$$f(x,y) = \begin{cases} x^2 y^2 \cos \dfrac{1}{x^4+y^4} & \text{dacă } x^2+y^2 \neq 0 \\ 0 & , x = y = 0 \end{cases}$$

a) Continuitate

Obs. că f e continuă pe $\mathbb{R}^2 \setminus \{(0,0)\}$

$$\lim_{x\to 0} f(x,x) = \lim_{x\to 0} x^4 \cdot \cos \frac{1}{2x^4} = 0$$

$$\lim_{x\to 0} f(x,-x) = \lim_{x\to 0} x^4 \cdot \cos \frac{1}{2x^4} = 0$$

$$\lim_{x\to 0} f(x,x^2) = \lim_{x\to 0} x^2 \cdot x^4 \cdot \cos \frac{1}{x^4+x^8} = 0$$

Dem. că $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$

Evaluăm $|f(x,y) - 0| = |f(x,y)|$

$$0 \le |f(x,y)| = \left| x^2 \cdot y^2 \cdot \cos \frac{1}{x^4+y^4} \right| = x^2 \cdot y^2 \cdot \left| \cos \frac{1}{x^4+y^4} \right|$$

$$-1 \le \cos \frac{1}{x^4+y^4} \le 1 \quad \Rightarrow \left| \cos \frac{1}{x^4+y^4} \right| \le 1 \quad \Big| \cdot (x^2 \cdot y^2) \ge 0$$

$$0 \le x^2 \cdot y^2 \left| \cos \frac{1}{x^4+y^4} \right| \le x^2 y^2 \cdot 1$$

$$\lim_{(x,y)\to(0,0)} x^2 \cdot y^2 = 0$$

$$\begin{tikzcd} 0 \le \arrow[dr, "(x,y)\to(0,0)"'] & {|f(x,y)|} \le \arrow[d, dashed] & x^2 \cdot y^2 \arrow[dl, "(x,y)\to(0,0)"] \\ & 0 & \end{tikzcd}$$

Conform. "cleşte" avem:

$$\left.\begin{array}{l} \Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y) = 0 \\ f(0,0) = 0 \end{array}\right\} \Rightarrow f \text{ continuă în } (0,0)$$

$\Rightarrow f$ continuă pe $\mathbb{R}^2$

b)

$$\frac{df}{dx}(x,y) = 2xy^2\cos\frac{1}{x^4+y^4} + x^2\cdot y^2\cdot\left(-\sin\frac{1}{x^4+y^4}\right)\cdot\frac{-4\cdot x^3}{(x^4+y^4)^2}$$
$\forall (x,y) \neq (0,0)$

$\frac{df}{dx}(x,y)$ este continuă pe $\mathbb{R}^2 \setminus \{(0,0)\}$ ...

$$\lim_{t\to 0}\frac{f((0,0)+(t,0))-f(0,0)}{t} = \lim_{t\to 0}\frac{f(t,0)}{t} = \lim_{t\to 0}\frac{t^2\cdot 0^2 \cos\frac{1}{t^4+0^4}}{t} = 0$$

$$\Rightarrow \exists\ \frac{df}{dx}(0,0) = 0$$

Studiez continuitatea în $(0,0)$

$$\left|\frac{df}{dx}(x,y) - \frac{df}{dx}(0,0)\right| = \left|\frac{df}{dx}(x,y)\right| =$$

$$= \left|2xy^2\cos\frac{1}{x^4+y^4} + x^2y^2\cdot\left(+\sin\frac{1}{x^4+y^4}\right)\cdot\frac{4x^3}{(x^4+y^4)^2}\right| \le$$

$$\le \left|2xy^2\cdot 1 + x^2\cdot y^2\cdot\frac{4x^2\cdot|x|}{(x^4+y^4)^2}\right| \le |2xy^2| + \left|x^2y^2\cdot\frac{4x^2|x|}{(x^4+y^4)^2}\right|$$

$$\le 2y^2|x| + x^4\cdot\cancel{y^2}\cdot\cancel{4}\cdot\frac{|x|}{\cancel{2}x^2\cdot\cancel{y^2}} = 2y^2|x| + 2\cdot x^2\cdot|x| \xrightarrow[(x,y)\to(0,0)]{} 0$$

$\Rightarrow \frac{df}{dx}$ continuă în $(0,0) \Rightarrow \frac{df}{dx}$ continuă pe $\mathbb{R}^2$

$\lim_{t\to 0} \frac{f(0,t)}{t} = \lim_{t\to 0} \frac{t^2 \cdot \cos\frac{1}{t^4+0}}{t} = 0 \Rightarrow \exists \frac{df}{dy}(0,0) = 0$

c) $T(x,y) = 0$, $T:\mathbb{R}^2 \to \mathbb{R}$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-f(0,0)-T((x,y)-(0,0))}{\|(x,y)-(0,0)\|} =$$

$$= \lim_{(x,y)\to(0,0)} \frac{|f(x,y)|}{\sqrt{x^2+y^2}}$$

$$= \lim_{(x,y)\to(0,0)} \left| \frac{x^2 \cdot y^2 \cdot \cos\frac{1}{x^4+y^4}}{\sqrt{x^2+y^2}} \right|$$

$x^2+y^2 \geq x^2 \Rightarrow \sqrt{x^2+y^2} \geq \sqrt{x^2} \Rightarrow \sqrt{x^2+y^2} \geq |x|$

$\Rightarrow \frac{1}{\sqrt{x^2+y^2}} \leq \frac{1}{|x|} \quad | \cdot x^2 \cdot y^2 \geq 0$

$$\frac{x^2y^2}{\sqrt{x^2+y^2}} \leq \frac{x^2y^2}{|x|} = |x| \cdot y^2$$

$$\frac{x^2y^2}{\sqrt{x^2+y^2}} \left|\cos\frac{1}{x^4+y^4}\right| \leq |x| y^2 \left|\cos\frac{1}{x^4+y^4}\right| \leq |x| \cdot y^2 \cdot 1 = |x| \cdot y^2$$

$(x,y) \to (0,0) \Rightarrow |x| \cdot y^2 \longrightarrow 0$

$$0 \leq \left| \frac{x^2y^2\cos\frac{1}{x^4+y^4}}{\sqrt{x^2+y^2}} \right| \leq |x| \cdot y^2$$

0

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{f(x,y)-f(0,0)-T((x,y)-(0,0))}{\|(x,y)-(0,0)\|} = 0$$

$\Rightarrow f$ derivabilità in $(0,0)$