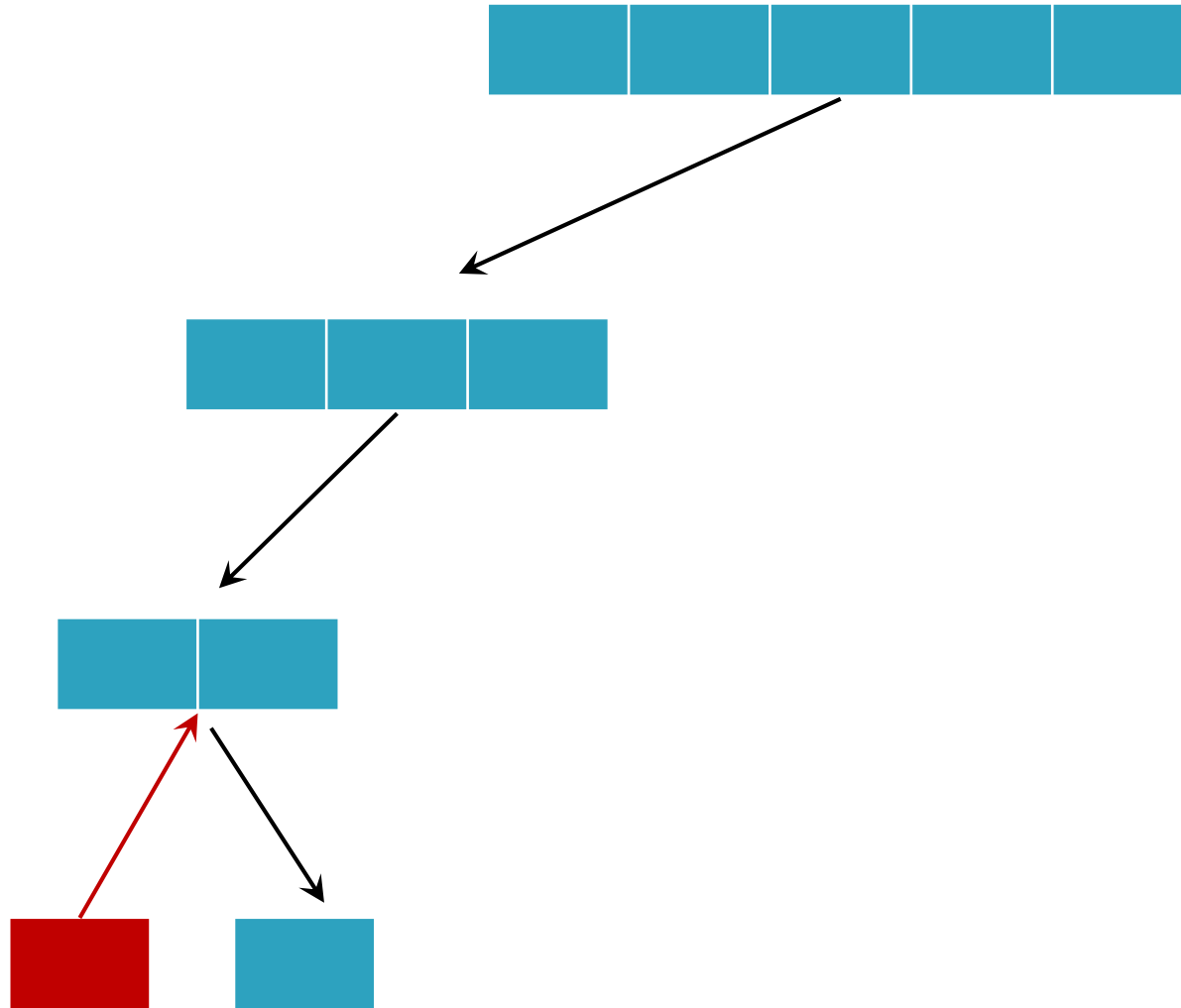
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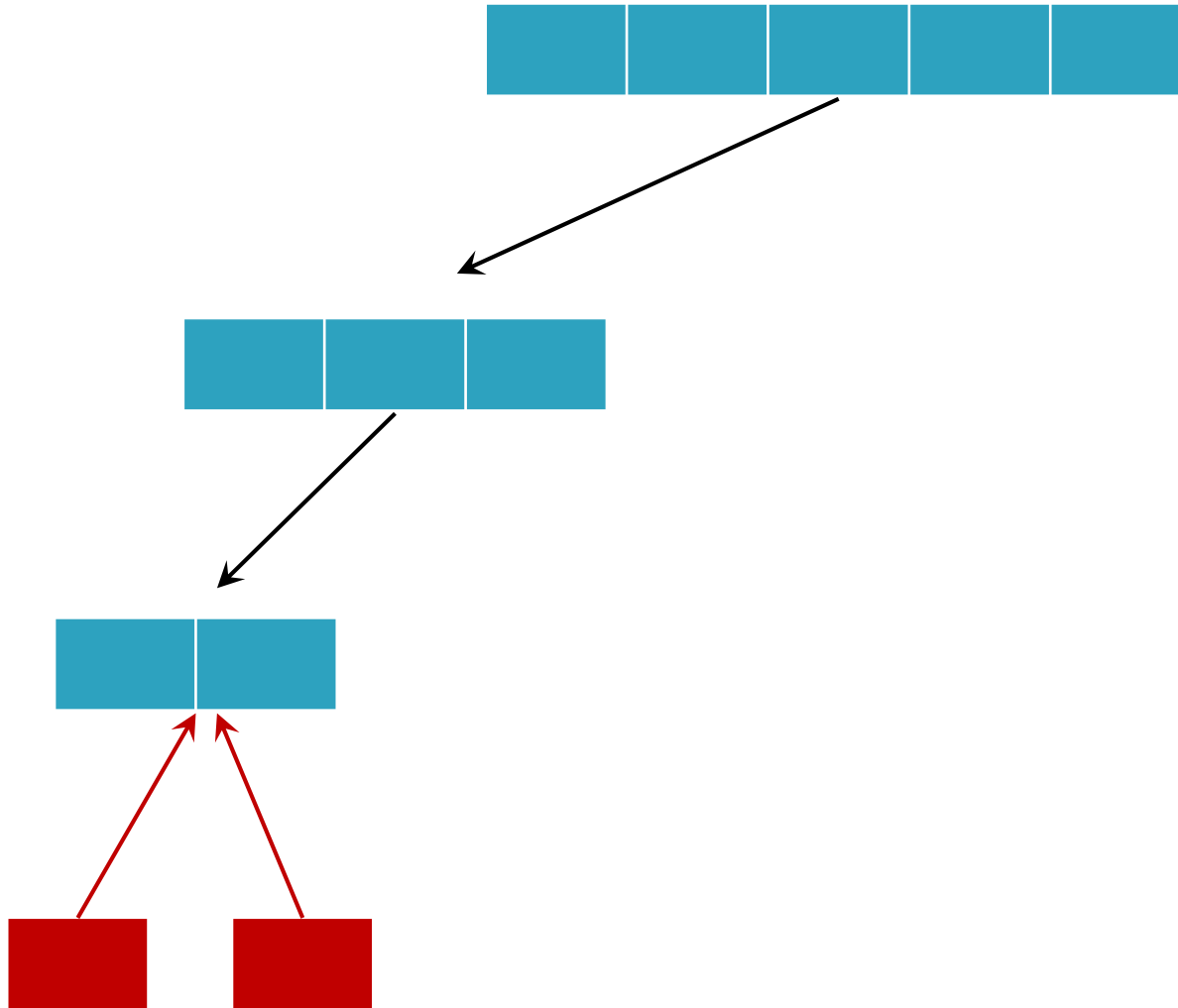
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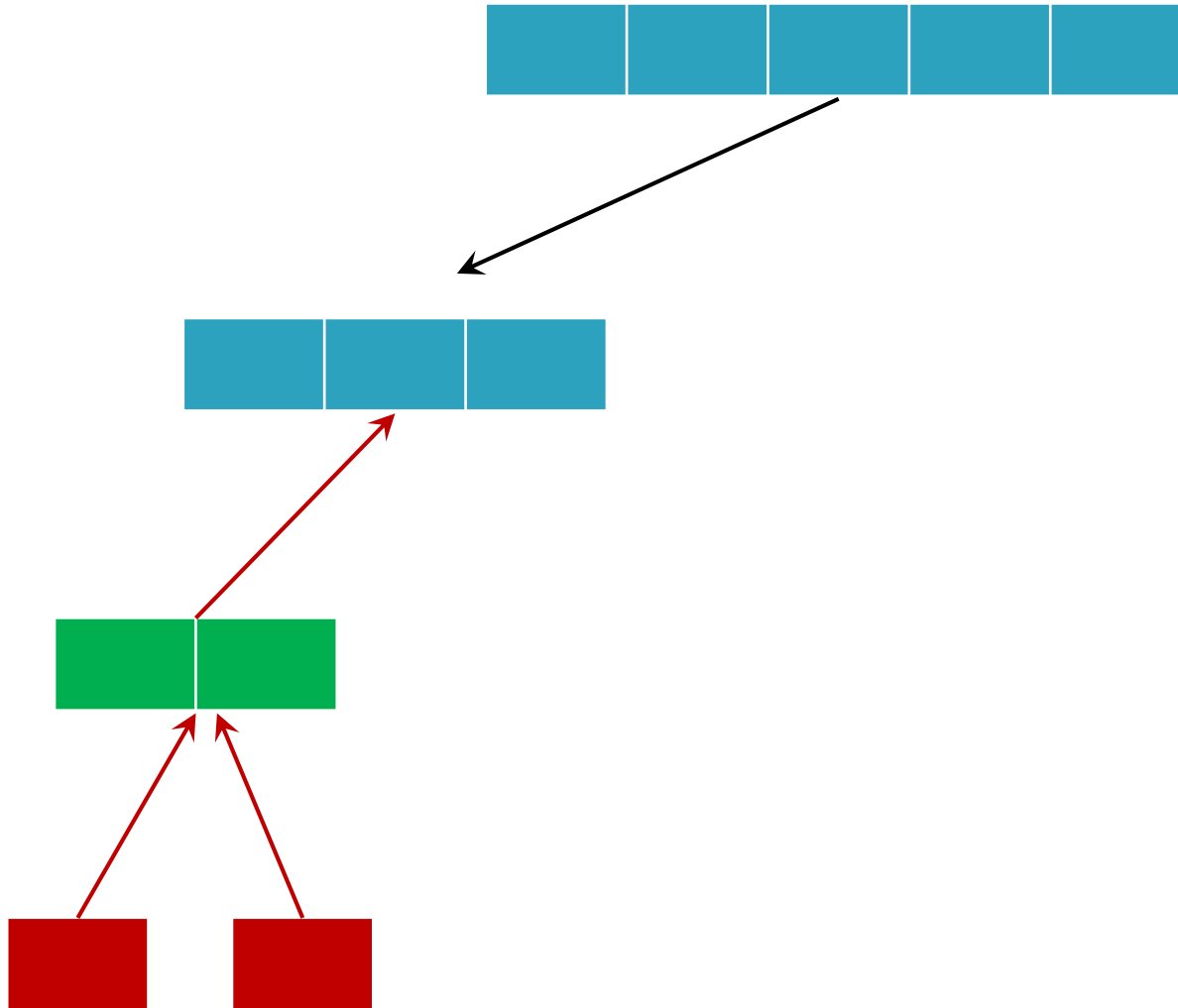
Metoda Divide et Impera



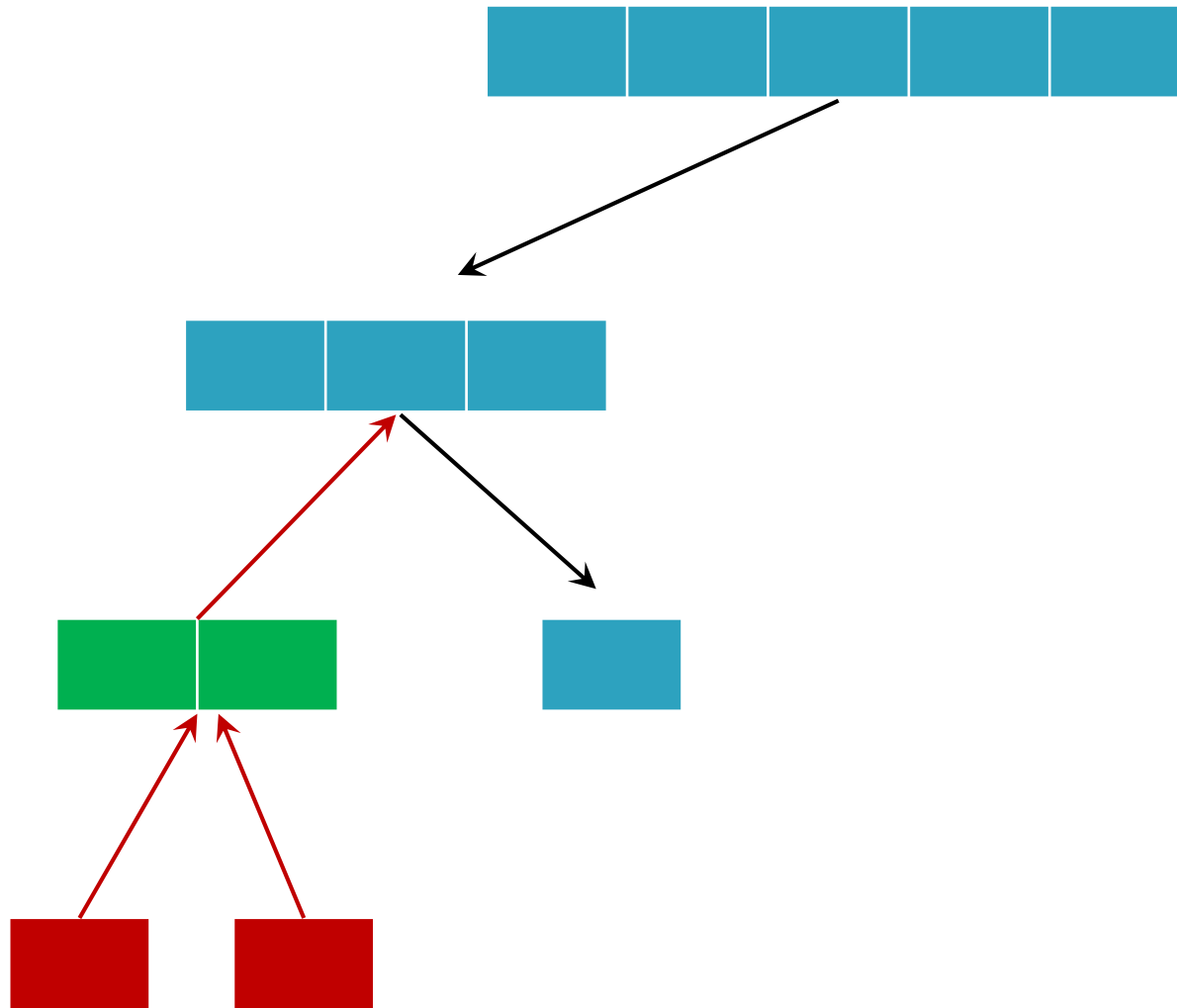
Metoda Divide et Impera



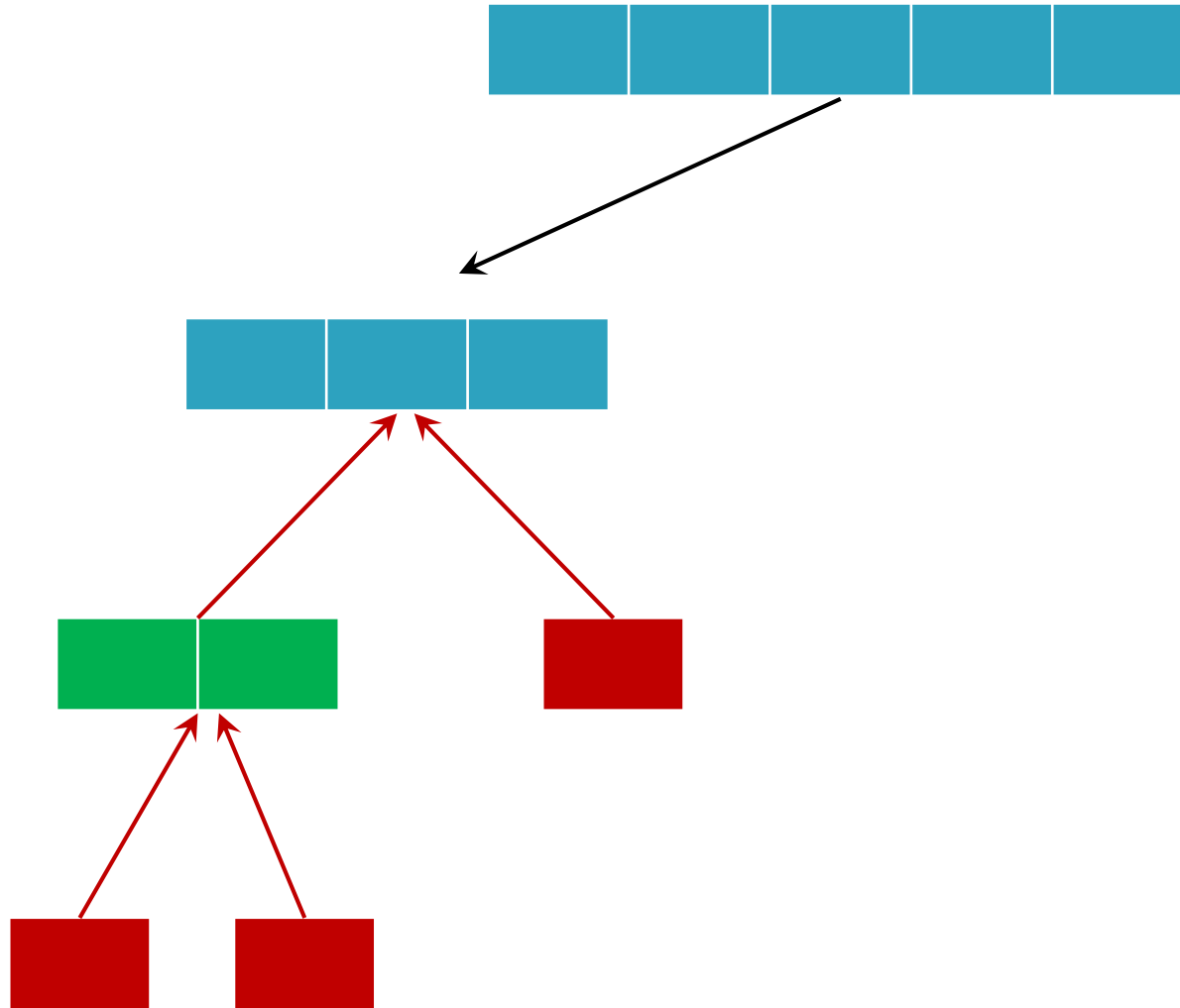
Metoda Divide et Impera



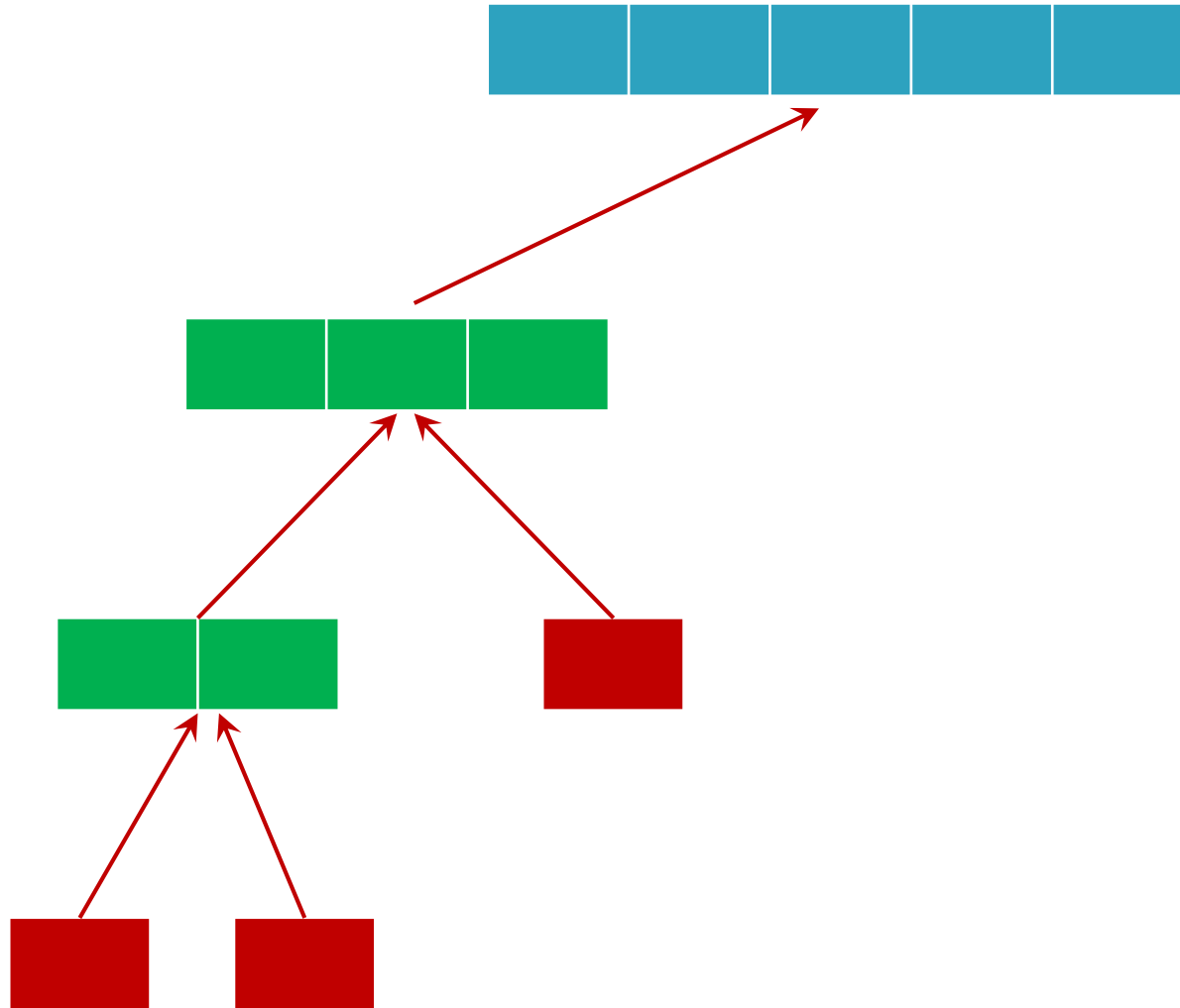
Metoda Divide et Impera



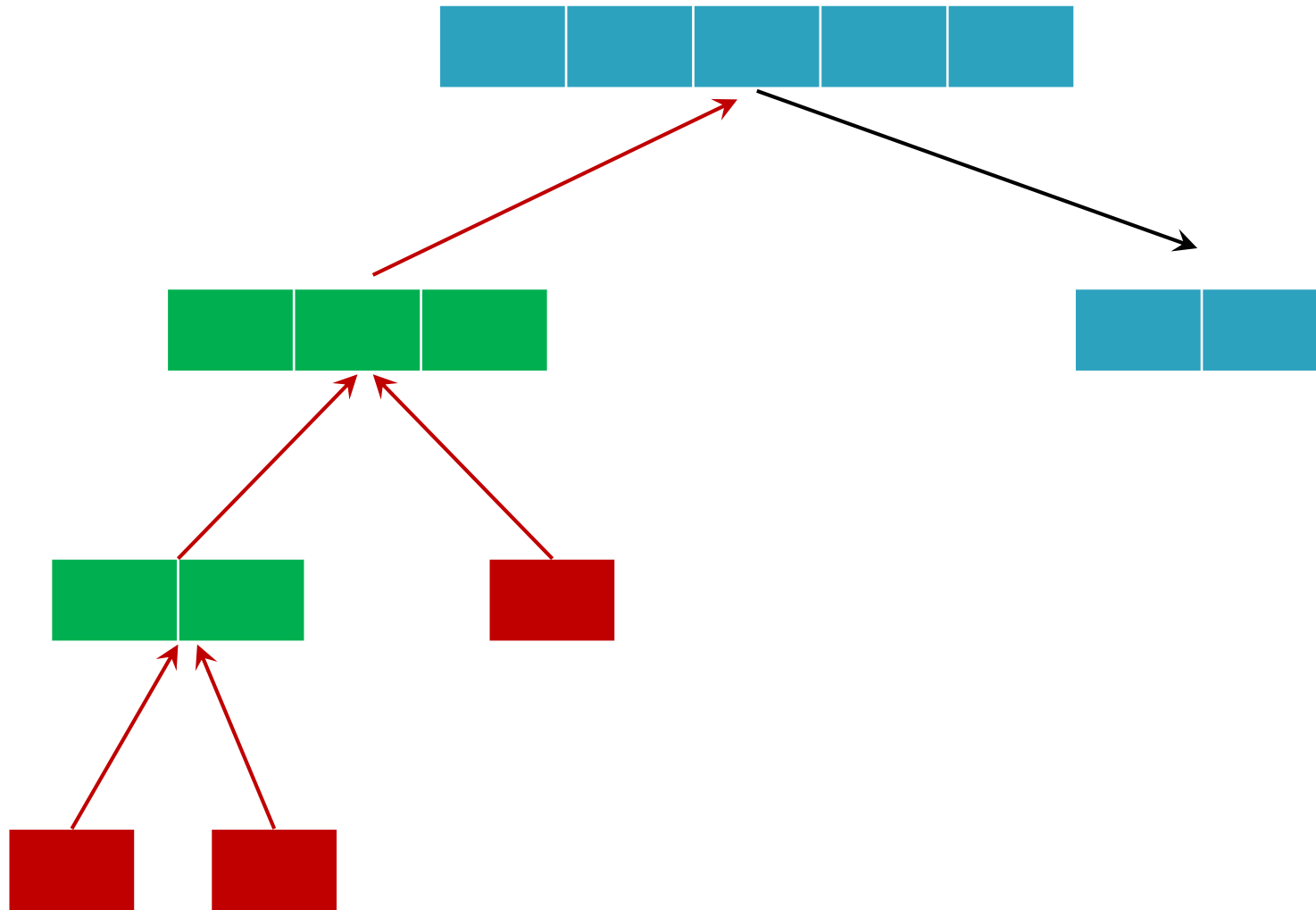
Metoda Divide et Impera



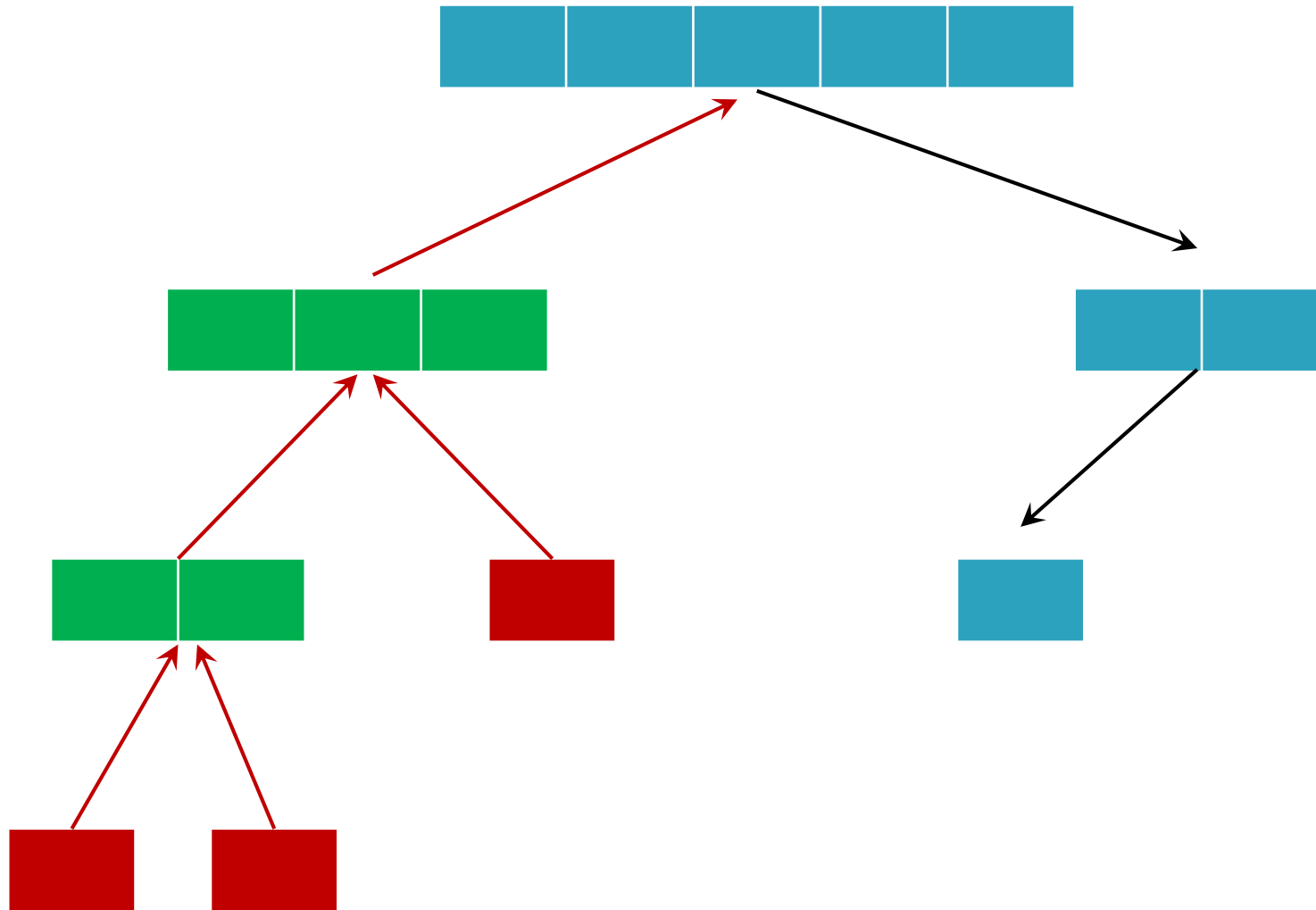
Metoda Divide et Impera



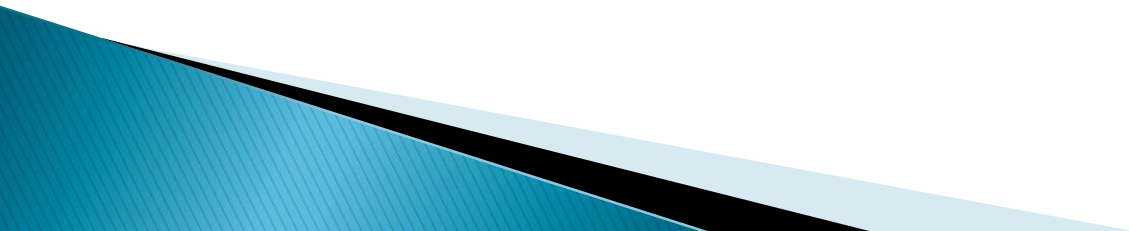
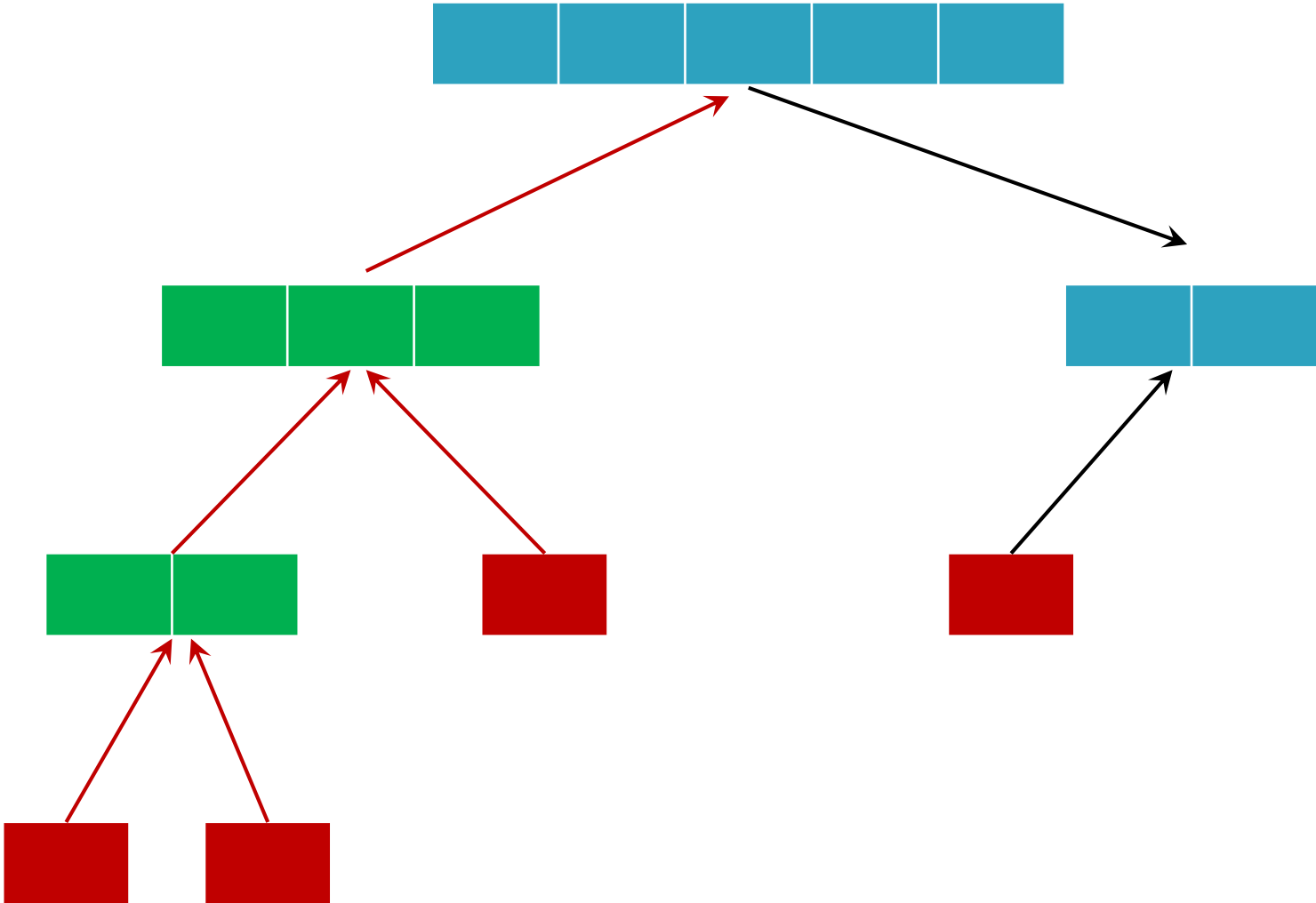
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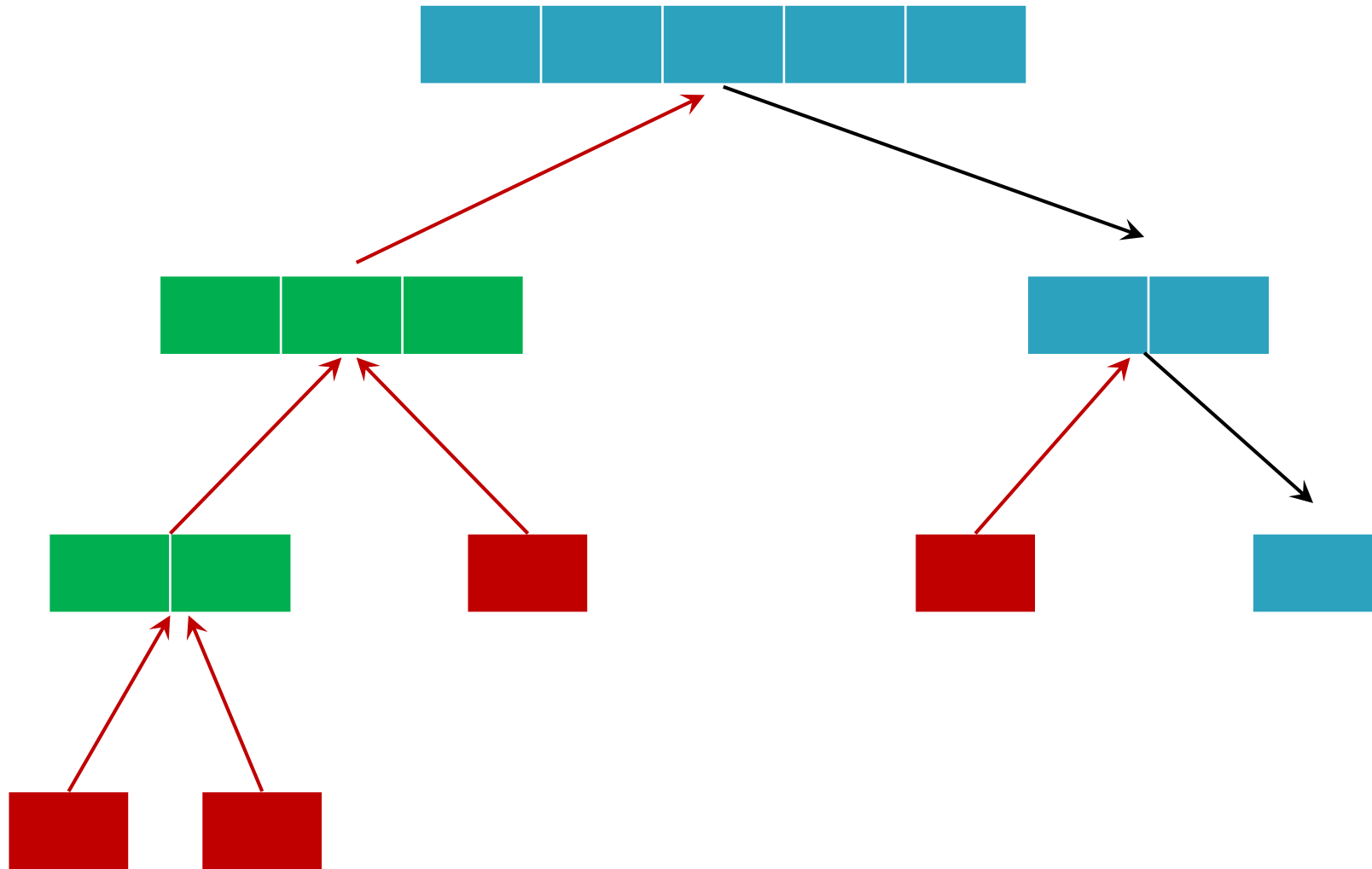
Metoda Divide et Impera



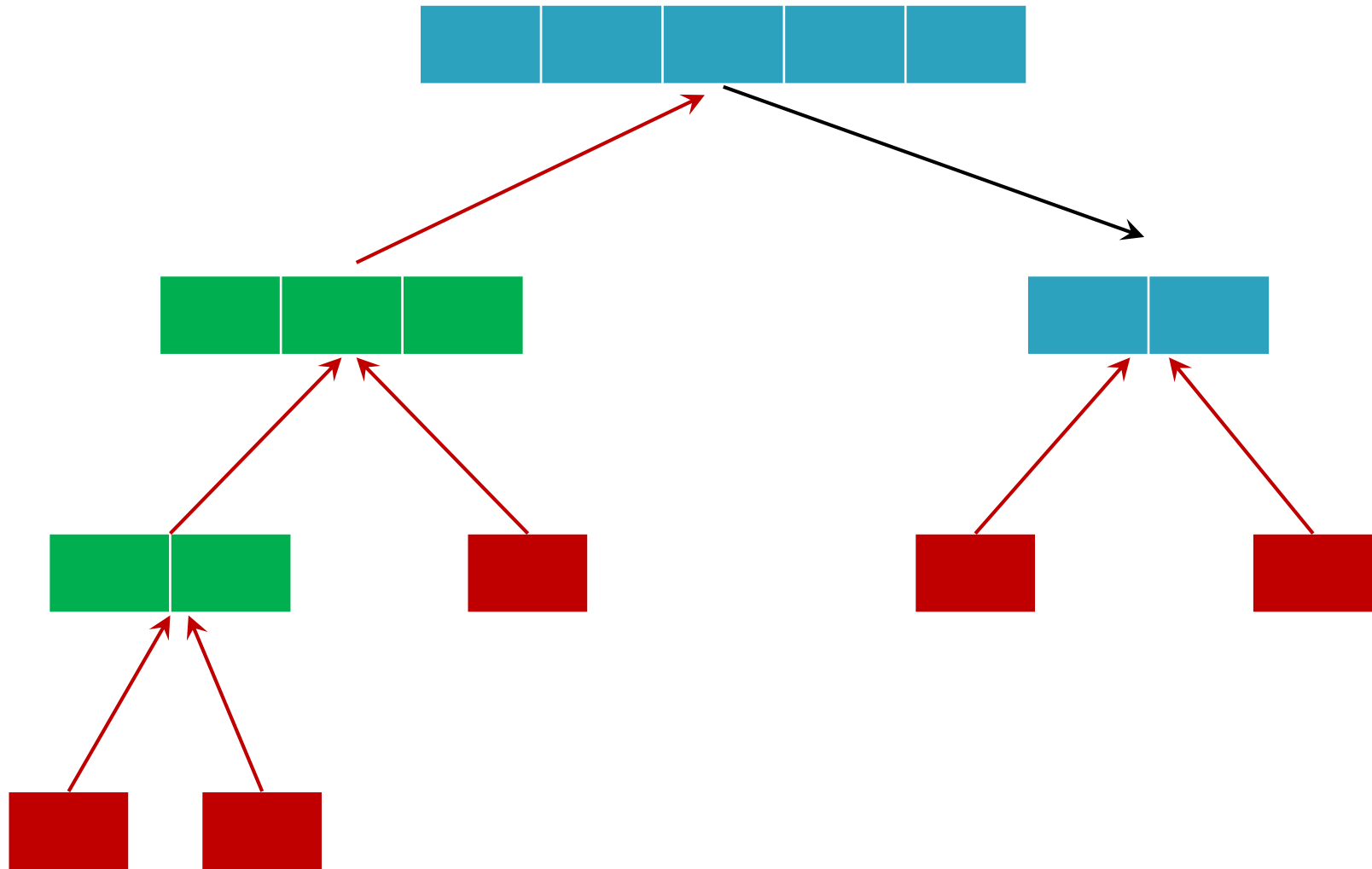
Metoda Divide et Impera



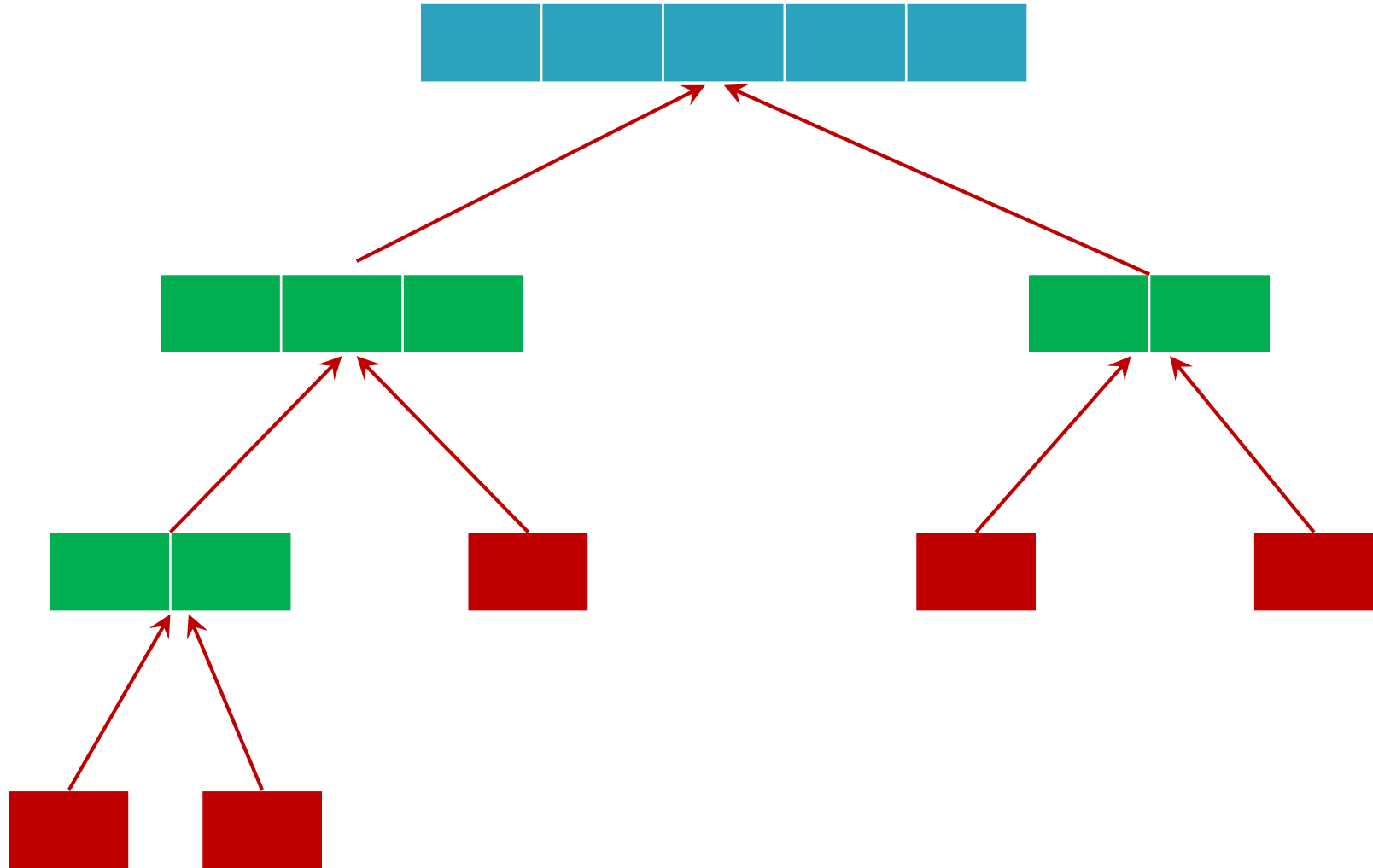
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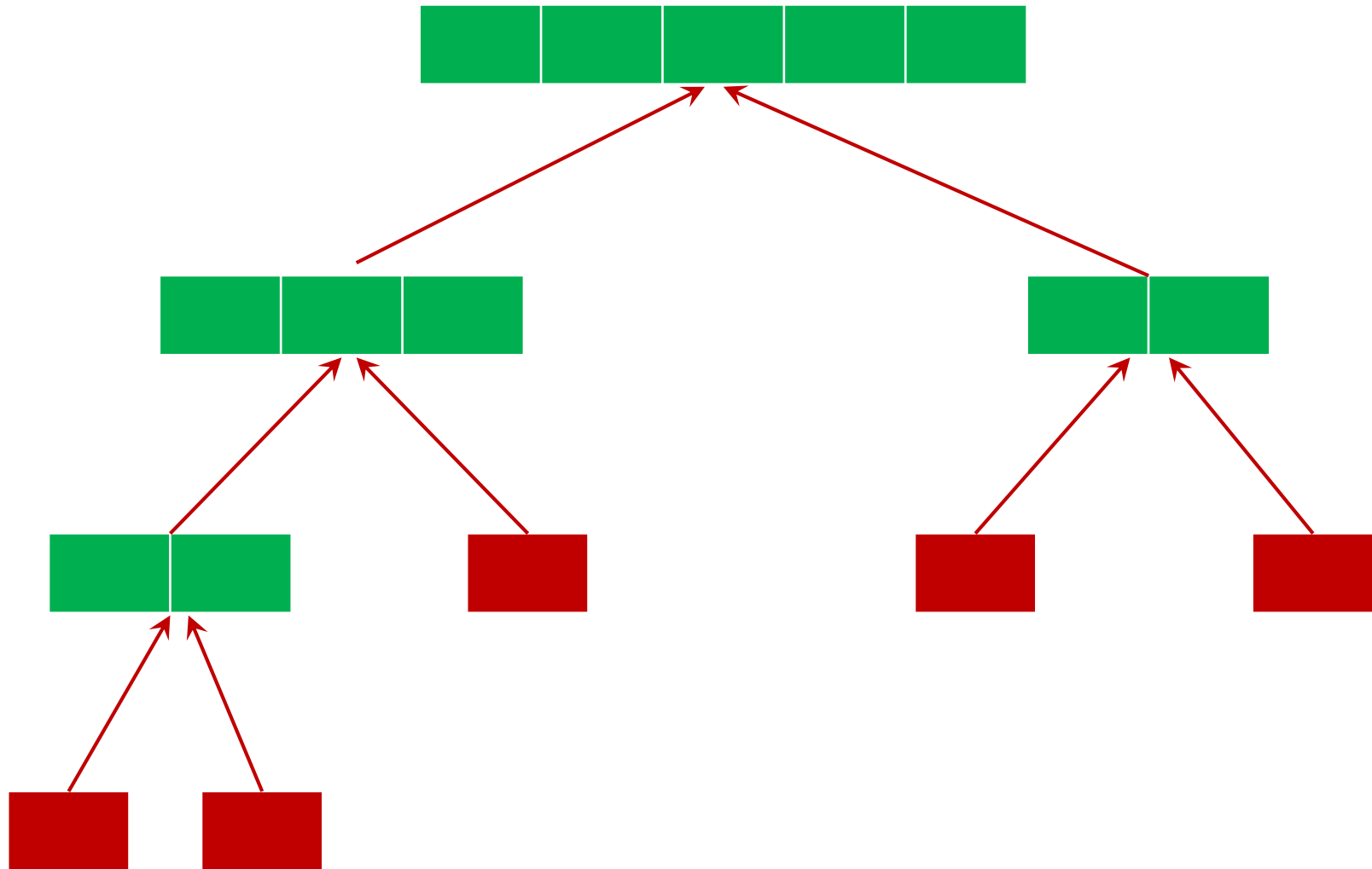
Metoda Divide et Impera



Metoda Divide et Impera



Metoda Divide et Impera



Exemplu –maximul elementelor unui vector

```
function DivImp(p,u)
    if u==p
        r ← a[p]
    else
        m ← ⌊(p+u)/2⌋
        r1 ← DivImp(p,m)
        r2 ← DivImp(m+1,u)
        if r1>r2
            r ← r1
        else
            r ← r2
    return r
```

Apel: DivImp(0, n-1)

Metoda Divide et Impera

3

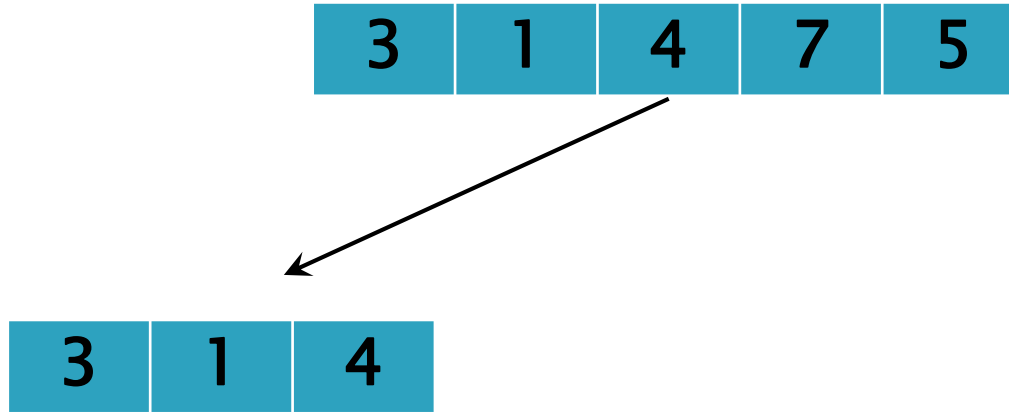
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4

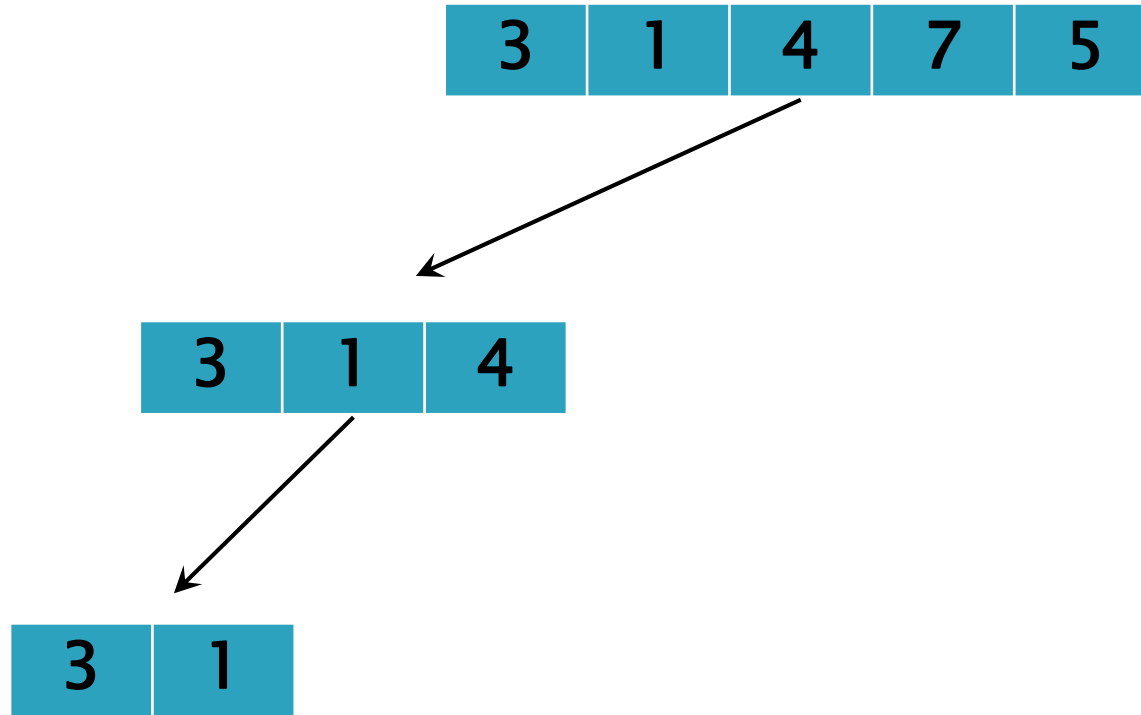
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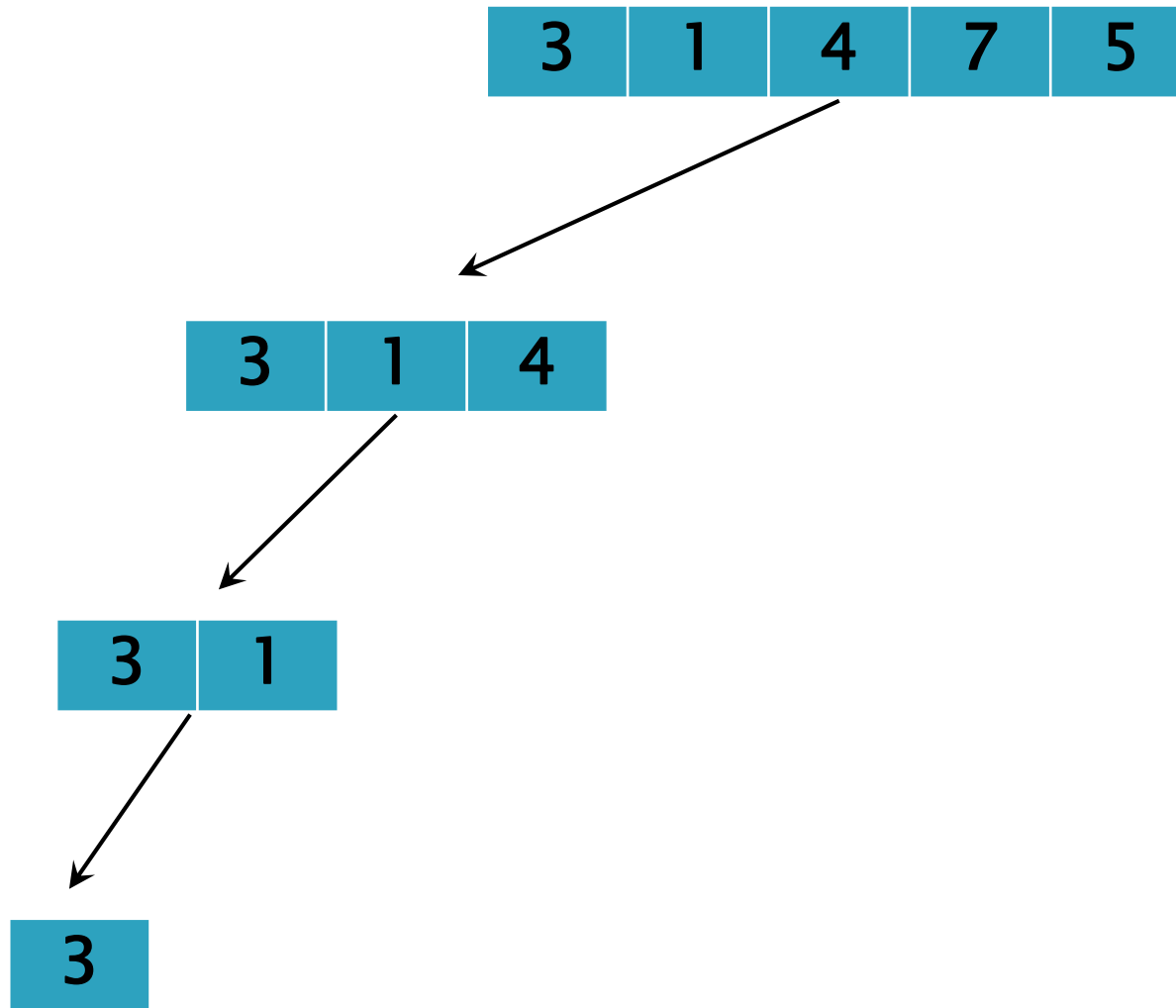
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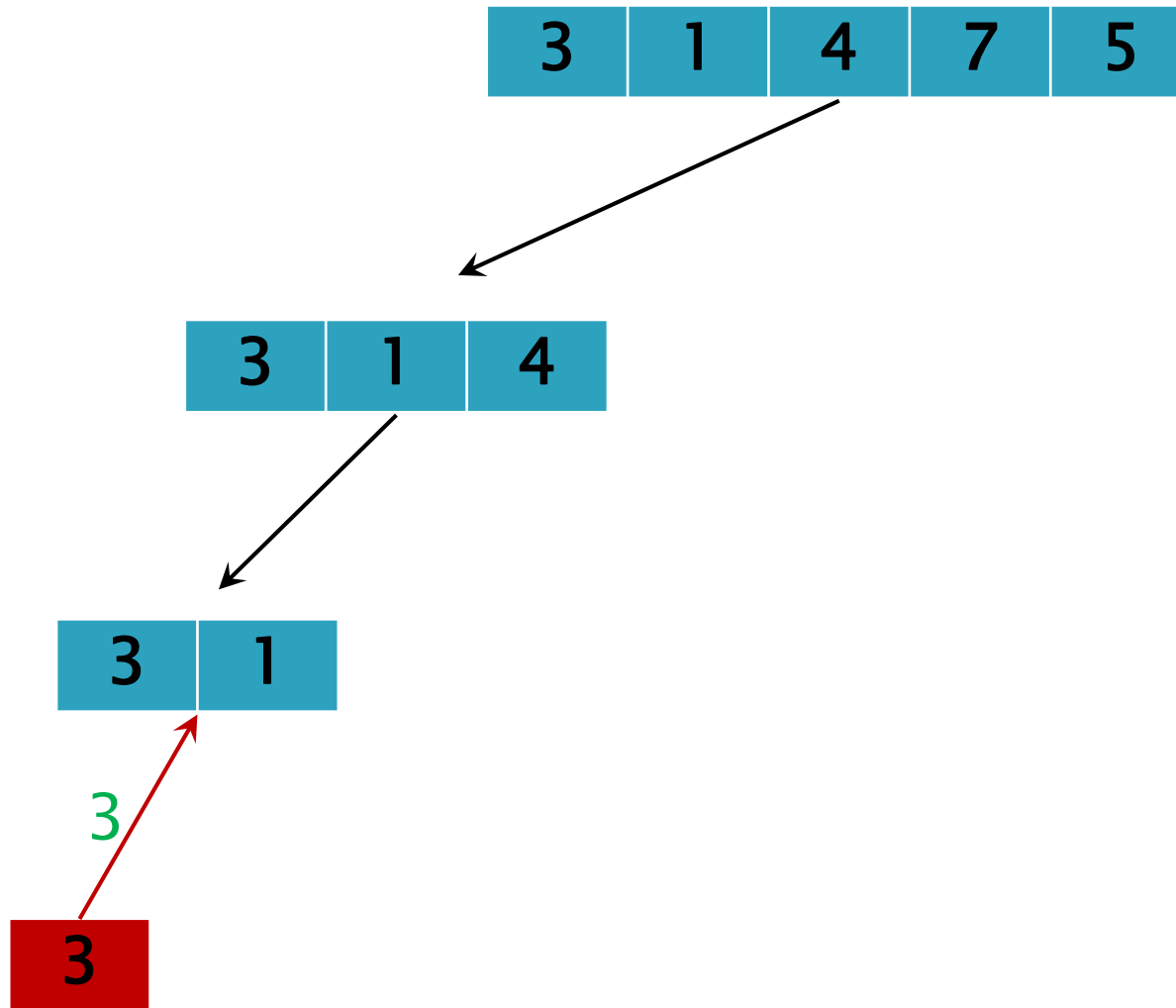
Metoda Divide et Impera



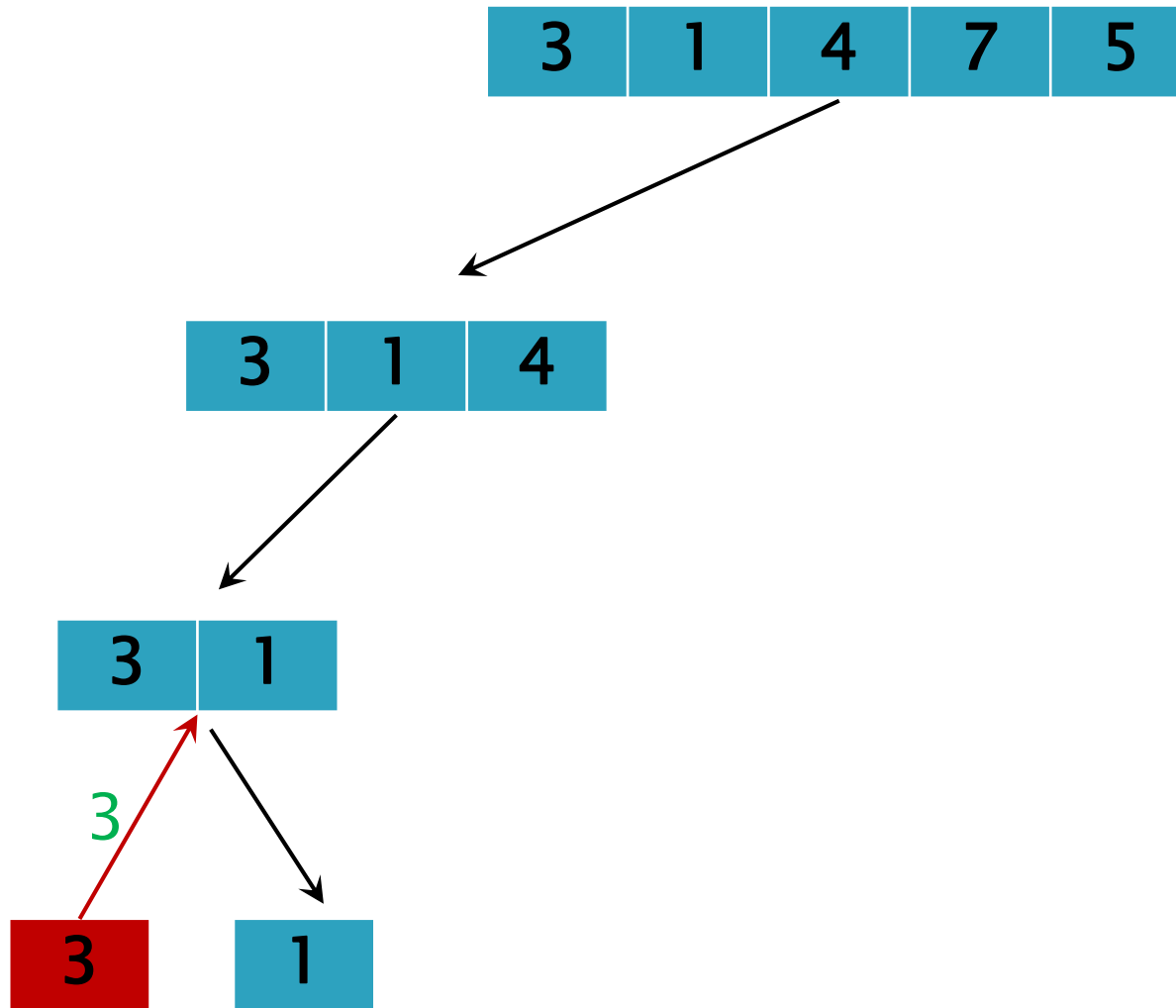
Metoda Divide et Impera



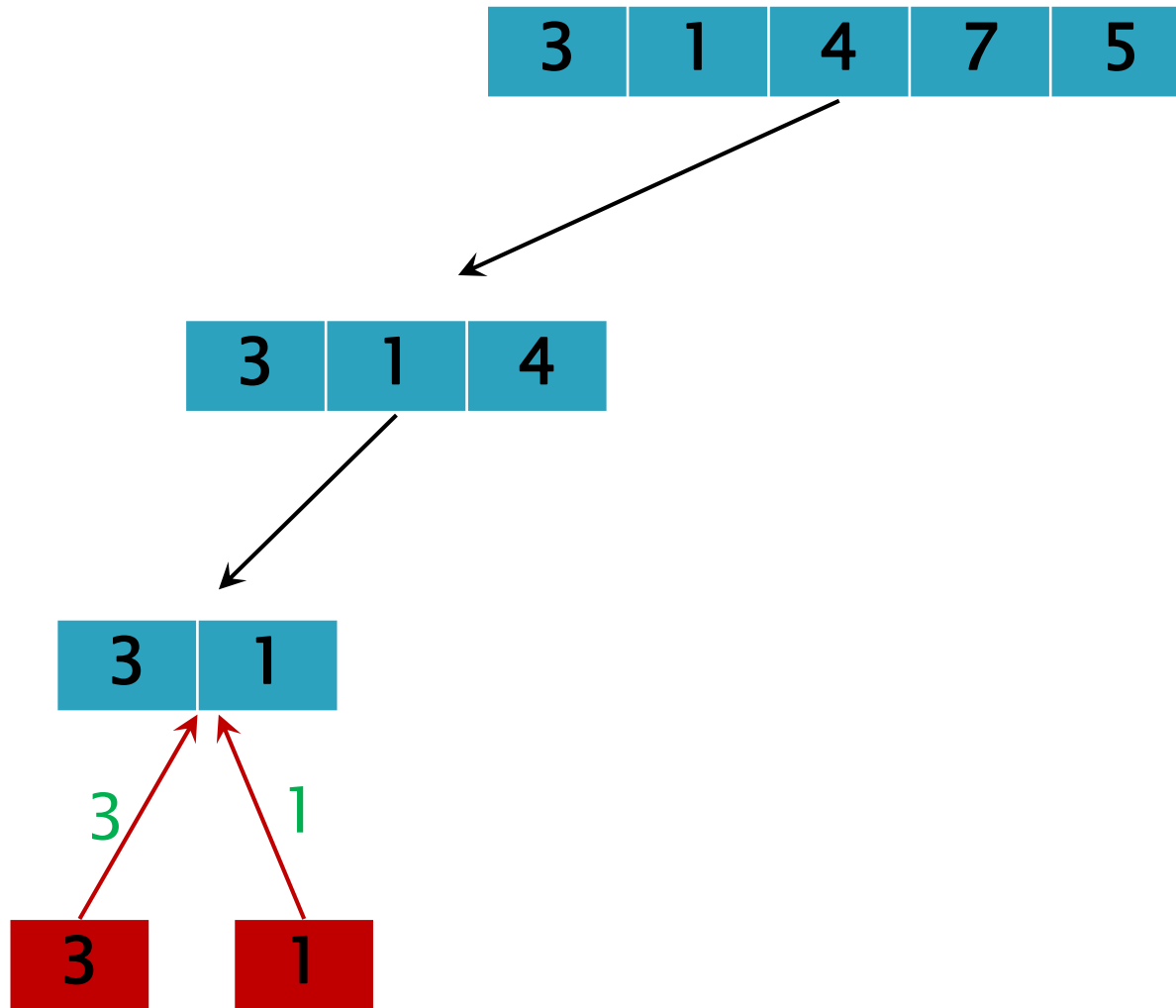
Metoda Divide et Impera



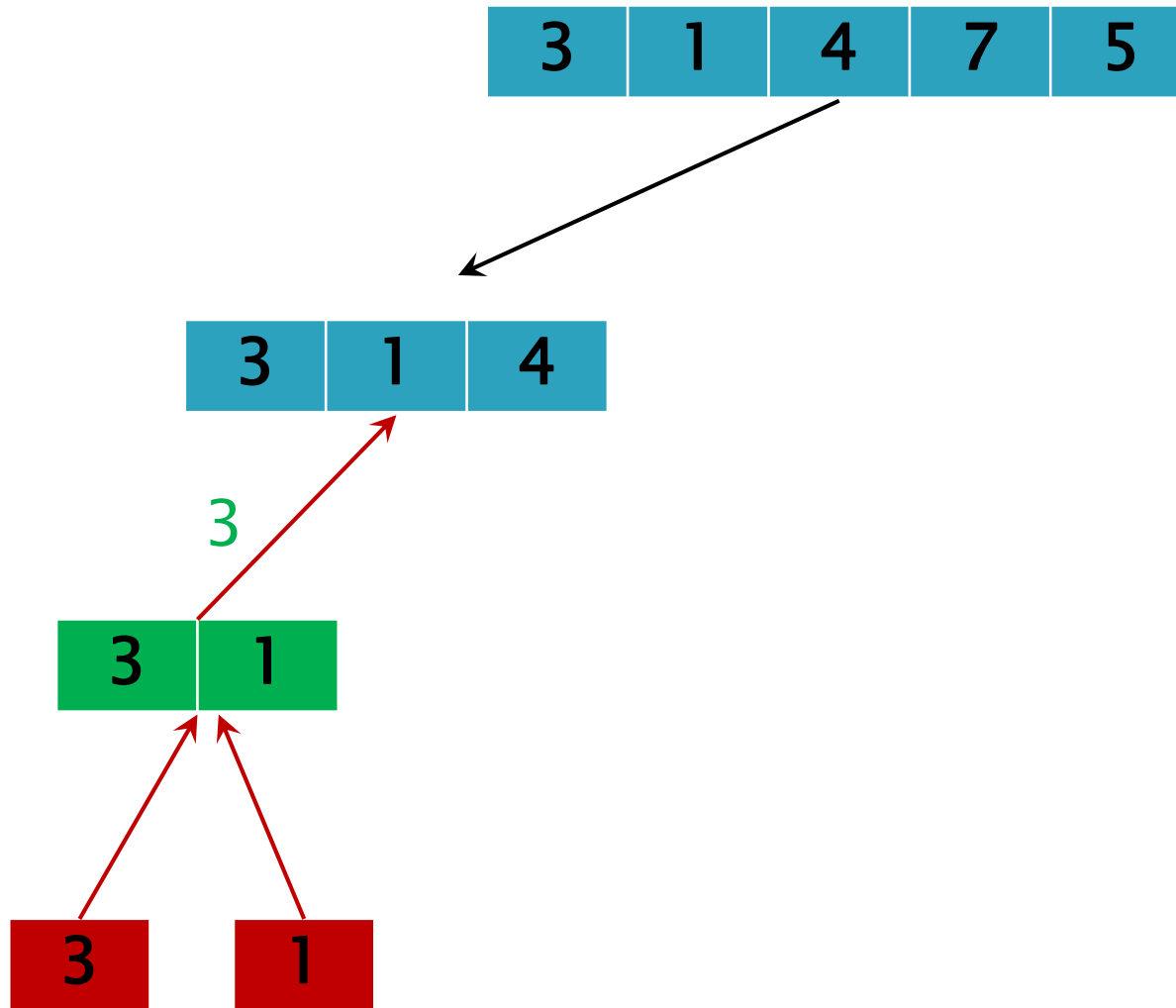
Metoda Divide et Impera



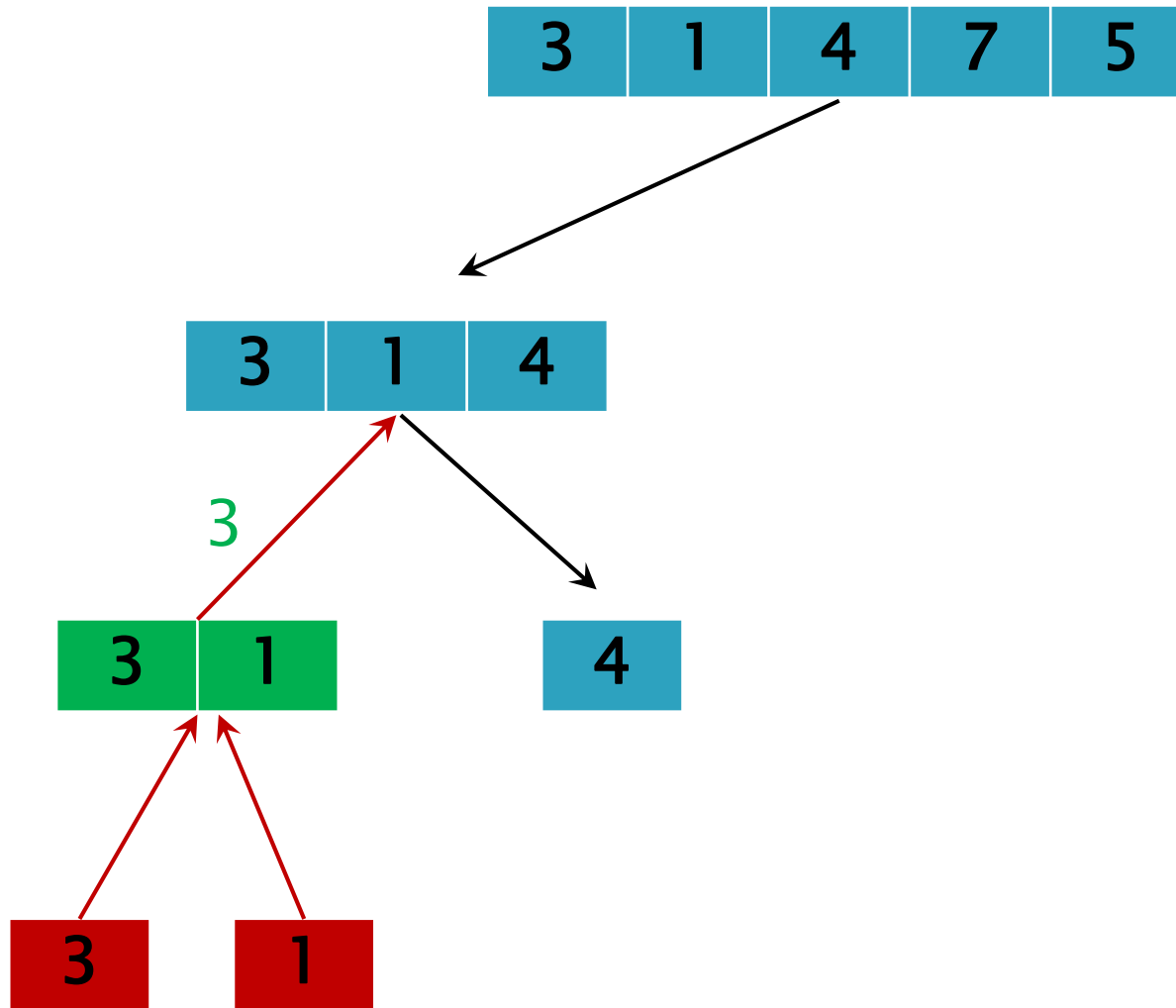
Metoda Divide et Impera



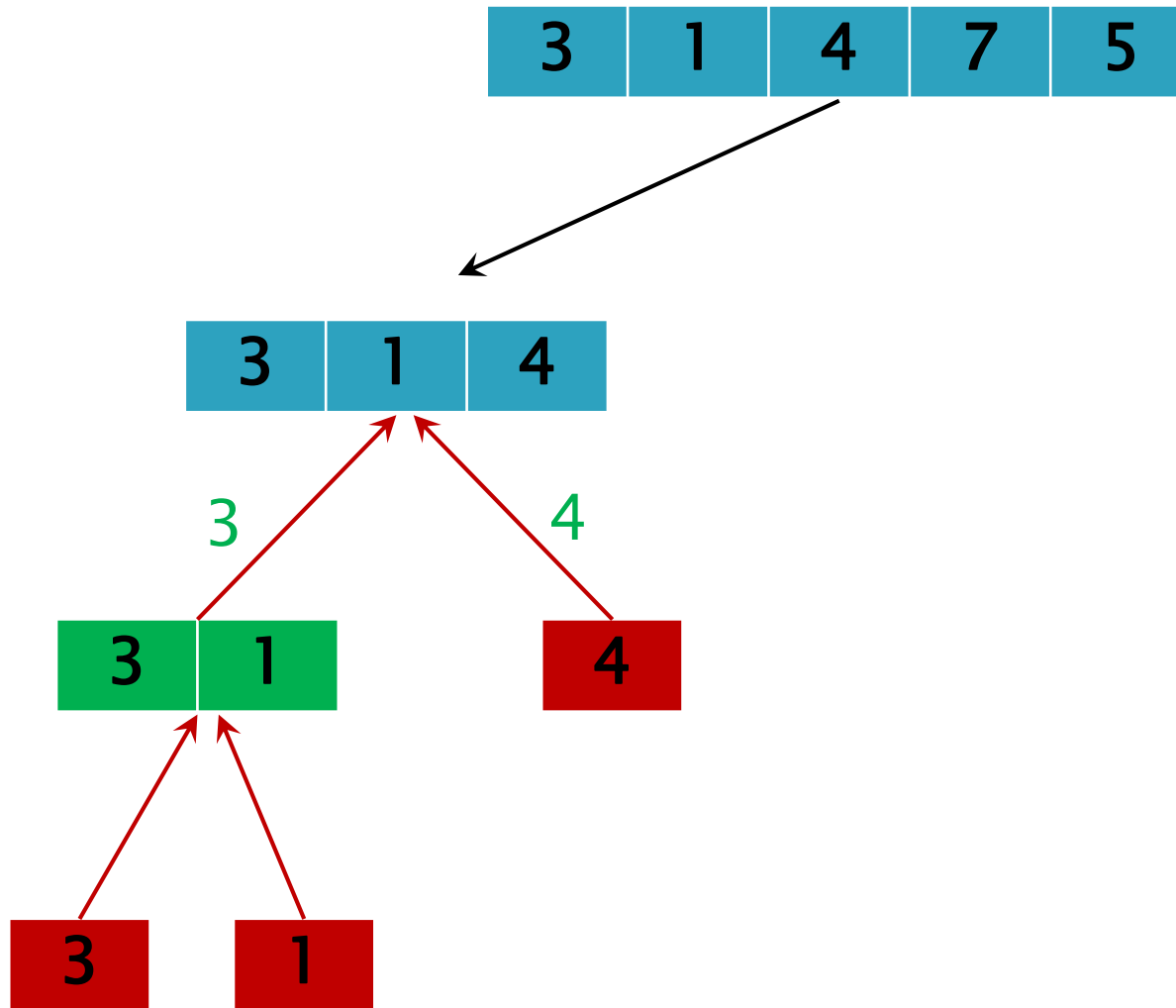
Metoda Divide et Impera



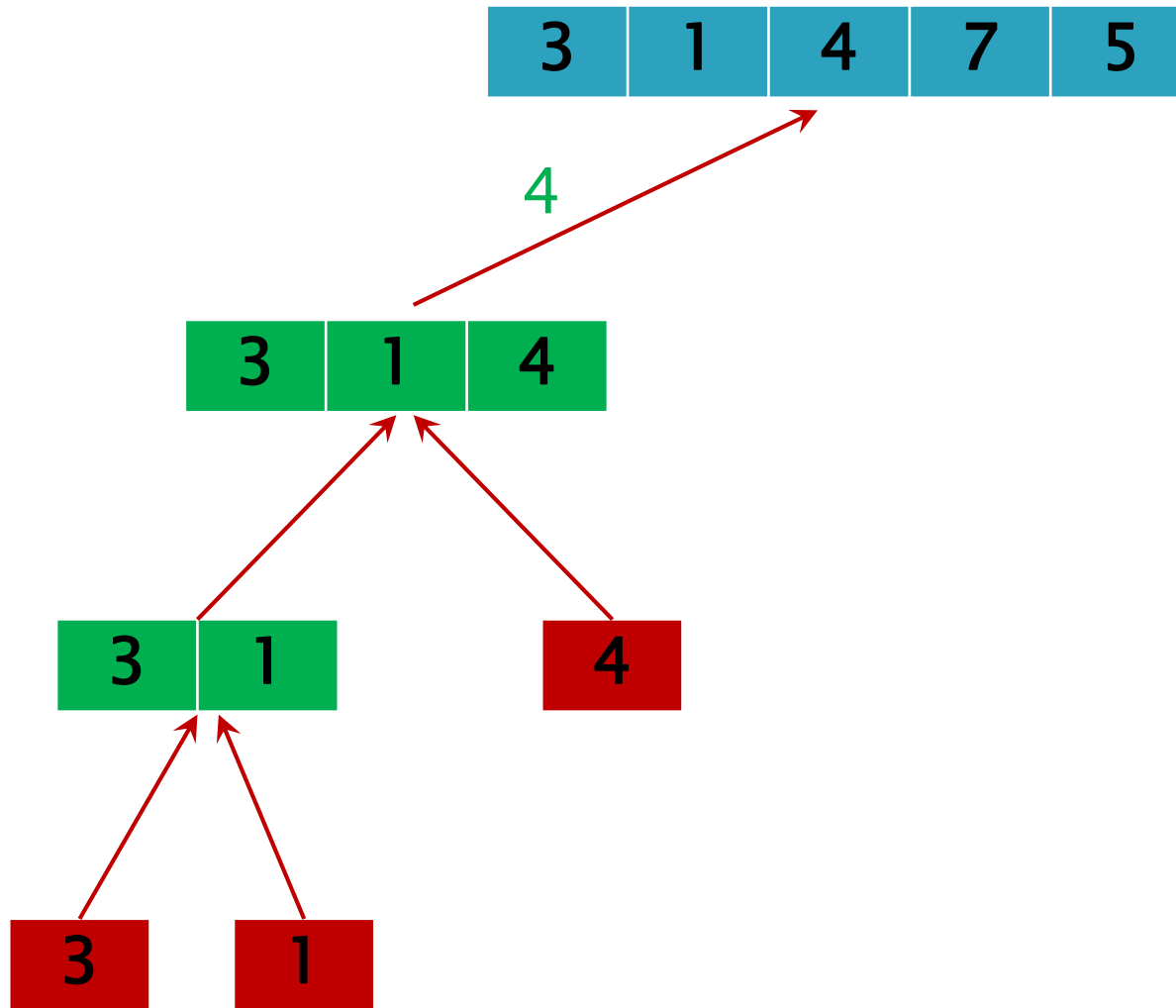
Metoda Divide et Impera



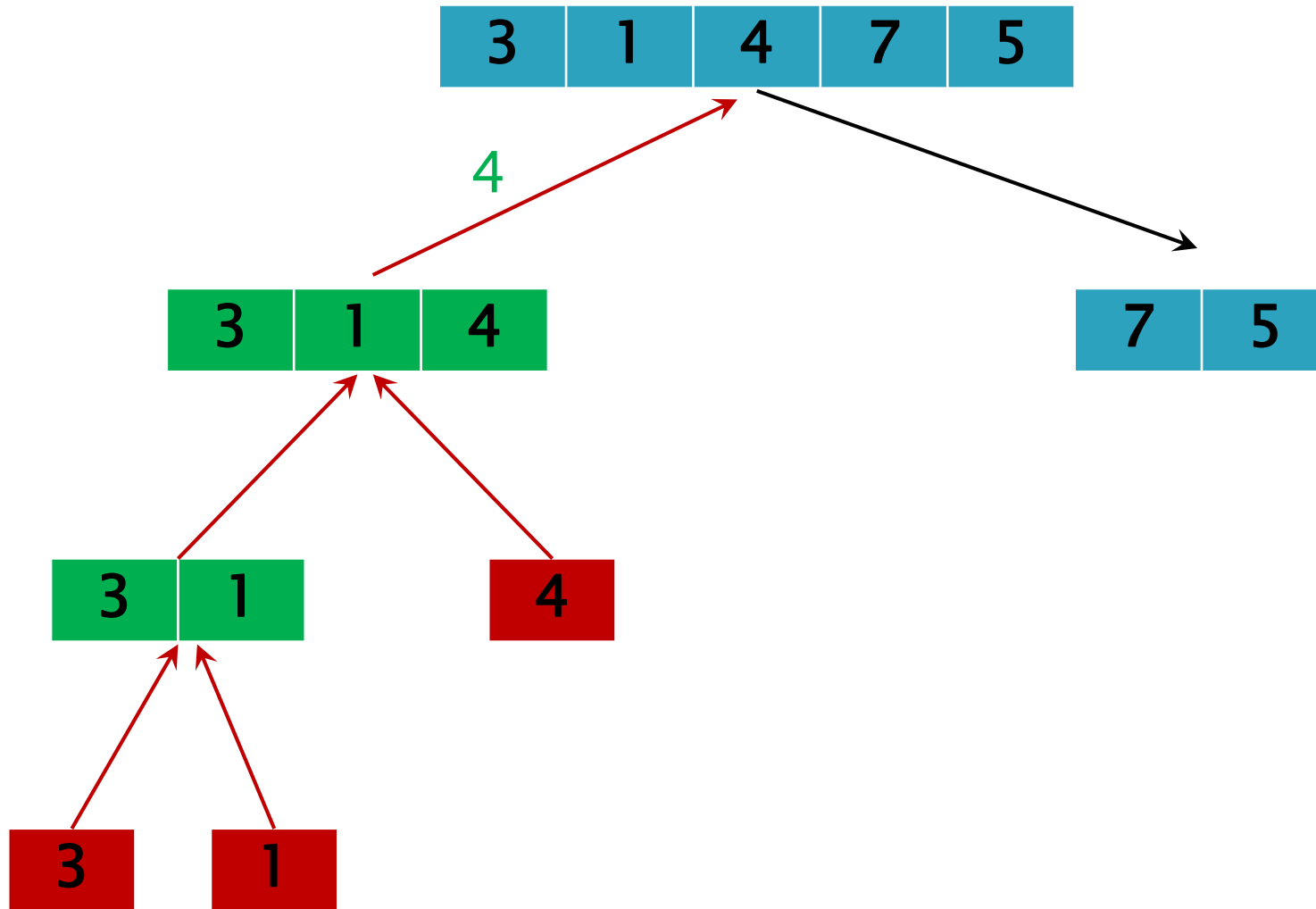
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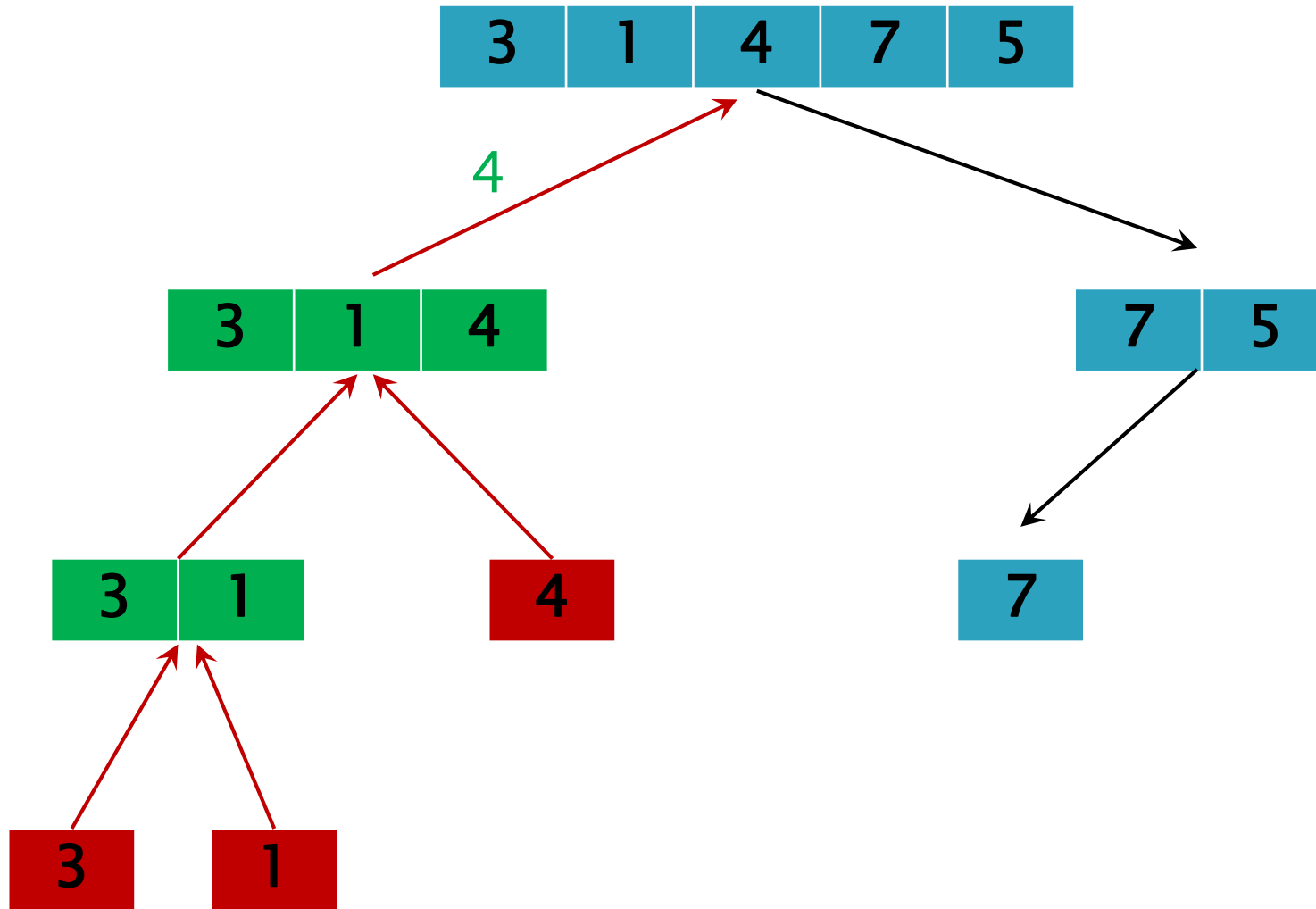
Metoda Divide et Impera



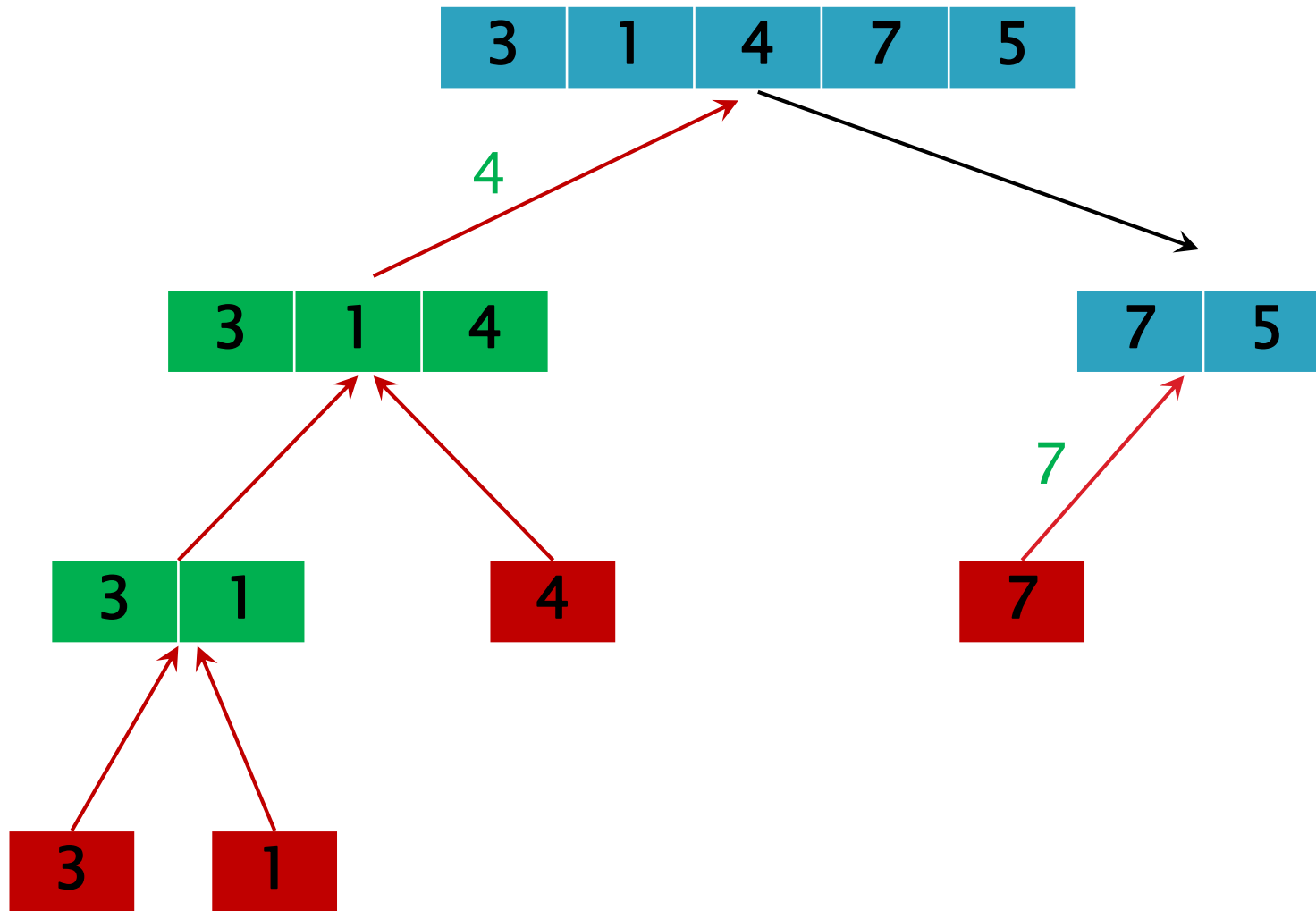
Metoda Divide et Impera



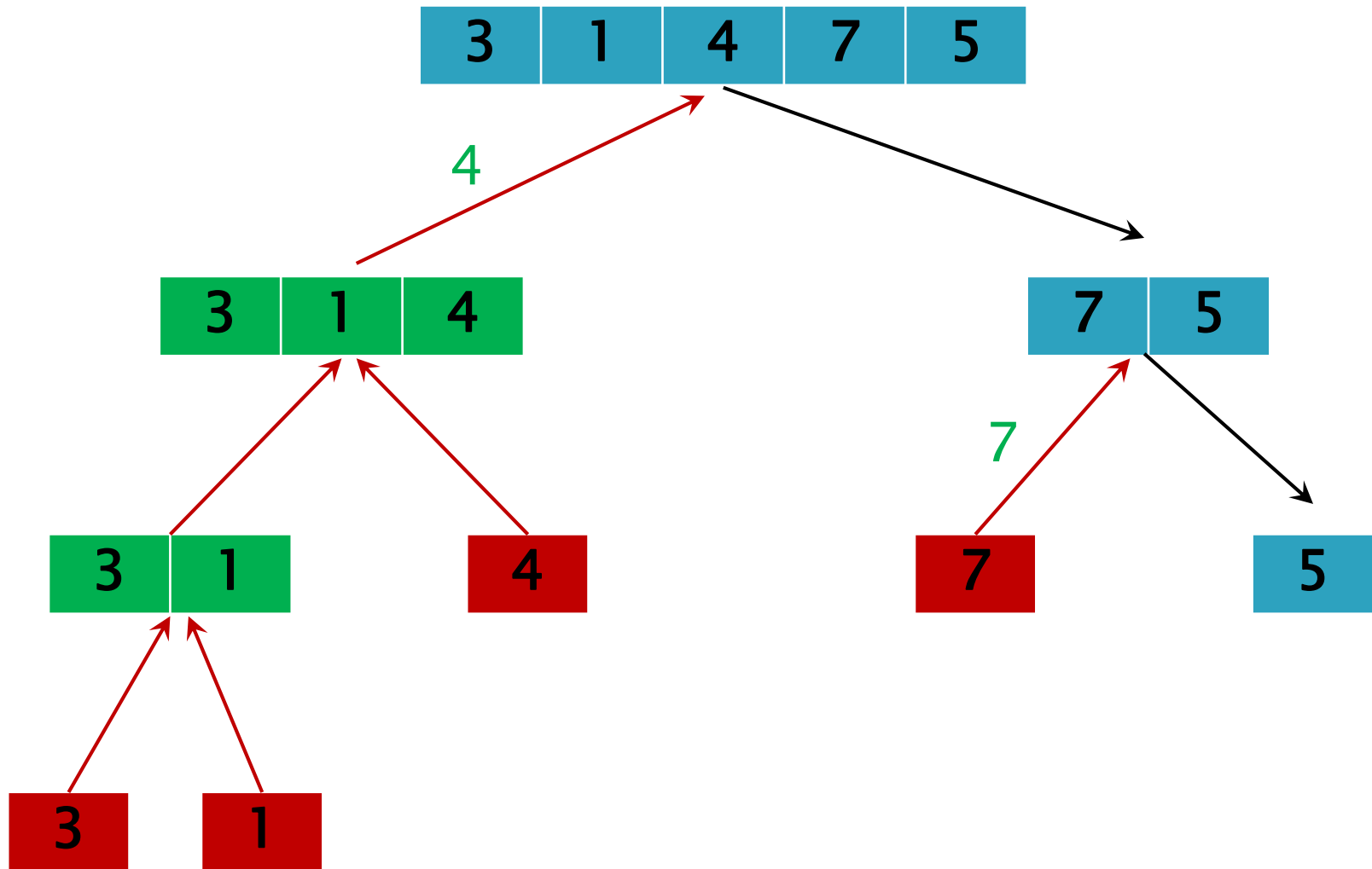
Metoda Divide et Impera



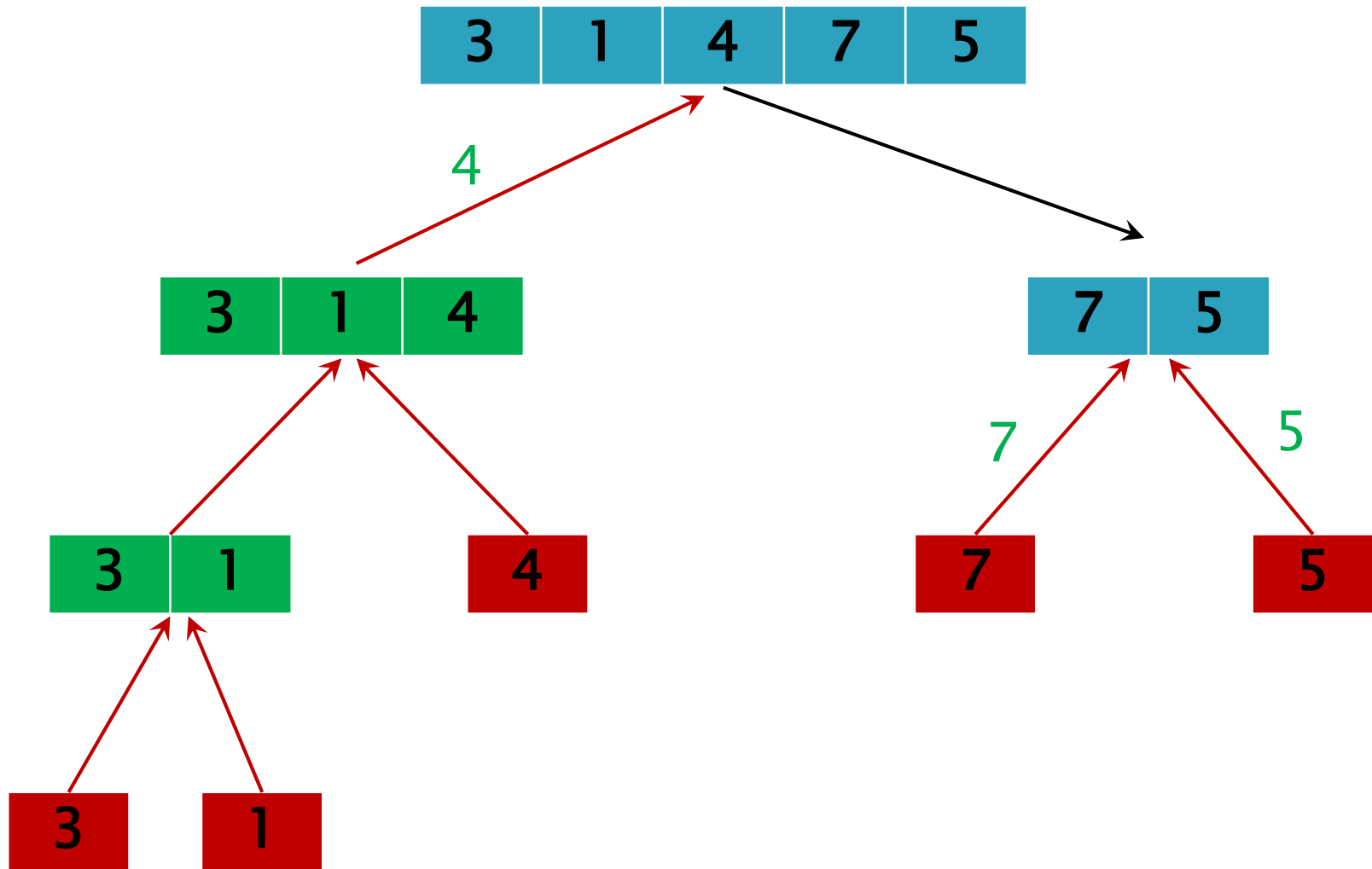
Metoda Divide et Impera



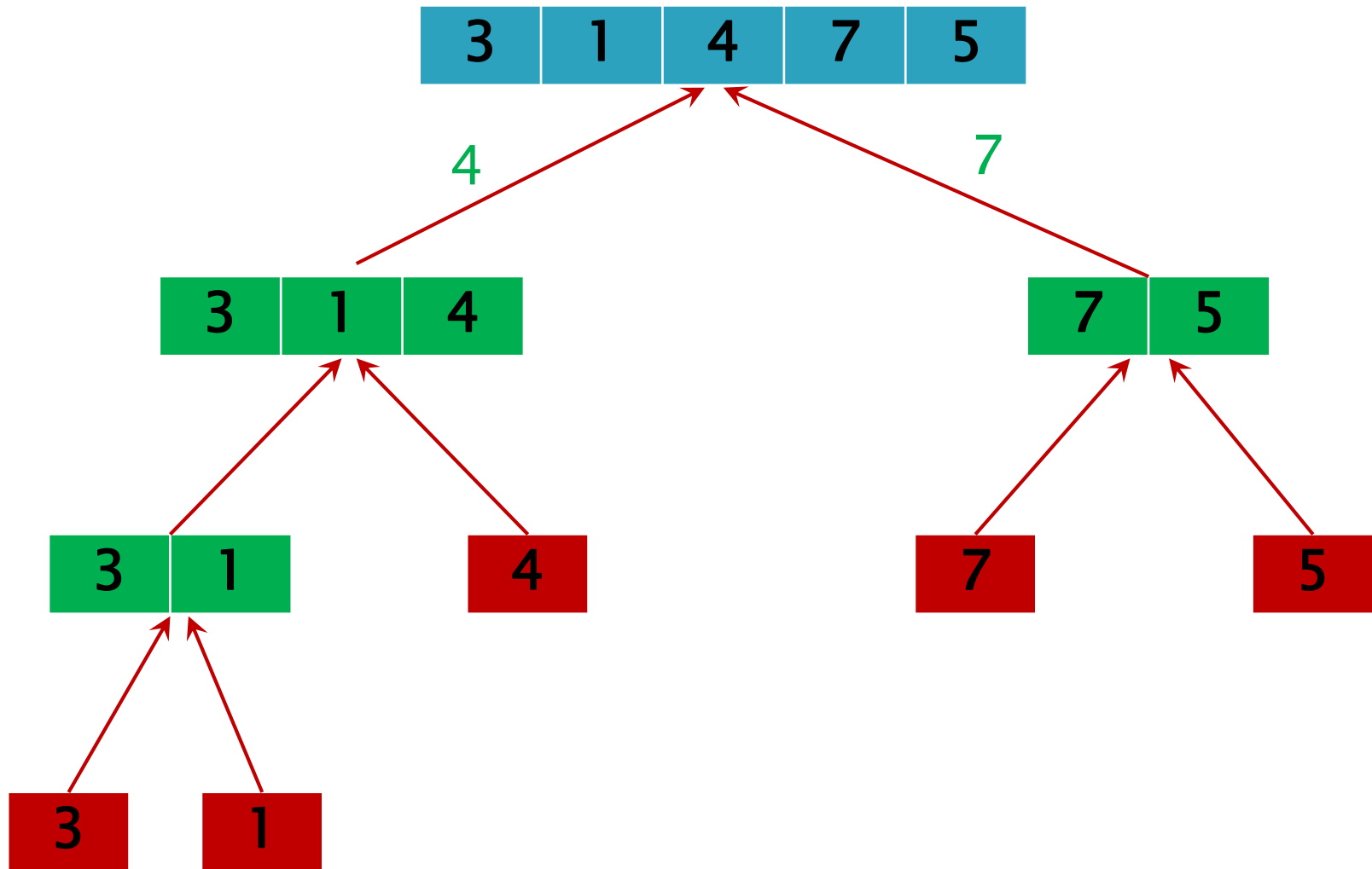
Metoda Divide et Impera



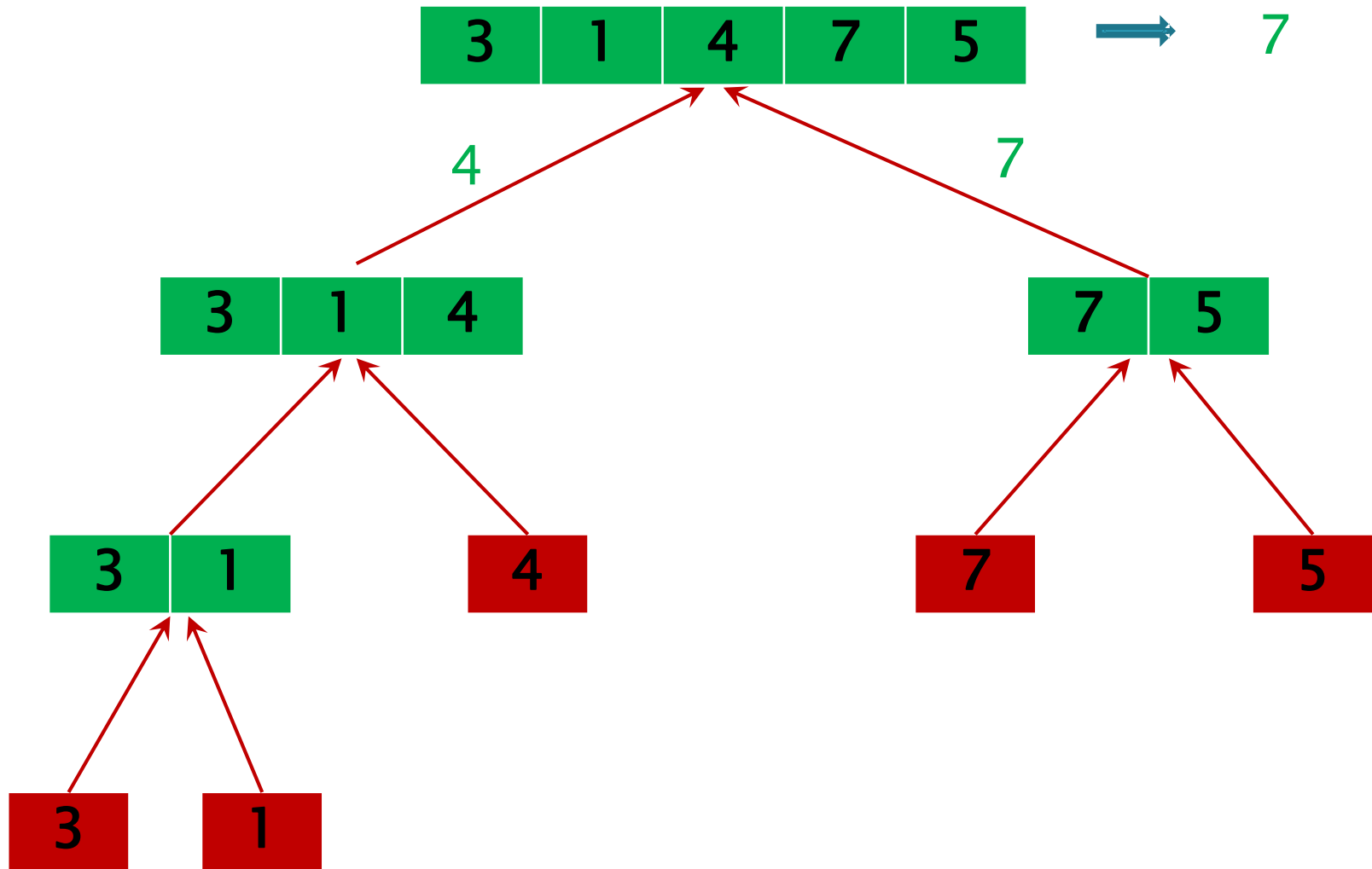
Metoda Divide et Impera



Metoda Divide et Impera



Metoda Divide et Impera



Exemplu –maximul elementelor unui vector

Complexitate

```
function DivImp(p,u)
```

```
    if u == p  $\longrightarrow$  subproblema de dimensiune 1:  $u-p < 1$ 
```

```
        r  $\leftarrow$  a[p]
```

```
    else
```

```
        m  $\leftarrow$   $\lfloor (p+u) / 2 \rfloor$ 
```

```
        r1  $\leftarrow$  DivImp(p,m)  $\longleftarrow$  Subproblema de dimensiune  $n/2$ 
```

```
        r2  $\leftarrow$  DivImp(m+1,u)  $\longleftarrow$  Subproblema de dimensiune  $n/2$ 
```

```
        if r1 > r2
```

```
            r  $\leftarrow$  r1
```

\longleftarrow **Combinarea rezultatelor celor două subprobleme – timp constant c ($O(1)$)**

```
        else
```

```
            r  $\leftarrow$  r2
```

```
    return r
```

Apel: DivImp(0, n-1) \longrightarrow **$T(n) = 2T(n/2) + O(1) \Rightarrow$
 $T(n) = 2T(n/2) + 1$ (putem considera $c=1$)**

Metoda Divide et Impera

- ▶ Complexitate – relație de recurență

$$T(n) = \begin{cases} c, & \text{pentru } n = 1 \\ T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + c, & \text{pentru } n > 1 \end{cases}$$

Metoda Divide et Impera

► Complexitate – relație de recurență

$$T(n) = \begin{cases} c, & \text{pentru } n = 1 \\ T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + c, & \text{pentru } n > 1 \end{cases}$$

- Suficient să presupunem că $n = 2^k$, $c=1$
(apoi înlocuim $=$ cu \leq în relații)
- Rezolvarea recurenței
 - metoda substituțiilor repetate
 - arbori de recurențe

=> demonstrație Teorema master

Metoda Divide et Impera

$$T(n) = \begin{cases} 1, & \text{pentru } n = 1 \\ 2T(n/2) + 1, & \text{pentru } n > 1 \end{cases}$$

- Suficient să presupunem că $n = 2^k$

Metoda 1:

- Rezolvarea recurenței prin metoda substituțiilor repetate:

Putem înlocui de la început n cu 2^k sau lucra cu n , $n/2$, $n/2^2$...
Prezentăm ambele variante:

$$T(n) = 2T(n/2) + 1$$

VARIANTA 1 – lucrăm cu n , $n/2$, $n/2^2$...



$$T(n) = 2T(n/2) + 1$$

$$T(n) = 2 \cdot T(n/2) + 1 =$$

$$= 2 \cdot [2T(n/2^2) + 1] + 1 =$$

$$T(n) = 2T(n/2) + 1$$

$$T(n) = 2 \cdot T(n/2) + 1 =$$

$$= 2 \cdot [2T(n/2^2) + 1] + 1 =$$

$$= 2^2 \cdot T(n/2^2) + (1+2) =$$

$$=$$

$$T(n) = 2T(n/2) + 1$$

$$T(n) = 2 \cdot T(n/2) + 1 =$$

$$= 2 \cdot [2T(n/2^2) + 1] + 1 =$$

$$= 2^2 \cdot T(n/2^2) + (1+2) =$$

$$= 2^2 \cdot [2T(n/2^3) + 1] + (1+2) =$$

$$= 2^3 \cdot T(n/2^3) + (1+2+2^2) = \dots$$

$$= 2^k \cdot T(n/2^k) + (1+2+2^2+\dots+2^{k-1}) =$$

$$= 2^k \cdot T(1) + (2^k - 1) = n + (n-1) = (2n-1) \Rightarrow$$

$k = \log_2(n)$



$$\Rightarrow T(n) = O(n)$$

$$T(n) = 2T(n/2) + 1$$

VARIANTA 2 – înlocuim de la început n cu 2^k

$$T(n) = 2T(n/2) + 1$$

$$T(n) = T(2^k) = 2 \cdot T(2^{k-1}) + 1 =$$

$$T(n) = 2T(n/2) + 1$$

$$\begin{aligned} T(n) = T(2^k) &= 2 \cdot T(2^{k-1}) + 1 = \\ &= 2 \cdot [2T(2^{k-2}) + 1] + 1 = \end{aligned}$$

$$T(n) = 2T(n/2) + 1$$

$$\begin{aligned} T(n) &= T(2^k) = 2 \cdot T(2^{k-1}) + 1 = \\ &= 2 \cdot [2T(2^{k-2}) + 1] + 1 = \\ &= 2^2 \cdot T(2^{k-2}) + (1+2) = \\ &= \end{aligned}$$

$$T(n) = 2T(n/2) + 1$$

$$T(n) = T(2^k) = 2 \cdot T(2^{k-1}) + 1 =$$

$$= 2 \cdot [2T(2^{k-2}) + 1] + 1 =$$

$$= 2^2 \cdot T(2^{k-2}) + (1+2) =$$

$$= 2^2 \cdot [2T(2^{k-3}) + 1] + (1+2) =$$

$$T(n) = 2T(n/2) + 1$$

$$\begin{aligned} T(n) &= T(2^k) = 2 \cdot T(2^{k-1}) + 1 = \\ &= 2 \cdot [2T(2^{k-2}) + 1] + 1 = \\ &= 2^2 \cdot T(2^{k-2}) + (1+2) = \\ &= 2^2 \cdot [2T(2^{k-3}) + 1] + (1+2) = \\ &= 2^3 \cdot T(2^{k-3}) + (1+2+2^2) = \dots \\ &= \end{aligned}$$

$$T(n) = 2T(n/2) + 1$$

$$\begin{aligned} T(n) &= T(2^k) = 2 \cdot T(2^{k-1}) + 1 = \\ &= 2 \cdot [2T(2^{k-2}) + 1] + 1 = \\ &= 2^2 \cdot T(2^{k-2}) + (1+2) = \\ &= 2^2 \cdot [2T(2^{k-3}) + 1] + (1+2) = \\ &= 2^3 \cdot T(2^{k-3}) + (1+2+2^2) = \dots \\ &= 2^k \cdot T(2^0) + (1+2+2^2+\dots+2^{k-1}) = \end{aligned}$$

$$T(n) = 2T(n/2) + 1$$

$$T(n) = T(2^k) = 2 \cdot T(2^{k-1}) + 1 =$$

$$= 2 \cdot [2T(2^{k-2}) + 1] + 1 =$$


$$= 2^2 \cdot T(2^{k-2}) + (1+2) =$$

$$= 2^2 \cdot [2T(2^{k-3}) + 1] + (1+2) =$$

$$= 2^3 \cdot T(2^{k-3}) + (1+2+2^2) = \dots$$

$$= 2^k \cdot T(2^0) + (1+2+2^2+\dots+2^{k-1}) =$$

$$= 2^k \cdot T(1) + (2^k-1) = n + (n-1) = 2n-1 \Rightarrow$$

$$k = \log_2(n), \quad 2^k = n$$


$$\Rightarrow T(n) = O(n)$$

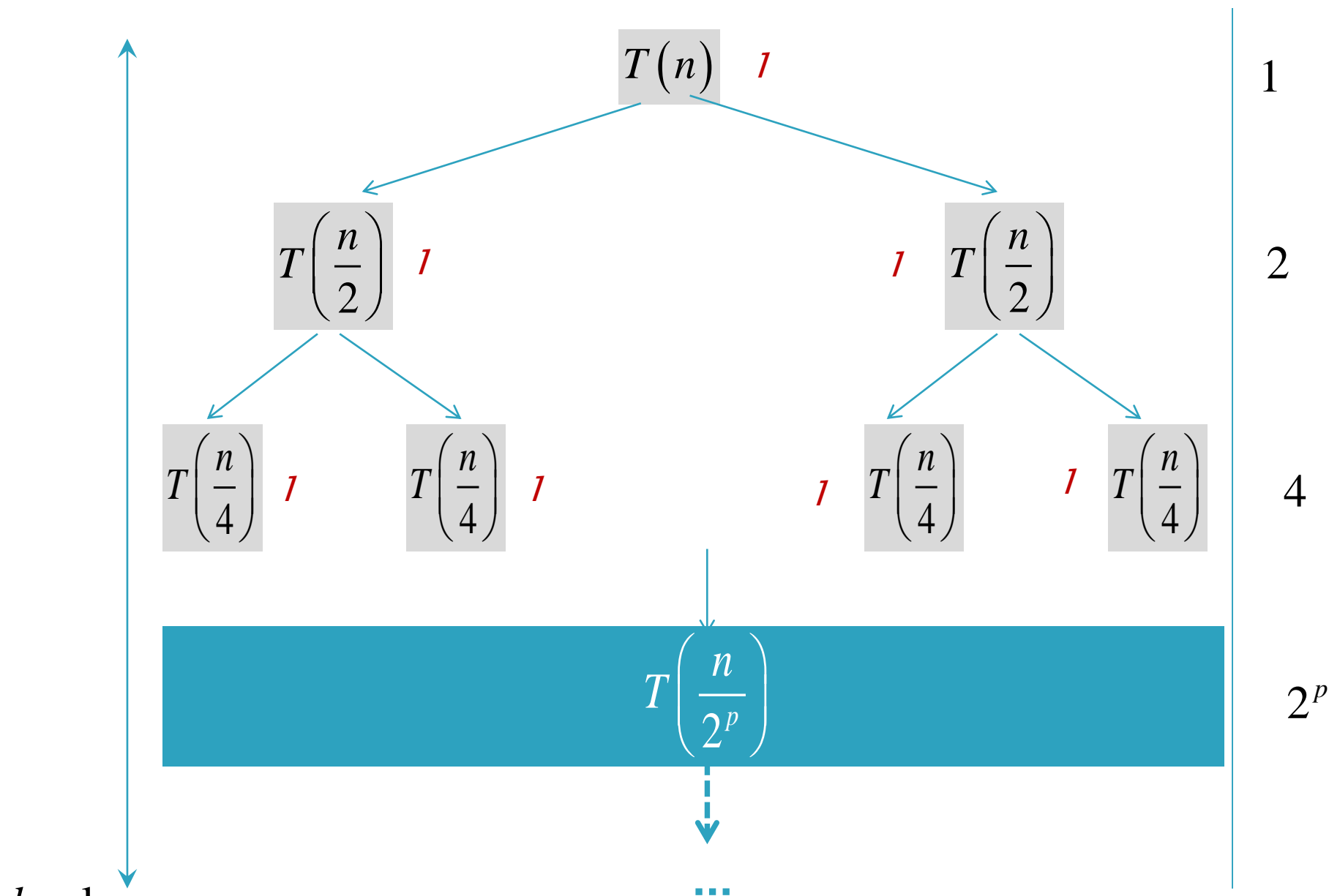
Metoda Divide et Impera

$$T(n) = \begin{cases} 1, & \text{pentru } n = 1 \\ 2T(n/2) + 1, & \text{pentru } n > 1 \end{cases}$$

- Suficient să presupunem că $n = 2^k$
(apoi înlocuim $=$ cu \leq în relații)

Metoda 2:

- Rezolvarea recurenței folosind **arbori de recurențe**
 \Rightarrow demonstrație Teorema master



$$1 + 2 + 4 + \dots + 2^k = (2^{k+1} - 1) = 2n - 1 \Rightarrow O(n)$$

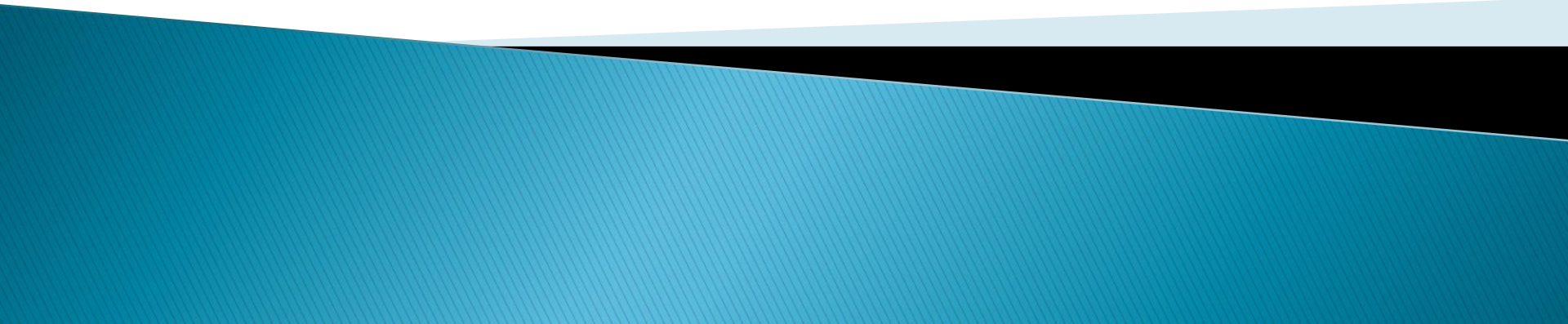
Metoda Divide et Impera

- ▶ În termeni de arbori, DI constă în
 - **construirea dinamică a unui arbore** (prin împărțirea în subprobleme)urmată de
 - **parcursarea în postordine a arborelui** (prin asamblarea rezultatelor parțiale).

Exemple clasice

- ▶ Căutare binară
- ▶ Sortarea prin interclasare (Merge Sort)
- ▶ Sortarea rapidă (Quick Sort)
- ▶ Problema turnurilor din Hanoi

Căutarea binară



Căutarea binară

- ▶ Se consideră vectorul $a=(a_1,\dots,a_n)$ **ordonat crescător** și o valoare x . Se cere să se determine dacă x apare printre componentele vectorului.
- ▶ Mai exact căutăm perechea (b,i) dată de:
 - (True, i) dacă $a_i = x$;
 - (False, i) dacă **$a_{i-1} < x < a_i$** ,unde, prin convenție,
$$a_0 = -\infty, a_{n+1} = +\infty.$$

Căutarea binară

```
def cautare_binara(x,ls,p,u):  
    if p > u:  
        return (False, u)  
    else:  
        mij = (p + u) // 2  
        if x == ls[mij]:  
            return (True,mij)  
        elif x < ls[mij]:  
            return cautare_binara(x,ls, p, mij-1)  
        else:  
            return cautare_binara(x,ls, mij+1, u)  
  
def cautare(x,ls):  
    n = len(ls)  
    return cautare_binara(x,ls,0,n-1)
```

Căutarea binară

Implementare nerecursivă

```
def cautare_binara_nerecursiv(x,ls):  
    p = 0  
    u = len(ls) - 1  
    while p<=u:  
        mij = (p + u) // 2  
        if x == ls[mij]:  
            return (True, mij)  
        elif x < ls[mij]:  
            u = mij - 1  
        else:  
            p = mij + 1  
    return (False,u)
```

Căutarea binară

► **Complexitate:** $O(\log n)$

- $T(n) = T(n/2) + c$

Căutarea binară

- ▶ **Complexitate:** $O(\log n)$
 - $n=2^k$
 - $T(1) = 1, T(n) = T(n/2) + 1$

Căutarea binară

► Complexitate: $O(\log n)$

- $n=2^k$
- $T(1) = 1, T(n) = T(n/2)+1$

Metoda substituțiilor repetate

- $T(n) = T(n/2)+1 = [T(n/2^2)+1]+1 = T(n/2^2)+2 =$

Căutarea binară

► Complexitate: $O(\log n)$

- $n=2^k$
- $T(1) = 1, T(n) = T(n/2)+1$

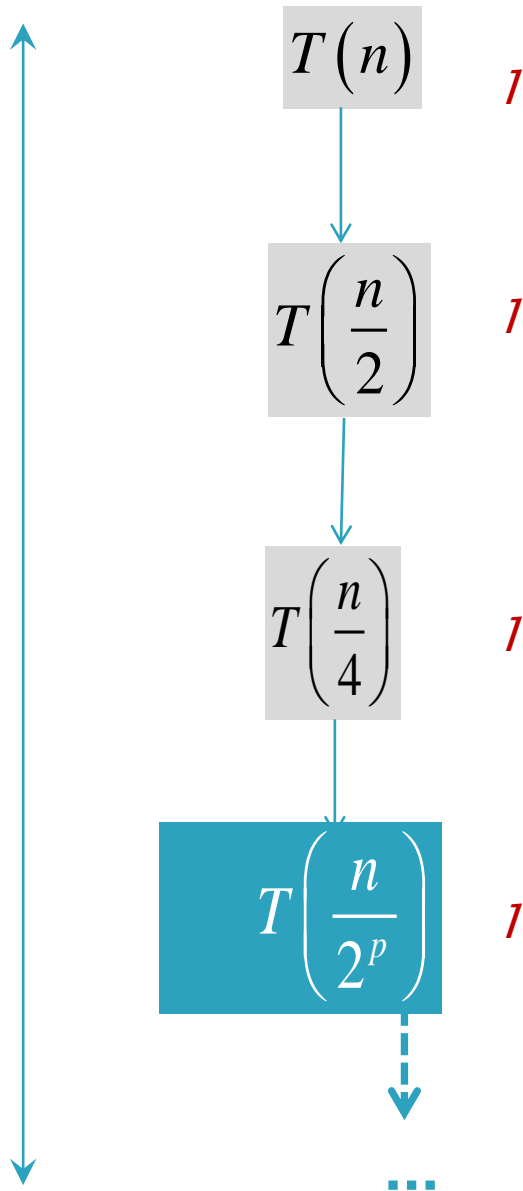
Metoda substituțiilor repetate

$$\begin{aligned} \circ T(n) &= T(n/2)+1 = [T(n/2^2)+1]+1 = T(n/2^2)+2 = \\ &= \dots = T(n/2^k)+k = 1 + \log_2 n \end{aligned}$$

$$k = \log_2 n$$

$$\Rightarrow O(\log_2 n)$$

Înălțime
 $k = \log_2 n$
($k+1$ niveluri)



$$1 + 1 + 1 + \dots + 1 = k + 1 = \log(n) + 1 \Rightarrow O(\log(n))$$

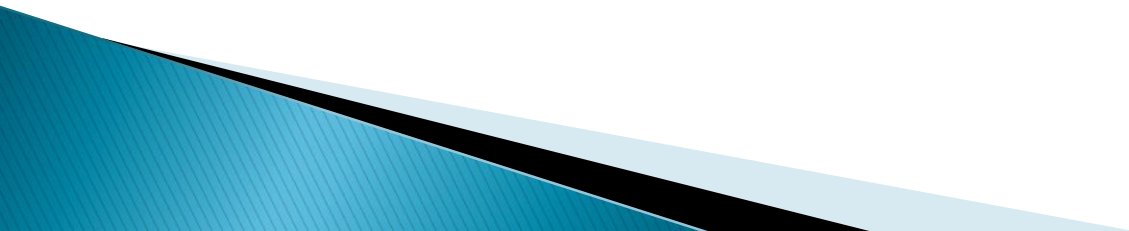
Sortarea prin interclasare

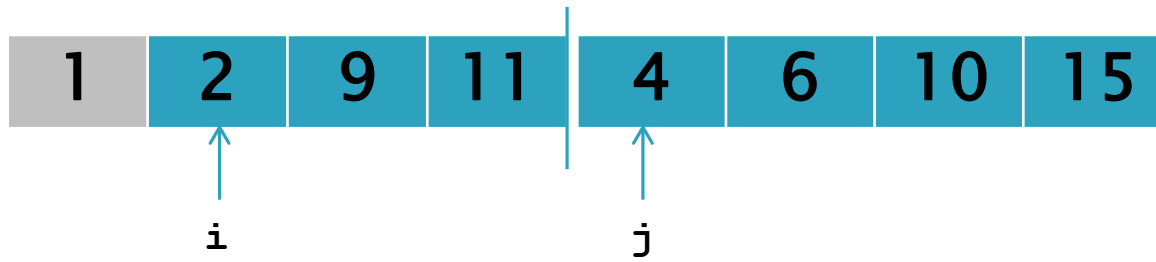


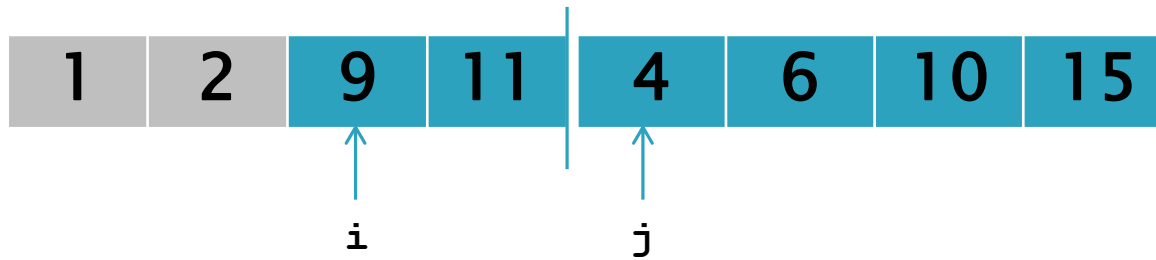
Sortare prin interclasare

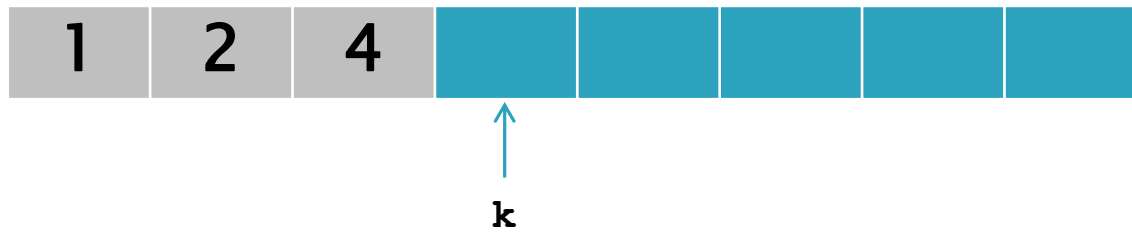
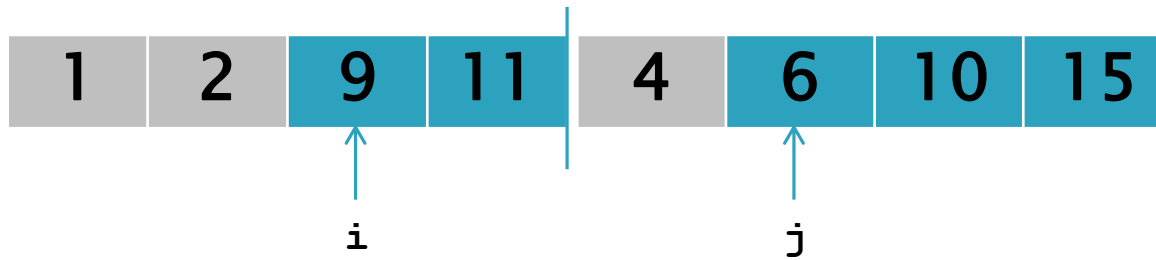
▶ Idee:

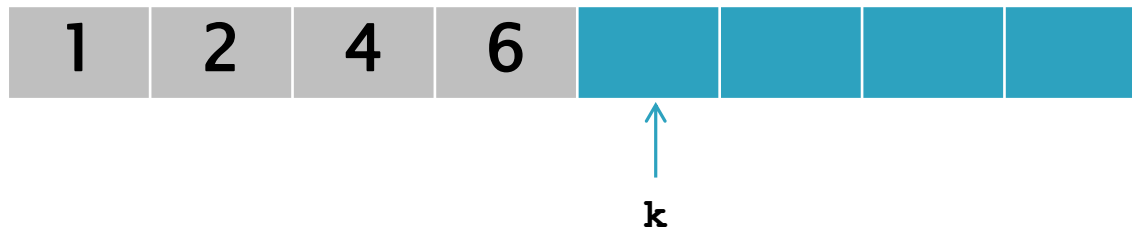
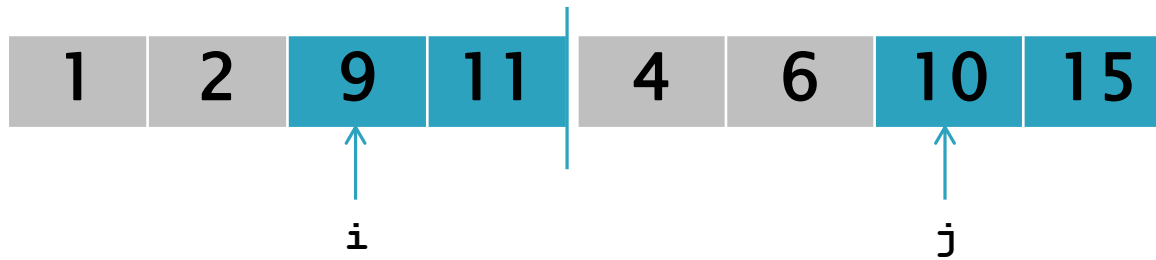
- împărțim vectorul în doi subvectori
- ordonăm crescător fiecare subvector
- asamblăm rezultatele prin *interclasare*

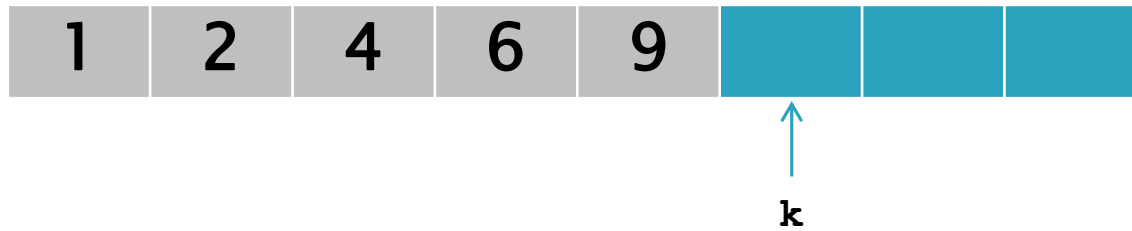
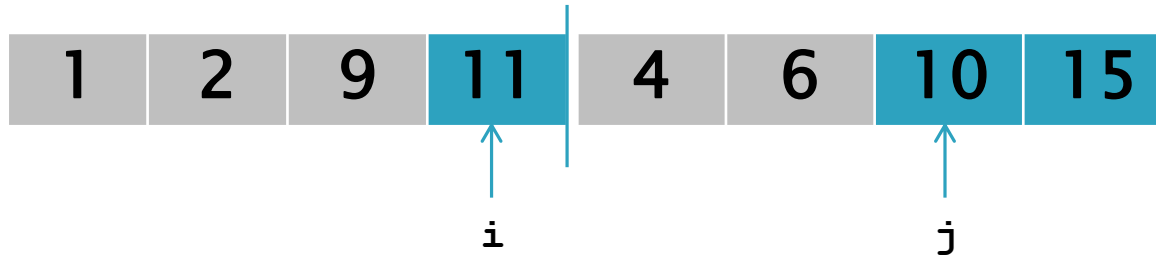


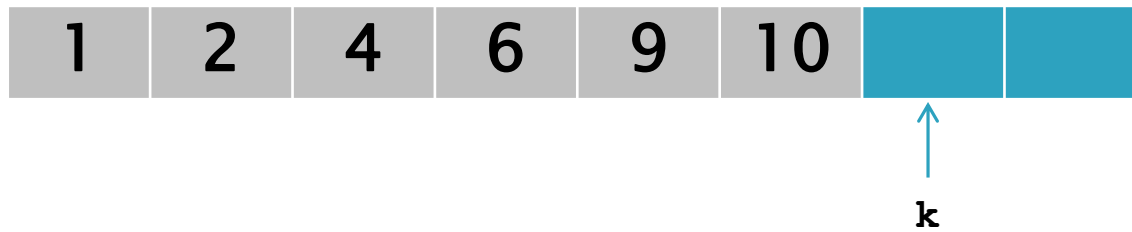
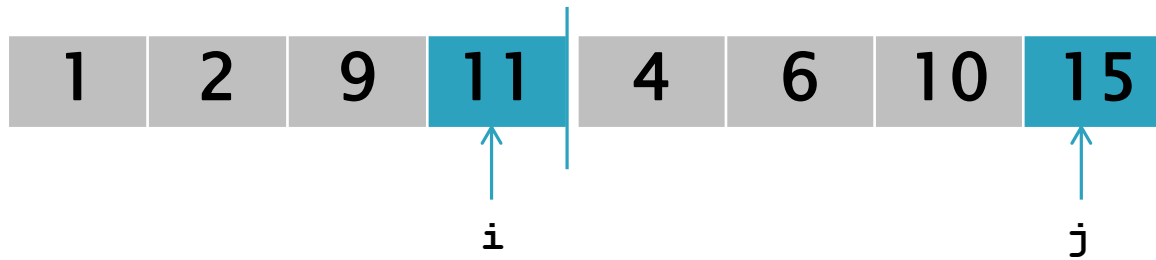


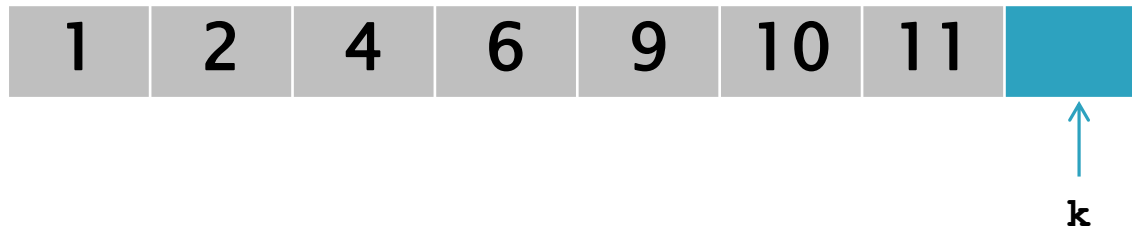
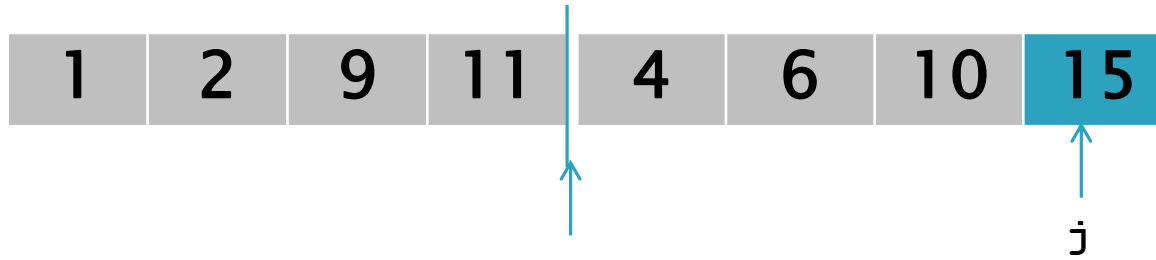










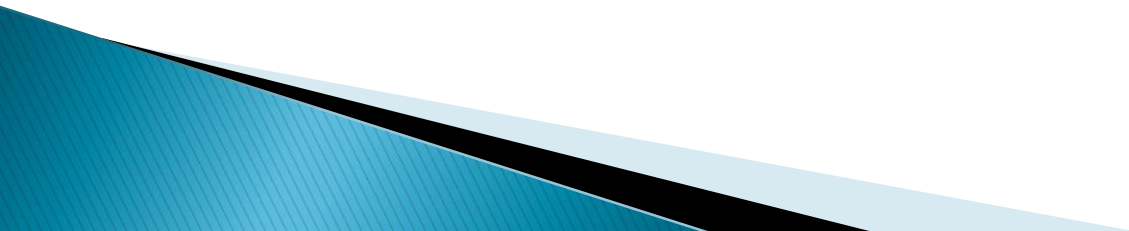




1	2	9	11	4	6	10	15
---	---	---	----	---	---	----	----

1	2	4	6	9	10	11	15
---	---	---	---	---	----	----	----

copiat



Sortare prin interclasare

```
def sort_interclasare(v, p, u):  
    if p == u:  
        pass  
    else:  
        m = (p+u) // 2  
        sort_interclasare(v, p, m)  
        sort_interclasare(v, m+1, u)  
        interclaseaza(v, p, m, u)
```

```

def interclaseaza(a, p, m, u):
    b = [None]*(u-p+1)
    i = p
    j = m + 1
    k = 0
    while (i<=m) and (j <= u):
        if a[i] <= a[j]:
            b[k] = a[i]; i += 1
        else:
            b[k] = a[j]; j+= 1
        k+=1

    while i<=m:
        b[k] = a[i]; k += 1; i += 1

    while j<=u:
        b[k] = a[j]; k += 1; j += 1

    for i in range(p,u+1):
        a[i] = b[i-p]

```

Sortare prin interclasare

➤Complexitate:

def sort_interclasare(v, p, u): **Problema de dimensiune n**

 if **p == u**:

 pass

 else:

 m = (p+u) // 2

 sort_interclasare(v, **p**, **m**)

 sort_interclasare(v, **m+1**, **u**)

 interclaseaza(v, p, m, u)

Timp constant $O(1)$

← **Două subprobleme
de dimensiune $n/2$**

← **$O(n)$
(interclaseaza cele
doua jumatati, adica
doi vectori de
dimensiune $n/2$)**

$$\Rightarrow T(n) = 2T(n/2) + O(n)$$

$$T(n) = 2T(n/2) + n$$

Sortare prin interclasare

➤ Complexitate:

$$T(n) = 2T(n/2) + n, \text{ pentru } n > 1$$

$$T(n) = 2T(n/2) + n, \quad n = 2^k$$

Varianta 1

$$\begin{aligned} T(n) &= 2 T(n/2) + n = \\ &= 2 [2T(n/2^2) + n/2] + n = \\ &= \end{aligned}$$

$$T(n) = 2T(n/2) + n, n = 2^k$$

Varianta 1

$$T(n) = 2 T(n/2) + n =$$

$$= 2 [2T(n/2^2) + n/2] + n =$$

$$= 2^2 T(n/2^2) + 2 \cdot n/2 + n = 2^2 T(n/2^2) + n + n =$$

$$= 2^2 T(n/2^2) + 2 \cdot n =$$

$$= \dots =$$


$$T(n) = 2T(n/2) + n, n = 2^k$$

Varianta 1

$$T(n) = 2 T(n/2) + n =$$

$$= 2 [2T(n/2^2) + n/2] + n =$$

$$= 2^2 T(n/2^2) + 2 \cdot n/2 + n = 2^2 T(n/2^2) + n + n =$$

$$= 2^2 T(n/2^2) + 2 \cdot n =$$

$$= \dots = 2^k T(n/2^k) + k \cdot n =$$

$$T(n) = 2T(n/2) + n, n = 2^k$$

Varianta 1

$$T(n) = 2 T(n/2) + n =$$

$$= 2 [2T(n/2^2) + n/2] + n =$$

$$= 2^2 T(n/2^2) + 2 \cdot n/2 + n = 2^2 T(n/2^2) + n + n =$$

$$= 2^2 T(n/2^2) + 2 \cdot n =$$

$$= \dots = 2^k T(n/2^k) + k \cdot n = 2^k T(1) + k \cdot n = n + n \cdot \log_2 n$$

$$\Rightarrow O(n \log(n))$$

$k = \log_2(n), 2^k = n$

$$T(n) = 2T(n/2) + n, \quad n = 2^k$$

Varianta 2 – inlocuim de la inceput n cu 2^k

$$T(n) = T(2^k) = 2 \, T(2^{k-1}) + 2^k =$$

$$= 2 \, [2T(2^{k-2}) + 2^{k-1}] + 2^k = 2^2 T(2^{k-2}) + 2 \cdot 2^k$$

$$= \dots =$$


$$T(n) = 2T(n/2) + n, n = 2^k$$

Varianta 2 – inlocuim de la inceput n cu 2^k


$$T(n) = T(2^k) = 2 T(2^{k-1}) + 2^k =$$

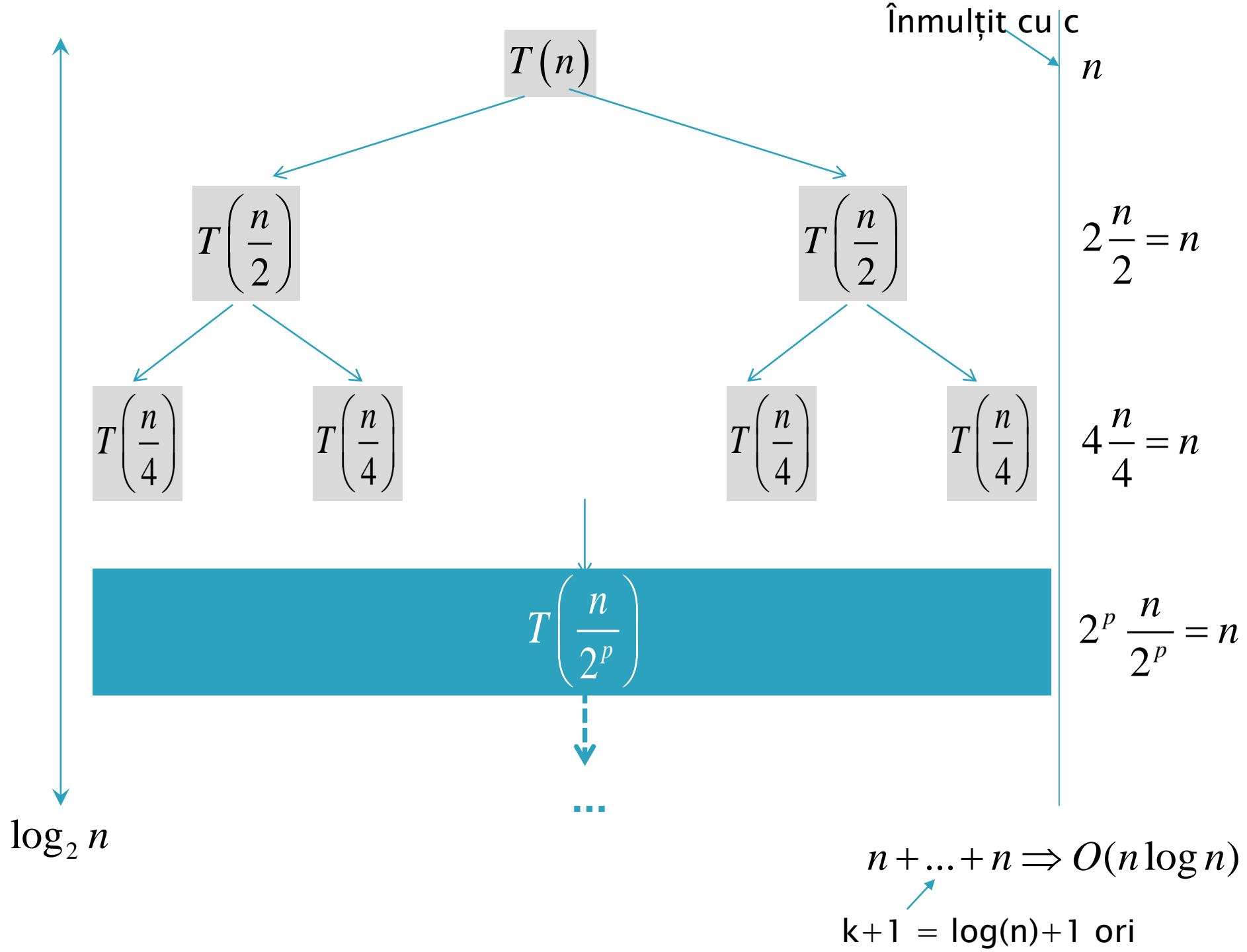
$$= 2 [2T(2^{k-2}) + 2^{k-1}] + 2^k = 2^2 T(2^{k-2}) + 2 \cdot 2^k$$

$$= \dots = 2^i T(2^{k-i}) + i \cdot 2^k =$$

$$= 2^k T(1) + k \cdot 2^k = n + n \cdot \log_2 n$$

$$\Rightarrow O(n \log(n))$$

$$k = \log_2(n), 2^k = n$$




Aplicație

Numărare inversiuni

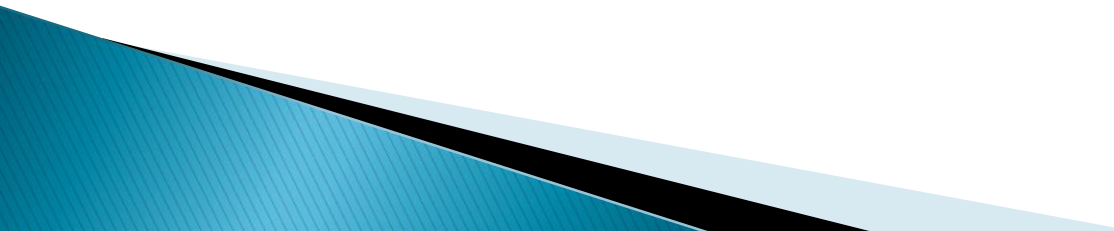


Problemă

Se consideră un vector cu n elemente distincte.
Să de determine numărul de inversiuni din acest vector

- Inversiune = pereche (i, j) cu proprietatea că $i < j$ și $a_i > a_j$
- Exemplu $1, 2, 11, 9, 4, 6 \Rightarrow 5$ inversiuni
 $((11, 9), (11, 4), (9, 4), (11, 6), (9, 6))$

Aplicații

- ▶ **Măsură a diferenței între două liste ordonate**
 - ▶ **“Gradul de ordonare” al unui vector**
 - ▶ **Probleme de analiză a clasificărilor (ranking)**
 - Asemănarea între preferințele a doi utilizatori – sugestii de utilizatori cu preferințe similare
 - Asemănări dintre rezultatele întoarse de motoare diferite de căutare pentru aceeași cerere
 - collaborative filtering
- 

Aplicații

- ▶ Suficient să presupunem că prima clasificare este

$1, 2, 3, \dots, n$

- ▶ Gradul de asemănare dintre clasificări = numărul de inversiuni din a doua clasificare

Aplicații

Preferințe
utilizator 1

Arghezi



Bacovia



Blaga



Barbu



Preferințe
utilizator 2



Blaga



Arghezi



Barbu



Bacovia

Aplicații

Preferințe
utilizator 1

Arghezi

Bacovia

Blaga

Barbu



Preferințe
utilizator 2

Blaga

Arghezi

Barbu

Bacovia

3 inversiuni

Numărare inversiuni



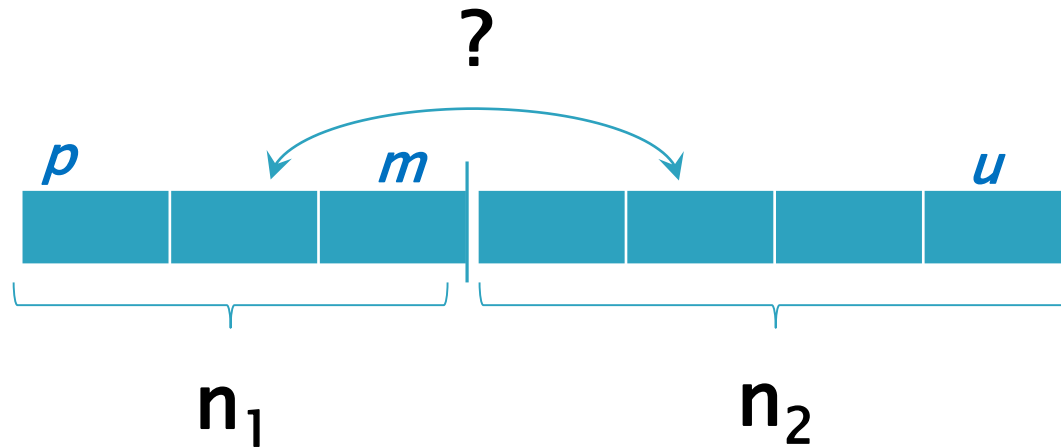
Numărul maxim de inversiuni pentru un vector cu n elemente?

Numărare inversiuni

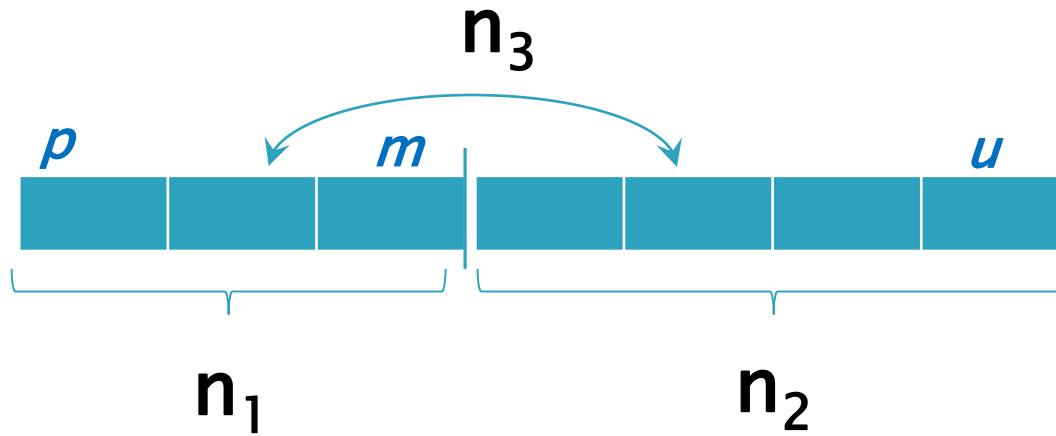
- ▶ Algoritm $\Theta(n^2)$ – evident

Numărare inversiuni

► Algoritm Divide et Impera



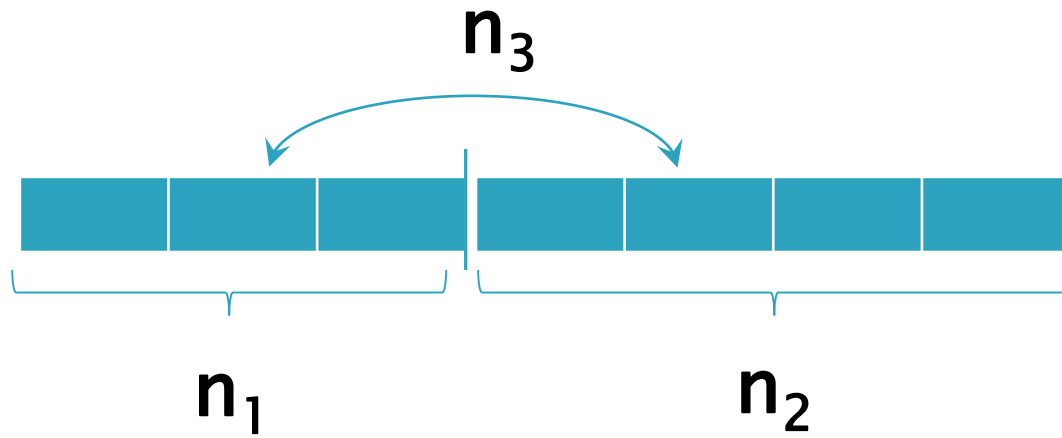
$$n_1 + n_2 + ?$$



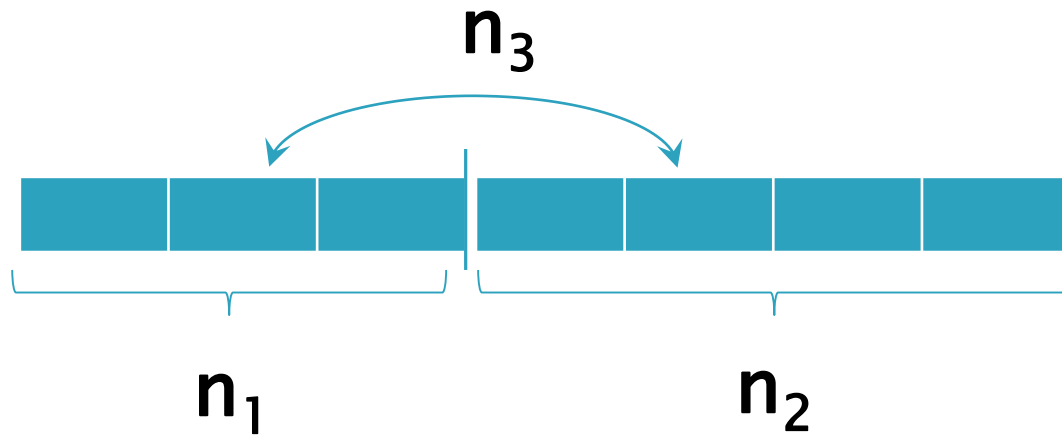
$$n_1 + n_2 + n_3$$



Cum calculăm eficient n_3 ?

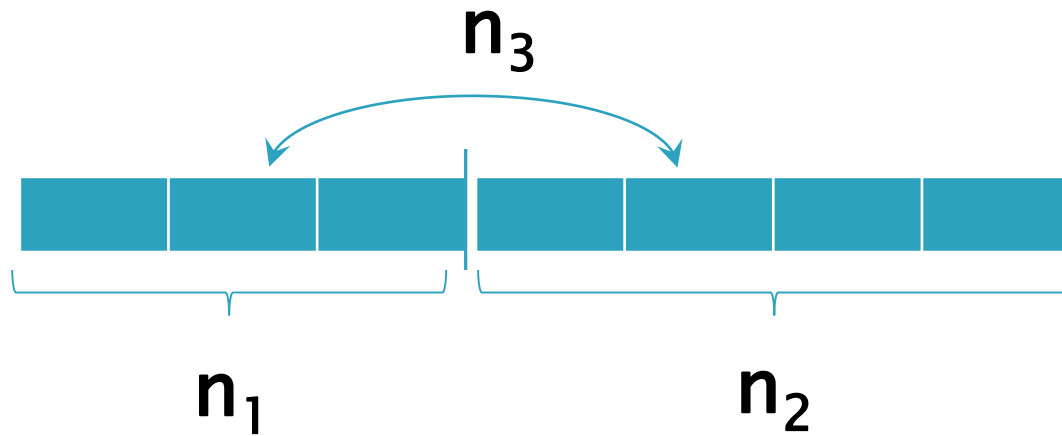


Cum calculăm eficient n_3 ?



Cum calculăm eficient n_3 ?

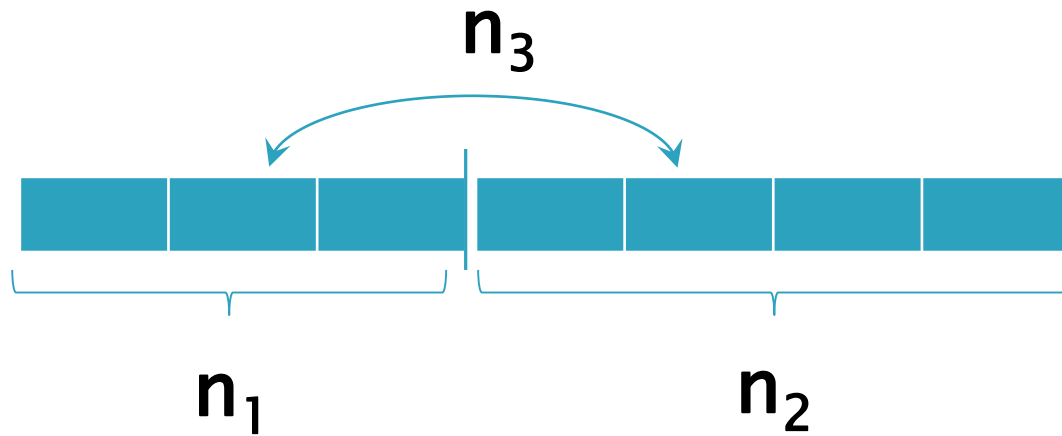
- **Încercare:** Considerăm fiecare pereche (i,j) cu i în subvectorul stâng și j în cel drept



Cum calculăm eficient n_3 ?

- **Încercare:** Considerăm fiecare pereche (i,j) cu i în subvectorul stâng și j în cel drept

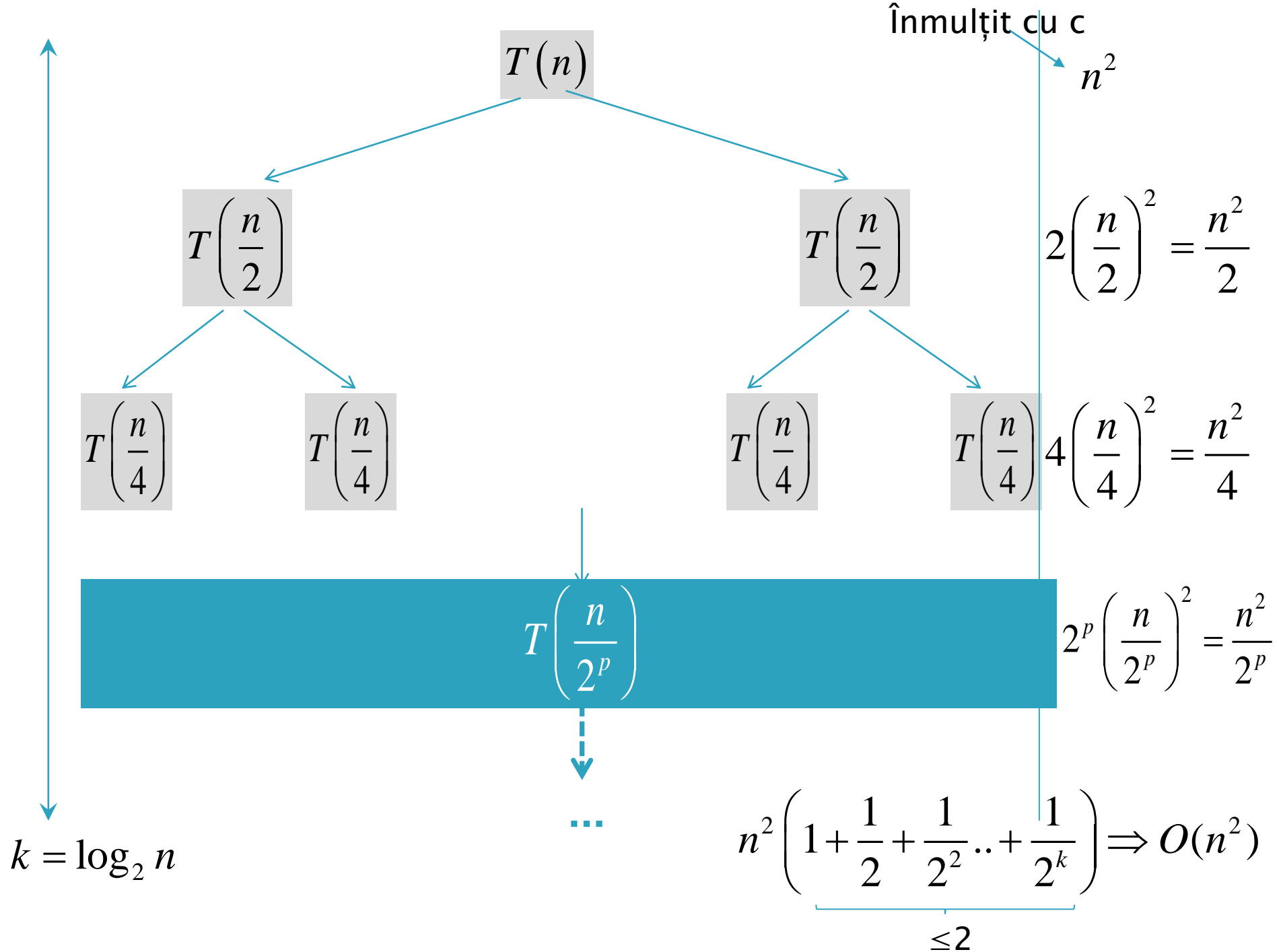
$$T(n) = 2T(n/2) + cn^2$$



Cum calculăm eficient n_3 ?

- **Încercare:** Considerăm fiecare pereche (i,j) cu i în subvectorul stâng și j în cel drept

$$T(n) = 2T(n/2) + cn^2 \Rightarrow O(?)$$



Complexitate

$$T(n) = 2T(n/2) + n^2 =$$

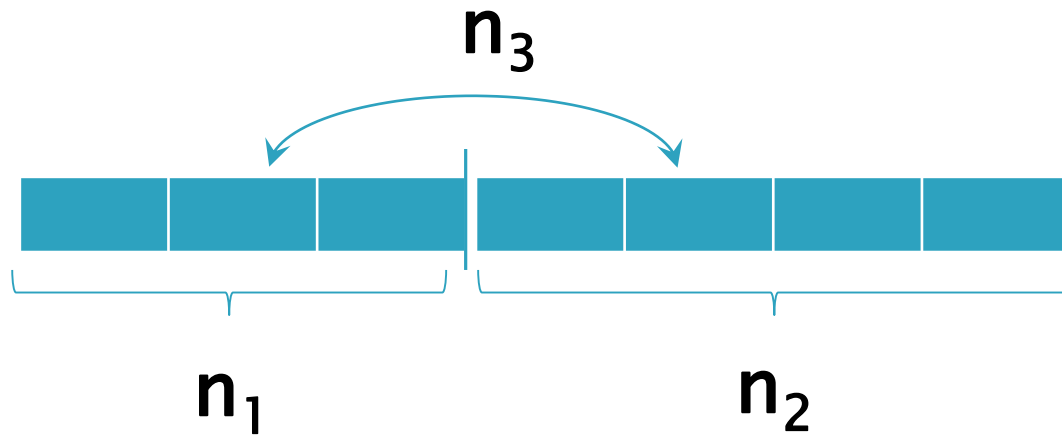
$$= 2[2T(n/2^2) + (n/2)^2] + n^2 =$$

$$= 2^2T(n/2^2) + n^2(1 + 1/2)$$

$$= \dots =$$

$$= 2^kT(n/2^k) + n^2(1 + 1/2 + \dots + 1/2^{k-1})$$

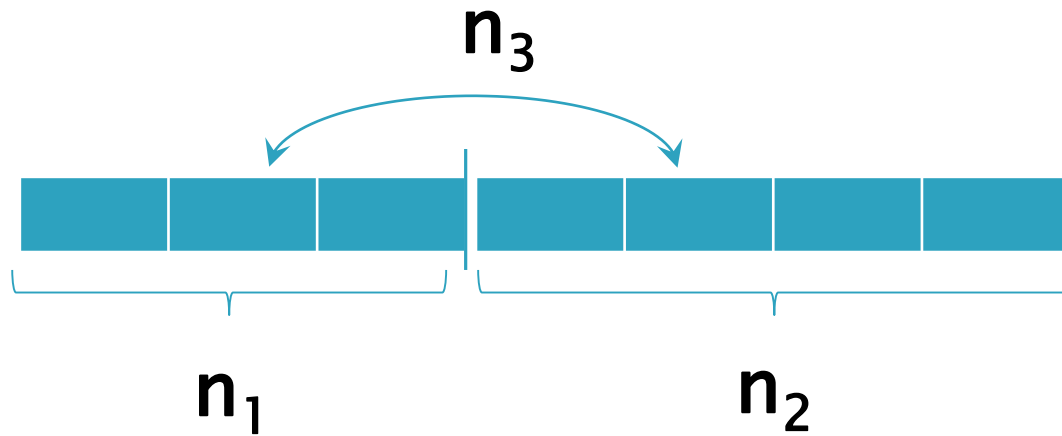
$$n = 2^k \Rightarrow n^2/2^k \Rightarrow \text{tot suma anterioară}$$



Cum calculăm eficient n_3 ?

- **Încercare:** Considerăm fiecare pereche (i,j) cu i în subvectorul stâng și j în cel drept

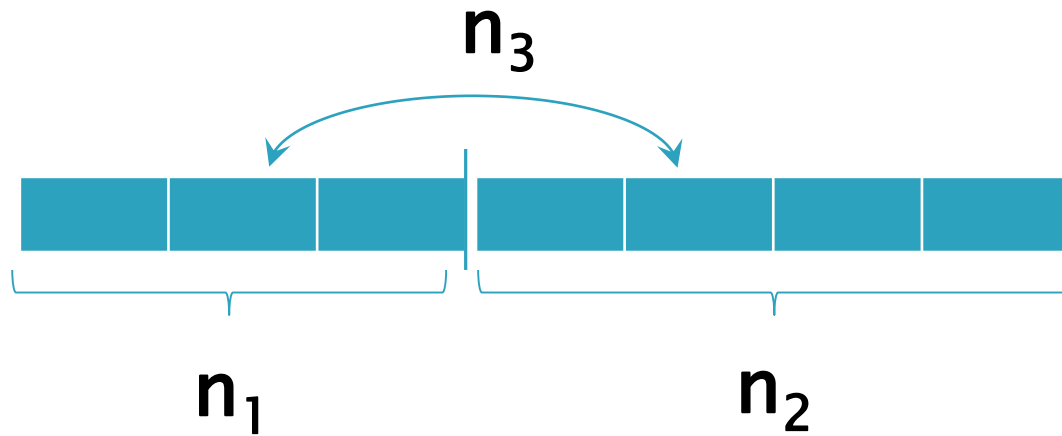
$$T(n) = 2T(n/2) + cn^2 \quad - \text{tot } O(n^2)$$



Cum calculăm eficient n_3 ?



Dacă subvectorii stâng și drept sunt **sortați crescător**, numărarea inversiunilor (i,j) date de elemente din subvectori diferiți **se poate face la interclasare**

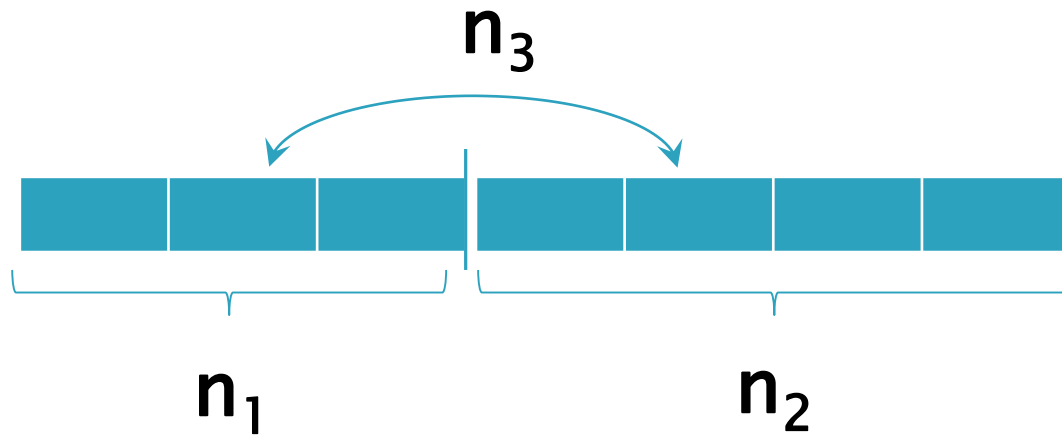


Cum calculăm eficient n_3 ?



Dacă subvectorii stâng și drept sunt **sortați crescător**, numărarea inversiunilor (i,j) date de elemente din subvectori diferiți **se poate face la interclasare**

$$T(n) = 2T(n/2) + cn \Rightarrow$$

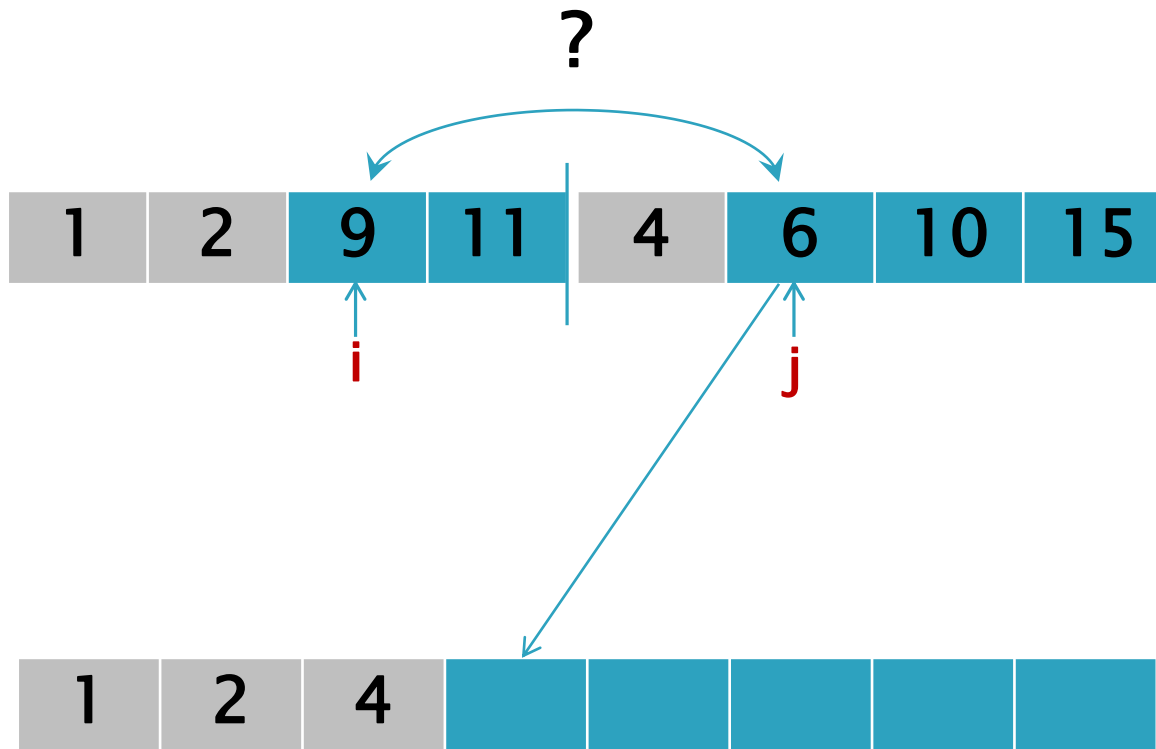


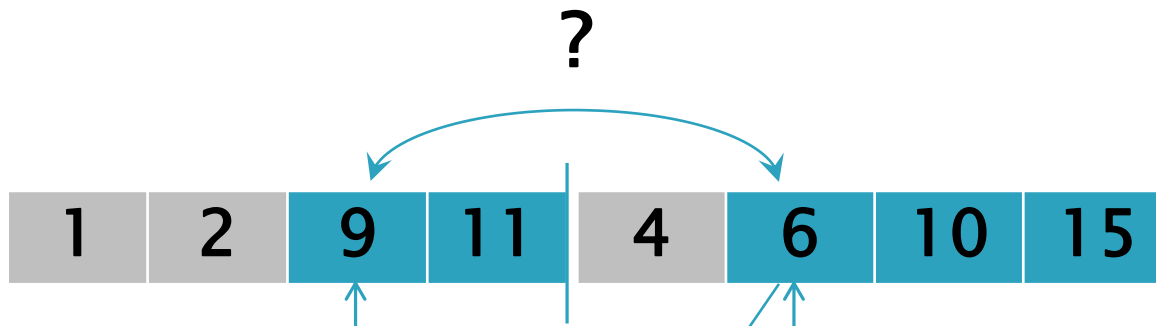
Cum calculăm eficient n_3 ?



Dacă subvectorii stâng și drept sunt **sortați crescător**, numărarea inversiunilor (i,j) date de elemente din subvectori diferiți **se poate face la interclasare**

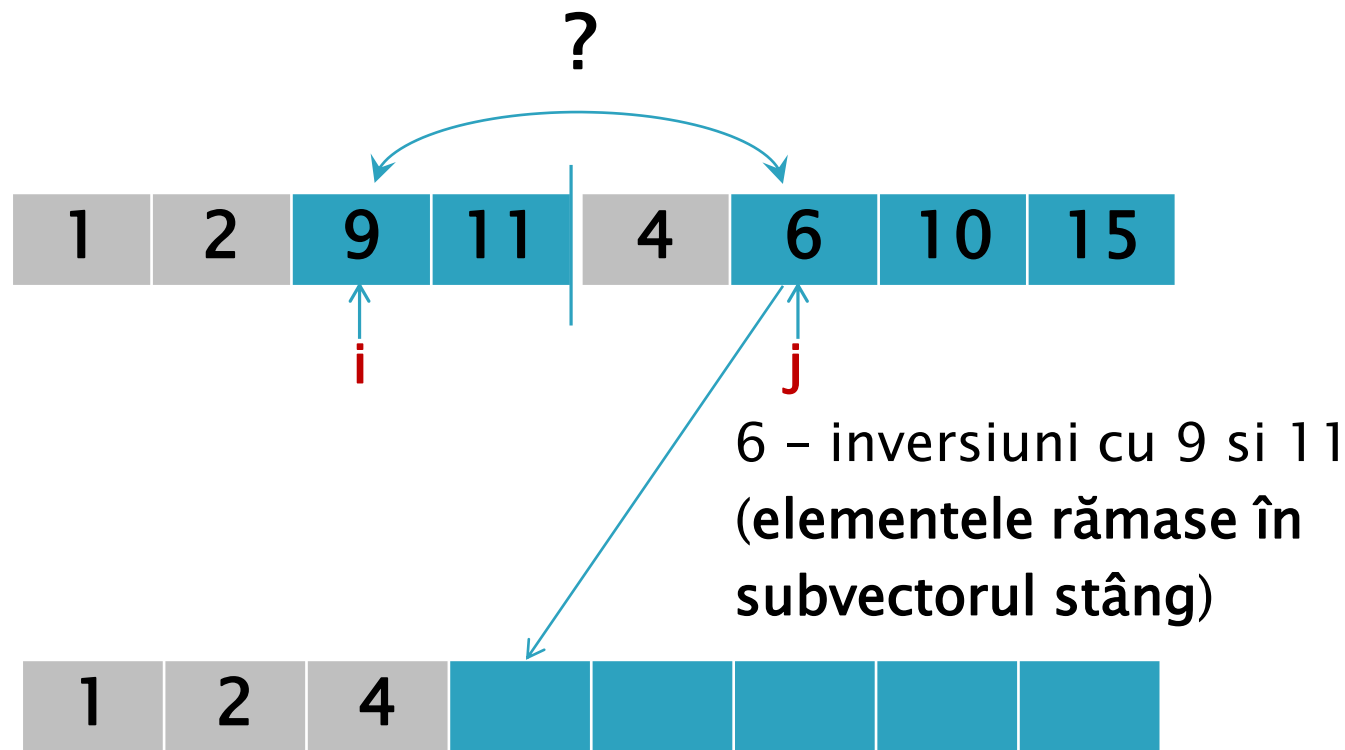
$$T(n) = 2T(n/2) + cn \Rightarrow T(n) = O(n \log n)$$





6 – inversiuni cu 9 si 11
(**elementele rămase în
subvectorul stâng**)

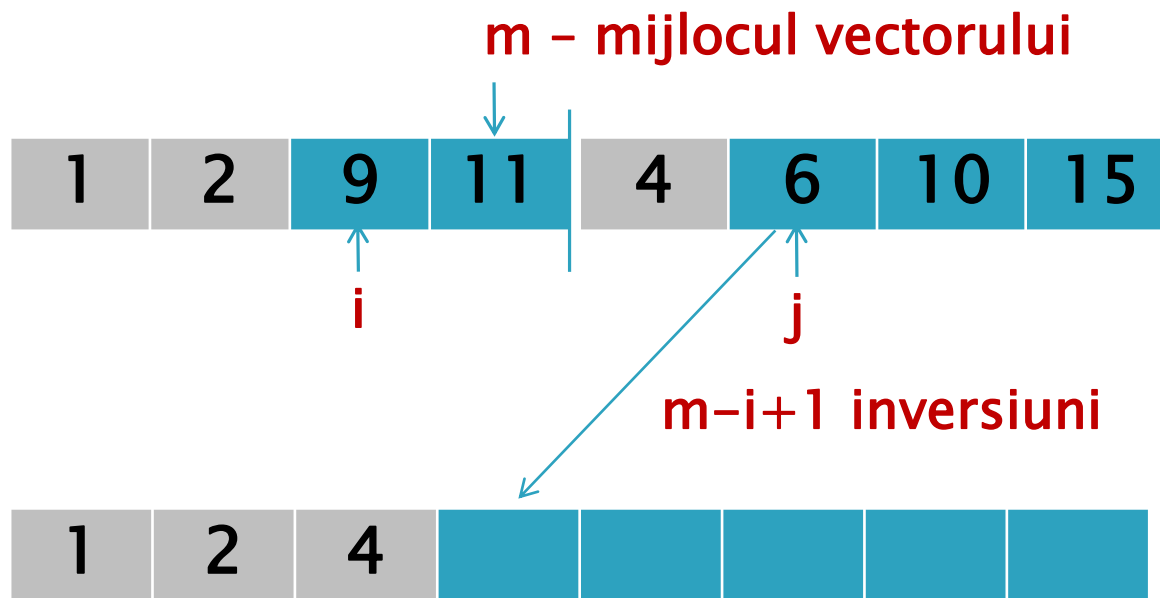




Când $a[j]$ cu $j > m$ este adăugat în vectorul rezultat, el este **mai mic (doar)** decât toate elementele din subvectorul stâng **neadăugate** încă în vectorul rezultat

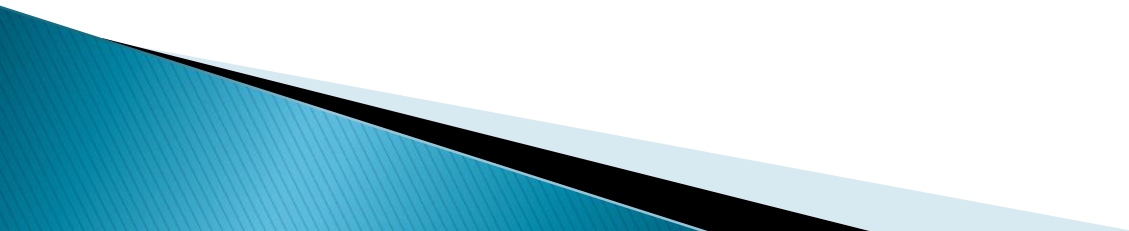


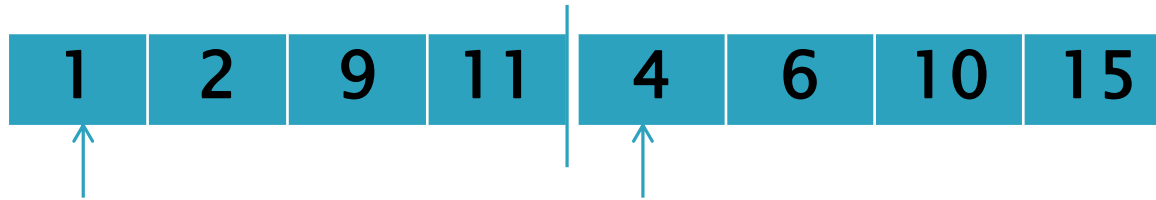
Câte inversiuni determină deci $a[j]$ cu elementele din subvectorul stâng?

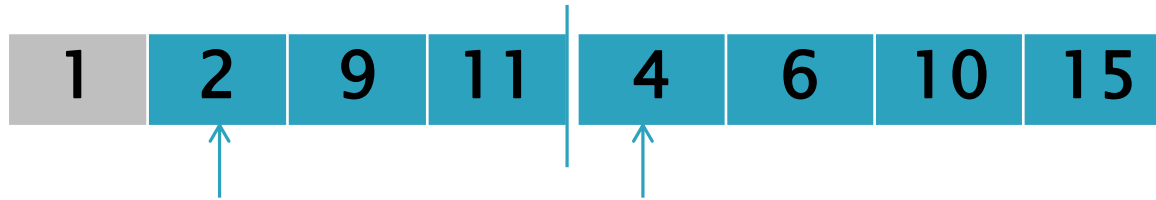


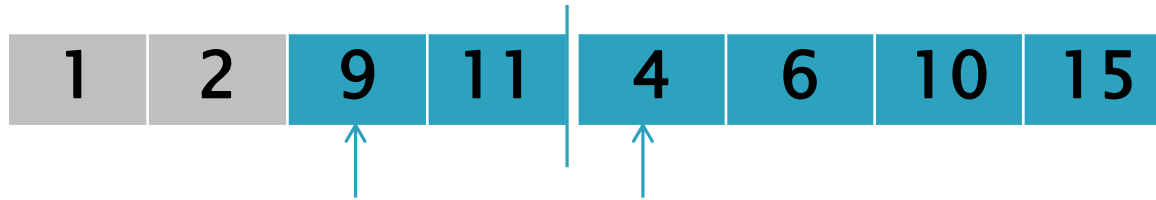
- $a[j]$ determină $m - i + 1$ inversiuni

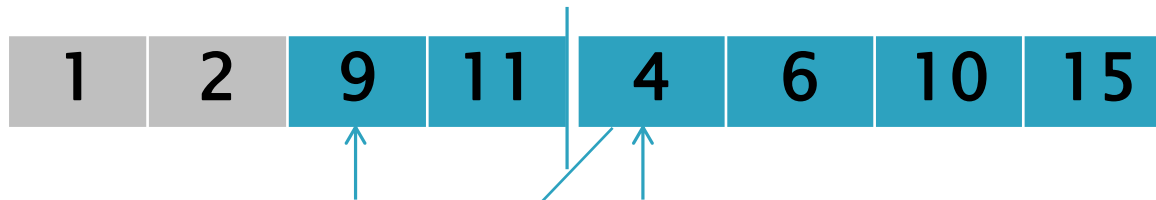
Exemplu – numărarea inversiunilor la interclasare











4 – inversiuni cu 9 si 11



Inversiuni = 2



Inversiuni = 2



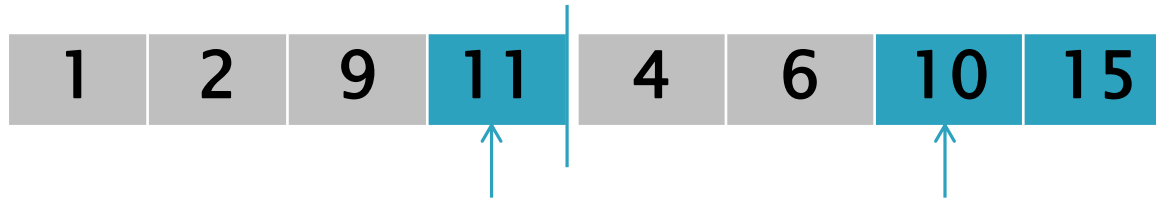
6- inversiuni cu 9 si 11



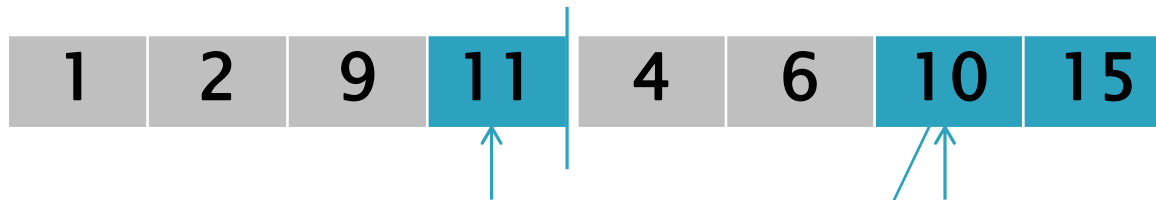
Inversiuni = 2 + 2



Inversiuni = 2 + 2



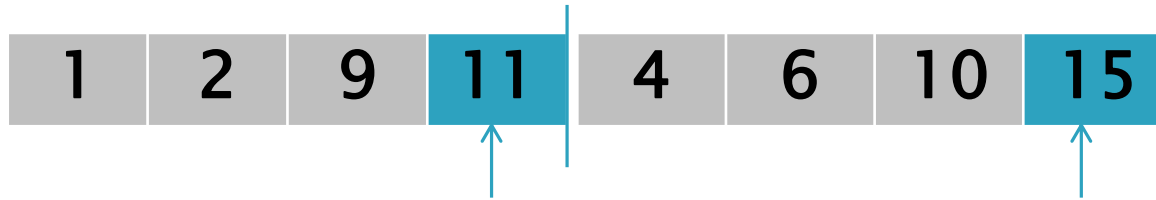
Inversiuni = 2 + 2



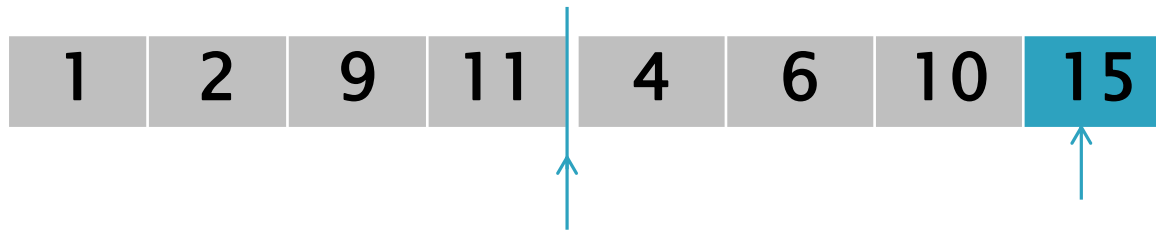
10- inversiuni cu 11



$$\text{Inversiuni} = 2 + 2 + 1$$



$$\text{Inversiuni} = 2 + 2 + 1$$



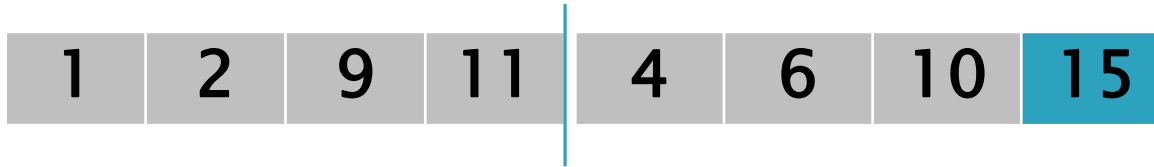
$$\text{Inversiuni} = 2 + 2 + 1$$



15- nicio inversiune



$$\text{Inversiuni} = 2 + 2 + 1 + 0$$



$$\text{Inversiuni} = 2 + 2 + 1 + 0 = 5$$

```
def nr_inversiuni(v, p, u):  
    if p==u:  
        return 0  
    else:  
        m = (p+u)//2  
        n1 = nr_inversiuni(v, p, m)  
        n2 = nr_inversiuni(v, m+1, u)  
        return n1+n2+interclaseaza(v, p, m, u)
```

► **Apel:** **x** = nr_inversiuni(v,0, len(v)-1)

```

def interclaseaza(a, p, m, u):
    b = [None]*(u-p+1)
    nr = 0
    i = p; j = m + 1; k = 0
    while (i<=m) and (j <= u):
        if a[i] <= a[j]:
            b[k] = a[i]; i += 1
        else:
            b[k] = a[j]; j += 1; nr += (m-i+1)
        k+=1

    while i<=m:
        b[k] = a[i]; k += 1; i += 1

    while j<=u:
        b[k] = a[j]; k += 1; j += 1

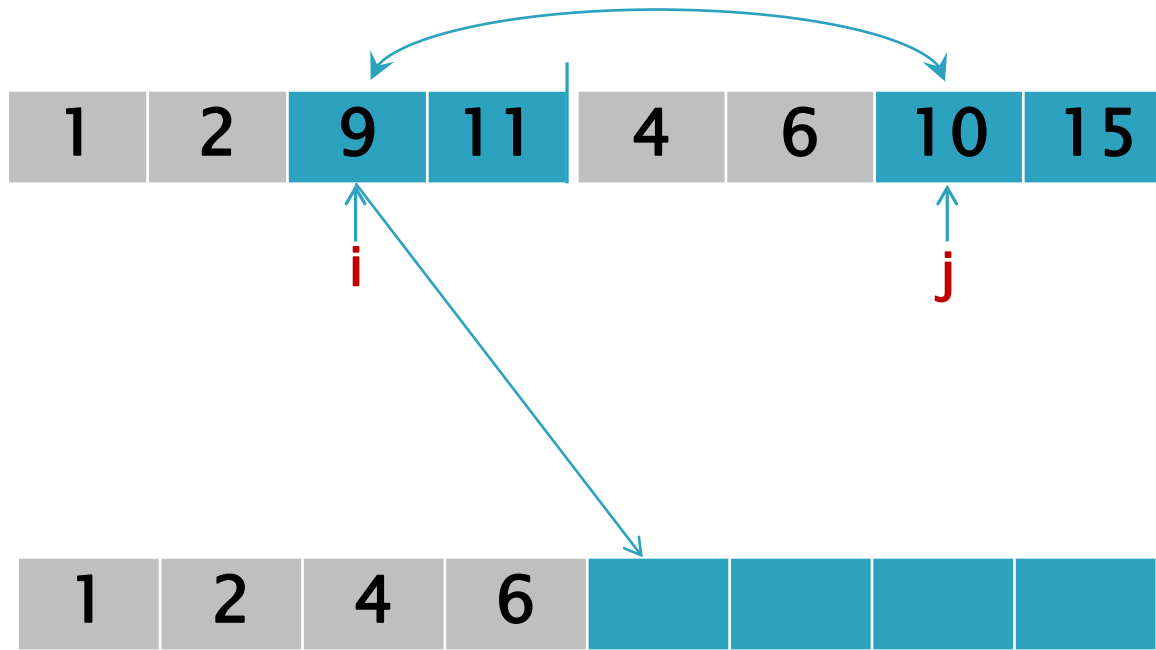
    for i in range(p,u+1):
        a[i] = b[i-p]

    return nr

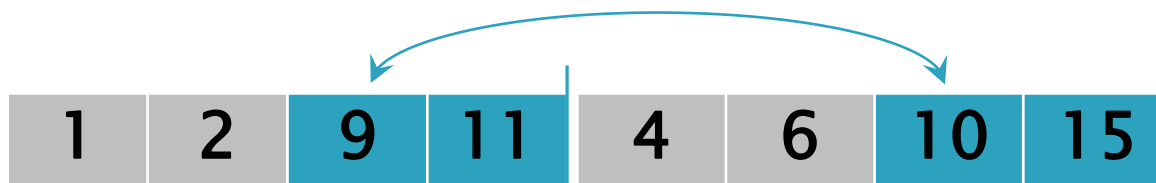
```

- ▶ **Varianta 2** – puteam număra inversiunile dintre subvectori și atunci când un element $a[i]$ cu $i \leq m$ (din subvectorul stâng) este adăugat în vectorul rezultat.

?

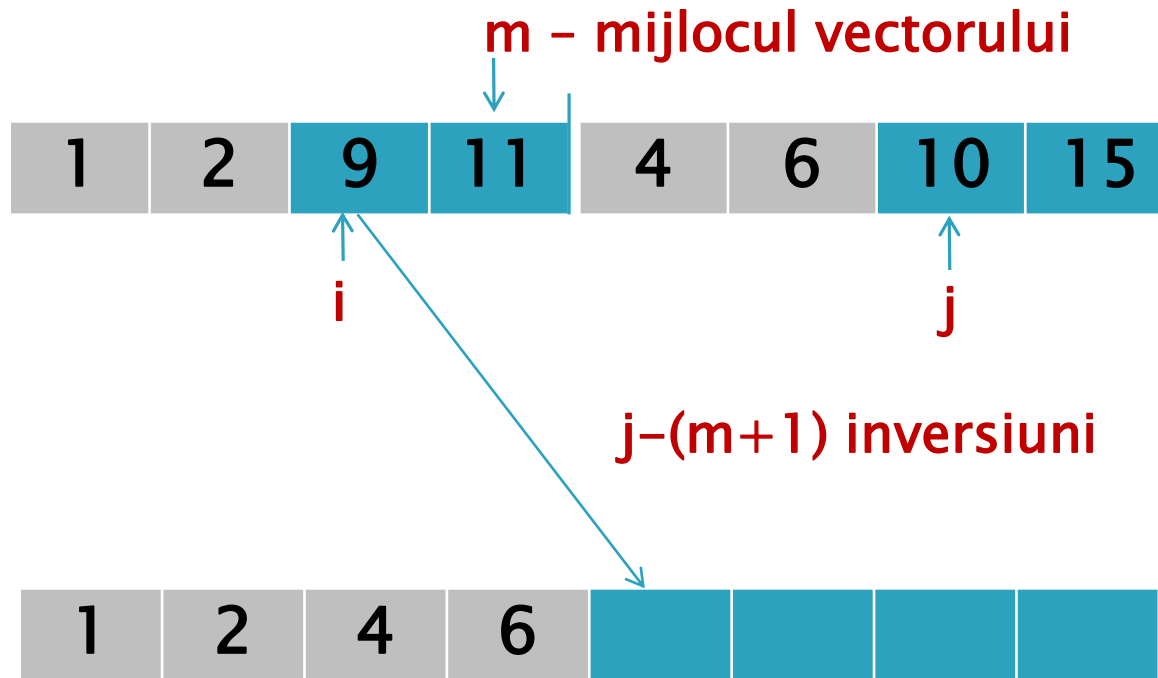


?



9 – inversiuni cu 4 si 6
(elementele deja adăugate
din subvectorul drept)





- $a[i]$ determină $j - m - 1$ inversiuni

- ▶ **Temă** – Propuneți un algoritm similar pentru determinarea numărului de inversiuni ale unui vector oarecare (ale cărui elemente **nu sunt neapărat distincte**)

Quicksort

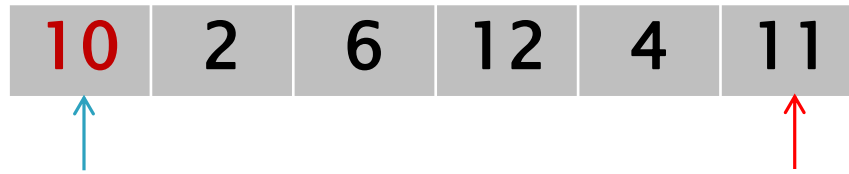
Sortarea rapidă

Quicksort

► Idee:

- poziționăm primul element al secvenței (**pivotul**) pe poziția sa finală = astfel încât elementele din stânga sa sunt mai mici, iar cele din dreapta mai mari
- ordonăm crescător elementele din stânga
- ordonăm crescător elementele din dreapta

Exemplu – poziționare pivot



10	2	6	12	4	11
----	---	---	----	---	----

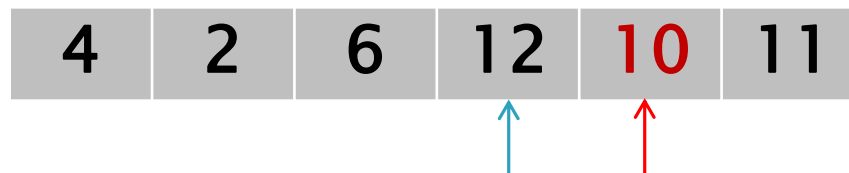
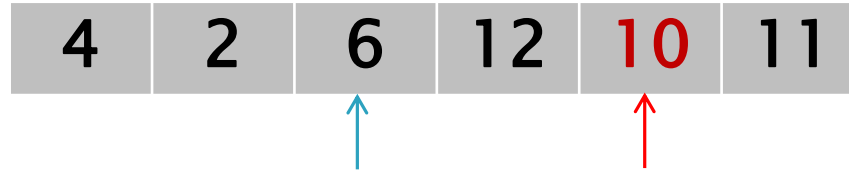
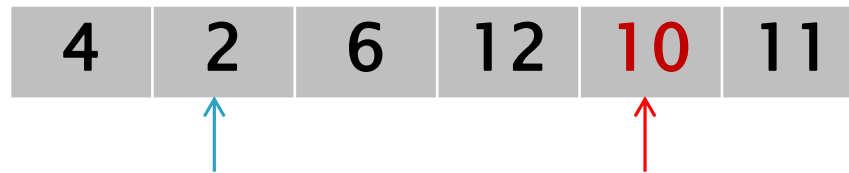
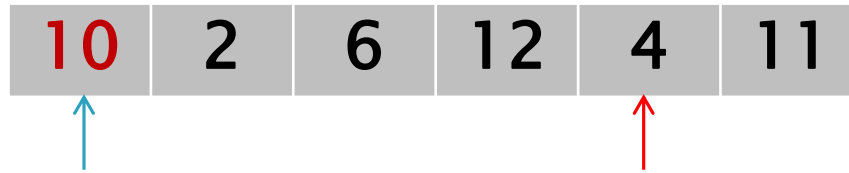
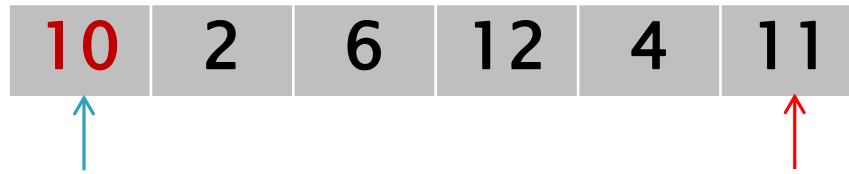


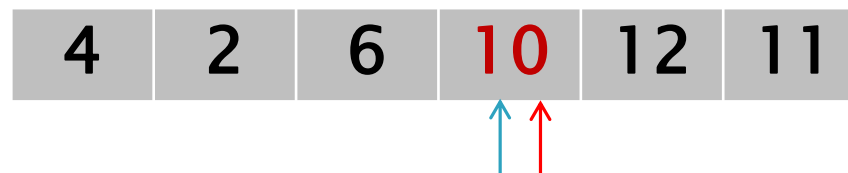
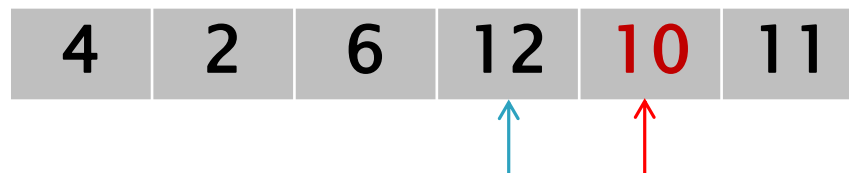
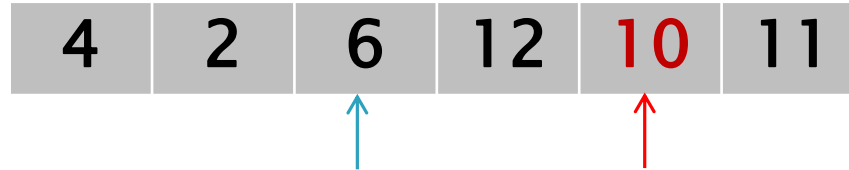
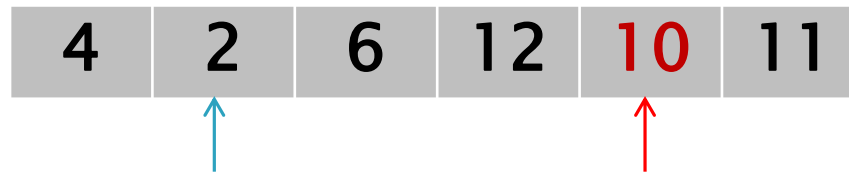
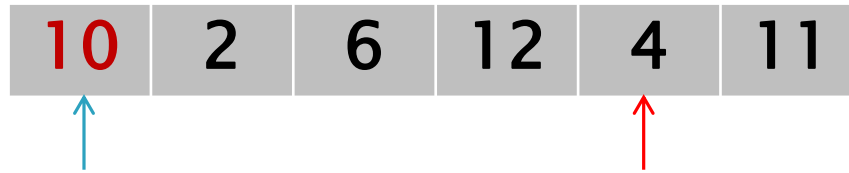
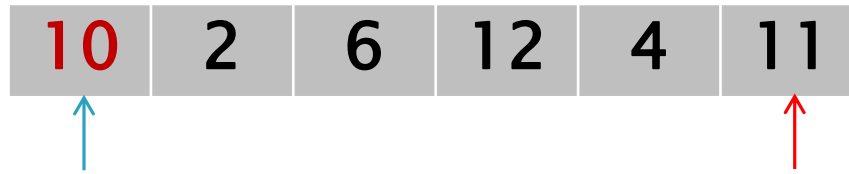
10	2	6	12	4	11
----	---	---	----	---	----











Quicksort

```
def quick_sort_di(v, p, u):  
    if p >= u:  
        return  
    m = poz(v, p, u)  
    quick_sort_di(v, p, m - 1)  
    quick_sort_di(v, m + 1, u)
```

```
def poz(v, p, u):  
    i = p  
    j = u  
    depli = 0  
    deplj = -1  
    while i < j:  
        if v[i] > v[j]:  
            v[i], v[j] = v[j], v[i]  
            depli, deplj = -deplj, -depli  
            #aux= depli; depli= -deplj; deplj= -aux;  
        i += depli  
        j += deplj  
    return i
```

Quicksort

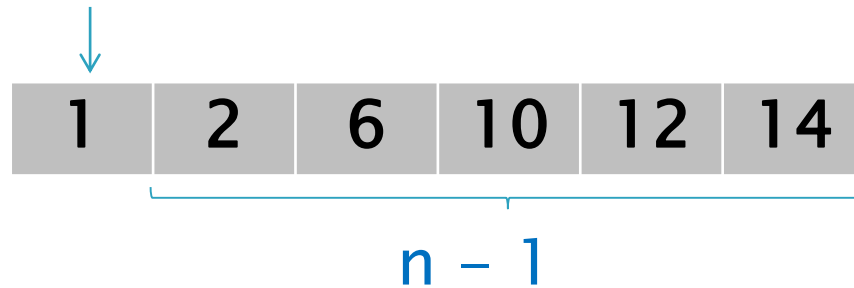
► Complexitate:

- Defavorabil: $O(n^2)$

- pentru vector deja sortat \Rightarrow
- una dintre subprobleme are dimensiune $n-1$
- pivotarea $n-1$

$$T(n) = T(n-1) + n-1$$

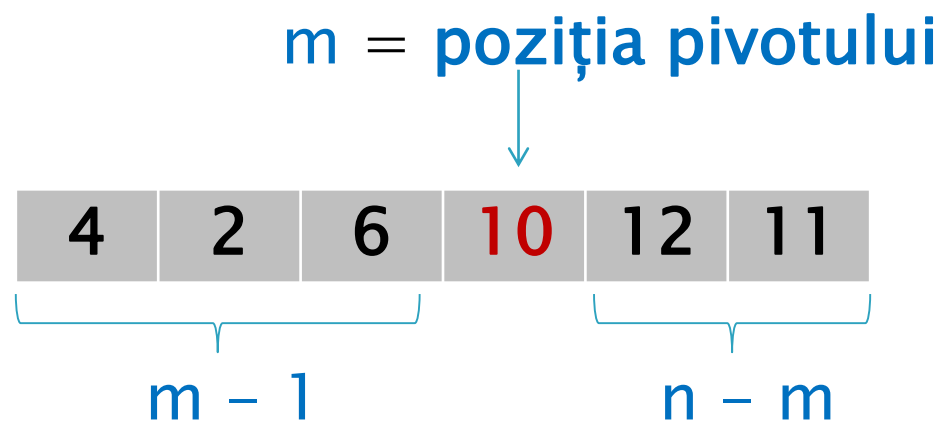
$m = 1 =$ poziția pivotului



Quicksort

► Complexitate:

- **Mediu:** $O(n \log n)$ – cu **pivot ales aleator**
- Alegem aleator un element ca pivot, îl interschimbăm cu primul element și folosim procedura de pivotare anterioară



Quicksort – pivot aleator

```
def poz_rand(v, p, u):  
    r = random.randint(p, u)  
    v[r], v[p] = v[p], v[r]  
    return poz(v, p, u)
```

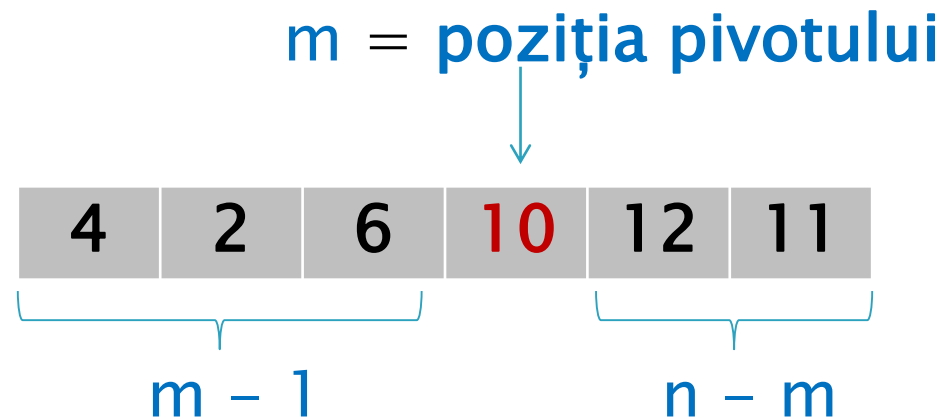
Quicksort – pivot aleator

```
def quick_sort_di(v, p, u):  
    if p >= u:  
        return  
  
    m = poz_rand(v, p, u)  
    quick_sort_di(v, p, m - 1)  
    quick_sort_di(v, m + 1, u)
```

Quicksort

► Complexitate:

- Mediu: $O(n \log n)$ – cu **pivot ales aleator**
- Justificarea complexității caz mediu –
SUMPLIMENTAR



Quicksort

Timp mediu



$$\begin{cases} T(n) = n - 1 + \frac{1}{n} \sum_{k=1}^n [T(k-1) + T(n-k)] \\ T(1) = T(0) = 0 \end{cases}$$

Quicksort

$$\begin{cases} T(n) = n - 1 + \frac{1}{n} \sum_{k=1}^n [T(k-1) + T(n-k)] \\ T(1) = T(0) = 0 \end{cases}$$

$$nT(n) = n(n-1) + 2[T(0) + T(1) + \dots + T(n-1)]$$

Quicksort

$$\begin{cases} T(n) = n - 1 + \frac{1}{n} \sum_{k=1}^n [T(k-1) + T(n-k)] \\ T(1) = T(0) = 0 \end{cases}$$

$$nT(n) = n(n-1) + 2[T(0) + T(1) + \dots + T(n-1)]$$

$$(n-1)T(n-1) = (n-1)(n-2) + 2[T(0) + \dots + T(n-2)]$$

Quicksort

$$\begin{cases} T(n) = n - 1 + \frac{1}{n} \sum_{k=1}^n [T(k-1) + T(n-k)] \\ T(1) = T(0) = 0 \end{cases}$$

$$nT(n) = n(n-1) + 2[T(0) + T(1) + \dots + T(n-1)]$$

$$(n-1)T(n-1) = (n-1)(n-2) + 2[T(0) + \dots + T(n-2)]$$

Scădem cele două relații:

$$nT(n) - (n-1)T(n-1) = 2(n-1) + 2T(n-1)$$

Quicksort

$$\begin{cases} T(n) = n - 1 + \frac{1}{n} \sum_{k=1}^n [T(k-1) + T(n-k)] \\ T(1) = T(0) = 0 \end{cases}$$

$$nT(n) = n(n-1) + 2[T(0) + T(1) + \dots + T(n-1)]$$

$$(n-1)T(n-1) = (n-1)(n-2) + 2[T(0) + \dots + T(n-2)]$$

$$nT(n) - (n-1)T(n-1) = 2(n-1) + 2T(n-1)$$

$$nT(n) = (n+1)T(n-1) + 2(n-1)$$

Quicksort

$$\begin{cases} T(n) = n - 1 + \frac{1}{n} \sum_{k=1}^n [T(k-1) + T(n-k)] \\ T(1) = T(0) = 0 \end{cases}$$

$$nT(n) = n(n-1) + 2[T(0) + T(1) + \dots + T(n-1)]$$

$$(n-1)T(n-1) = (n-1)(n-2) + 2[T(0) + \dots + T(n-2)]$$

$$nT(n) - (n-1)T(n-1) = 2(n-1) + 2T(n-1)$$

$$nT(n) = (n+1)T(n-1) + 2(n-1) \quad | \quad :n(\mathbf{n+1})$$

Quicksort

$$\frac{T(n)}{n+1} = \frac{T(n-1)}{n} + \frac{2(n-1)}{n(n+1)}$$

Quicksort

$$\frac{T(n)}{n+1} = \frac{T(n-1)}{n} + \frac{2(n-1)}{n(n+1)}$$

$$\frac{T(n)}{n+1} = \frac{T(n-1)}{n} + 2\left(\frac{2}{n+1} - \frac{1}{n}\right)$$

Quicksort

$$\frac{T(n)}{n+1} = \frac{T(n-1)}{n} + \frac{2(n-1)}{n(n+1)}$$

$$\frac{T(n)}{n+1} = \frac{T(n-1)}{n} + 2\left(\frac{2}{n+1} - \frac{1}{n}\right)$$

$$\frac{T(n-1)}{n} = \frac{T(n-2)}{n-1} + 2\left(\frac{2}{n} - \frac{1}{n-1}\right)$$

.....

$$\frac{T(2)}{3} = \frac{T(1)}{2} + 2\left(\frac{2}{3} - \frac{1}{2}\right)$$

Quicksort

$$\frac{T(n)}{n+1} = \frac{T(n-1)}{n} + \frac{2(n-1)}{n(n+1)}$$

$$\frac{T(n)}{n+1} = \frac{T(n-1)}{n} + 2\left(\frac{2}{n+1} - \frac{1}{n}\right)$$

$$\frac{T(n-1)}{n} = \frac{T(n-2)}{n-1} + 2\left(\frac{2}{n} - \frac{1}{n-1}\right)$$

.....

$$\frac{T(2)}{3} = \frac{T(1)}{2} + 2\left(\frac{2}{3} - \frac{1}{2}\right)$$

+

$$\frac{T(n)}{n+1} = 2\left(\frac{1}{n+1} + \frac{1}{n} + \dots + \frac{1}{3}\right) + \frac{2}{n+1} - 1$$

Quicksort

$$\begin{aligned}\frac{T(n)}{n+1} &= 2 \left(\frac{1}{n+1} + \frac{1}{n} + \dots + \frac{1}{3} \right) + \frac{2}{n+1} - 1 \\ &\leq 2 \left(\frac{1}{n+1} + \frac{1}{n} + \dots + \frac{1}{3} \right)\end{aligned}$$

sume Riemann inferioare pentru funcția $f(x)=1/x$

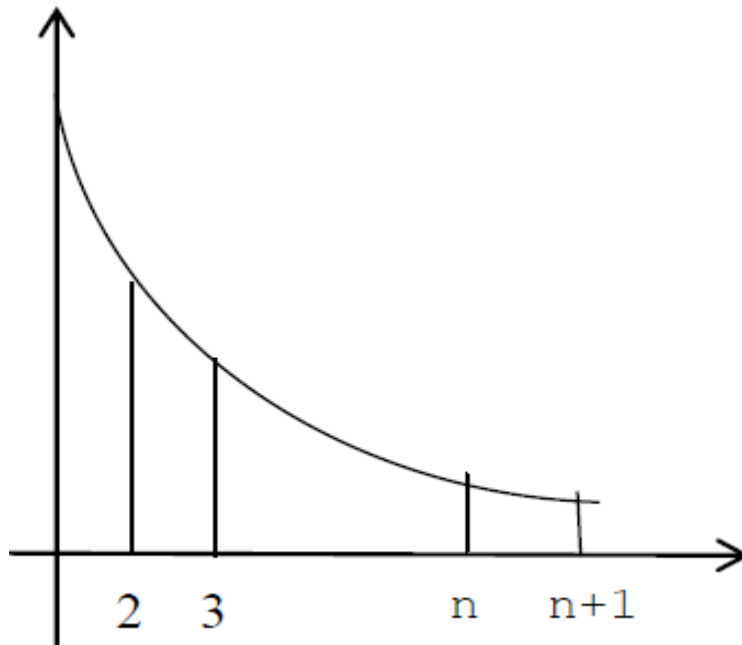
$$\Delta = \{2, 3, \dots, n+1\}$$

Quicksort

$$\begin{aligned}\frac{T(n)}{n+1} &= 2 \left(\frac{1}{n+1} + \frac{1}{n} + \dots + \frac{1}{3} \right) + \frac{2}{n+1} - 1 \\ &\leq 2 \left(\frac{1}{n+1} + \frac{1}{n} + \dots + \frac{1}{3} \right)\end{aligned}$$

sume Riemann inferioare pentru funcția $f(x) = 1/x$

$$\Delta = \{2, 3, \dots, n+1\}$$



Quicksort

$$\begin{aligned}\frac{T(n)}{n+1} &= 2 \left(\frac{1}{n+1} + \frac{1}{n} + \dots + \frac{1}{3} \right) + \frac{2}{n+1} - 1 \\ &\leq 2 \left(\frac{1}{n+1} + \frac{1}{n} + \dots + \frac{1}{3} \right)\end{aligned}$$

sume Riemann inferioare pentru funcția $f(x)=1/x$

$$\Delta = \{2, 3, \dots, n+1\}$$

$$\frac{T(n)}{n+1} \leq 2 \int_2^{n+1} \frac{1}{x} dx = 2 \ln x \Big|_2^{n+1} \leq 2 \ln(n+1)$$

Statistici de ordine



Dat un vector a de n numere și un indice k ,
 $1 \leq k \leq n$, să se determine al k -lea cel mai mic
element din vector.

Statistici de ordine

A i -a statistică de ordine a unei mulțimi = al i -lea cel mai mic element.

- ▶ **Minimul** = prima statistică de ordine
- ▶ **Maximul** = a n -a statistică de ordine

Statistici de ordine

- ▶ **Mediana** = punctul de la jumătatea unei mulțimi
= o valoare v a.î. numărul de elemente din mulțime mai mici decât v este egal cu numărul de elemente din mulțime mai mari decât v .

Statistici de ordine

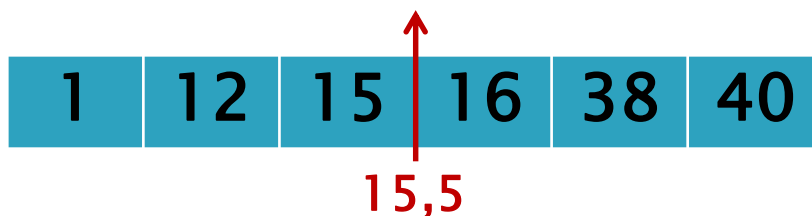
► Mediana

Dacă n este impar, atunci mediana este

a $\lceil n/2 \rceil$ -a statistică de ordine, altfel, prin

convenție mediana este media aritmetică dintre a

$\lfloor n/2 \rfloor$ -a statistică și a $(\lfloor n/2 \rfloor + 1)$ -a statistică de ordine



► Mediană inferioară / superioară

Statistici de ordine – utilitate

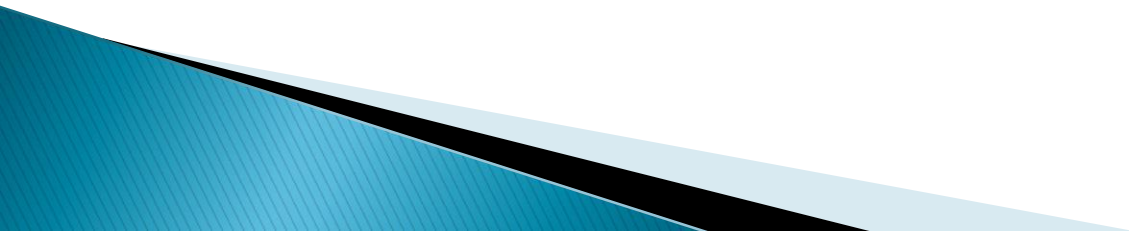
- ▶ **Statistică**
- ▶ **Mediana** – pentru o mulțime $A=\{a_1, \dots, a_n\}$ valoarea μ care minimizează expresia

$$\sum_{i=1}^n |\mu - a_i|$$

Statisticile de ordine

Idee

Al k -lea minim



Statistici de ordine

Idee

Al k-lea minim – folosim poziționarea de la quicksort (**pivot aleator**)

Statistici de ordine

Fie m poziția pivotului

- Dacă $m = k$, pivotul este al k -lea minim
-
-

Statistici de ordine

Fie m poziția pivotului

- Dacă $m = k$, pivotul este al k -lea minim
- Dacă $m > k$, al k -lea minim este în **stânga** pivotului (al k -lea minim din stanga)
-

Statistici de ordine

Fie m poziția pivotului

- Dacă $m = k$, pivotul este al k -lea minim
- Dacă $m > k$, al k -lea minim este în **stânga** pivotului (al k -lea minim din stanga)
- Dacă $m < k$, al k -lea minim este în **dreapta** pivotului (al $(k-m)$ -lea minim din dreapta)

$k = 2$

10	2	6	12	4	11
----	---	---	----	---	----

poziționare pivot

4	2	6	10	12	11
---	---	---	----	----	----

$m = 4 > k \Rightarrow$ stânga

$k = 2$

10	2	6	12	4	11
----	---	---	----	---	----

poziționare pivot

4	2	6	10	12	11
---	---	---	----	----	----

$m = 4 > k \longrightarrow$ stânga

4	2	6
---	---	---

poziționare pivot

2	4	6
---	---	---

$m = 2 = k \longrightarrow$ stop;
pivotul este al k -lea minim

#pentru numerotare de la 0


```
def sel_k_min(v, k, p, u):  
    m = poz_rand(v, p, u)  
    if m == k - 1:  
        return v[m]  
    if m < k - 1:  
        return sel_k_min(v, k, m + 1, u)  
    return sel_k_min(v, k, p, m - 1)
```

Apel: `x = sel_k_min(v, k, 0, len(v) - 1)`

Complexitate

- Timpul mediu

$m = \text{poziția pivotului}$


$$\left\{ \begin{array}{l} T(n) \leq n - 1 + \frac{1}{n} \sum_{m=1}^n T(\max\{m-1, n-m\}) \\ \leq n - 1 + \frac{2}{n} \sum_{m=\frac{n}{2}}^{n-1} T(m) \end{array} \right.$$

$$T(n) \leq cn, c \geq 4$$

– se demonstrează prin inducție

Complexitate

- Detalii pas inducție

$$T(n) \leq n - 1 + \frac{2}{n} \sum_{m=\frac{n}{2}}^{n-1} T(m) \leq n - 1 + \frac{2}{n} \sum_{m=\frac{n}{2}}^{n-1} (cm)$$

Complexitate

- Detalii pas inducție

$$\begin{aligned} T(n) &\leq n-1 + \frac{2}{n} \sum_{m=\frac{n}{2}}^{n-1} T(m) \leq n-1 + \frac{2}{n} \sum_{m=\frac{n}{2}}^{n-1} (cm) \\ &\leq n-1 + \frac{2}{n} c \left[\frac{1}{2} n(n-1) - \frac{1}{2} \frac{n}{2} \left(\frac{n}{2} - 1 \right) \right] = \\ &= \end{aligned}$$

Complexitate

- Detalii pas inducție

$$\begin{aligned}T(n) &\leq n-1 + \frac{2}{n} \sum_{m=\frac{n}{2}}^{n-1} T(m) \leq n-1 + \frac{2}{n} \sum_{m=\frac{n}{2}}^{n-1} (cm) \\&\leq n-1 + \frac{2}{n} c \left[\frac{1}{2} n(n-1) - \frac{1}{2} \frac{n}{2} \left(\frac{n}{2} - 1 \right) \right] = \\&= n-1 + \frac{2}{n} c \frac{n}{2} \left(n-1 - \frac{n}{4} + \frac{1}{2} \right) = n-1 + c \left(\frac{3}{4} n - \frac{1}{2} \right)\end{aligned}$$

Dar $c \geq 4 \Rightarrow$

Complexitate

- Detalii pas inducție

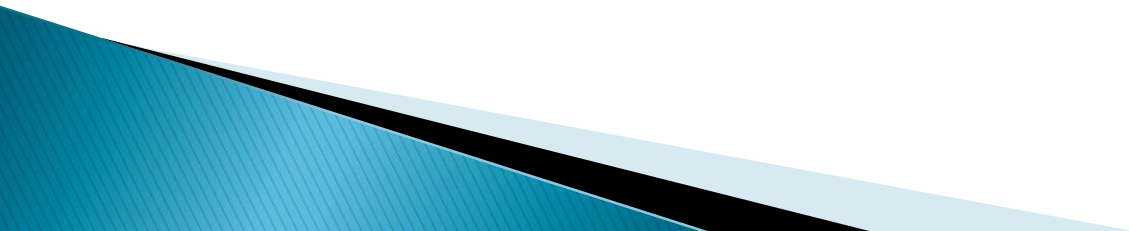
$$\begin{aligned}T(n) &\leq n-1 + \frac{2}{n} \sum_{m=\frac{n}{2}}^{n-1} T(m) \leq n-1 + \frac{2}{n} \sum_{m=\frac{n}{2}}^{n-1} (cm) \\&\leq n-1 + \frac{2}{n} c \left[\frac{1}{2} n(n-1) - \frac{1}{2} \frac{n}{2} \left(\frac{n}{2} - 1 \right) \right] = \\&= n-1 + \frac{2}{n} c \frac{n}{2} \left(n-1 - \frac{n}{4} + \frac{1}{2} \right) = n-1 + c \left(\frac{3}{4} n - \frac{1}{2} \right)\end{aligned}$$

Dar $c \geq 4 \Rightarrow$

$$T(n) \leq c \left(\frac{n-1}{4} + \frac{3}{4} n - \frac{1}{2} \right) = c \left(n - \frac{1}{4} - \frac{1}{2} \right) \leq cn$$

Statistici de ordine

Algoritm $O(n)$ caz defavorabil (suplimentar)



Statistici de ordine

Algoritm $O(n)$ caz defavorabil

Idee

În algoritmul anterior `sel_k_min` se folosește în loc de pivot aleatoriu un pivot calculat astfel:

Statistici de ordine

Algoritm $O(n)$ caz defavorabil

Idee

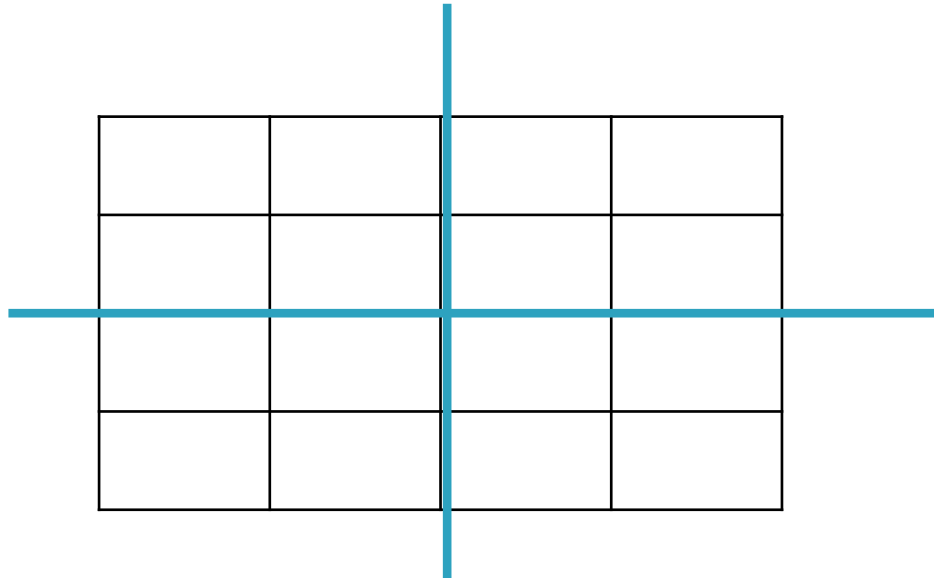
În algoritmul anterior `sel_k_min` se folosește în loc de pivot aleatoriu un pivot calculat astfel:

- _ se împarte vectorul în grupe de 5 (cu cel mult o excepție – ultimul grup)
- se formează un vector `mediane[]` având ca elemente mediana fiecărui grup
- se calculează mediana acestui vector folosind aceeași funcție `sel_k_min mediane, $\left\lceil \frac{n}{5} \right\rceil, \left\lceil \frac{n}{10} \right\rceil$` \Rightarrow
 \Rightarrow **aceasta este pivotul**

Divide et impera – Matrice



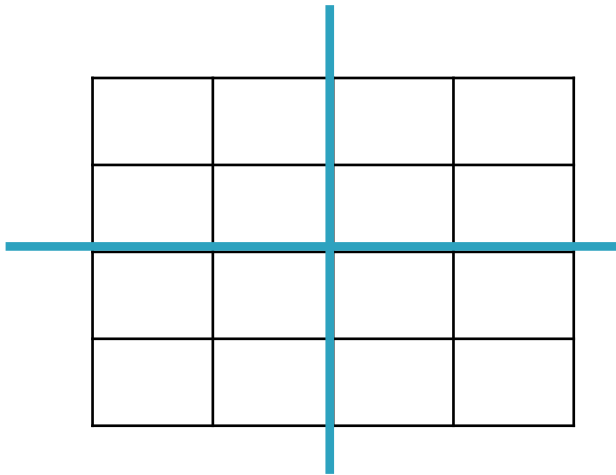
Să se calculeze suma elementelor unei matrice pătratice de dimensiune n unde n este putere a lui 2 folosind Divide et Impera (împărțind matricea succesiv în 4 subcadrane)



Divide et impera – Matrice



Să se calculeze suma elementelor unei matrice pătratice de dimensiune n unde n este putere a lui 2 folosind Divide et Impera (împărțind matricea succesiv în 4 subcadrame)



O subproblemă este identificată de:

- Două colțuri opuse ale submatricei

`divide(x1, y1, x2, y2)`

sau

- Un colț al submatricei și dimensiunea ei

`divide(x, y, dim)`

Divide et impera – Matrice

```
def suma(m,x,y,n):
```

```
    if n==1:
```

```
        return m[x][y]
```

```
    s1 = suma(m, x, y, n//2)
```

```
    s2 = suma(m, x+n//2, y, n//2)
```

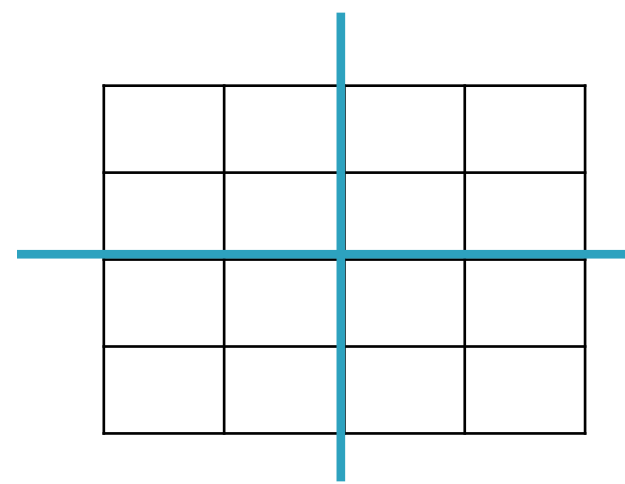
```
    s3 = suma(m, x, y+n//2, n//2)
```

```
    s4 = suma(m, x+n//2, y+n//2, n//2)
```

```
    return s1+s2+s3+s4
```

```
def suma_matrice(m):
```

```
    return suma(m,0,0,len(m))
```



Divide et impera – Matrice

```
def suma(m,x,y,n):
```

Problema matrice de latura n

```
    if n==1:
```

```
        return m[x][y]
```

Timp constant O(1)

```
    s1 = suma(m, x, y, n//2)
```

```
    s2 = suma(m, x+n//2, y, n//2)
```

```
    s3 = suma(m, x, y+n//2, n//2)
```

```
    s4 = suma(m, x+n//2, y+n//2, n//2)
```

```
    return s1+s2+s3+s4
```

O(1)

4 subprobleme
cu matrice de
dimensiune n/2

```
def suma_matrice(m):
```

```
    return suma(m,0,0,len(m))
```

=> $T(n) = 4T(n/2) + O(1)$

$T(n) = 4T(n/2) + 1$

$$T(n) = 4T(n/2) + 1, n = 2^k$$

$$T(n) = 4 T(n/2) + 1 =$$

$$= 4 [4T(n/2^2) + 1] + 1 =$$

$$= 4^2 T(n/2^2) + 4 + 1 =$$

$$= 4^2 [4T(n/2^3) + 1] + 4 + 1 =$$

$$= 4^3 T(n/2^3) + 4^2 + 4 + 1 =$$

$$= \dots = 4^k T(n/2^k) + 4^{k-1} + \dots + 4^2 + 4 + 1 =$$

$$= 4^k + 4^{k-1} + \dots + 4^2 + 4 + 1 = (4^{k+1} - 1)/3 = (4n^2 - 1)/3$$

$$\Rightarrow O(n^2)$$

$$n = 2^k \Rightarrow 4^k = n$$

$$T(n/2^k) = T(1) = 1$$

Mediana a doi vectori sortați



Mediana a doi vectori sortați



Se dau doi vectori a și b **de lungime n** , cu elementele **ordonate crescător**. Să se determine mediana vectorului obținut prin interclasarea celor doi vectori.

Mediana a doi vectori sortați

Exemplu: $n = 5$

1 12 15 16 38

2 13 17 30 45

Mediana a doi vectori sortați

Exemplu: $n = 5$

1 12 15 16 38

2 13 17 30 45



1 2 12 13 15 16 17 30 38 45

Mediana $(15+16)/2 = 15,5$

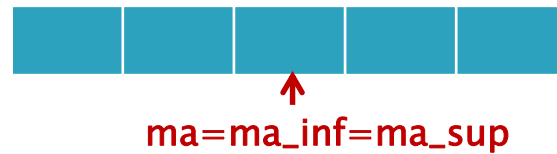
Mediana a doi vectori sortați

- **Algoritm $O(n)$** – interclasăm vectorii și apoi aflăm mediana în timp constant (din elementele de la mijlocul vectorului, conform definiției)

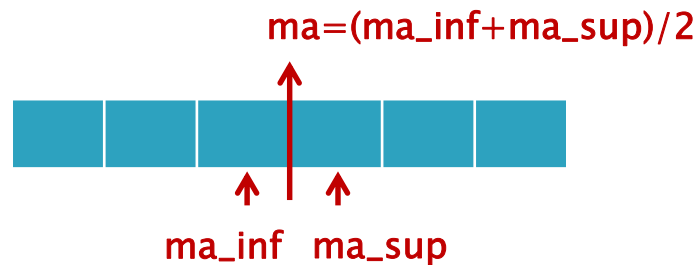
Mediana a doi vectori sortați

- Algoritm $O(\log(n))$

- ▶ Fie ma_inf , ma_sup , ma mediana inferioară, superioară, respectiv mediana vectorului a
- ▶ mb_inf , mb_sup , mb – similar pentru vectorul b
- ▶ c – vectorul obținut prin interclasare
- ▶ n impar:



- ▶ n par:





► Comparăm ma și mb



ma

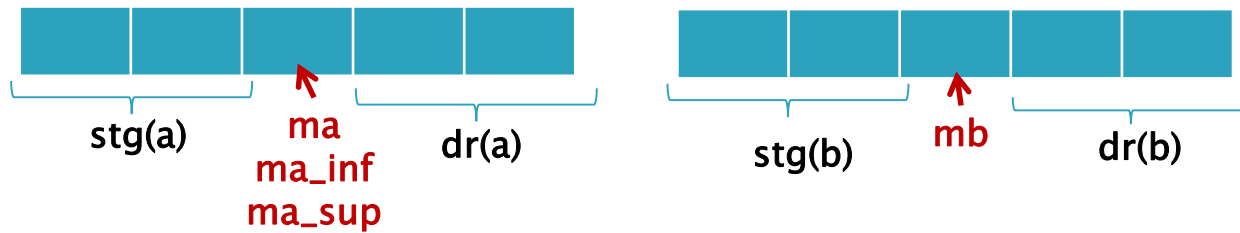


mb

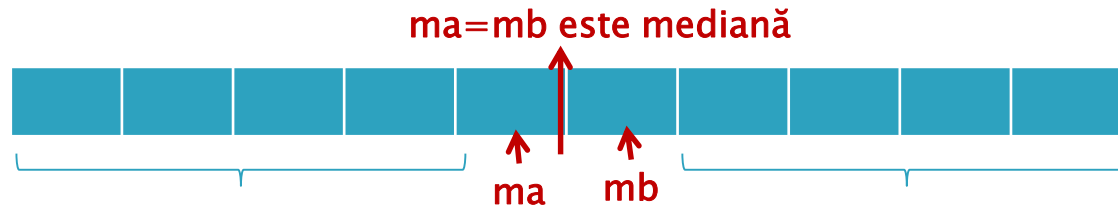
▶ $ma = mb \Rightarrow mc = ma = mb?$

► $ma = mb \Rightarrow mc = ma = mb$

n impar

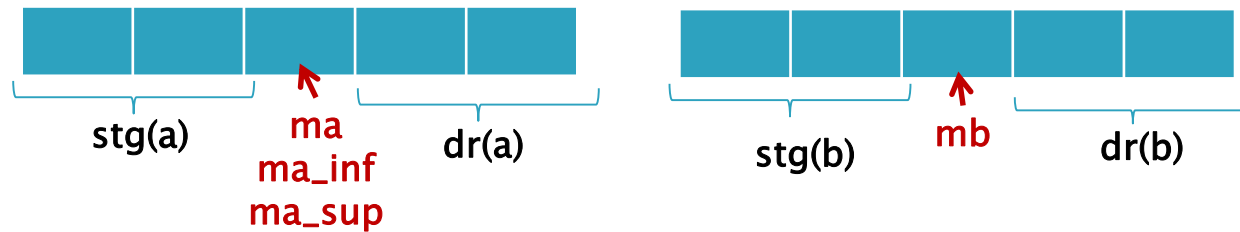


c:

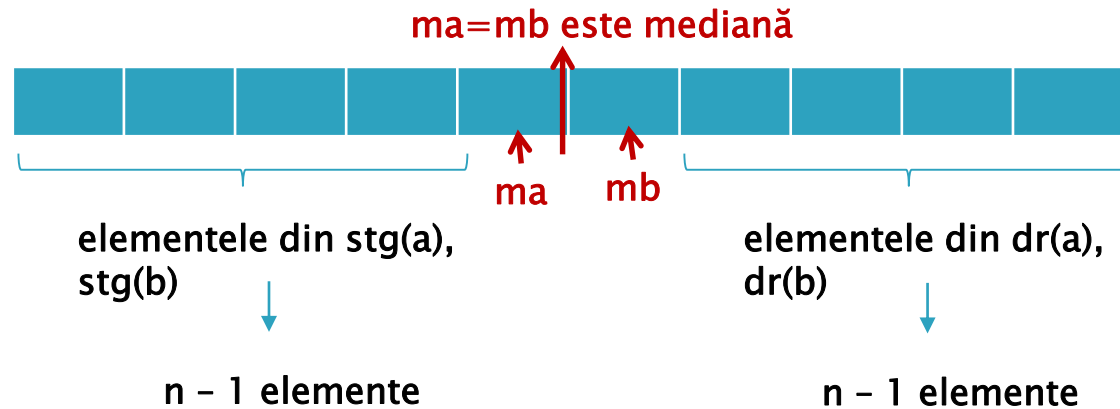


► $ma = mb \Rightarrow mc = ma = mb$

n impar

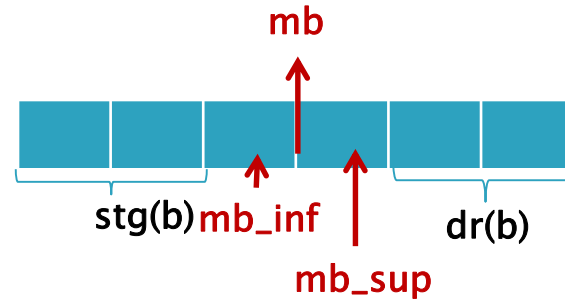
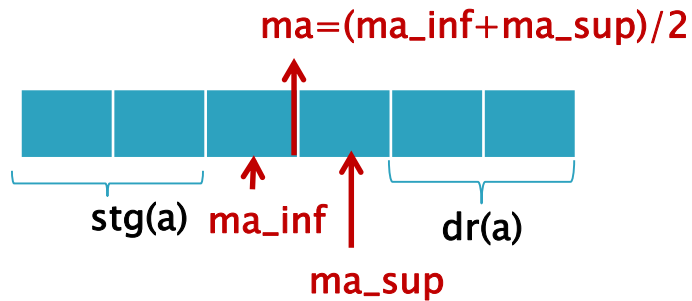


c:

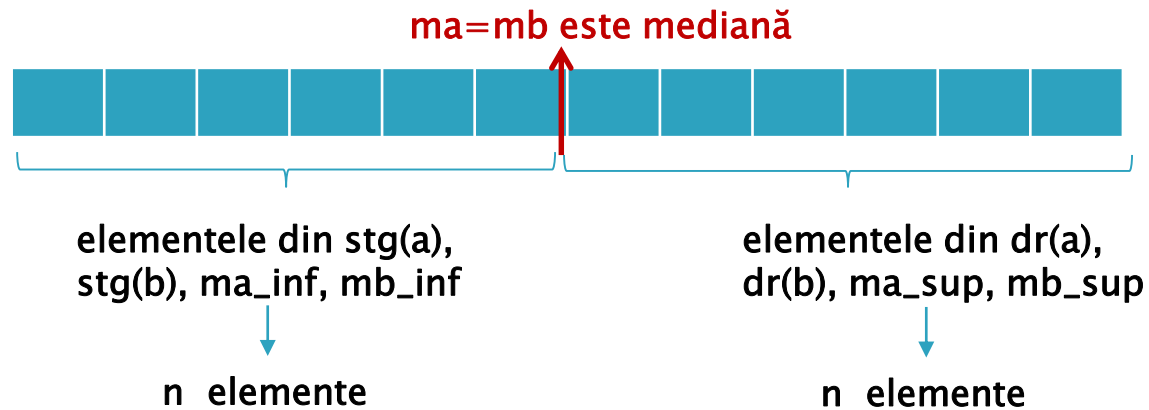


▶ $ma = mb \Rightarrow mc = ma = mb$

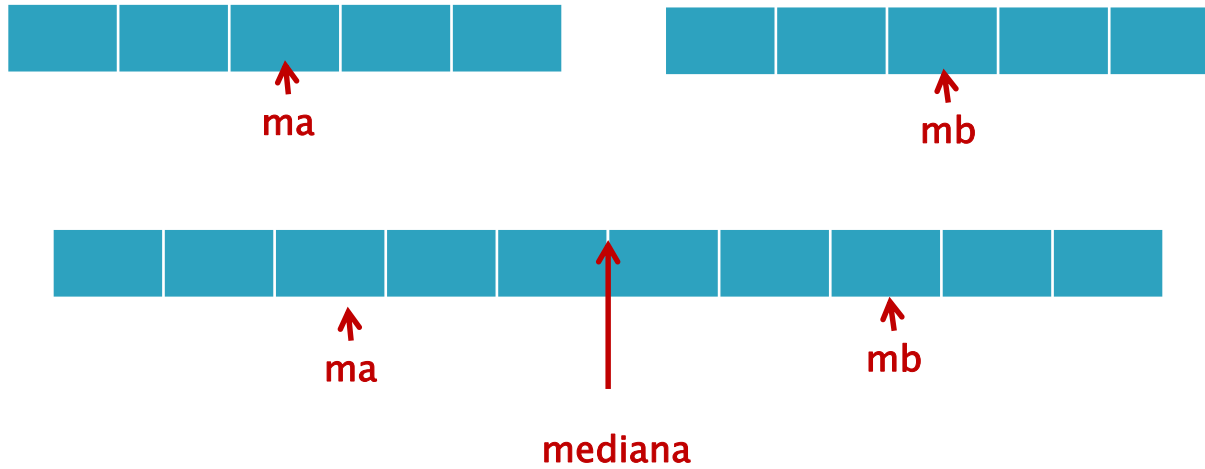
n par



c:

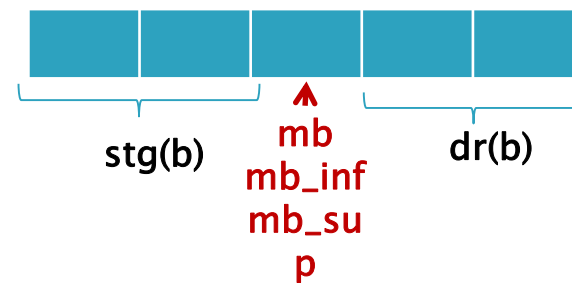
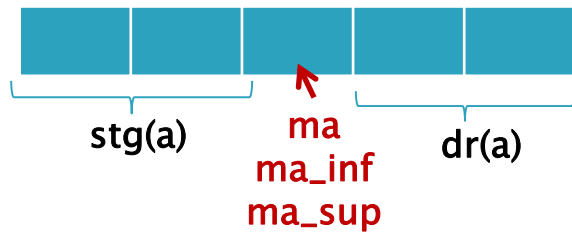


- ▶ $ma < mb \Rightarrow mc \in [ma, mb]$?



- ▶ $ma < mb \Rightarrow mc, mc_inf, mc_sup \in [ma_inf, mb_sup]$

n impar



c:

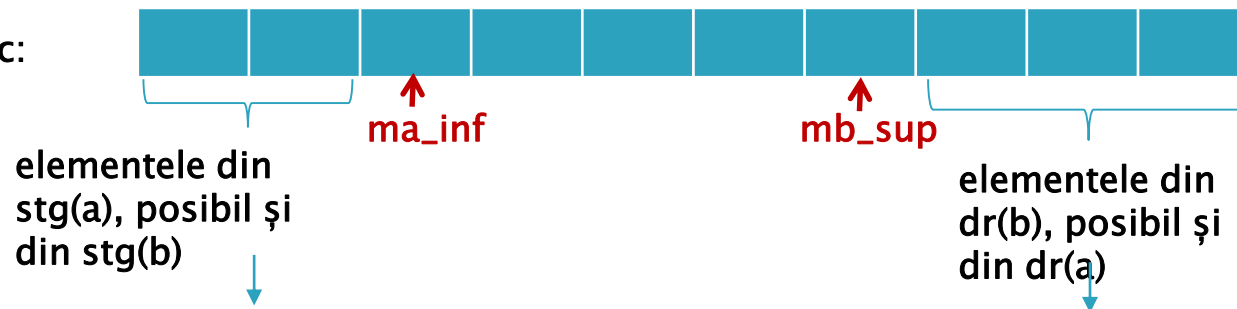


- ▶ $ma < mb \Rightarrow mc, mc_inf, mc_sup \in [ma_inf, mb_sup]$

n impar

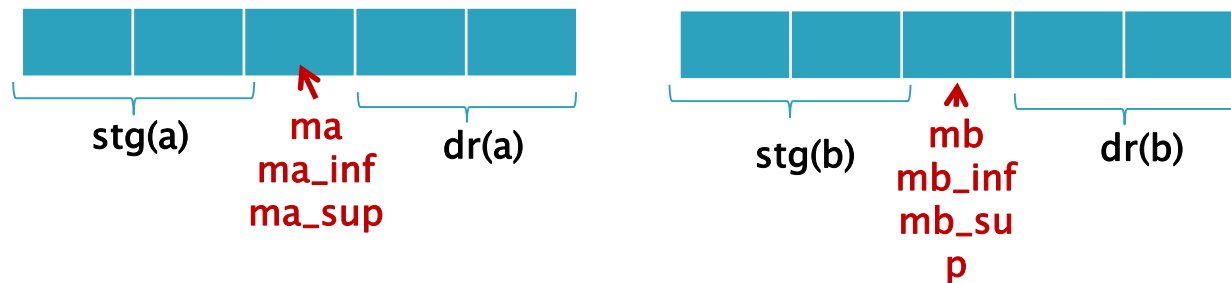


c:

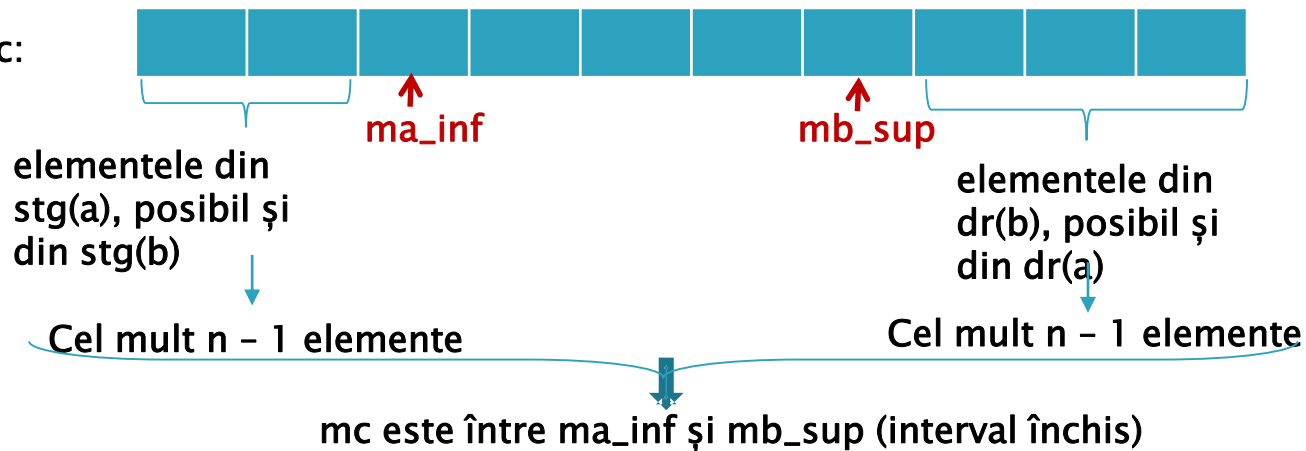


- ▶ $ma < mb \Rightarrow mc, mc_inf, mc_sup \in [ma_inf, mb_sup]$

n impar

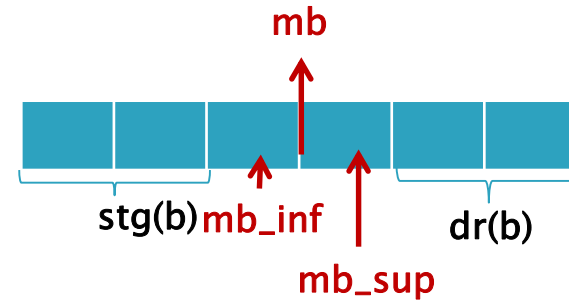
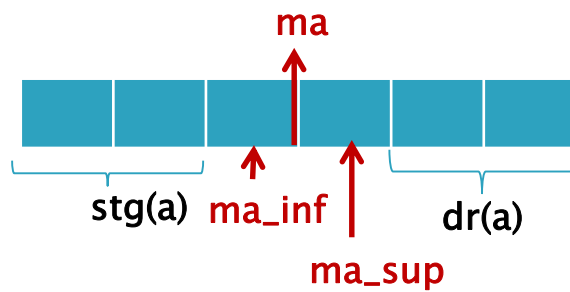


c:



- ▶ $ma < mb \Rightarrow mc, mc_inf, mc_sup \in [ma_inf, mb_sup]$

n par

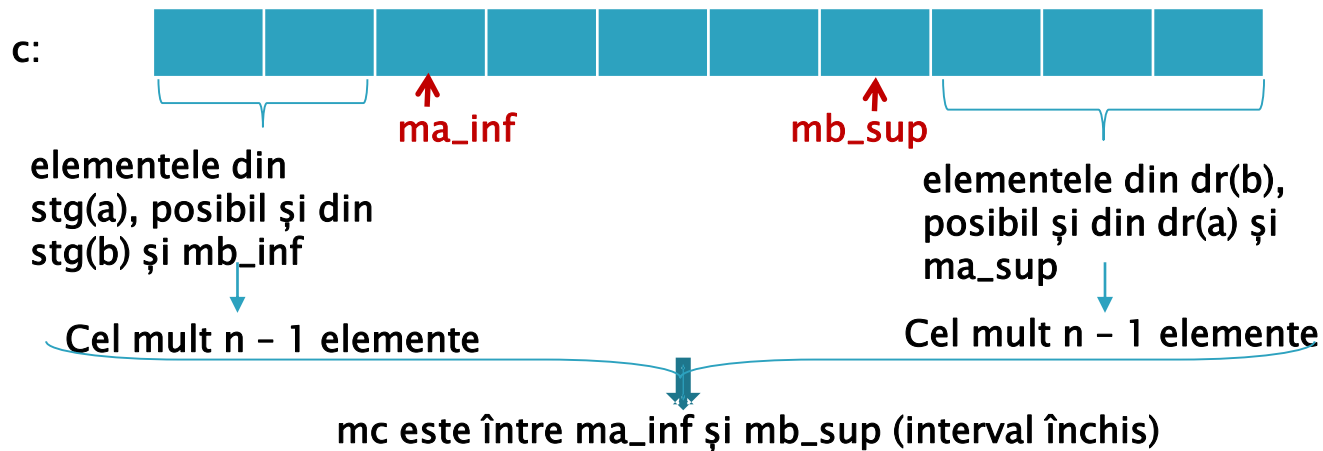
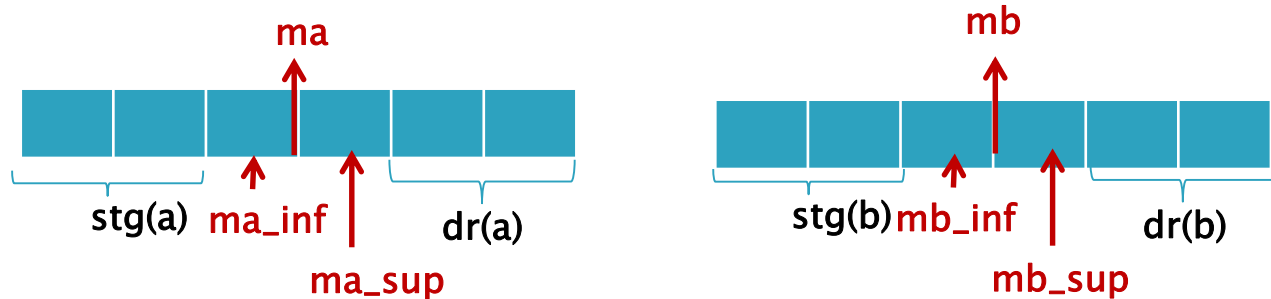


c:



▸ $ma < mb \Rightarrow mc, mc_inf, mc_sup \in [ma_inf, mb_sup]$

n par



- ▶ $ma < mb \Rightarrow mc, mc_inf, mc_sup \in [ma_inf, mb_sup]$

Rezultă:

Pentru a determina mediana este suficient să considerăm:

- Subvectorul drept din primul vector (inclusiv mediana inferioară)
- Subvectorul stâng din al doilea vector (inclusiv mediana superioară)

- ▶ $ma < mb \Rightarrow mc, mc_inf, mc_sup \in [ma_inf, mb_sup]$

Rezultă:

Pentru a determina mediana este suficient să considerăm:

- Subvectorul drept din primul vector (inclusiv mediana inferioară)
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Astfel

- din vectorul a renunțăm la $[(n-1)/2]$ elemente care sunt înaintea lui ma_inf (deci și a lui mc_inf) în c
- din vectorul b renunțăm **tot la** $[(n-1)/2]$ elemente care sunt după lui mb_sup (deci și după mc_sup) în c

deci media noilor vectori este egală cu mediana lui c

- ▶ $m_a > m_b$ – Similar

Mediana a doi vectori sortați

Corectitudine:

mediana noii probleme = mediana problemei inițiale

Pseudocod

- ▶ Fie m_a mediana vectorului a și m_b mediana vectorului b
 - Dacă $m_a = m_b$ atunci această valoare este mediana

- ▶ Fie m_a mediana vectorului a și m_b mediana vectorului b
 - Dacă $m_a = m_b$ atunci această valoare este mediana
 - Dacă $m_a > m_b$ atunci mediana = mediana subvectorilor

$$a[0..\lfloor n/2 \rfloor], \quad b[\lfloor (n-1)/2 \rfloor..n-1]$$

- Dacă $m_a < m_b$ atunci mediana = mediana subvectorilor

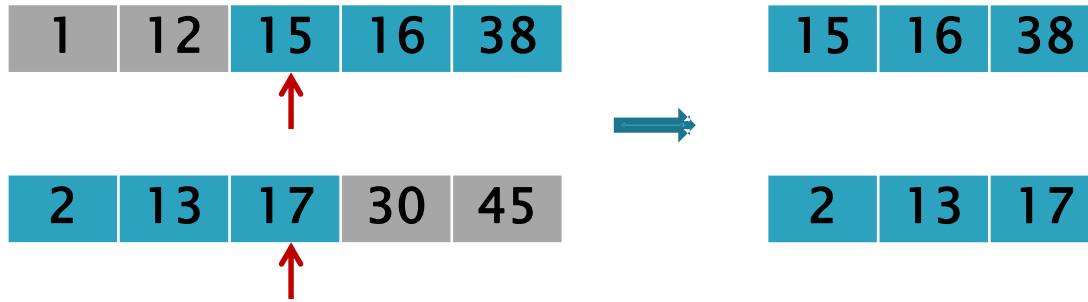
$$a[\lfloor (n-1)/2 \rfloor..n-1], \quad b[0..\lfloor n/2 \rfloor]$$

1	12	15	16	38
---	----	----	----	----



2	13	17	30	45
---	----	----	----	----





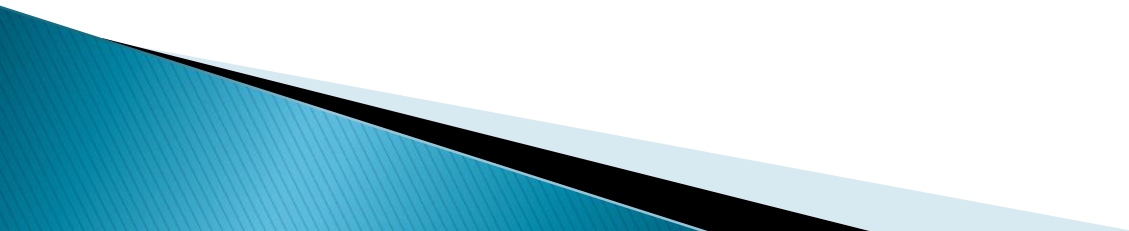
► **Știm să rezolvăm direct:**

- $n = 1: (a[1] + b[1]) / 2$

- $n = 2: \text{mediana lui } \{a[1], b[1], a[2], b[2]\}$

$$= (\max\{a[1], b[1]\} + \min\{a[2], b[2]\}) / 2$$

Exemplu



1	12	15	16	38
---	----	----	----	----

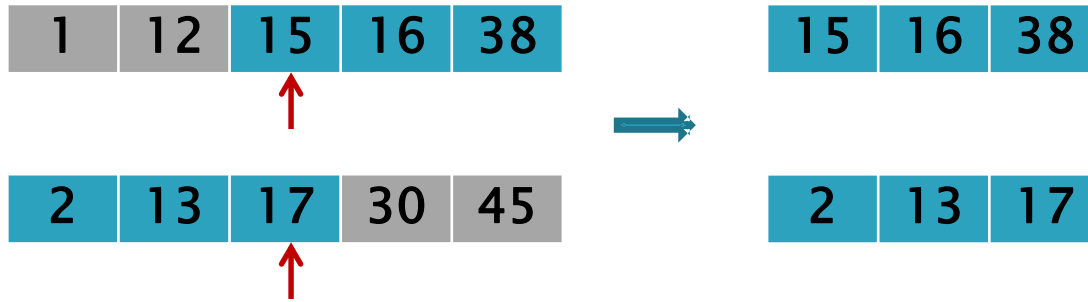
2	13	17	30	45
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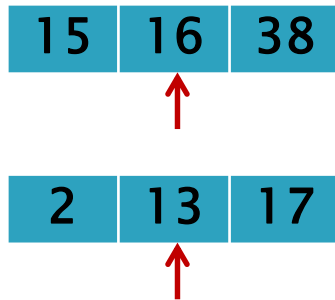
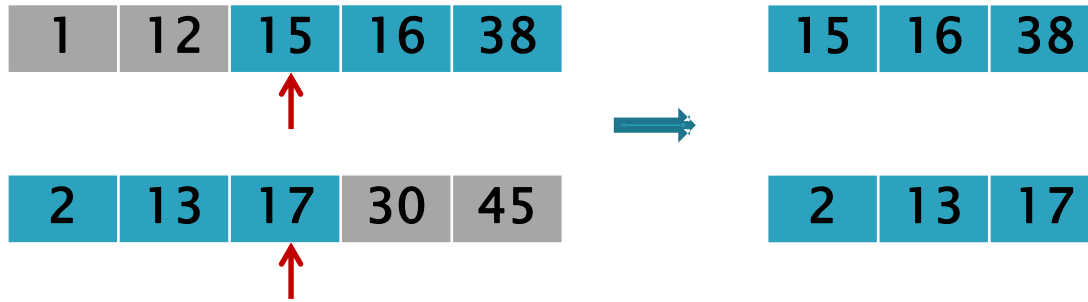
1	12	15	16	38
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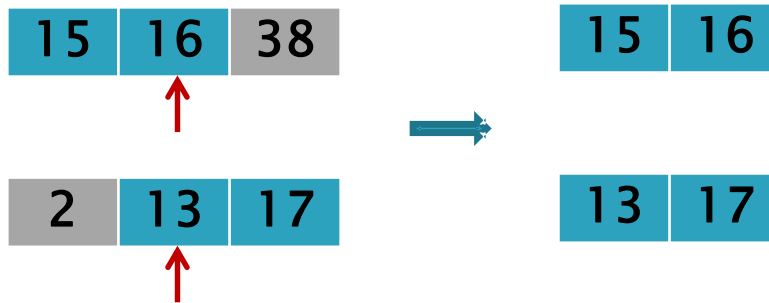
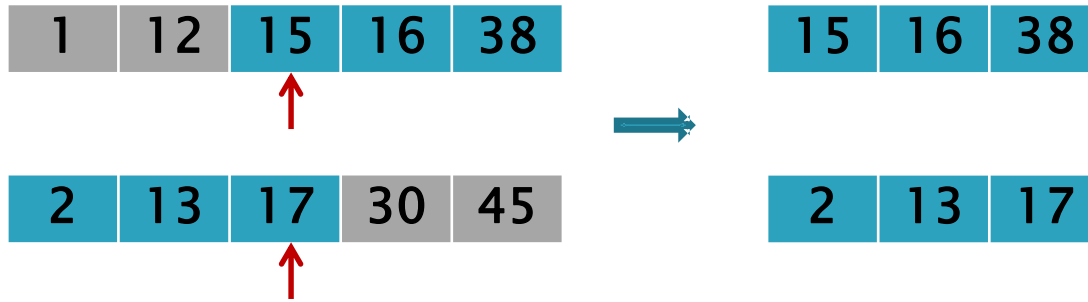


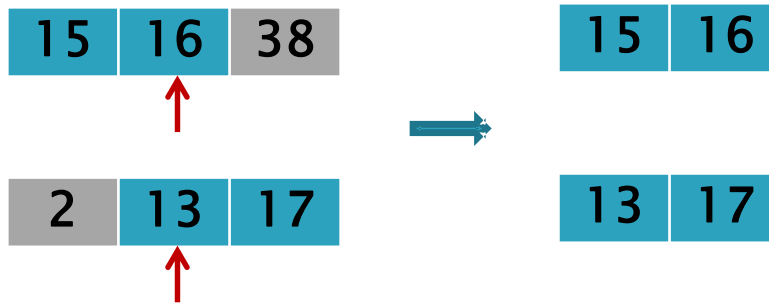
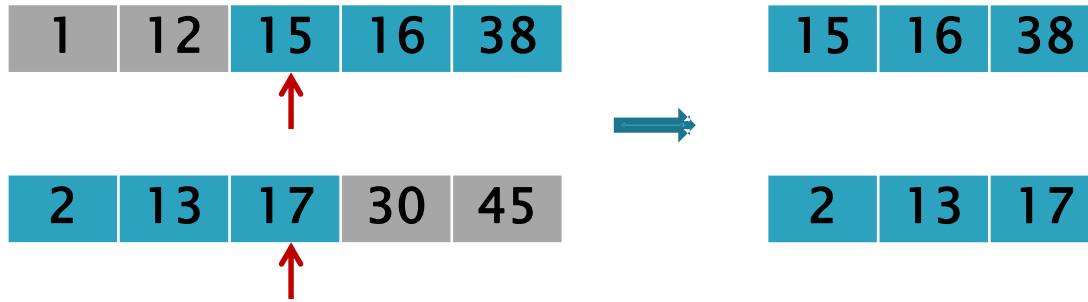
2	13	17	30	45
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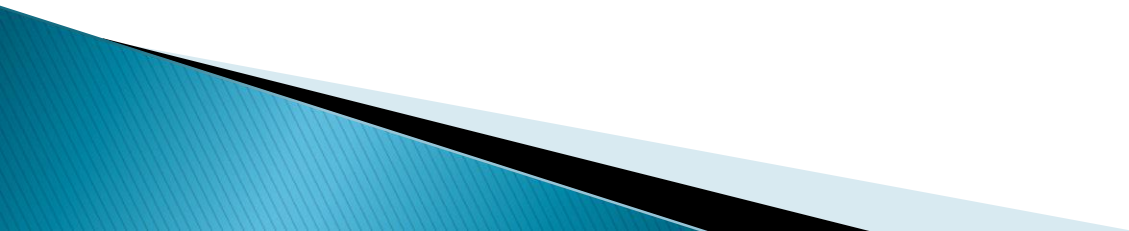






$$\begin{aligned} \text{Mediana} &= \frac{\max\{13,15\} + \min\{16,17\}}{2} = \frac{15+16}{2} \\ &= 15,5 \end{aligned}$$

Exemplul 2



1	12	15	16	38	40
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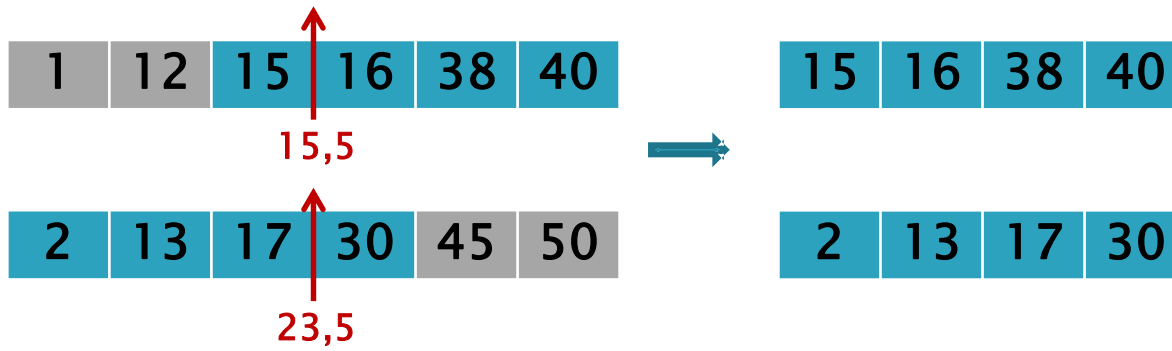
2	13	17	30	45	50
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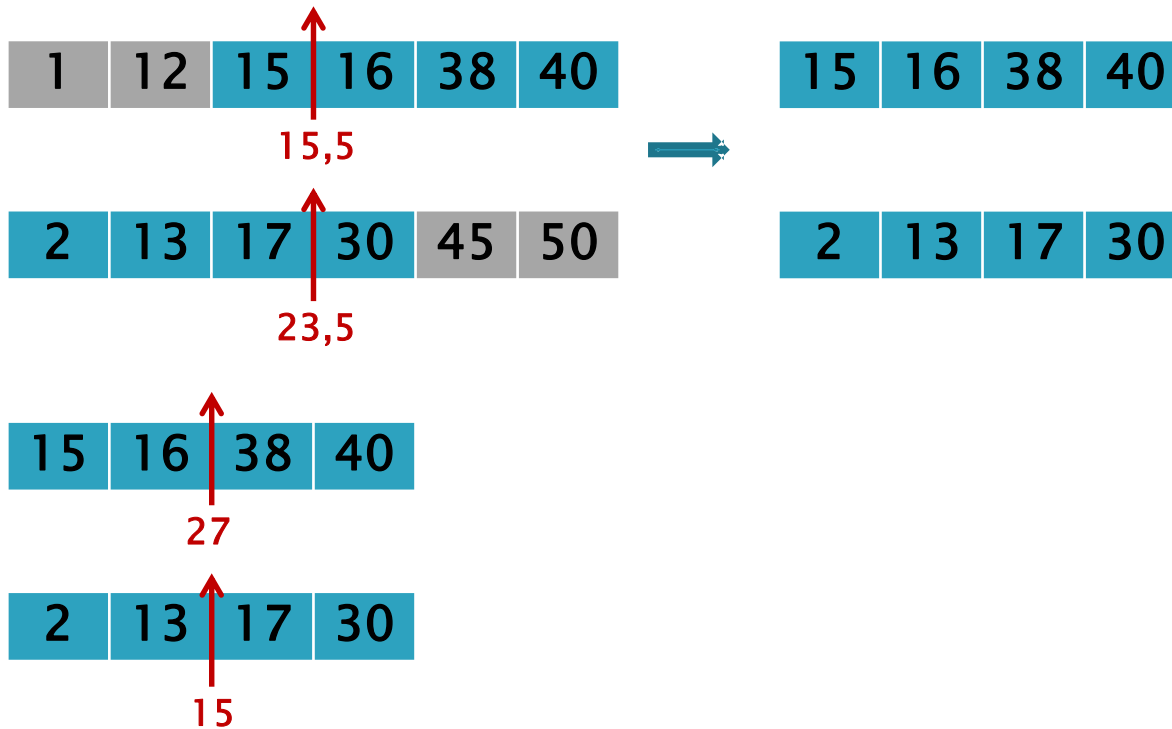
1	12	15	16	38	40
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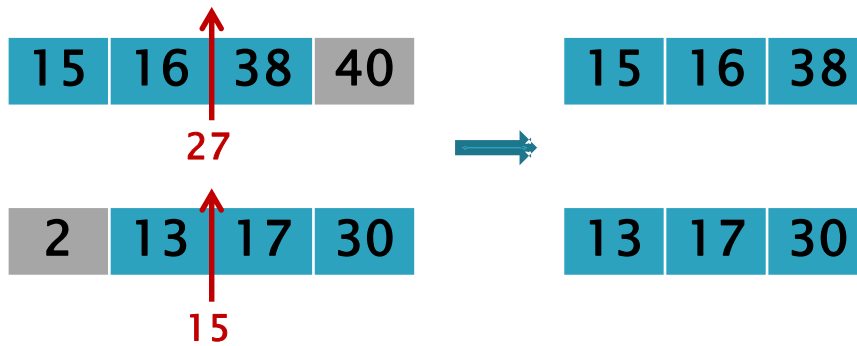
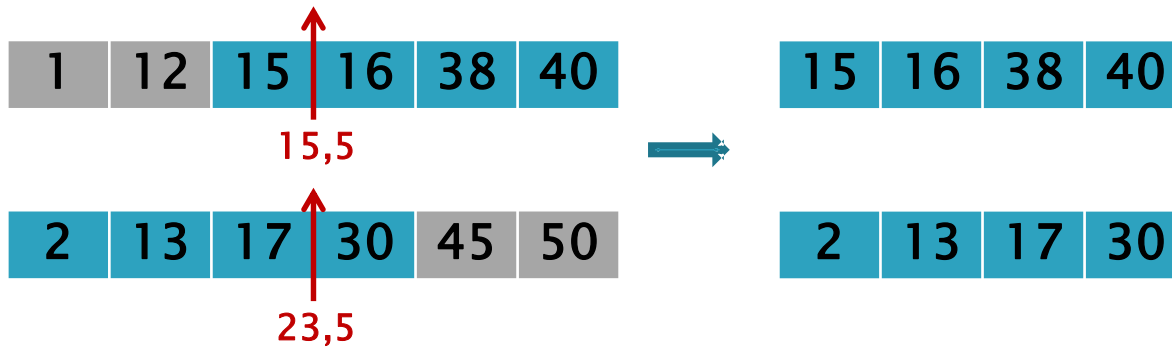
15,5

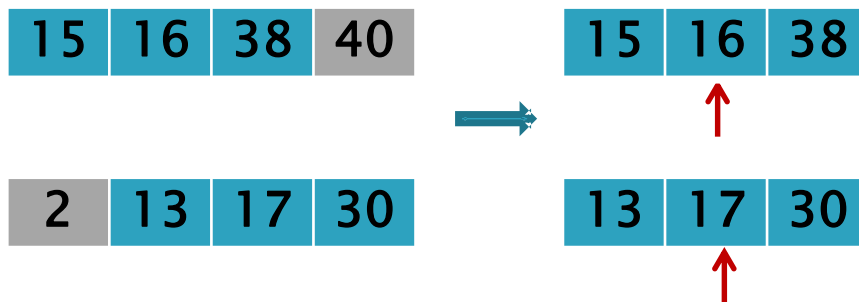
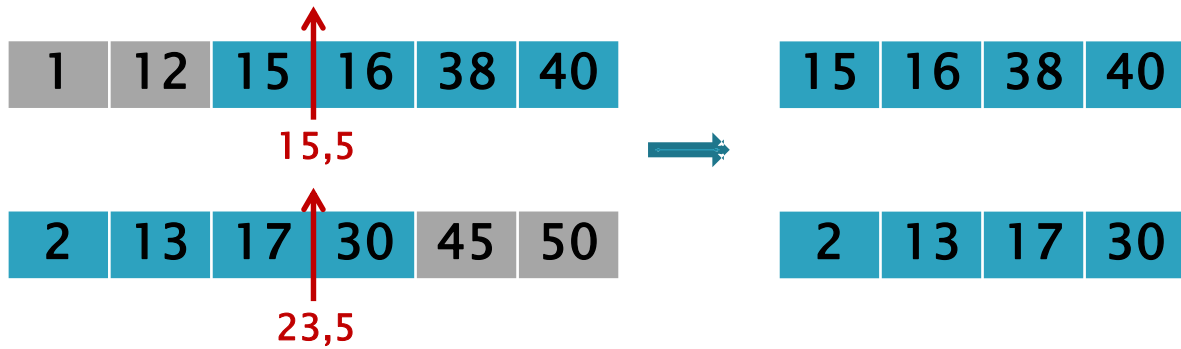
2	13	17	30	45	50
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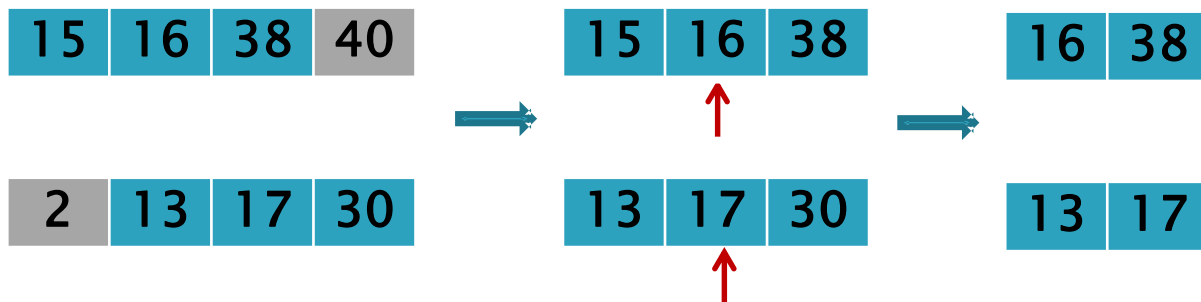
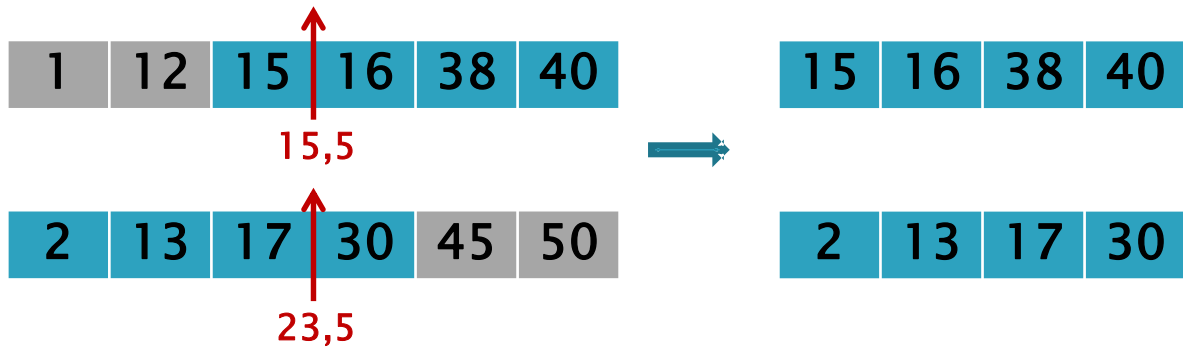
23,5

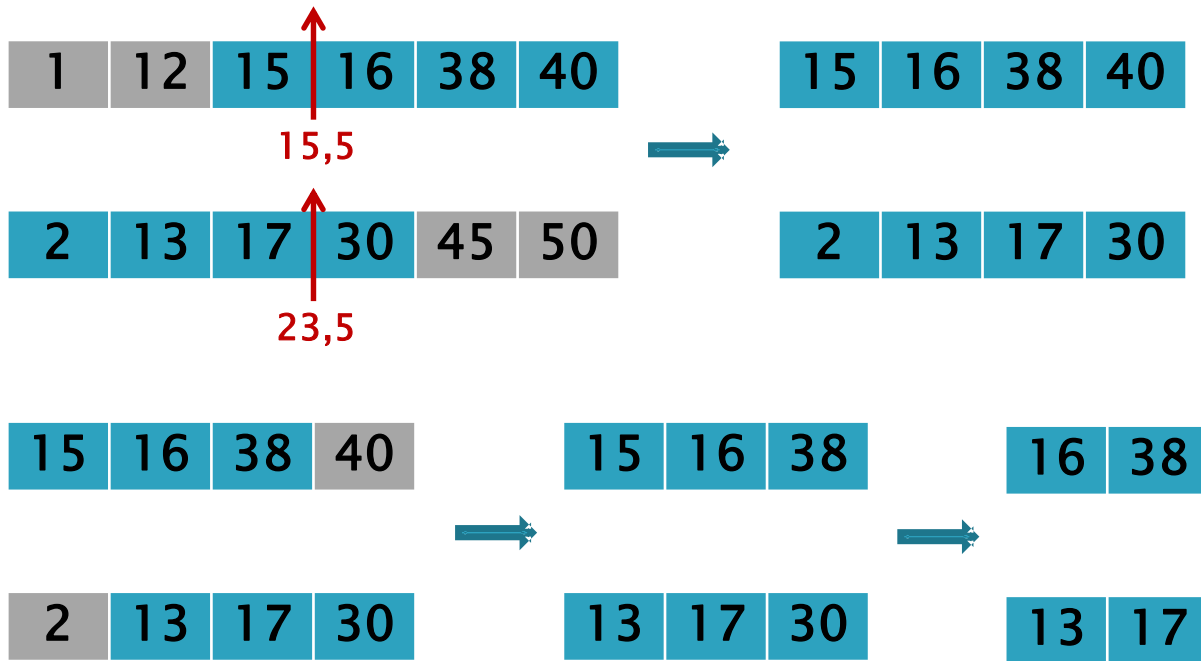












$$\begin{aligned} \text{Mediana} &= \frac{\max\{13, 16\} + \min\{17, 38\}}{2} = \frac{16 + 17}{2} \\ &= 16,5 \end{aligned}$$

Algoritm

```
mediana(v, pv, uv){  
    n = uv-pv+1  
    m =(uv+pv)/2  
    if (n%2==0)  
        return (v[m]+v[m+1])/2  
    else  
        return v[m]  
}
```



```
calculMediana(pa, ua, pb, ub){  
    n = ua-pa+1  
    if (n<=2) //rezolv direct  
        return (max(a[pa],b[pb])+min(a[ua],b[ub]))/2
```

```
calculMediana(pa, ua, pb, ub){  
    n = ua-pa+1  
    if (n<=2) //rezolv direct  
        return (max(a[pa],b[pb])+min(a[ua],b[ub]))/2  
  
    ma = mediana(a,pa,ua); //mediana lui a[pa..ua]  
    mb = mediana(b,pb,ub); //mediana lui b[pb..ub]  
  
    if(ma == mb) return ma
```

```

calculMediana(pa, ua, pb, ub){
    n = ua-pa+1
    if (n<=2) //rezolv direct
        return (max(a[pa],b[pb])+min(a[ua],b[ub]))/2
    ma = mediana(a,pa,ua); //mediana lui a[pa..ua]
    mb = mediana(b,pb,ub); //mediana lui b[pb..ub]
    if(ma == mb) return ma
    if (ma>mb)
        return calculMediana(pa, pa+n/2, pb+(n-1)/2,ub)
    else
        return calculMediana(pa+(n-1)/2, ua, pb,pb+n/2)
}

```

```

calculMediana(){
    return calculMediana(0,n-1,0,n-1)
}

```

Mediana a doi vectori sortați

- ▶ Complexitate: $O(\log n)$

Mediana a doi vectori sortați



Mai este valabilă ideea pentru vectori de lungimi diferite (reducem problema la o problemă de același tip păstrând jumătate din fiecare vector)?

