Serii Remarcabile

1) Seria geometrică de rație q: $\sum_{n\geq 0} q^n$, $q \in \mathbb{R}$, $q^0 = 1$

→ $q \in (-\infty; -1]$ seria divergentă

→ $q \in (-1; 1)$ seria convergentă cu suma $\sum_{n\geq 0} q^n = \frac{1}{1-q}$

→ $q \in [1; +\infty)$ seria divergentă

2) Seria armonică generalizată $\sum_{n\geq 1} \frac{1}{n^\alpha}$, $\alpha \in \mathbb{R}$

→ $\alpha > 1$, seria convergentă

→ $\alpha < 1$, seria divergentă

→ $\alpha = 1$ $\sum_{n\geq 1} \frac{1}{n}$ seria armonică → divergentă

Algoritm General

I Studiem absolut convergența (modul) $\sum_{n\geq 1} |x_n|$

1) C. Raportului / Radicalului

2) C. comparației / Raabe-Duhamel

II Calculăm $\lim_{n\to\infty} x_n$, dacă limita $\neq 0$ ⇒ seria divergentă

III C. Abel, C. seriei alternate

Ex. 1. $\sum_{n=1}^{\infty} \left(\frac{an^2+4n+9}{bn^2+3n+1} \right)^n$, $a>0, b>0$

Rezolvare:

Notăm $x_n = \left(\frac{an^2+4n+9}{bn^2+3n+1} \right)^n$

$$\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \frac{an^2+4n+9}{bn^2+3n+1} = \frac{a}{b} = L$$

I $a > b$, $L > 1$ ⇒ seria divergentă

II $a < b$, $L < 1$ ⇒ seria convergentă

III $a = b$, $L = 1$ ⇒ $x_n = \frac{an^2+4n+9}{an^2+3n+1} = 1 + \frac{n+8}{an^2+3n+1}$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \left(1 + \frac{n+8}{an^2+3n+1}\right)^n = \lim_{n\to\infty} \left[\left(1 + \frac{n+8}{an^2+3n+1}\right)^{\frac{an^2+3n+1}{n+8}} \right]^{n \cdot \frac{n+8}{an^2+3n+1}} =$$

$$= e^{\lim_{n\to\infty} \frac{n^2+8n}{an^2+3n+1}} = e^{\frac{1}{a}} \neq 0, \quad a > 0$$

$\sum x_n$ diverg.

$n \geq 1$

Ex 2. $x_n = \dfrac{a^n \cdot (n!)^2}{(2n)!}$, $a > 0$

Rezolvare:

$$\frac{x_{n+1}}{x_n} = \frac{a^{n+1}\,((n+1)!)^2}{(2n+2)!} \cdot \frac{2n!}{a^n \cdot (n!)^2} = \frac{a \cdot (n+1)^2}{(2n+1)\cdot(2n+2)} = \frac{a \cdot (n^2+2n+1)}{4n^2+6n+2}$$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{a}{4} = L$$

I $a < 4 \Rightarrow L < 1 \Rightarrow$ seria converg.

II $a > 4 \Rightarrow L > 1 \Rightarrow$ seria diverg.

III $a = 4 \Rightarrow L = 1$, aplicăm Raabe-Duhamel

$$\lim_{n\to\infty} n \cdot \left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n \cdot \left(\frac{4n^2+6n+2}{4n^2+8n+4} - 1\right) = \lim_{n\to\infty} n \cdot \frac{-2n-2}{4n^2+8n+4} =$$

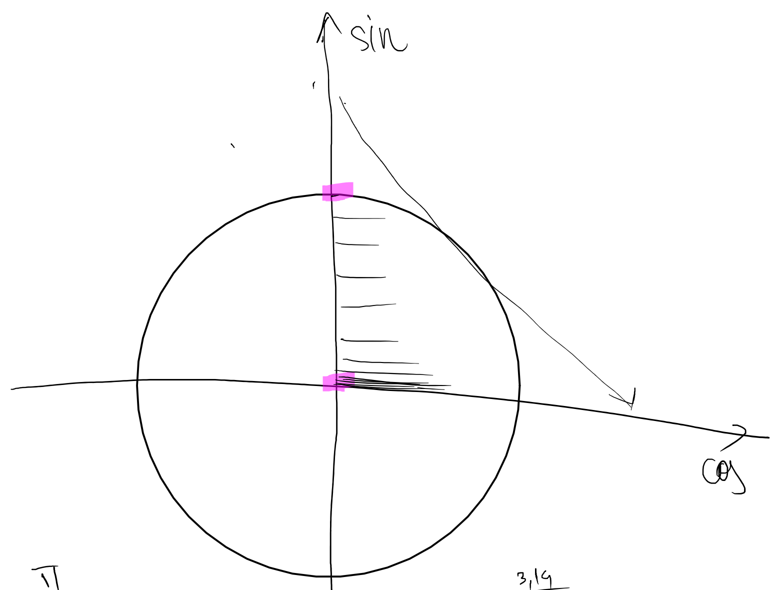
$$= \lim_{n\to\infty} \frac{-2n^2-2n}{4n^2+8n+4} = -\frac{2}{4} = -\frac{1}{2} < 1 \xrightarrow{C.R.D} \text{seria diverg.}$$

Ex 3. $x_n = \dfrac{\sqrt{1-\cos\frac{\pi}{n}}}{n \cdot \ln(n+1)}$, $n \geq 1$

$$1 - \cos\frac{\pi}{n} = 1 - \cos\frac{2\pi}{2n} = 2\sin^2\frac{\pi}{2n}$$

$$x_n = \frac{\sqrt{2\sin^2\frac{\pi}{2n}}}{n \cdot \ln(n+1)} = \frac{\sqrt{2}\cdot\sin\frac{\pi}{2n}}{n\cdot\ln(n+1)}$$

$$\boxed{\cos 2x = 1 - 2\sin^2 x \quad , \quad x = \frac{\pi}{2n}}$$



$$\sin\frac{\pi}{2n} < \frac{\pi}{2n}$$

$$x_n < \frac{\sqrt{2}\cdot\frac{\pi}{2n}}{n\cdot\ln(n+1)} = \frac{\pi\sqrt{2}}{2n^2\cdot\ln(n+1)} < \frac{1}{n^2} \quad (*)$$

$0 < \frac{\pi}{2n} \leq \frac{\pi}{2}$

$0 \leq \sin\frac{\pi}{2n} \leq 1$

$\frac{3,14}{2}$

$\pi > 3,14$

$\frac{\pi}{2n}$ ↘

n ↗

$(*) \Leftrightarrow \pi \cancel{n^2}\sqrt{2} < 2\cancel{n^2}\ln(n+1) \Big|_{:\sqrt{2}} \Leftrightarrow$

$\pi < \sqrt{2}\cdot\ln(n+1) \Leftrightarrow e^{\pi} < e^{\sqrt{2}}\cdot(n+1) \Leftrightarrow n+1 > e^{\frac{\pi}{\sqrt{2}}} \approx e^{2,24}$

$n > e^{\frac{\pi}{\sqrt{2}}} - 1$

$\frac{\pi}{4}$ ... comparatie

$n > e^{\sqrt{2}} - 1$

Pt $n > e^{\frac{\pi}{\sqrt{2}}} - 1$ aplicăm criteriul comparației

$x_n < \frac{1}{n^2} \Rightarrow$ cum $\sum \frac{1}{n^2}$ conv $\Rightarrow$ $\sum x_n$ conv, $n > e^{\frac{\pi}{\sqrt{2}}} - 1$

Topologii

Definiția! Pt $X \neq \emptyset$ $\tau \subset P(X)$ = mulțimea submulțimilor

τ = topologie

1) $\emptyset, X \in \tau$

2) $D_1, D_2 \in \tau \Rightarrow D_1 \cap D_2 \in \tau$

3) $(D_i)_{i \in J} \subset \tau \Rightarrow \bigcup_{i=J} D_i \in \tau$ $D_1, D_2, D_3 \dots$ familie de mulțimi

$D_1, D_2, \dots D_n \subset \tau$

$D_1 \cup D_2 \cup \dots \cup D_n \subset \tau$

$P(X)$ = mulțimea tuturor submulțimilor $\{1,2\} = \{2,1\}$

$(D_i)_{i \in J}$ = familie de mulțimi

$B(x, r) = (x - r, x + r)$

$B[x, r] = [x - r, x + r]$

$\mathring{A}$ = interiorul lui A $x \in \mathring{A} \Rightarrow \exists r > 0$ a.î. $B(x, r) \subset A$

$\bar{A}$ = închiderea lui A $x \in \bar{A} \Rightarrow \forall r > 0$ $B(x, r) \cap A \neq \emptyset$

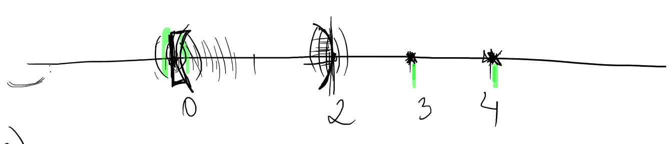
A' = mulțimea punctelor de acumulare ale lui A

$x \in A' \Rightarrow \forall r > 0$ $B(x, r) \cap (A \setminus \{x\}) \neq \emptyset$

$Fr(A)$ = frontiera lui A $Fr(A) = \bar{A} \setminus \mathring{A}$

$iz(A)$ = mulțimea punctelor izolate $iz(A) = A \setminus A'$

$A = [0;2) \cup \{3;4\}$



$\mathring{A} = ? \Rightarrow \mathring{A} = (0;2)$ $\mathring{A} \subset \bar{A}$

$\bar{A} = ?$ $(0;2) \subset \bar{A}$

$B(0, r) \cap A \neq \emptyset$ $\Rightarrow \{0, 2, 3, 4\} \in \bar{A}$ $\Rightarrow \bar{A} = [0,2] \cup \{3,4\}$

$B(2, r) \cap A \neq \emptyset$

$B(0,r) \cap A \neq \emptyset$
$B(3,r) \cap A \neq \emptyset$
$B(4,r) \cap A \neq \emptyset$
$B(2,r) \cap A \neq \emptyset$
$\Rightarrow \{0,2,3,4\} \subset \bar{A}$

$(0;2) \subset A'$
$B(0,r) \cap (A \setminus \{0\}) \neq \emptyset$
$B(2,r) \cap (A \setminus \{2\}) \neq \emptyset$
$B(3,r), B(4,r)$ nu mai are loc proprietatea
$\Rightarrow A' = [0;2]$

$Fr(A) = \bar{A} \setminus \mathring{A} = \{0,2,3,4\}$
$iz(A) = A \setminus A' = \{3,4\}$

Convergența simplă și uniformă

Def! Fie A mulțime și (X,d) spațiu metric.
$f_n, f : A \longrightarrow X$
Spunem că f_n CONVERGE SIMPLU (punctual) la f dacă $\forall x \in A \Rightarrow f_n(x) \to f(x)$.
Spunem că f_n CONVERGE UNIFORM la f pt $\forall \varepsilon > 0 \ \exists n_\varepsilon$ a.î. $\forall n \geq n_\varepsilon$
$\Rightarrow d(f_n(x), f(x)) \leq \varepsilon \ \forall x \in A$

Algoritm de rezolvare:

1) Calculăm $\lim\limits_{n \to \infty} f_n(x)$ (considerăm x constantă)

2) Găsim A pt $f : A \to \mathbb{R}$ $f(x) = \lim\limits_{n \to \infty} f_n(x)$

$f_n \xrightarrow[A]{s} f$ C. SIMPLU

3) Fixăm $n \in \mathbb{N}$. $\sup\limits_{x \in A} |f_n(x) - f(x)| = \sup\limits_{x \in A} g(x)$, $g(x) = |f_n(x) - f(x)|$

4) Calculăm g' + tabelul de semne $\Rightarrow \sup\limits_{x \in A} g(x)$

5) Dacă $\sup g(x) \in \mathbb{R} \Rightarrow f_n \xrightarrow[A]{u} f$
$\sup g(x) \notin \mathbb{R} \Rightarrow f_n \not\xrightarrow[A]{u} f$

C. UNIFORM

EX Studiați c.s și c.u. a șirului de funcții

$f_n(x) = \frac{nx}{\ldots}$ $\forall x \in (0, \infty)$

Ex Studiați c.s și c.u. a șirului de funcții

$$f_n:(0;+\infty)\to\mathbb{R}, \quad f_n(x)=\frac{nx}{n+x} \quad \forall x\in(0;+\infty)$$

Rezolvare:

Fie $x\in(0;+\infty)$.

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\frac{nx}{n+x} = x\in\mathbb{R} \quad \forall x\in(0;+\infty)$$

$$A=(0;+\infty)$$

$$f:(0;+\infty)\to\mathbb{R} \qquad f(x)=x \qquad f_n \xrightarrow[(0;+\infty)]{s} f$$

Fixăm $n\in\mathbb{N}$.

$$\sup_{x\in(0;+\infty)} |f_n(x)-f(x)| = \sup_{x\in(0;+\infty)}\left|\frac{nx}{n+x}-x\right| = \sup_{x\in(0;+\infty)}\left|\frac{\cancel{nx}-\cancel{nx}-x^2}{n+x}\right| = \sup_{x\in(0;+\infty)}\frac{x^2}{n+x} \overset{not}{=} \sup_{x\in(0;+\infty)} g(x)$$

$$g:(0;+\infty)\to\mathbb{R} \qquad g(x)=\frac{x^2}{n+x}$$

$$g'(x)=\frac{2x\cdot(n+x)-x^2\overset{1}{(n+x)'}}{(n+x)^2} = \frac{2xn+2x^2-x^2}{n^2+2nx+x^2} = \frac{x^2+2xn}{x^2+2nx+n^2}$$

$$g'(x)=0 \Leftrightarrow x^2+2xn=0 \Leftrightarrow x(x+2n)=0 \Rightarrow x=0, \; x=-2n$$

	0					$+\infty$
$g'(x)$	0	$+$	$+$	$+$	$+$	$+$
$g(x)$	0	$\nearrow$				$+\infty$

$$g'(1)=\frac{1+n}{1+2n+n^2}=\frac{n+1}{(n+1)^2}=\frac{1}{n+1}>0$$

$$\lim_{x\to\infty} g(x)=\lim_{x\to\infty}\frac{x^2}{n+x}=+\infty$$

$$\lim_{x\to 0} g(x)=\frac{0}{n}=0$$

$$\sup_{x\in(0;+\infty)} g(x)\notin\mathbb{R} \Rightarrow f_n \underset{(0;+\infty)}{\overset{u}{\not\longrightarrow}} f$$

$$+\infty\notin\mathbb{R} \qquad \mathbb{R}\cup\{\pm\infty\}=\overline{\mathbb{R}}$$