Tipuri de ex.

① Serii → convergență

Alg general:

Pas 1: Studiem absolut convergența (modul) $\sum_{n\geq 1}^{\infty} |x_n|$

Subpas 1: C. raportului / C. radicalului

Subpas 2: C. comp / C. R-D

Pas 2: Calc. $\boxed{\lim_{n\to\infty} x_n}$. Dacă este $\neq 0$ $\Rightarrow$ serie dvg.

Pas 3: C. Abel, C. seriei alternante ($\sum (-1)^n a_n$ cvg $\Leftarrow$ $a_n \downarrow 0$)

Ex 1 $\boxed{\sum_{n\geq 1} x^n \frac{1}{n^\alpha}}$ $x\in\mathbb{R}$, $\alpha\in\mathbb{R}$

St. abs cvg $\to$ $\sum_{n\geq 1} \left| x^n \frac{1}{n^\alpha} \right| = \sum_{n\geq 1} |x|^n \frac{1}{n^\alpha}$

I Dacă $|x| = 0 \Rightarrow$ s cvg (suma va fi 0)

II. Dacă $|x| \neq 0$. Fie $a_n = |x|^n \frac{1}{n^\alpha}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{|x|^{\cancel{n+1}}}{(n+1)^\alpha} \cdot \frac{n^\alpha}{\cancel{|x|^n}} = \lim_{n\to\infty} |x| \underbrace{\left(\frac{n}{n+1}\right)^\alpha}_{\downarrow\ 1} = |x|$$

i) $|x| > 1 \Rightarrow$ serie dvg ($a_n \nearrow \infty$)

ii) $|x| < 1 \Rightarrow$ serie abs cvg

iii) $|x| = 1$

$\to x = 1 \Rightarrow \sum_{n\geq 1} \frac{1}{n^\alpha} = \begin{cases} \text{cvg}, & \alpha > 1 \\ \text{dvg}, & \alpha \leq 1 \end{cases}$

$\to x = -1 \Rightarrow \sum_{n\geq 1} \frac{(-1)^n}{n^\alpha} = \begin{cases} \text{abs cvg}, & \alpha > 1 \\ ?, & \alpha \leq 1 \end{cases}$

$$\lim_{n\to\infty} \frac{1}{n^\alpha} = \begin{cases} 0, & \alpha > 0 \\ 1, & \alpha = 1 \\ \infty, & \alpha < 0 \end{cases} \Bigg| \Rightarrow s \text{ dvg}$$

Pt $1 \geq \alpha > 0 \Rightarrow \frac{1}{n^\alpha} \downarrow 0 \Rightarrow$ serie cvg

Ex 2: $\boxed{\sum_{n\geq 1} x^n \operatorname{arctg} \frac{1}{n^\alpha}}$

St abs cvg: $\sum_{n\geq 1} |x|^n \operatorname{arctg} \frac{1}{n^\alpha}$

1) Pt $|x| = 0 \Rightarrow$ s cvg

2) Pt $|x| \neq 0$: Fie $a_n = |x|^n \operatorname{arctg} \frac{1}{n^\alpha}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{|x|^{\cancel{n+1}} \operatorname{arctg} \frac{1}{(n+1)^\alpha}}{\cancel{|x|^n} \operatorname{arctg} \frac{1}{n^\alpha}} = |x|$$

$$\lim_{n\to\infty} \frac{1}{n^\alpha} = \begin{cases} 0, & \alpha > 0 \\ 1, & \alpha = 0 \\ \infty, & \alpha < 0 \end{cases}$$

$$\alpha > 0 \Rightarrow \lim_{n\to\infty} \frac{\operatorname{arctg} \frac{1}{(n+1)^\alpha}}{\operatorname{arctg} \frac{1}{n^\alpha}} = \lim_{n\to\infty} \underbrace{\frac{\operatorname{arctg} \frac{1}{(n+1)^\alpha}}{\frac{1}{(n+1)^\alpha}}}_{\downarrow\, 1} \cdot \underbrace{\frac{\frac{1}{n^\alpha}}{\operatorname{arctg} \frac{1}{n^\alpha}}}_{\downarrow\, 1} \cdot \underbrace{\frac{n^\alpha}{(n+1)^\alpha}}_{\downarrow\, 1} = 1$$

$$\alpha = 0 \Rightarrow \lim_{n\to\infty} \frac{\operatorname{arctg} 1}{\operatorname{arctg} 1} = 1$$

$$\alpha < 0 \Rightarrow \lim_{n\to\infty} \frac{\operatorname{arctg} \infty}{\operatorname{arctg} \infty} = 1$$

Revenind,

$|x| > 1 \Rightarrow$ s dvg

$|x| < 1 \Rightarrow$ s cvg (chiar absolut cvg)

$x = 1$: $\to$ $x = 1 \Rightarrow \sum_{n\geq 1}' \operatorname{arctg} \frac{1}{n^\alpha} \sim \sum_{n\geq 1} \frac{1}{n^\alpha}$, deoarece

$$\lim_{n\to\infty} \frac{\operatorname{arctg} \frac{1}{n^\alpha}}{\frac{1}{n^\alpha}} = 1 \in (0, \infty) \Rightarrow \sum_{n\geq 1} \operatorname{arctg} \frac{1}{n^\alpha} = \begin{cases} \text{cvg}, & \alpha > 1 \\ \text{dvg}, & \alpha \leq 1 \end{cases}$$

$\to$ $x = -1 \Rightarrow \sum' (-1)^n \operatorname{arctg} \frac{1}{n^\alpha} \sim \sum_{n\geq 1} (-1)^n \frac{1}{n^\alpha} = \begin{cases} \text{abs cvg}, & \alpha > 1 \\ \text{semi cvg}, & \alpha \in (1, 0) \\ \text{dvg}, & \alpha \leq 0 \end{cases}$

(2.) Cvg simplă și uniformă pt un șir de funcții

Teorie: Fie $f_n, f: A \to \mathbb{R}$

$f_n \xrightarrow{s} f \ \forall x \in A \iff \lim\limits_{n\to\infty} f_n(x) = f(x)$

$f_n \xrightarrow{u} f$ dacă $f_n \xrightarrow{s} f$ și $\sup\limits_{x\in A} |f_n(x) - f(x)| = a_n \to 0$

Ex: $f_n: [0,1] \to \mathbb{R}$ $f_n(x) = x^2(1-x)^n$

1) $\lim\limits_{\substack{n\to\infty \\ x\in[0,1]}} f_n(x) = \lim\limits_{n\to\infty} x^2(1-x)^n = 0 \Rightarrow f_n \xrightarrow{u} 0$

2) $a_n = \sup\limits_{x\in[0,1]} |f_n(x) - f(x)| = \sup\limits_{x\in[0,1]} x^2(1-x)^n = \sup\limits_{x\in[0,1]} f_n(x)$

$f_n'(x) = 2x(1-x)^n + x^2(-n)(1-x)^{n-1} = x(1-x)^{n-1}[2(1-x) - nx]$

$f_n'(x) = 0 \iff$ I $x = 0$

II $x = 1$

III $2(1-x) - nx = 0$

$\Downarrow$

$x(2+n) = 2$

$\Downarrow$

$x = \frac{2}{2+n}$

x	0		$\frac{2}{n+2}$		1
$f_n'(x)$	0	$+++$	0	$---$	0
f_n	0	$\nearrow$	$f\left(\frac{2}{n+2}\right)$	$\searrow$	0

$a_n = f\left(\frac{2}{n+2}\right) = \left(\frac{2}{n+2}\right)^2 \left(\frac{n}{n+2}\right)^n \longrightarrow 0 \Rightarrow f_n \xrightarrow{u} 0$

(3) Cont și deriv unei funcții

$f: \mathbb{R} \to \mathbb{R}$ $f(x) = \begin{cases} x^3 + x & , x \in \mathbb{Q} \cup (10, 20) = A \\ 2x^2 & , \text{în rest} = B \end{cases}$

$A' = \mathbb{R}$ $B' = (-\infty, 10] \cup [20, \infty)$

$f|_{(10,20)} = x^3+x \Rightarrow f$ cont și deriv pe $(10,20)$

$\downarrow$ mult deschisă

fie $a \in \mathbb{R} \setminus [10,20) \Rightarrow a \in A'$ și $a \in B'$

$$\left.\begin{array}{l}\lim\limits_{\substack{x\to a\\ x\in A\\ \Updownarrow\\ x\in \mathbb{Q}}} f(x) = \lim\limits_{x\to a} x^3+x = a^3+a \\ \lim\limits_{\substack{x\to a\\ x\in B}} f(x) = \lim\limits_{x\to a} 2x^2 = 2a^2\end{array}\right| \Rightarrow f(a) \in \{a^3+a,\ 2a^2\}$$

f cont în $a \Leftrightarrow a^3+a = 2a^2 \Rightarrow a(a-1)^2 = 0 \Leftrightarrow a = 0$ sau $a = 1$

Pt deriv: pe $\mathbb{R}\setminus(\{0,1\}\cup(10,20))$ – f disc $\Rightarrow$ nu e deriv

$$\left.\begin{array}{l}\lim\limits_{\substack{x\to 0\\ x\in \mathbb{Q}}} \frac{f(x)-f(0)}{x-0} = \lim\limits_{\substack{x\to 0\\ x\in \mathbb{Q}}} \frac{x^3+x}{x} = \lim\limits_{x\to 0} x^2+1 = 1 \\ \lim\limits_{\substack{x\to 0\\ x\notin \mathbb{Q}}} \frac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0} 2x = 0\end{array}\right| \begin{array}{l} 1\neq 0 \\ \Rightarrow f \text{ nu e} \\ \text{deriv în } 0\end{array}$$

Se procedează la fel pentru 1.

④ **Cont și deriv unei funcții de mai multe variabile**

$$f:\mathbb{R}^2 \to \mathbb{R} \quad f(x,y) = \begin{cases} \dfrac{x^6y^5}{x^{10}+y^{10}}, & x^2+y^2 \neq 0 \\ 0, & x=y=0 \end{cases}$$

f cont pe $\mathbb{R}^2 \setminus \{(0,0)\}$

$$\lim_{\substack{x\to 0\\ y=ax}} f(x,y) = \lim_{\substack{x\to 0\\ y=ax}} \frac{x^6\cdot a^5x^5}{x^{10}+a^{10}x^{10}} = \lim_{x\to 0} \frac{xa^5}{1+a^{10}} = 0$$

Din moment ce nu depinde de param „a" există șanse ca funcția să fie continuă.

$V_1: |f(x,y) - f(0,0)| = \left|\frac{x^6 y^5}{x^{10}+y^{10}}\right| = \left|\frac{x^5 y^5}{x^{10}+y^{10}}\right| |x| \le \frac{1}{2} \cdot |x| \xrightarrow[x\to 0]{} 0$

$V_2: \left|\frac{x^6 y^5}{x^{10}+y^{10}}\right| \le \left|\frac{x^{10}}{x^{10}+y^{10}}\right|^{\frac{6}{10}} \left|\frac{y^{10}}{x^{10}+y^{10}}\right|^{\frac{5}{10}} |x^{10}+y^{10}|^{\frac{6}{10}+\frac{5}{10}-1} \le$

$|x^{10}+y^{10}|^{\frac{1}{10}} \xrightarrow[y\to 0]{x\to 0} 0$

Așadar, f cont în $(0,0)$

f - der pe $\mathbb{R}^2 \setminus \{(0,0)\}$

f - der în $(0,0) \Leftrightarrow \exists T \in L(\mathbb{R}^2, \mathbb{R})$ aî

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{f(x,y) - \overset{=0}{f(0,0)} - T(x,y)}{\sqrt{x^2+y^2}} = 0$$

$T = ax + by$

$a = \frac{\partial f}{\partial x}(0,0) = 0$

$b = \frac{\partial f}{\partial y}(0,0) = 0$

$\Rightarrow T(x,y) = 0$

Deci limita devine $\lim\limits_{\substack{x\to 0\\ y = ax\\ x>0}} \frac{x^6 a^5 x^5}{x^{10}(1+a^{10}) x\sqrt{1+a^2}} = \frac{a^5}{(1+a^{10})\sqrt{1+a^5}}$

deci f nu e deriv în $(0,0)$, deoarece lim depinde de a

(5) $\boxed{\text{Serii de funcții}}$

Ex: $s(x) = \sum\limits_{n\ge 1} \frac{1}{n^2+x^n}$ $\quad s:(0,1) \to \mathbb{R}$

$\left|\frac{1}{n^2+x^n}\right| \le \frac{1}{n^2}$, $\sum\limits_{n\ge 1} \frac{1}{n^2}$ - cvg $\Rightarrow s(x)$ - unif cvg.

$$s_1(x) = \sum_{n\geq 1} \left(\frac{1}{n^2+x}\right)' = \sum_{n\geq 1} -\frac{1}{(n^2+x)^2}$$

$$\left| -\frac{1}{(n^2+x)^2} \right| \leq \frac{1}{n^4}, \quad \sum_{n\geq 1} \frac{1}{n^4} \text{ - cvg} \Rightarrow s_1 \text{ unif cvg}, \ s_1 = s'$$

$$s_2 = \sum_{n\geq 1} \left(\frac{1}{n^2+x}\right)'' = \sum_{n\geq 1}' \frac{2}{(n^2+x)^3}$$

$$\left| \frac{2}{(n^2+x)^3} \right| \leq \frac{1}{n^6}, \quad \sum_{n\geq 1}' \frac{1}{n^6} \text{ - cvg} \Rightarrow s_2 \text{ unif cvg}, \ s_2 = s''$$

În general:

$$s_K = \sum_{n\geq 1}' \left(\frac{1}{n^2+x}\right)^{(K)} = \sum_{n\geq 1}' \left((n^2+x)^{-1}\right)^{(K)} = \sum_{n\geq 1}' \frac{(-1)^k k!}{(n^2+x)^{k+1}}$$

$$\left| \frac{(-1)^k k!}{(n^2+x)^{k+1}} \right| \leq \frac{k!}{n^{2k+2}}, \quad \sum_{n\geq 1} \frac{k!}{n^{2k+2}} \text{ - cvg} \Rightarrow s_k \text{ unif cvg}, \ s_k = s^{(k)}$$

6 Dezvoltare în serie de puteri a unei funcții

$f: \mathbb{R} \to \mathbb{R} \quad f(x) = \cos x$

$f'(x) = -\sin x \quad f''(x) = -\cos x \quad f'''(x) = \sin(x)$

$f^{(n)}(x) = \cos\left(x + \frac{n\pi}{2}\right)$. Pt $a = 0$

$$\boxed{f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} \cdot x^k + \frac{f^{(n+1)}(\alpha)}{(n+1)!} \cdot x^{n+1}}$$

$$= \sum_{k=0}^{n} \frac{\cos\left(\frac{\pi}{2}k\right)}{k!} \cdot x^k + \underbrace{\frac{\cos\left(\alpha + \frac{\pi}{2}(n+1)\right)}{(n+1)!} \cdot x^{n+1}}$$

$$\left| R_{f,n,0}(x) \right| \leq \frac{M^{n+1}}{(n+1)!} \xrightarrow{n\to\infty} 0$$

$$|x| < M$$

$$\cos(x) = \sum_{n\geq 0}' \frac{\cos\left(\frac{\pi}{2}n\right)}{n!} \cdot x^n = \sum_{n\geq 0} \frac{\cos \pi n}{(2n)!} \cdot x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} = \ldots$$

7) Extreme locale pentru funcții de mai multe var.

$f:(\mathbb{R}^*)^2 \to \mathbb{R} \quad f(x,y) = \frac{1}{x} + \frac{1}{y} + xy$

$\frac{\partial f}{\partial x} = -\frac{1}{x^2} + y$

$\frac{\partial f}{\partial y} = -\frac{1}{y^2} + x.$

$$\begin{cases} \frac{\partial f}{\partial x} = 0 \Leftrightarrow -\frac{1}{x^2} + y = 0 \Leftrightarrow 1 = yx^2 \\ \frac{\partial f}{\partial y} = 0 \Leftrightarrow -\frac{1}{y^2} + x = 0 \Leftrightarrow 1 = xy^2 \end{cases} \Rightarrow x = y \Rightarrow x^3 = 1 \Rightarrow x = y = 1$$

$$f'' = \begin{pmatrix} \frac{2}{x^3} & 1 \\ 1 & \frac{2}{y^3} \end{pmatrix} \qquad f''(1,1) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\left.\begin{array}{l} \Delta_1 = 2 > 0 \\ \Delta_2 = 4-1 = 3 > 0 \end{array}\right| \Rightarrow (1,1) \text{ pct de minim}$$

În general:

- → dacă determinații $\Delta_1, \Delta_2 \dots \Delta_n$ au semnele $+ + + + \Rightarrow$ pct minim
- → dacă det au semnele $- + - + - + \Rightarrow$ pct maxim (trebuie să înceapă neapărat cu minus)
- → altceva $\Rightarrow$ pct șa

8) Integrala curbilinie de primul/al doilea tip

Fie $\gamma: [0, 2\pi] \to \mathbb{R}^2 \quad \gamma(t) = (r\cos t, r\sin t)$

$\|\gamma'(t)\| = \sqrt{r^2\cos^2 t + r^2\sin^2 t} = r$

(În general $\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}$, $x = (x_1 \dots x_n) \in \mathbb{R}^n$)

Fie $f: \mathbb{R}^2 \to \mathbb{R} \quad f(x,y) = x^2y^2$

$$\int_\gamma f\,dt = \int_0^{2\pi} f(\gamma(t))\,\|\gamma'(t)\|\,dt = \int_0^{2\pi} r^2\cos^2 t \cdot r^2\sin^2 t \cdot r\,dt$$

$$= r^5 \int_0^{2\pi} \frac{4\cos^2 t \sin^2 t}{4}\,dt = r^5 \int_0^{2\pi} \frac{(\sin 2t)^2}{4}\,dt = r^5 \int_0^{2\pi} \frac{1-\cos 4t}{8}\,dt =$$

$$= r^5 \int_0^{2\pi} \frac{1}{8} - \frac{\cos 4t}{8} = r^5 \cdot \frac{2\pi}{8} - r^5 \frac{\sin 4t}{4\cdot 8}\Big|_0^{2\pi} = \frac{r^5 \pi}{4}$$

$\boxed{\cos 2x = 1 - 2\sin^2 x}$ $\Rightarrow$ $\sin^2 x = \frac{1-\cos 2x}{2}$

9. Int duble cu sau fără sch de var.

Ex1: $I = \iint_A xy\,dxdy \quad A = \{x^2 \le y \le 3x-2\}$

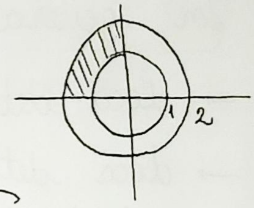
$\cancel{I = \int}$ $\quad x^2 - 3x + 2 \le 0$

$(x-1)(x-2) \le 0 \Rightarrow x \in [1,2]$

Deci $I = \int_1^2 \left(\int_{x^2}^{3x-2} xy\,dy \right) dx = \int_1^2 \frac{xy^2}{2}\Big|_{y=x^2}^{y=3x-2} dx =$

$$= \int_1^2 \frac{x(3x-2)^2}{2} - \frac{x^5}{2}\,dx = \frac{1}{2}\int_1^2 9x^3 - 12x^2 + 4x - x^5\,dx = \dots$$

Ex2: cu schimbare de var



$B = \{(x,y) \mid 1 \le x^2+y^2 \le 4\},\ x \le 0,\ y \ge 0$

Fie $\varphi : [0,\infty) \times \mathbb{R} \to \mathbb{R}^2 \quad \varphi(r,\theta) = (\overbrace{r\cos\theta}^{f_1}, \overbrace{r\sin\theta}^{f_2})$

Pentru mulțimea noastră B, $r \in [1,2]$, $\theta \in (\frac{\pi}{2}, \pi)$

(În general $r = \sqrt{x^2+y^2}$. Dacă nu iese din desen se rez ecuațiile)

$$\iint_B e^{x^2+y^2}dxdy = \int_A e^{(r\cos\theta)^2+(r\sin\theta)^2} |\det \varphi'|\,dr\,d\theta = \int_0^1 \left(\int_{\pi/2}^{\pi} e^{r^2}\cdot r\,d\theta\right) dr$$

$$= \int_0^1 \frac{\pi}{2} e^{r^2}\cdot r\,dr = \frac{\pi}{4} e^{r^2}\Big|_0^1 = \frac{\pi}{4}(e-1)$$

$$\varphi' = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \Rightarrow \det|\varphi'| = r\cos^2\theta + r\sin^2\theta = r$$

(aici $x = r$ și $y = \theta$)