30.11.2020

Monday, November 30, 2020 11:47 AM

Serii de puteri

Ex 1 $\sum_{n=0}^{+\infty} \frac{x^{2n+1}}{2n+1} = \sum_{n=0}^{+\infty} \frac{1}{2n+1} \cdot (x-0)^{2n+1} = \frac{1}{2\cdot 0+1} \cdot x^{2\cdot 0+1} + \frac{1}{2\cdot 1+1} \cdot x^{2\cdot 1+1} \ldots$

$= \frac{1}{1} \cdot x^1 + \frac{1}{3} \cdot x^3 + \ldots$

$= \frac{1}{1} \cdot x^1 + 0 \cdot x^2 + \frac{1}{3} \cdot x^3 + 0 \cdot x^4 + \ldots$

$x_0 = 0$

$a_{2n+1} = \frac{1}{2n+1}$

$a_{2n} = 0$

$l_1 = \lim_{n\to\infty} \sqrt[2n]{|a_{2n}|} = 0$

$l_2 = \lim_{n\to\infty} \sqrt[2n+1]{|a_{2n+1}|} = \lim_{n\to\infty} \sqrt[2n+1]{\left|\frac{1}{2n+1}\right|} = \lim_{n\to\infty} \sqrt[2n+1]{\frac{1}{2n+1}}$

$n \to 2n+1$ (calculăm raportul $\frac{a_{n+1}}{a_n} \to l = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \Rightarrow l = \lim_{n\to\infty} \sqrt[n]{a_n}$)

$\lim_{n\to\infty} \frac{a_{2n+3}}{a_{2n+1}} = \lim_{n\to\infty} \frac{\frac{1}{2n+3}}{\frac{1}{2n+1}} = \lim_{n\to\infty} \frac{2n+1}{2n+3} = 1 \Rightarrow l_2 = \lim_{n\to\infty} \sqrt[2n+1]{\frac{1}{2n+1}} = 1$

$\rho = \frac{1}{\max\{l_1, l_2\}} = \frac{1}{\max\{0,1\}} = \frac{1}{1} = 1$

Intervalul de convergență este: $(x_0-\rho, x_0+\rho) = (0-1, 0+1) = (-1;1)$

$\Rightarrow (-1;1) \subseteq D \subseteq [-1;1]$

$-1 \in D$?
$x = -1 \Rightarrow \sum_{n=0}^{+\infty} \frac{(-1)^{2n+1}}{2n+1} = \sum_{n=0}^{+\infty} \left(-\frac{1}{2n+1}\right) = -\sum_{n=0}^{+\infty} \frac{1}{2n+1}$ serie divergentă $\Rightarrow -1 \notin D$

$1 \in D$?
$x = 1 \Rightarrow \sum_{n=0}^{+\infty} \frac{1^{2n+1}}{2n+1} = \sum_{n=0}^{+\infty} \frac{1}{2n+1} \Rightarrow$ serie divergentă $\Rightarrow 1 \notin D$

$D = (-1;1)$

$f: (-1;1) \to \mathbb{R} \quad f(x) = \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{2n+1}$

$D = (x_0-\rho; x_0+\rho) \Rightarrow f$ funcție de clasă C^∞ pe $(-1;1)$ (integrabilă și derivab. de ∞ ori)

$f(x_0) = a_0 \Rightarrow f(0) = 0$

$f'(x) = \left(\sum_{n=0}^{+\infty} \frac{x^{2n+1}}{2n+1}\right)' = \sum_{n=0}^{+\infty} \left(\frac{x^{2n+1}}{2n+1}\right)' = \sum_{n=0}^{+\infty} \frac{\cancel{(2n+1)} \cdot x^{2n}}{\cancel{2n+1}} = \sum_{n=0}^{+\infty} x^{2n}$

$\sum_{n=0}^{+\infty} x^n = \frac{1}{1-x}$

$x \to x^2 \Rightarrow \sum x^{2n} = \frac{1}{1-x^2}$

$$\sum_{n=0} x^n = \frac{1}{1-x}$$

$$x \to x^2 \Rightarrow \sum_{n=0}^{+\infty} x^{2n} = \frac{1}{1-x^2}$$

$$\Rightarrow f'(x) = \frac{1}{1-x^2}$$

$$f(x) = \int \frac{1}{1-x^2}\,dx$$

$$\int \frac{1}{1-x^2}\,dx = (-1)\cdot\frac{1}{2}\cdot \ln\left|\frac{x-1}{x+1}\right| + C$$

$$f(x) = (-1)\cdot\frac{1}{2}\cdot \ln\left|\frac{x-1}{x+1}\right| + c$$

$$\left.\begin{array}{l} f(0) = (-1)\cdot\frac{1}{2}\cdot \ln\left|\frac{-1}{1}\right| + c = (-1)\cdot\frac{1}{2}\cdot \ln 1 + c = c \\ f(0) = 0 \end{array}\right\} \Rightarrow c = 0$$

$$f(x) = \left(-\frac{1}{2}\right)\cdot \ln\left|\frac{x-1}{x+1}\right|$$

Ex2 $\sum_{n=1}^{\infty} (-1)^{n+1}\cdot\frac{x^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1}\cdot\frac{1}{n}\cdot(x-0)^n = (-1)^{1+1}$

$x_0 = 0$

$a_n = (-1)^{n+1}\cdot\frac{1}{n}$

$$L = \lim_{n\to\infty} \sqrt[n]{\left|(-1)^{n+1}\cdot\frac{1}{n}\right|} = \lim_{n\to\infty}\sqrt[n]{\frac{1}{n}} = 0^{\frac{1}{\infty}} = [0^0]$$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{n+1} = 1 \Rightarrow \lim_{n\to\infty}\sqrt[n]{\frac{1}{n}} = 1$$

$$\rho = \frac{1}{1} = 1 \Rightarrow (x_0-\rho;\ x_0+\rho) = (0-1;\ 0+1) = (-1;1)$$

$$\rho = \frac{1}{L}$$

$$(-1;1) \subseteq D \subseteq [-1;1]$$

$-1 \in D$?

$$x=-1 \Rightarrow \sum_{n=1}^{+\infty}(-1)^{n+1}\cdot\frac{(-1)^n}{n} = \sum_{n=1}^{+\infty}\frac{(-1)^{2n+1}}{n} = \sum_{n=1}^{+\infty} -\frac{1}{n} = -\sum_{n=1}^{+\infty}\frac{1}{n} \text{ diverg.}$$

$$-1 \notin D$$

$$x=1 \Rightarrow \sum_{n=1}^{+\infty}(-1)^{n+1}\cdot\frac{1^n}{n} = \sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n} = \sum_{n=1}(-1)^n\cdot\frac{-1}{n}$$

$$\sum^{+\infty} |(-1)^{n+1}| \sum^{+\infty} \frac{}{} \text{ diver}$$

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} = \sum_{n=1}^{+\infty} (-1) \cdot \frac{(-1)^n}{n} = -\sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$$

$\frac{1}{n} \searrow 0 \quad \frac{1}{n} \geq 0$

$\Rightarrow \sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$ conv. $\Rightarrow \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n}$ conv.

$\sum_{n=1}^{+\infty} (-1)^{n+1} \cdot \frac{x^n}{n}$ - abs. conv. pe $(-1;1)$

- diverg. $(-\infty;-1] \cup (1;+\infty)$
- conv/semiconv. în 1

$D = (-1;1]$

$f : (-1;1] \to \mathbb{R} \qquad f(x) = \sum_{n=1}^{+\infty} (-1)^{n+1} \cdot \frac{x^n}{n}$

f pe $(-1;1)$ este de clasă C^∞

f continuă în $x=1$ $\qquad f(1) = \lim\limits_{\substack{x\to 1\\ x<1}} f(x)$

$$f'(x) = \left(\sum_{n=1}^{+\infty} (-1)^{n+1} \cdot \frac{x^n}{n}\right)' = \sum_{n=1}^{+\infty} \left((-1)^{n+1} \cdot \frac{x^n}{n}\right)' = \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{\cancel{n} \cdot x^{n-1}}{\cancel{n}} = \sum_{n=1}^{+\infty} (-1)^{n+1} x^{n-1} =$$

$$= \sum_{n=1}^{+\infty} (-1)^{n-1} \cdot (-1)^2 \cdot x^{n-1} = \sum_{n=1}^{+\infty} (-1)^{n-1} \cdot x^{n-1}$$

$\left.\sum_{m=0}^{+\infty} (-1)^m x^m = \frac{1}{1+x}\right\} \Rightarrow \sum_{n=1}^{+\infty} (-1)^{n-1} \cdot x^{n-1} = \frac{1}{1+x}$

$m = n-1$

$f'(x) = \frac{1}{1+x} \Rightarrow f(x) = \int \frac{1}{1+x}\, dx$

$\int \frac{1}{1+x}\, dx = \int (1+x)^{-1}\, dx = \ln(1+x) + \mathcal{C}$

$f(x) = \ln(1+x) + c$

$\left.\begin{array}{l} f(0) = \ln(1+0) + c = \ln 1 + c = c \\ f(x_0) = a_0 \Rightarrow f(0) = 0 \end{array}\right\} \Rightarrow c = 0$

$f(x_0) = a_0 \Rightarrow f(0) = 0$

$$\sum_{n=1}^{+\infty} (-1)^{n+1} \cdot \frac{x^n}{n} = (-1)^2 \cdot \frac{x}{1} + (-1)^3 \cdot \frac{x^2}{2} + \dots$$
$$= \underset{a_0}{0} \cdot x^0 + (-1)^2 \cdot \frac{x}{1} + (-1)^3 \cdot \frac{x^2}{2} + \dots$$

$$f(x) = \ln(1+x)$$

$$f(1) = \lim_{\substack{x \to 1 \\ x < 1}} f(x) = \lim_{\substack{x \to 1 \\ x < 1}} \ln(1+x) = \ln 2,$$

Ex 3 $\quad S : (0;1) \longrightarrow \mathbb{R}$

$$S(x) = \sum_{n \geq 1} \frac{1}{n^2 + x}$$

Demonstrați că S este funcție de clasă C^∞

Rezolvare

S = funcție de clasă $C^\infty \Rightarrow S$ integrabilă / derivabilă de ∞ ori

$$\frac{1}{n^2+x} \leq \frac{1}{n^2} \quad \frac{1}{n^2} \text{ converg.} \Rightarrow \sum_{n \geq 1} \frac{1}{n^2+x} \quad \text{convergentă}$$

$$S'(x) = S_1(x) = \sum_{n \geq 1} \left(\frac{1}{n^2+x}\right)' = \sum_{n \geq 1} \frac{-1}{(n^2+x)^2}$$

$$\left| -\frac{1}{(n^2+x)^2} \right| = \frac{1}{(n^2+x)^2} = \frac{1}{n^4 + 2n^2 x + x^2} \leq \frac{1}{n^4}$$

$$\sum_{n \geq 1} \frac{1}{n^4} \text{ converg} \Rightarrow \sum_{n \geq 1} \frac{-1}{(n^2+x)^2} \text{ abs. conv}$$

$$S''(x) = S_2(x) = \sum_{n \geq 1} \left(\frac{1}{n^2+x}\right)'' = \sum_{n \geq 1} \frac{2}{(n^2+x)^3}$$

$$\left| \frac{2}{(n^2+x)^3} \right| = \frac{2}{(n^2+x)^3} \leq \frac{2}{n^6}$$

$$\sum_{n \geq 1} \frac{2}{n^6} = 2 \cdot \sum_{n \geq 1} \frac{1}{n^6} \text{ conv}$$

$$\Rightarrow \sum_{n \geq 1} \frac{2}{(n^2+x)^3} \text{ conv.}$$

Generalizare: $S_k = S^{(k)}(x) = \sum_{n \geq 1} \left(\frac{1}{(n^2+x)}\right)^{(k)} = \sum_{n \geq 1} \left[(n^2+x)^{-1}\right]^{(k)} = \sum_{n \geq 1} \frac{(-1)^k k!}{(n^2+x)^{k+1}}$

$\left| (-1)^k \, k! \right. \quad \quad k! \quad \quad k!$

$$\left|\frac{(-1)^k\, k!}{(n^2+x)^{k+1}}\right| \leq \frac{k!}{n^{2(k+1)}} = \frac{k!}{n^{2k+2}}$$

$$\sum_{n\geq 1} \frac{k!}{n^{2k+2}} = k! \cdot \sum_{n\geq 1} \frac{1}{n^{2k+2}} \text{ conv}$$

$\Rightarrow S_k$ conv

Prin inducție, vom avea că S_k conv. $\forall\, k \geq 1$ $k \in \mathbb{N}$

$\Rightarrow$ S derivabilă de oricâte ori $\Rightarrow$ S funcție de clasă C^∞

Ex 5) $f: \mathbb{R} \to \mathbb{R}$ $f(x) = \cos x$ în jurul $c=0$

Dezvoltați în serie de puteri

Rezolvare:

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

Inducție: $f^{(n)}(x) = \cos\left(x + \frac{n\pi}{2}\right)$

$P(n) \Rightarrow P(n+1)$

$f^{(n)}(x) = \cos\left(x + \frac{n\pi}{2}\right)$

$$f^{(n+1)}(x) = \left(\cos\left(x + \frac{n\pi}{2}\right)\right)' = -\sin\left(x + \frac{n\pi}{2}\right) \cdot 1 = -\sin\left(x + \frac{n\pi}{2}\right)$$

$\sin x = -\cos\left(x + \frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2} - x\right)$

$$f^{(n+1)}(x) = -\left(-\cos\left(x + \frac{n\pi}{2} + \frac{\pi}{2}\right)\right) = \cos\left(x + \frac{(n+1)\pi}{2}\right)$$

Pt. $c = 0$

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} \cdot x^k + \frac{f^{(n+1)}(\alpha)}{(n+1)!} \cdot x^{n+1}$$

$$= \sum_{k=0}^{n} \frac{\cos\left(\frac{\pi}{2} \cdot k\right)}{k!} \cdot x^k + \frac{\cos\left(\alpha + \frac{\pi}{2}\cdot(n+1)\right)}{(n+1)!} \cdot x^{n+1}$$

$$= \underbrace{\sum_{k=0} \frac{\cos(\frac{\pi}{2} k)}{k!} \cdot x^k}_{} + \underbrace{\frac{\cos(\alpha + \frac{\pi}{2} \cdot (n+1))}{(n+1)!} \cdot x^{n+1}}_{}$$

$$\left| R_{f,n,0}(x) \right| \le \frac{M^{n+1}}{(n+1)!} \xrightarrow{n \to \infty} 0$$

$$f(x) = \cos x = \sum_{n \ge 0} \frac{\cos(\frac{\pi}{2} \cdot n)}{n!} \cdot x^n$$