Seminar 7

1. Studiați convergența simplă și uniformă pentru următoarele șiruri de funcții:

a) $f_n : [0,\infty) \to \mathbb{R}$, $f_n(x) = \dfrac{x}{x+n}$ $\forall n \in \mathbb{N}^*$.

<u>Soluție</u>. <u>Convergența simplă</u>

Fie $x \in [0,\infty)$.

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{x+n} = 0 \Longrightarrow f_n \xrightarrow[n\to\infty]{\Delta} f,$$

unde $f : [0,\infty) \to \mathbb{R}$, $f(x) = 0$.

<u>Convergența uniformă.</u>

$$\sup_{x\in[0,\infty)} |f_n(x) - f(x)| = \sup_{x\in[0,\infty)} \left|\frac{x}{x+n} - 0\right| = \sup_{x\in[0,\infty)} \frac{x}{x+n} \geq$$

$$\underset{x=n}{\geq} \frac{n}{n+n} = \underset{\substack{\vee \\ 0}}{\frac{1}{2}} \not\xrightarrow[n\to\infty]{} 0 \Rightarrow f_n \not\xrightarrow[n\to\infty]{u} f. \quad \square$$

b) $f_n : [2,3] \to \mathbb{R}$, $f_n(x) = \dfrac{x}{x+n}$ $\forall n \in \mathbb{N}$.

Soluție. **Convergența simplă**

Fie $x \in [2,3]$.

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{x+n} = 0 \Rightarrow f_n \xrightarrow[n\to\infty]{\Delta} f,$$

unde $f : [2,3] \to \mathbb{R}$, $f(x) = 0$.

Convergența uniformă

$$\sup_{x\in[2,3]} |f_n(x) - f(x)| = \sup_{x\in[2,3]} \left|\frac{x}{x+n} - 0\right| = \sup_{x\in[2,3]} \frac{x}{x+n}.$$

Fie $f_n : [2,3] \to \mathbb{R}$, $f_n(x) = \frac{x}{x+n}$ $\forall n \in \mathbb{N}$.

$$f_n'(x) = \frac{\cancel{x}+n-\cancel{x}}{(x+n)^2} = \frac{n}{(x+n)^2} \geq 0 \ \forall x \in [2,3], \forall n \in \mathbb{N}.$$

Deci f_n este crescătoare $\forall n \in \mathbb{N}$.

Așadar $\sup_{x\in[2,3]} \frac{x}{x+n} = \frac{3}{3+n} \xrightarrow[n\to\infty]{} 0$, i.e.

$$f_n \xrightarrow[n\to\infty]{u} f. \qquad \square$$

c) $f_n : [0,\infty) \to \mathbb{R}$, $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$ $\forall\, n \in \mathbb{N}^*$.

<u>Soluție.</u> <u>Convergența simplă</u>

Fie $x \in [0,\infty)$.

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \sqrt{x^2 + \frac{1}{n}} = \sqrt{x^2 + 0} = \sqrt{x^2} = |x| \underset{x\in[0,\infty)}{\overset{\uparrow}{=}} x \Rightarrow$$

$$\Rightarrow f_n \xrightarrow[n\to\infty]{s} f, \text{ unde } f : [0,\infty) \to \mathbb{R},\ f(x) = x.$$

<u>Convergența uniformă</u>

$$\sup_{x\in[0,\infty)} |f_n(x) - f(x)| = \sup_{x\in[0,\infty)} \left| \sqrt{x^2 + \frac{1}{n}} - x \right| =$$

$$= \sup_{x\in[0,\infty)} \left| \frac{\cancel{x^2} + \frac{1}{n} - \cancel{x^2}}{\sqrt{x^2 + \frac{1}{n}} + x} \right| = \sup_{x\in[0,\infty)} \frac{\frac{1}{n}}{\sqrt{x^2 + \frac{1}{n}} + x} =$$

$$= \frac{\frac{1}{n}}{\sqrt{\frac{1}{n}}} = \sqrt{\frac{1}{n}} \xrightarrow[n\to\infty]{} 0 \Rightarrow f_n \xrightarrow[n\to\infty]{u} f. \ \square$$

d) $f_n : [0,\infty) \to \mathbb{R}$, $f_n(x) = \dfrac{n}{n+x}$ $\forall\, n \in \mathbb{N}^*$.

<u>Soluție.</u> <u>Convergența simplă</u>

Fie $x \in [0,\infty)$.

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{n}{n+x} = 1 \Rightarrow f_n \xrightarrow[n\to\infty]{s} f, \text{ unde}$$

$f : [0,\infty) \to \mathbb{R},\ f(x) = 1.$

<u>Convergența uniformă.</u>

$$\sup_{x\in[0,\infty)} |f_n(x) - f(x)| = \sup_{x\in[0,\infty)} \left| \frac{n}{n+x} - 1 \right| =$$

$$= \sup_{x\in[0,\infty)} \left| \frac{\cancel{n} - \cancel{n} - x}{n+x} \right| = \sup_{x\in[0,\infty)} \left| \frac{-x}{n+x} \right| = \sup_{x\in[0,\infty)} \frac{x}{n+x}.$$

Fie $g_n : [0,\infty) \to \mathbb{R},\ g_n(x) = \dfrac{x}{n+x}\ \forall n \in \mathbb{N}^*.$

$$g_n'(x) = \frac{n + \cancel{x} - \cancel{x}}{(n+x)^2} = \frac{n}{(n+x)^2} > 0\ \forall n \in \mathbb{N}^*$$

Deci g_n este crescătoare $\forall n \in \mathbb{N}^*$.

Așadar $\sup_{x\in[0,\infty)} \dfrac{x}{n+x} = \lim_{x\to\infty} \dfrac{x}{n+x} = 1 \underset{n\to\infty}{\not\longrightarrow} 0$,

i.e. $f_n \overset{u}{\underset{n\to\infty}{\not\longrightarrow}} f$. $\square$

e) $f_n : (0,1] \to \mathbb{R},\ f_n(x) = x^n\ \forall n \in \mathbb{N}.$

<u>Soluție</u>. <u>Convergența simplă</u>

Fie $x \in (0,1]$.

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \begin{cases} 0 ; & x \in (0,1) \\ 1 ; & x = 1 \end{cases} \Rightarrow f_n \xrightarrow[n\to\infty]{s} f,$$

unde $f : (0,1] \to \mathbb{R}$, $f(x) = \begin{cases} 0 ; & x \in (0,1) \\ 1 ; & x = 1. \end{cases}$

Convergența uniformă

$\left.\begin{array}{l} f_n \text{ continuă } \forall\, n \in \mathbb{N} \\ f \text{ nu este continuă (în 1)} \end{array}\right| \Rightarrow f_n \cancel{\xrightarrow[n\to\infty]{u}} f. \ \square$

f) $f_n : \left[\frac{1}{2}, 1\right] \longrightarrow \mathbb{R}$, $f_n(x) = \dfrac{(1+x)^n}{e^{2nx}}$ $\forall\, n \in \mathbb{N}^*$.

Soluție. Convergența simplă

Fie $x \in \left[\frac{1}{2}, 1\right]$.

Avem $f_n(x) = \dfrac{(1+x)^n}{e^{2nx}} = \left(\dfrac{1+x}{e^{2x}}\right)^n = \left(f_1(x)\right)^n$ $\forall\, n \in \mathbb{N}^*$.

Fie $g : \left[\frac{1}{2}, 1\right] \to \mathbb{R}$, $g(x) = 1 + x - e^{2x}$.

$g'(x) = 1 - 2e^{2x} < 0$ $\forall\, x \in \left[\frac{1}{2}, 1\right]$.

x	$\frac{1}{2}$			1
$g'(x)$	$-$	$-$	$-$	$-$
$g(x)$	$\frac{3}{2} - e$	$\searrow$	$\searrow$	$2 - e^2$

Deci $g(x) < 0$ $\forall\, x \in \left[\frac{1}{2}, 1\right]$.

Prin urmare $0 < f_1(x) < 1 \ \forall x \in [\frac{1}{2}, 1]$.

Așadar $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} (f_1(x))^n = 0$, deci

$f_n \xrightarrow[n\to\infty]{\Delta} f$, unde $f: [\frac{1}{2}, 1] \to \mathbb{R}$, $f(x) = 0$.

<u>Convergența uniformă</u>

Avem: 1) $[\frac{1}{2}, 1]$ mulțime compactă.

2) f_n continuă $\forall n \in \mathbb{N}^*$, f continuă

3) $(f_n)_n$ descrescător.

4) $f_n \xrightarrow[n\to\infty]{\Delta} f$.

Conform Teoremei lui Dini $f_n \xrightarrow[n\to\infty]{u} f$. $\square$

g) $f_n: [\frac{1}{2}, \frac{\pi}{2}] \to \mathbb{R}$, $f_n(x) = \cos^n x \ \forall n \in \mathbb{N}^*$.

<u>Soluție.</u> <u>Convergența simplă</u>

Fie $x \in [\frac{1}{2}, \frac{\pi}{2}]$.

$\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \cos^n x = 0 \Rightarrow f_n \xrightarrow[n\to\infty]{\Delta} f$, unde

$f: [\frac{1}{2}, \frac{\pi}{2}] \to \mathbb{R}$, $f(x) = 0$.

Convergența uniformă

$$\begin{matrix} x & \longrightarrow & \cos x \\ \cap & & \cap \\ \left[\frac{1}{2}, \frac{\pi}{2}\right] & & \mathbb{R} \end{matrix}$$ descrescătoare $\Rightarrow f_n$ descrescătoare $\forall n \in \mathbb{N}^*$.

Avem : 1) $f_n : \left[\frac{1}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$, $f_n(x) = \cos^n x \ \forall n \in \mathbb{N}^*$.

2) f_n descrescătoare $\forall n \in \mathbb{N}^*$.

3) $f_n \xrightarrow[n\to\infty]{s} f$

4) f continuă.

Conform Teoremei lui Polya $f_n \xrightarrow[n\to\infty]{u} f$. $\square$

2. Studiați convergența simplă și uniformă pentru $(f_n)_n$ și $(f_n')_n$, unde:

a) $f_n : [0,\pi] \to \mathbb{R}$, $f_n(x) = \dfrac{\cos nx}{n} \ \forall n \in \mathbb{N}^*$.

Soluție. Pentru $(f_n)_n$

Convergența simplă

Fie $x \in [0,\pi]$. Avem $\underbrace{-\frac{1}{n}}_{\to 0} \le \underbrace{f_n(x)}_{\to 0} \le \underbrace{\frac{1}{n}}_{\to 0} \ \forall n \in \mathbb{N}^*$, deci $\lim\limits_{n\to\infty} f_n(x)=0$, i.e. $f_n \xrightarrow[n\to\infty]{s} f$, unde

$f:[0,\pi]\to\mathbb{R}$, $f(x)=0$.

Convergența uniformă

$$\sup_{x\in[0,\pi]} |f_n(x)-f(x)| = \sup_{x\in[0,\pi]} \left|\frac{\cos nx}{n}-0\right| =$$

$$= \sup_{x\in[0,\pi]} \frac{|\cos nx|}{n} \le \frac{1}{n} \xrightarrow[n\to\infty]{} 0 \Rightarrow f_n \xrightarrow[n\to\infty]{u} f.$$

Pentru $(f_n')_n$

$$f_n'(x)=\left(\frac{\cos nx}{n}\right)' = -\sin nx \quad \forall\, x\in[0,\pi],\ \forall\, n\in\mathbb{N}^*.$$

Convergența simplă

Fie $x=\frac{\pi}{2}$.

$$f_{4n}'\left(\frac{\pi}{2}\right) = -\sin\left(4n\frac{\pi}{2}\right) = -\sin(2n\pi)=0 \xrightarrow[n\to\infty]{} 0.$$

$$f_{4n+1}'\left(\frac{\pi}{2}\right) = -\sin\left(4n\frac{\pi}{2}+\frac{\pi}{2}\right) = -\sin\left(2n\pi+\frac{\pi}{2}\right) =$$

$$= -\sin\frac{\pi}{2} = -1 \xrightarrow[n\to\infty]{} -1.$$

Deci $\nexists \lim\limits_{n\to\infty} f_n'\left(\frac{\pi}{2}\right)$.

Prin urmare $(f_n')_n$ nu este simplu convergent.

<u>Convergența uniformă</u>

$(f_n')_n$ nu este simplu convergent $\Rightarrow (f_n')_n$ nu este uniform convergent. $\square$

b) $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \frac{\operatorname{arctg} nx}{n}$ $\forall n \in \mathbb{N}^*$.

<u>Soluție.</u> <u>Pentru $(f_n)_n$</u>

<u>Convergența simplă</u>

Fie $x \in \mathbb{R}$. Avem $\frac{-\frac{\pi}{2}}{n} \leq f_n(x) \leq \frac{\frac{\pi}{2}}{n}$ $\forall n \in \mathbb{N}^*$, deci

$f_n \xrightarrow[n\to\infty]{\Delta} f$, unde $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 0$.

<u>Convergența uniformă</u>

$$\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = \sup_{x\in\mathbb{R}} \left| \frac{\operatorname{arctg} nx}{n} - 0 \right| =$$

$$= \sup_{x\in\mathbb{R}} \frac{|\operatorname{arctg} nx|}{n} \leq \frac{\frac{\pi}{2}}{n} \xrightarrow[n\to\infty]{} 0 \Rightarrow f_n \xrightarrow[n\to\infty]{u} f.$$

<u>Pentru $(f_n')_n$</u>

$$f_n'(x) = \left(\frac{1}{n} \cdot \operatorname{arctg} nx\right)' = \frac{1}{\cancel{n}} \cdot \frac{1}{1+n^2x^2} \cdot \cancel{n} = \frac{1}{1+n^2x^2} \ \forall x \in$$

$\in \mathbb{R}, \forall n \in \mathbb{N}^*$.

Convergența simplă

Fie $x \in \mathbb{R}$.

$$\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} \frac{1}{1+n^2x^2} = \begin{cases} 1 ; & x = 0 \\ 0 ; & x \in \mathbb{R}^* \end{cases} \Rightarrow$$

$$\Rightarrow f_n' \xrightarrow[n\to\infty]{s} g, \text{ unde } g : \mathbb{R} \to \mathbb{R},\ g(x) = \begin{cases} 1 ; & x = 0 \\ 0 ; & x \in \mathbb{R}^* \end{cases}.$$

Convergența uniformă

$$\left.\begin{array}{l} f_n' \text{ continuă } \forall n \in \mathbb{N}^* \\ g \text{ nu e continuă (în 0)} \end{array}\right| \Rightarrow f_n' \overset{u}{\underset{n\to\infty}{\not\longrightarrow}} g.$$