Seminar 2 - 143

Ex. 1: $\varphi(n) = n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_k}\right)$,

unde $n = p_1^{a_1} \cdots p_k^{a_k}$, φ ind. lui Euler, p_i nr. prime

$a_i \geq 1$.

Rez:

$\varphi(n) = |\{a \in \mathbb{N} \mid a \leq n, (a,n) = 1\}|$

$(a,n) = \text{cmmdc}(a,n)$

$= n - |\underbrace{\{b \in \mathbb{N}^* \mid b \leq n, (b,n) \neq 1\}}_{B}|$

$A_i = \{a \in \mathbb{N}^* \mid a \leq n, p_i \mid a\}$.

$(a \,\vdots\, p_i)$

$|A_i| = \left[\frac{n}{p_i}\right] = \frac{n}{p_i}$

$|A_i| =$ nr. multiplilor de p_i din $\{1,\dots,n\}$

$B = A_1 \cup A_2 \cup \dots \cup A_k$

$\varphi(n) = n - |B| = n - |A_1 \cup A_2 \cup \dots \cup A_k| \overset{P.I.E.}{=}$

$= n - \left(\sum_{i=1}^{k} |A_i| - \sum_{1\leq i<j\leq k} |A_i \cap A_j| + \dots + (-1)^{k-1} |A_1 \cap \dots \cap A_k|\right)$

$= n - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| - \dots + (-1)^k |A_1 \cap \dots \cap A_k|$

$|A_i| = \frac{n}{p_i}$

$|A_i \cap A_j| = \frac{n}{p_i p_j}$

$\vdots$

$|A_1 \cap A_2 \cap \dots A_k| = \frac{n}{p_1 p_2 \cdots p_k}$

$\varphi(n) = n - \sum_i \frac{n}{p_i} + \sum \frac{n}{p_i p_j} - \dots + (-1)^k \frac{n}{p_1 \cdots p_k} =$

$= n\left(1 - \sum \frac{1}{p_i} + \sum \frac{1}{p_i p_j} - \dots + (-1)^k \frac{1}{p_1 \cdots p_k}\right)$

$\overset{?}{=} n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_k}\right)$

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$$\left(1-\sum\frac{1}{p_i}+\sum\frac{1}{p_ip_j}-\dots+(-1)^k\frac{1}{p_1\dots p_k}\right)=\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\dots\left(1-\frac{1}{p_k}\right)$$

Dem. prin. ind.

$k=1$: $1-\frac{1}{p_1}=1-\frac{1}{p_1}$

$k=2$: $\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)=\left[1-\left(\frac{1}{p_1}+\frac{1}{p_2}\right)+\frac{1}{p_1\cdot p_2}\right]$ OK.

Ex. 2: Să se studieze inj., surj. și bij. funcției $f:\mathbb{R}\to\mathbb{R}$, în funcție de parametrul real m:

$$f(x)=\begin{cases} x^2+m, & x\le 0 \\ mx, & x\in(0,1) \\ m^2-x, & x\ge 1\end{cases}$$

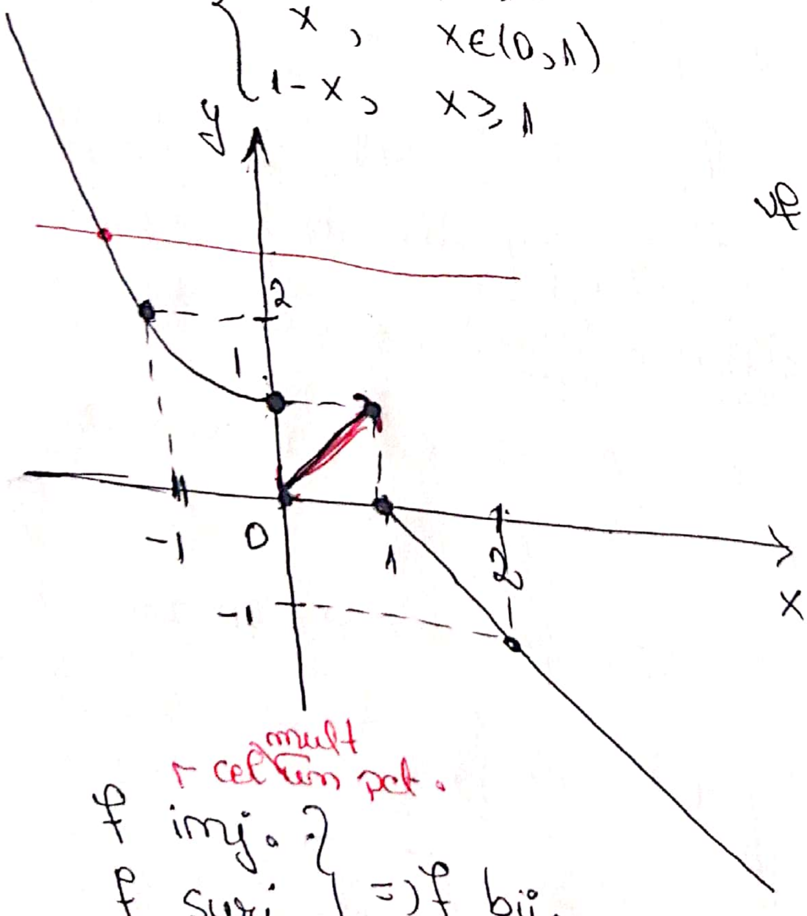
$[m,+\infty)$

$(0,m)$ sau $(m,0)$

$(-\infty,m^2-1]$

Rez: $m=1$

$$f(x)=\begin{cases} x^2+1, & x\le 0 \\ x, & x\in(0,1) \\ 1-x, & x\ge 1\end{cases}$$



vf. parabolei

$x_v=-\frac{b}{2a}=0$

$\left.\begin{array}{l} f \text{ inj.} \\ f \text{ surj}\end{array}\right\} \Rightarrow f \text{ bij.}$

cel mult un pct.

cel puțin un pct.

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Ex. 3: Fie M o mulțime și $A, B \subseteq M$. Definim

$f: \mathcal{P}(M) \to \mathcal{P}(A) \times \mathcal{P}(B)$, $f(X) = (X \cap A, X \cap B)$.

Arătați că:

a. f inj $\Leftrightarrow A \cup B = M$.

b. f surj $\Leftrightarrow A \cap B = \emptyset$

c. f bij $\Leftrightarrow A = C_M B$. În acest caz, aflați f^{-1}.

Rez:

a. "$\Rightarrow$" f inj.

Pp. că $A \cup B \neq M$

$A \cup B \subsetneq M \Rightarrow (\exists)\, x \in M$, $x \notin A \cup B$.

$\{x\} \in \mathcal{P}(M)$

$f(\{x\}) = (\{x\} \cap A, \{x\} \cap B) = (\emptyset, \emptyset)$

$f(\emptyset) = (\emptyset, \emptyset)$

$\Rightarrow f$ nu este inj. ⨳

"$\Leftarrow$" $A \cup B = M$

Pp. că f nu este inj $\Rightarrow \exists\, X_1 \neq X_2 \in \mathcal{P}(M)$ a.î.

$f(X_1) = f(X_2) \Rightarrow (X_1 \cap A, X_1 \cap B) = (X_2 \cap A, X_2 \cap B)$

$X_1 \neq X_2 \Leftrightarrow (X_1 \setminus X_2 \neq \emptyset$ sau $X_2 \setminus X_1 \neq \emptyset)$

(Ex: $X_1 = \{1, 2\}$, $X_2 = \{1, 3\}$, $A = \{1\}$)

Pp. că $X_1 \setminus X_2 \neq \emptyset \Rightarrow (\exists)\, x \in X_1$, $x \notin X_2$.

$X_1 \cap A = X_2 \cap A$, $x \in X_1$, $x \notin X_2$ $\Rightarrow x \notin A$

[Dacă $x \in A$, $x \in X_1 \Rightarrow x \in A \cap X_1 = X_2 \cap A \Rightarrow x \in X_2$ ⨳]

La fel, $X_1 \cap B = X_2 \cap B$, $x \in X_1$, $x \notin X_2$ $\Rightarrow x \notin B$

$x \notin A$, $x \notin B \Rightarrow x \notin A \cup B = M$ ⨳.

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b. „$\Rightarrow$" f surj.

Pp. că $A \cap B \neq \emptyset \Rightarrow (\exists)\ x \in A \cap B$.

f surj $\Leftrightarrow Imf = \mathcal{P}(A) \times \mathcal{P}(B)$

? $f(\{x\}) = (\{x\} \cap A, \{x\} \cap B) = (\{x\}, \{x\})$

Vrem $(C, D) \in (\mathcal{P}(A) \times \mathcal{P}(B)) \setminus Imf$

$(\{x\}, \emptyset) \notin Imf$

$(\exists) X \subseteq M$, $f(X) = (\{x\}, \emptyset)$

$\Rightarrow \quad \left.\begin{array}{l} X \cap A = \{x\} \Rightarrow x \in A,\ x \in X \\ X \cap B = \emptyset \end{array}\right\} \Rightarrow x \notin B$ absurd

„$\Leftarrow$" $A \cap B = \emptyset$

(Pp. că f nu este surj. $\Rightarrow (C, D) \in (\mathcal{P}(A) \times \mathcal{P}(B)) \setminus Imf$)

Vrem să arătăm că f este surj.

Fie $(C, D) \in \mathcal{P}(A) \times \mathcal{P}(B)$. Vrem $X \in \mathcal{P}(M)$ a.î.

$f(X) = (C, D)$

$(X \cap A, X \cap B) = (C, D)$

$X \cap A = C \qquad C \subseteq A$

$X \cap B = D \qquad D \subseteq B$

$f(C \cup D) = ((C \cup D) \cap A, (C \cup D) \cap B) =$

$= (\underbrace{(C \cap A)}_{=C} \cup \underbrace{(D \cap A)}_{\emptyset}, \underbrace{(C \cap B)}_{\emptyset} \cup \underbrace{(D \cap B)}_{D}) = (C, D)$

$D \cap A \subseteq B \cap A = \emptyset$

Luăm $X = C \cup D$.

Exemplu: $M = \{1,2,3,4,5\}$, $A = \{1,2\}$, $B = \{5\}$

$A \cap B = \emptyset$, $A \cup B = \{1,2,5\}$

$f(A \cup B) = (\{1,2\}, \{5\}) = (A, B)$

$f(M) = (A, B)$

Ex. 4: Fie $f: M \to M$, M mult. finită. Arătați că UAE (următ. afirm. sunt echivalente):

a. f inj.

b. f surj. (f inj $\Leftrightarrow$ surj $\Leftrightarrow$ bij)

c. f bij.

Rez: inj $\Leftrightarrow$ surj.

"$\Rightarrow$"

$$\left.\begin{array}{l} f \text{ inj.} \\ |M| = n \end{array}\right\} \Rightarrow \left.\begin{array}{l} |\operatorname{Im} f| = n \\ \operatorname{Im} f \subseteq M \\ |M| = n \ (\hookrightarrow \text{codomeniu}) \end{array}\right\} \Rightarrow \operatorname{Im} f = M \Rightarrow f \text{ surj.}$$

"$\Leftarrow$" f surj. $\Rightarrow$ $|\operatorname{Im} f| = n$, $|M| = n$ ($\hookrightarrow$ (domeniu)) $\Rightarrow$ f inj.

Ex. 5: Fie M, N două mult. finite, $|M| = m$, $|N| = n$. Se cere: (Să se studieze în fct. de m și n)

a. nr. funcțiilor $f: M \to N$ $\quad n^m = \underbrace{n \cdot \ldots \cdot n}_{m}$

b. nr. fct. inj. $f: M \to N$ $(m \leq n)$ $\quad A_n^m$

c. — " — surj — " — $(m \geq n)$

d. — " — bij — " — $(m = n)$ $\quad P_n$ $\quad A_n^m$ $\quad B$

P.i.e. (n^m − |{fct. care nu sunt surj}|)

Hint: $B = A_1 \cup \ldots \cup A_n$, $A_i = \{f: M \to N \mid i \notin \operatorname{Im} f\}$ $= \{f: M \to N \setminus \{i\}\}$

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