9) Amintim identitatea lui Hermite:
Lista 1

$(\forall) x \in \mathbb{R}, (\forall) n \in \mathbb{N}^*, \sum_{k=0}^{n-1} [x+\frac{k}{n}] = [nx]$.

$n=2 \Rightarrow (\forall) x \in \mathbb{R}, [x] + [x+\frac{1}{2}] = [2x]$

$n=3 \Rightarrow (\forall) x \in \mathbb{R}, [x] + [x+\frac{1}{3}] + [x+\frac{2}{3}] = [3x]$

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$f: \mathbb{N} \to \mathbb{N}$

$f(n) = [\frac{n}{p}] + [\frac{n+1}{p}] + [\frac{n+2}{p}]$

$p = ?$ a.î. f bij.

I. $p = 1 \Rightarrow f(n) = [n] + [n+1] + [n+2] =$

$= [n] \cdot 3 + 1 + 2$

$= 3[n] + 3$

$= 3n + 3 \Rightarrow f$ nu e surj (nu atinge valoarea $y=0$, de exemplu)

II $p=2 \Rightarrow f(n) = \underbrace{\left[\frac{n}{2}\right] + \left[\frac{n}{2}+\frac{1}{2}\right]} + \left[\frac{n}{2}+1\right]$

Aplic Hermite pt $x=\frac{n}{2}$

$$f(n) = \left[2\cdot\frac{n}{2}\right] + \left[\frac{n}{2}\right]+1$$
$$= n+1+\left[\frac{n}{2}\right] \geq 1, (\forall)\, n$$

f nu e surj ($f(n)=0$ nu are sol.)

III $\cancel{f(n)}$ $p=3 \Rightarrow f(n) = \underbrace{\left[\frac{n}{3}\right]+\left[\frac{n}{3}+\frac{1}{3}\right]+\left[\frac{n}{3}+\frac{2}{3}\right]}$

Aplic Hermite pt $x=\frac{n}{3}$

$$f(n)=\left[3\cdot\frac{n}{3}\right]=n=id_{\mathbb{N}} \Rightarrow \text{DA},$$

f bij pt $p=3$

IV $p\geq 4 \Rightarrow f(n)=$ $\cancel{\left[\frac{n}{4}\right]}$

$\cancel{\left[\frac{n}{p}\right]+2\left[\frac{n+1}{p}\right]}$

$\left[\frac{n}{p}\right]+\left[\frac{n}{p}+\frac{1}{p}\right]+\left[\frac{n}{p}+\frac{2}{p}\right]$

$$f(n) = 1 + \left[1+\frac{1}{n}\right] + \left[1+\frac{2}{n}\right] = \\ = 3 + \left[\frac{1}{n}\right] + \left[\frac{2}{n}\right] \quad \Rightarrow \underline{f(n)=3}$$

$n > 2$

$$f(n+1) = \left[\frac{n}{n}+\frac{1}{n}\right] + \left[\frac{n}{n}+\frac{2}{n}\right] + \left[\frac{n}{n}+\frac{3}{n}\right] = \\ = 3 + \underbrace{\left[\frac{1}{n}\right]}_{0} + \underbrace{\left[\frac{2}{n}\right]}_{0} + \underbrace{\left[\frac{3}{n}\right]}_{0}$$

pt că $n > 4$

$\Rightarrow f(n+1) = f(n) = 3 \Rightarrow \underline{(\forall)\, n \geq 4,\ f \text{ nu e bij}}$

Sol: $n = 2$

4) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x + m^2\{x\}$, $m \in \mathbb{R}$

Liste 1 $m = ?$ a.î. f inj.

$x = [x] + \{x\} \Rightarrow \{x\} = x - [x]$

$f(x) = x + m^2(x - [x])$

$= x(1+m^2) - m^2[x]$

Obs. că pt $m = 0$, $f(x) = x$ care e inj.

Pt $m \neq 0$:

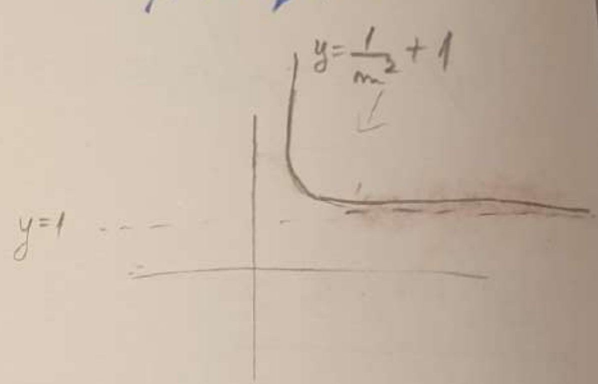
Căutăm valorile lui $m \in \mathbb{R}^*$ pt care f nu e inj.

Adică, căutăm $m \in \mathbb{R}^*$ a.î. $\exists\, x, y \in \mathbb{R}, x \neq y, f(x) = f(y)$.

$\Rightarrow \exists\, x \neq y$ a.î. $x(1+m^2) - m^2[x] = y(1+m^2) - m^2[y]$

$$(x-y)(1+m^2) = m^2([x]-[y]) \quad /:(x-y) \neq 0 \quad /:m^2 \neq 0$$

$$\frac{1+m^2}{m^2} = \frac{[x]-[y]}{x-y}$$



Notez $g = \frac{1+m^2}{m^2} = 1 + \frac{1}{m^2}$

A căuta $m \in \mathbb{R}^*$ înseamnă a căuta $g \in (1, +\infty)$.

$\Rightarrow g = \frac{[x]-[y]}{x-y} \in (1, +\infty)$

Fie $x \in [0, 1)$, ~~$y \in [1, 2)$~~ $y = 1$ $\Rightarrow [x] = 0, [y] = 1$

~~$g = \frac{-1}{x-y} = \frac{1}{y-x}$~~

~~$1 \leq y < 2$~~

$\cancel{1 \leq y < 2 \quad (1)}$

$0 \leq x < 1 \; /\cdot(-1)$

$-1 < -x \leq 0 \quad (2)$

$(1)+(2)$

$0 < y - x < 2$

$$\Rightarrow g = \cdot \frac{-1}{x-1} = \frac{1}{1-x} \; (*)$$

Fie $g : [0;1) \to \mathbb{R},\ g(t) = \frac{1}{1-t}$

Dem. că g este strict cresc: $(\forall) x_1, x_2 \in [0;1)$ cu $x_1 < x_2 \Rightarrow g(x_1) < g(x_2)$

$$\text{Fie } x_1 < x_2 \Rightarrow \frac{1}{1-x_1} \overset{?}{<} \frac{1}{1-x_2} \; /\cdot (1-x_2)(1-x_1) > 0$$

$$1 - x_2 \overset{?}{<} 1 - x_1$$

$$x_1 \overset{?}{<} x_2 \quad \text{DA} \Rightarrow \underline{g \text{ strict cresc}}$$

(sau se putea deriva)

$$\lim_{t \to 0} g(t) = g(0) = \frac{1}{1} = 1$$

$$\lim_{\substack{t \to 1 \\ t<1}} g(t) = \lim_{t \to 1} \frac{1}{1-t} = \frac{1}{+0} = +\infty$$

$$\underset{g \text{ strict cresc.}}{\Rightarrow} \underline{\operatorname{Im} g = (1; +\infty)}$$

Ne întoarcem la (*):

$q = \frac{1}{1-x}$ cu $q \in (1,+\infty)$ și $x \in [0,1)$.

Căutam până acum $q \in (1,+\infty)$ a.î. să $\exists\, x \in [0,1)$ care să îndeplinească că $q = \frac{1}{1-x}$.

$\Rightarrow$ Căutăm soluția ec. $g(x) = q$.

Cum $Im\, g = (1,+\infty)$ și $q \in (1,+\infty)$, $\exists\, x \in [0,1)$ a.î. $q = \frac{1}{1-x}$.

Deci $(\forall)\, q \in (1,+\infty)$, am găsit o pereche $(x, y) \in \mathbb{R}\times\mathbb{R}$ pt care $f(x) = f(y)$ și $x \neq y$

mai precis, $(x,y) = (x', 1)$ cu x' sol. ec. $g(x') = q$.

$\Rightarrow (\forall)\, q \in (1,+\infty)$, f nu e inj.

$\Downarrow$

$(\forall)\, m \in \mathbb{R}^*$, f nu e inj.

Sol: $m = 0$ pt. ca f să fie inj

7) $f(x) = x - 4[x] + [2x]$.

f bij, $f^{-1} = ?$ (def., nu preimagine)

• inj: Fie $x, y \in \mathbb{R}$ cu $f(x) = f(y)$

Vrem $x = y$

$\Rightarrow x - 4[x] + [2x] = y - 4[y] + [2y]$

$$\underbrace{x - y}_{} - 4([x] - [y]) + [2x] - [2y] = 0$$

$$[x] - [y] + (\{x\} - \{y\}) - 4([x] - [y]) + [2x] - [2y] = 0 \;/ - (\{x\} - \{y\})$$

$$\Rightarrow \underbrace{-3([x] - [y]) + [2x] - [2y]}_{\in \mathbb{Z}} = \underbrace{\{y\} - \{x\}}_{\in \mathbb{R}}$$

$\Rightarrow \{y\} - \{x\} \in \mathbb{Z}$

Dar $0 \le \{y\} < 1$

$-1 < -\{x\} \le 0$ "+"

$-1 < \{y\} - \{x\} \le 1$

$\Rightarrow \{y\} - \{x\} = 0$

$\Downarrow$

$\{y\} = \{x\}$

Cazul I. $x \in [k, k+\frac{1}{2})$ cu $k \in \mathbb{Z}$
$y \in [l; l+\frac{1}{2})$ cu $l \in \mathbb{Z}$

$x - y - 4([x]-[y]) + [2x] - [2y] = 0$

$\Rightarrow x - y - 4([x] - [y]) + [2x] - [2y] = 0$

Devine: $[x] - [y] - 4([x] - \overbrace{[y]) + [x] + [x+\frac{1}{2}]}^{\text{Hermite}} - [y] - [y+\frac{1}{2}] = 0$

$(*)$ $-3([x] - [y]) + [x] + [x+\frac{1}{2}] - [y] - [y+\frac{1}{2}] = 0$

$-3(k-l) + k + k - l - l = 0$

$-(k-l) = 0 \Rightarrow k = l \Rightarrow \left.\begin{array}{l} [x] = [y] \\ \text{Dar } \{x\} = \{y\} \end{array}\right\} \Rightarrow \underline{x = y}$

Caz II $x \in [k+\frac{1}{2}; k+1)$
$y \in [k+\frac{1}{2}; k+1)$, $k, l \in \mathbb{Z}$

$(*)$ Devine: $-3(k-l) + k + k + 1 - \cancel{k+1} \, l - l - 1 = 0$

$-(k-l) = 0 \Rightarrow [x] = [y] \Rightarrow x = y$

Obs. $\{x\} = \{y\}$, deci nu are sens cazul $x \in [k+\frac{1}{2}; k+1)$, $l \in [l; l+\frac{1}{2})$

Din I și II $\Rightarrow f$ inj.

Surj: ~~$(\forall) x \in$~~

Pt $\{x\} \in [0; \tfrac{1}{2})$, $f(x) = x - 4[x] + [2x]$

$$= x - 4[x] + \underbrace{[x]}_{[x]} + \underbrace{[x + \tfrac{1}{2}]}_{[x]}$$

$$= x - 2[x] = \{x\} - [x]$$

Pt $\{x\} \in [\tfrac{1}{2}; 1) \Rightarrow f(x) = x - 2[x] + 1 = \{x\} - [x] + 1$

$$\Rightarrow f(x) = \begin{cases} \{x\} - [x], & \{x\} \in [0; \tfrac{1}{2}) \\ \{x\} - [x] + 1, & \{x\} \in [\tfrac{1}{2}; 1) \end{cases}$$

Fie $y \in \mathbb{R}$ cu $\{y\} \in [0; \tfrac{1}{2})$

$\Rightarrow \exists\, x \in \mathbb{R}$ cu $\{x\} \in [0; \tfrac{1}{2})$ a.î. $f(x) = y$

$\Rightarrow \{x\} - [x] = y$

$\{x\} - [x] = [y] + \{y\}$

$$\underbrace{[y] + [x]}_{\in \mathbb{Z}} = \underbrace{\{x\} - \{y\}}_{\in \mathbb{R}}$$

$$\Rightarrow \begin{cases} \{x\} - \{y\} = 0 \\ [y] + [x] = 0 \end{cases} \Leftrightarrow \begin{cases} \{x\} = \{y\} \\ [x] = -[y] \end{cases}$$

$$x = \{y\} - [y]$$

Fie $y \in \mathbb{R}$ cu $\{y\} \in [\frac{1}{2}; 1)$

$\Downarrow$

$\{x\} - [x] + 1 = [y] + \{y\}$

$\Rightarrow 1 - [x] = [y]$ și $\{x\} = \{y\}$ $\Bigg| \Rightarrow [x] = 1 - [y]$

$\Downarrow$

$x = 1 - [y] + \{y\}$

$\Rightarrow$ ~~$\exists f^{-1}(x)$~~

$\Rightarrow (\forall)\, y \in \mathbb{R}, \exists x \in \mathbb{R}$ a.î. $f(x) = y$

$\left.\begin{array}{l} \Rightarrow f \text{ surj} \\ \text{dar } f \text{ inj deja} \end{array}\right\} \Rightarrow f \text{ bij}$

~~$f^{-1}(x) = \{y\} - [y]$~~ $\neq \{x\} - [x]$

Obs. că $\underbrace{f(f(x)) = x}_{\text{se mai numește și idempotentă}} \Rightarrow f(x) = f^{-1}(x)$