Să se studieze continuitatea în $(0,0)$

a funcției $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad$ 1-

$$f(x,y) = \begin{cases} \dfrac{x^n y^m}{x^{2p}+y^{2p}} & x^2+y^2 \neq 0 \\ 0 & x=y=0 \end{cases} \qquad n,m,p \in \mathbb{N}^*$$

$$\lim_{\substack{x\to 0 \\ y=ax}} f(x,y) = \lim_{x\to 0} f(x, ax) =$$

$$= \lim_{x\to 0} \frac{x^n a^m x^m}{x^{2p}(1+a^{2p})} = \lim_{x\to 0} x^{m+n-2p} \frac{a^m}{(1+a^{2p})} =$$

$$= \begin{cases} 0 & m+n > 2p \\ \dfrac{a^m}{1+a^{2p}} & m+n = 2p \\ 0 & m+n<2p \text{ și } a=0 \\ \nexists & m+n<2p \quad a\neq 0 \quad m \text{ impar} \\ \infty & m+n<2p \quad a \neq 0 \quad m \text{ par} \end{cases}$$

$\Rightarrow m+n \leq 2p \quad \nexists \lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y) \Rightarrow f$ disc. în $(0,0)$.

$m + n > 2p$

<u>Metoda 1</u> Inegalitatea mediilor generalizate

$$\alpha x + (1-\alpha) y \geq \cancel{\alpha} x^{\alpha} \cdot y^{1-\alpha} \qquad \alpha \in [0,1]$$

$$x \geq 0 \quad y \geq 0$$

Dem se consideră $f:(0,\infty) \to (0,\infty)$

$f(x) = \alpha x - x^{\alpha} \cdot y^{1-\alpha} + (1-\alpha) y$ cu y fixat

și se face tabelul de variație.

<u>CAZ 1</u> $m, n \leq 2p$

$$x^{2p} + y^{2p} \geq \frac{m}{2p} x^{2p} + \frac{2p-m}{2p} y^{2p} \geq$$

$$\geq |x|^m |y|^{2n-p} \Rightarrow \frac{|x|^m |y|^{2n-p}}{x^{2p}+y^{2p}} \leq 1.$$

$$\Rightarrow \left| \frac{x^m y^n}{x^{2p}+y^{2p}} \right| = \underbrace{\frac{|x|^m |y|^{2p-n}}{x^{2p}\ y^{2p}}}_{\leq 1} \cdot |y|^{m+n-2p} \leq |y|^{m+n-2p} \xrightarrow[y \to 0]{x \to 0} 0$$

CAZ 2 $n \geq 2p$

$$\frac{|x^n y^m|}{x^{2p}+y^{2p}} \leq \overbrace{\frac{x^{2p}}{x^{2p}+y^{2p}}}^{\leq 1} \cdot |x^{n-2p} y^m| \leq$$

$$\leq |x^{n-2p} y^m| \xrightarrow[\substack{x \to 0 \\ y \to 0}]{} 0$$

Metoda 2

$$\left| \frac{x^n y^m}{x^{2p}+y^{2p}} \right| \leq \overbrace{\left(\frac{x^{2p}}{x^{2p}+y^{2p}} \right)^{\frac{n}{2p}}}^{\leq 1} \cdot \overbrace{\left(\frac{y^{2p}}{x^{2p}+y^{2p}} \right)^{\frac{m}{2p}}}^{\leq 1} \cdot$$

$$\cdot \left(x^{2p}+y^{2p} \right)^{\frac{n}{2p}+\frac{m}{2p}-1} \leq \left(x^{2p}+y^{2p} \right)^{\frac{n+m-2p}{2p}}.$$

Exerciții propuse Să se studieze continuitatea în (0,0) a funcțiilor

a) $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^4+y^4} & x^2+y^2 \neq 0 \\ 0 & x=y=0 \end{cases}$

b) $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = \begin{cases} \dfrac{x^4 y^4}{x^6+y^8} & x^2+y^2 \neq 0 \\ 0 & x=y=0 \end{cases}$

c) $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = \begin{cases} \dfrac{x^3 y^4}{x^6+y^{12}} & x^2+y^2 \neq 0 \\ 0 & x=y \end{cases}$

d) $f: \mathbb{R}^3 \to \mathbb{R}$ $\quad f(x,y,z) = \begin{cases} \dfrac{x^2y^2z}{x^4+y^4+z^4} & x^2+y^2+z^2 \neq 0 \\ 0 & x=y=z=0 \end{cases}$

e) $f: \mathbb{R}^3 \to \mathbb{R}$ $\quad f(x,y,z) = \begin{cases} \dfrac{xy^2z}{x^4+y^4+z^4} & x^2+y^2+z^2 \neq 0 \\ 0 & x=y=z=0 \end{cases}$

f) $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = \begin{cases} x^3y^2 \sin \dfrac{1}{x^4+y^4} & x^2+y^2 \neq 0 \\ 0 & x=y=0 \end{cases}$

9) $f_n : \mathbb{R} \to \mathbb{R}$ $\quad f_n(x) = \dfrac{x^n}{1 + x^{2n}}$

10) $f_n : \mathbb{R} \to \mathbb{R}$ $\quad f_n(x) = \sqrt{x^2 + \dfrac{1}{n^3}}$ $\quad n \geq 1$

11) $f_n : \mathbb{R} \to \mathbb{R}$ $\quad f_n(x) = \dfrac{\cos nx}{n}$

Se cer şi c.u şi c.s pentru f_n'.

12) $f_n : [0,1] \to \mathbb{R}$ $\quad f_n(x) = n x (1-x)^{2n}$

13) $f_n : \mathbb{R} \to \mathbb{R}$ $\quad f_n(x) = \dfrac{e^{nx} - 2}{e^{nx} + 3}$

14) $f_n : \mathbb{R}^2 \to \mathbb{R}$ $\quad f_n(x, y) = \dfrac{x y n^2}{x^6 + y^6 + n^6}$

15) $f_n : \mathbb{R}^2 \to \mathbb{R}$ $\quad f_n(x, y) = \dfrac{y^2 y^2 n^2}{x^6 + y^6 + n^6}$

16) $f_n : [0,1] \to \mathbb{R}$ $\quad f_0(x) = x$ $\quad f_{n+1}(x) = f_n(x) - f_n^2(x)$

f_n şi $g_n = n f_n$.

17) Exercițiile din curs.

Să se studieze convergența simplă și uniformă a următoarelor șiruri de funcții: (dacă nu are loc convergența simplă sau uniformă să se determine mulțimile pe care au loc)

1) $f_n : [0,1] \to \mathbb{R}$ $\quad f_n(x) = x^n(1-x)^3$

2) $g_n : [0,1] \to \mathbb{R}$ $\quad g_n(x) = x^{7n}(1-x^{3n})$

3) $f_n : \mathbb{R} \to \mathbb{R}$ $\quad f_n(x) = \dfrac{x^2 n^3}{x^4 + n^4}$

4) $g_n : \mathbb{R} \to \mathbb{R}$ $\quad g_n(x) = \dfrac{x\, n^4}{x^4 + n^4}$

5) $f_n : \mathbb{R} \to \mathbb{R}$ $\quad f_n(x) = e^{-nx}$

6) $f_n : \mathbb{R} \to \mathbb{R}$ $\quad f_n(x) = x^3 e^{-nx}$

7) $f_n : (0,1) \to \mathbb{R}$ $\quad f_n(x) = \dfrac{1}{nx+1}$, $\quad n \geq 0$

8) $f_n : [-1,1] \to \mathbb{R}$ $\quad f_n(x) = \dfrac{x}{1+n^2x^2}$

Să se studieze convergența simplă și uniformă

pentru $f_n : \mathbb{R} \to \mathbb{R}$ $\qquad f_n(x) = \dfrac{x n^2}{x^4 + n^4}$

$$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{x n^2}{x^4 + n^4} = 0$$

$\Rightarrow f_n \to 0$

$$a_n = \sup_{x \in \mathbb{R}} |f_n(x) - f(x)| = \sup_{x \in \mathbb{R}} \left| \underbrace{\frac{x n^2}{x^4 + n^4}}_{f_n(x)} \right|$$

$$f_n'(x) = \frac{n^2(x^4 + n^4) - x n^2 \cdot 4x^3}{(x^4 + n^4)^2} =$$

$$= \frac{n^6 - 3n^2 x^4}{(x^4 + n^4)^2} = \frac{n^2(n^4 - 3x^4)}{(x^4 + n^4)^2} = 0$$

$$x_1 = \frac{n}{\sqrt[4]{3}} \qquad x_2 = -\frac{n}{\sqrt[4]{3}}$$

x	$-\infty$		x_2		x_1		∞
f_n'		$- - -$	0	$+ + + +$	0	$- - - -$	
f_n	0	$\searrow$	$-\frac{1}{2\sqrt[4]{3}\,n}$	$\nearrow$	$\frac{1}{2\sqrt[4]{3}\,n}$	$\searrow$	0

$$a_n = \frac{1}{2\sqrt[4]{3}\,n} \to 0 \Rightarrow f_n \xrightarrow{u} 0.$$

Fie $f_n : \mathbb{R} \to \mathbb{R}$ dat de:

$$f_n(x) = \frac{\sin x + a e^{nx}}{1 + e^{nx}}, \text{ unde } a \in \mathbb{R}$$

Se cer c.s. și c.u.

$$\lim_{n\to\infty} nx = \begin{cases} \infty & x > 0 \\ 0 & x = 0 \\ -\infty & x < 0 \end{cases} \Rightarrow \lim_{n\to\infty} e^{nx} = \begin{cases} \infty & x > 0 \\ 1 & x = 0 \\ 0 & x < 0 \end{cases}$$

$x > 0$ $$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\cancel{e^{nx}}\left(\frac{\sin nx}{e^{nx}} + a\right)}{\cancel{e^{nx}}\left(\frac{1}{e^{nx}} + 1\right)} = a$$

$x = 0$ $$f_n(0) = \frac{a}{2} \to \frac{a}{2}$$

$x < 0$ $$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\sin x + a \overbrace{e^{nx}}^{\to 0}}{1 + \underbrace{e^{nx}}_{\to 0}} = \sin x$$

$$f : \mathbb{R} \to \mathbb{R} \quad f(x) = \begin{cases} a & x > 0 \\ \frac{a}{2} & x = 0 \\ \sin x & x < 0 \end{cases}$$

$\Rightarrow f_n \xrightarrow{s} f$.

f este cont $\Leftrightarrow$ $a = 0$

$a \neq 0 \Rightarrow f_n \not\rightrightarrows f$ $\quad$ (f_n cont $\quad f$ disc.

$a = 0$

$$a_n = \sup_{x \in \mathbb{R}} |f_n(x) - f(x)| = \max\Big(\underbrace{\sup_{x \geq 0} |f_n(x) - f(x)|}_{b_n}, \underbrace{\sup_{x<0} |f_n(x) - f(x)|}_{c_n}\Big)$$

$$b_n = \sup_{x \geq 0} \left| \frac{\sin x}{1 + e^{nx}} - 0 \right| = \sup_{x \geq 0} \frac{|\sin x|}{1 + e^{nx}} \leq \sup_{x \geq 0} x e^{-nx}$$

$$c_n = \sup_{x<0} \left| \frac{\sin x}{1 + e^{nx}} - \sin x \right| = \sup_{x<0} \left| \frac{\sin x \, e^{nx}}{1 + e^{nx}} \right| \leq$$

$y = -x$

$$\leq \sup_{y \geq 0} y e^{-ny} = t_n$$

$g_n(x) = x e^{-nx}$ $\quad g_n : [0, \infty) \to \mathbb{R}$ $\quad x_0 = \frac{1}{n}$

$g_n'(x) = e^{-nx} + x(-n) e^{-nx} = e^{-nx}(1 - nx) = 0 \Rightarrow$

$t_n = g_n\left(\frac{1}{n}\right) = \frac{1}{n} e^{-1} \to 0$

x	0	$\frac{1}{n}$	∞
g_n'	$+ + + +$	0	$- - - -$
g_n	0 $\nearrow$		$\searrow$ 0

$\Rightarrow f_n \xrightarrow{u} f.$