Să se studieze convergența următoarelor serii: ①

ⓐ Criteriul comparației

1. $\sum_{n\geq 1} \frac{\sqrt{n+1}-\sqrt{n}}{n}$ 2. $\sum_{n\geq 1} \frac{n\sqrt{n}}{3n^2+2}$

3. $\sum_{n\geq 1} \frac{1+\frac{1}{2}+\ldots+\frac{1}{n}}{n}$ 4. $\sum_{n\geq 1} \frac{1+\sqrt{2}+\ldots+\sqrt{n}}{n^3}$

5. $\sum_{n\geq 1} \sqrt{n^4+3n+1}-n^2$ 6. $\sum_{n\geq 1} 2^n \sin\frac{\pi}{4^n}$

ⓑ Criteriul raportului

7. $\sum_{n\geq 1} \frac{x^n}{\sqrt[n]{C_{3n}^n}}$ $x>0$ 8. $\sum_{n\geq 1} x^{2n} \frac{n\sqrt{n}}{n^2+1}$ $x>0$

9 $\sum_{n\geq 1} \frac{2^n n!}{n^n}$ 10 $\sum_{n\geq 1} \frac{4^n n!}{n^n}$

11 $\sum_{n\geq 1} \frac{(n!)^2}{(2n)!}$

ⓒ Criteriul radicalului

12. $\sum_{n\geq 1} x^n \cdot \left(1+\frac{1}{n}\right)^{n^2+n}$ $x>0$ 13. $\sum_{n\geq 1} \left(\frac{3+(-1)^n\cdot 2}{4}\right)^n$

14. $\sum_{n\geq 1} \left(\frac{an+1}{bn+1}\right)^n$ $a,b>0$ 15. $\sum_{n\geq 1} n^{(-1)^{n+1}} a^n$ $a>0$.

(d) Raabe - Duhamel

16). $\sum_{n\geq 1} \frac{\sqrt{n!}}{(3+\sqrt{1})(3+\sqrt{2})\cdots(3+\sqrt{n})}$

17). $\sum_{n\geq 1} \frac{1!+2!+\cdots+n!}{(n+2)!}$

18). $\sum_{n\geq 1} \frac{1}{n^a}$ $a\in\mathbb{R}$ 19) $\sum_{n\geq 1} \frac{a(a+1)\cdots(a+n)}{(n+6)!}$ $a>0$

20) $\sum_{n\geq 1} \left(\frac{1\cdot 4\cdot 7\cdots(3n-2)}{3\cdot 6\cdot 9\cdots(3n)}\right)^2$

(e) Serii cu termeni oarecare:

21) $\sum_{n\geq 1} x^n \sin\frac{1}{n^\alpha}$ $\alpha>0$ 22) $\sum_{n\geq 1} x^n \operatorname{arctg}\frac{1}{n^\alpha}$ $\alpha\in\mathbb{R}$

23) $\sum_{n\geq 1} x^n \ln^\alpha\left(1+\frac{1}{n^2}\right)$ 24) $\sum_{n\geq 1} x^n \cdot \frac{(n+5)!}{a(a+1)\cdots(a+n)}$ $x\in\mathbb{R}$ $a>0$

25) $\sum_{n\geq 1} x^n \cdot \frac{\sqrt{n!}}{a(a+\sqrt{1})\cdots(a+\sqrt{n})}$ $x\in\mathbb{R}$ $a>0$

26) $\sum_{n\geq 1} x^n \left(\frac{a}{2}-\sqrt{a}\right)\left(2-\sqrt[3]{a}\right)\cdots\left(2-\sqrt[n]{a}\right)$ $a>0$ $x\in\mathbb{R}$.

27) $\sum_{n\geq 1} \frac{x^n\cdot n!}{n^{n+\alpha}}$ $x\in\mathbb{R}$ $\alpha\in\{-2,5\}$.

Rezolvări

3) $\lim\limits_{n\to\infty} \dfrac{1+\frac{1}{2}+\dots+\frac{1}{n}}{\ln n} = 1$

$a_n = \dfrac{1+\frac{1}{2}+\dots+\frac{1}{n}}{n}$ $\qquad b_n = \dfrac{\ln n}{n}$

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 1 \Rightarrow$

$\sum\limits_{n\geq 1} a_n \sim \sum\limits_{n\geq 1} b_n \geq \sum\limits_{n\geq 1} \dfrac{1}{n}$ - divergentă

9) $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \dfrac{2^{n+1}(n+1)!}{(1+n)^{n+1}} \cdot \dfrac{n^n}{2^n n!} = 2\left(\dfrac{n}{n+1}\right)^n \to$

$\to \dfrac{2}{e} < 1 \Rightarrow$ c. convergentă.

14) $\lim\limits_{n\to\infty} \sqrt[n]{\left(\dfrac{a_{n+1}}{b_{n+1}}\right)^n} = \lim\limits_{n\to\infty} \dfrac{a_{n+1}}{b_{n+1}} = \dfrac{a}{b}$.

Dacă $a > b \Rightarrow$ divergentă. Dacă $a < b \Rightarrow$ convergentă

Dacă $a = b \Rightarrow \left(\dfrac{a_{n+1}}{b_{n+1}}\right)^n = 1 \not\to 0 \Rightarrow$ c. divergentă.

19) $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{n+7}{a+n+1} - 1\right) =$

(4)

$= \lim_{n\to\infty} n \frac{6-a}{a+n+1} = 6-a$

Dacă $6-a > 1 \Leftrightarrow 5 > a$ $\Rightarrow$ divergentă

$6-a < 1 \Leftrightarrow 5 < a$ $\Rightarrow$ convergentă

$a = 5 \quad \sum_{n\geq 1} \frac{5 \cdot 6 \cdots (a+5)}{(n+6)!} = \sum_{n\geq 1} \frac{1}{4!\,(n+6)}$ divergentă.

22) $\sum_{n\geq 1} x^n \sin\frac{1}{n^\alpha} \quad \alpha > 0$

Studiem absolut convergența (PAS 1)

$\sum_{n\geq 1} |x|^n \cdot \sin\frac{1}{n^\alpha} \qquad a_n := |x|^n \sin\frac{1}{n^\alpha}$

$x = 0; \quad \sum_{n\geq 1} 0 \geq 0 \Rightarrow$ seria este absolut convergentă.

$\frac{a_{n+1}}{a_n} = \frac{|x|^{n+1} \sin\frac{1}{(n+1)^2}}{|x|^n \sin\frac{1}{n^\alpha}} =$

$= |x| \; \frac{\sin\frac{1}{(n+1)^\alpha}}{\frac{1}{(n+1)^\alpha}} \cdot \frac{\frac{1}{n^\alpha}}{\sin\frac{1}{n^\alpha}} \cdot \frac{n^\alpha}{(n+1)^\alpha} \to |x|$

$\frac{\sin\frac{1}{(n+1)^\alpha}}{\frac{1}{(n+1)^\alpha}} \to 1 \qquad \frac{\frac{1}{n^\alpha}}{\sin\frac{1}{n^\alpha}} \to 1 \qquad \frac{n^\alpha}{(n+1)^\alpha} \to 1$

Dacă $|x|<1$ seria este abs. conv

$|x|>1$ seria este divergentă ($a_n \to \infty$)

(5)

$|x|=1$ ($\Leftrightarrow x=1$)

$$\sum_{n\geq 1} \sin\frac{1}{n^\alpha} \sim \sum \frac{1}{n^\alpha}$$

Conv $\alpha > 1$

div $\alpha \leq 1$

$\Rightarrow$)

$$\lim_{n\to\infty} \frac{\frac{1}{n^\alpha}}{\sin\frac{1}{n^\alpha}} = 1$$

$x=-1$ $\quad \sum_{n\geq 1} (-1)^n \sin\frac{1}{n^\alpha}$

$\frac{1}{n^\alpha} \downarrow 0 \Rightarrow \sin\frac{1}{n^\alpha} \downarrow 0 \Rightarrow$

$\alpha > 1$ absolut convergentă

$0 < \alpha \leq 1$ semi convergentă.