Tutoriat 5

$$f(x) = T_{f,n,x_0}(x) + \underbrace{R_{f,n,x_0}(x)}_{<\,>0}$$

① $\underbrace{\ln(1+x)}_{f} < x - \dfrac{x^2}{2} + \dfrac{x^3}{3}, \ \forall x \in (0, \infty)$ //

Alegem funcția $f : (0, \infty) \longrightarrow \mathbb{R}$, $f(x) = \ln(1+x) \in C^{\infty}$

f este de $\underline{4}$ ori derivabilă.

$f'(x) = (\ln(1+x))' = \dfrac{1}{1+x}, \ \forall x \in (0, \infty)$

$f''(x) = \left(\dfrac{1}{1+x}\right)' = -\dfrac{1}{(1+x)^2}, \ \forall x \in (0, \infty)$

$f'''(x) = \left(-\dfrac{1}{(1+x)^2}\right)' = \dfrac{2}{(1+x)^3}, \ \forall x \in (0, \infty)$

$f^{(4)}(x) = \left(\dfrac{2}{(1+x)^3}\right)' = -\dfrac{6}{(1+x)^4}, \ \forall x \in (0, \infty)$

Aplicăm formula lui Taylor cu restul sub forma lui Lagrange

$\forall x \in (0, \infty)$, $\exists\, c \in (0, x)$ a.î. $f(x) = \underbrace{T_{f,3,0}(x)}\ + \underbrace{\dfrac{f^{(4)}(c)}{4!} \cdot x^4}$ //

$$T_{f,3,0}(x) = f(0) + \frac{f'(0)}{1!} x^1 + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} \cdot x^3$$

$$= 0 + \frac{1}{1} \cdot x + \frac{-1}{2!} x^2 + \frac{2}{3!} x^3$$

$$= x + \frac{x^2}{2} + \frac{x^3}{3}$$

Avem deci că $\underline{f(x)} = \underbrace{x - \frac{x^2}{2} + \frac{x^3}{3}} + \underbrace{\frac{f^{(4)}(c)}{4!} \cdot x^4}$ $\Rightarrow \ln(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}$, $\forall x \in (0, \infty)$

$\gtrless$

$$\frac{f^{(4)}(c)}{4!} x^4 = -\frac{6}{(1+c)^4} \cdot \underbrace{\frac{1}{4!}}_{24} \cdot x^4 = -\frac{1}{4} \cdot \frac{x^4}{(1+c)^4} < 0$$

① $\underline{\ln(1+x)}$ ...

f

Alegem f...

f este de...

$f'(x) = ($...

$f''(x) =$ ...

$f'''(x)$ ...

$f^{(4)}($...

$52| \leq \left| \frac{n \cdot x^{52}}{n} - x^{52} \right|$

(2) $f_n : [1,29] \longrightarrow \mathbb{R}$, $f_n(x) = \frac{[n \cdot x^{52}]}{n}$, $n \in \mathbb{N}$

conv. simplă și uniformă

! $k-1 < [k] \leq k$

CS: fie $x \in [1,29]$

$$n \cdot x^{52} - 1 < [n \cdot x^{52}] \leq n \cdot x^{52} \quad | : n$$

$$\frac{n \cdot x^{52} - 1}{n} < \frac{[n \cdot x^{52}]}{n} \leq \frac{n \cdot x^{52}}{n}$$

$$\frac{n \cdot x^{52} - 1}{n} \xrightarrow{n \to \infty} x^{52}, \qquad \frac{[n \cdot x^{52}]}{n} \to x^{52}, \qquad \frac{n \cdot x^{52}}{n} \xrightarrow{n \to \infty} x^{52}$$

$$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{[n \cdot x^{52}]}{n} = x^{52}$$

deci $f_n \xrightarrow[x \in [1,29]]{s} f$, $f(x) = x^{52}$, $f : [1,29] \longrightarrow \mathbb{R}$

CU: $\lim\limits_{n\to\infty} \sup\limits_{x\in[1,29]} |f_n(x)-f(x)|$ $\left|\frac{[n\cdot x^{52}]}{n} - x^{52}\right| \le \left|\frac{n\cdot x^{52}}{n} - x^{52}\right|$

$\sup\limits_{x\in[1,29]} |f_n(x)-f(x)| = \sup\limits_{x\in[1,29]}$

$= 0$

$\lim\limits_{n\to\infty} \sup\limits_{x\in[1,29]} |f_n(x)-f(x)| = 0$

$\Rightarrow f_n \xrightarrow[x\in[1,29]]{u} f$

② $f_n: [1,2$

Conv. simpl

CS: fie $x \in$

$)^{\frac{2}{4}} \cdot (x^8+y^4)^{\frac{5}{8}+\frac{2}{4}-1}$

$= (x^8+y^4)^{\frac{1}{8}} \xrightarrow[y\to 0]{x\to 0} 0$

$\mathbb{R}^2$

$r^{\frac{5}{8}+\frac{2}{4}} \cdot r^{-1}$

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① $f(x,y) = \begin{cases} \dfrac{x^5 \cdot y^2}{x^8+y^4}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}, \quad f:\mathbb{R}^2 \to \mathbb{R}$

f continuă pe $\mathbb{R}^2 \setminus \{(0,0)\}$ (operații cu funcții elementare)

studiem cont în $(0,0)$

$l = \lim\limits_{\substack{x\to 0\\ y\to 0}} |f(x,y) - f(0,0)| = 0 \to$ în forma asta verificăm continuitatea

$|f(x,y) - f(0,0)| = \left| \dfrac{x^5 y^2}{x^8+y^4} - 0 \right|$

$= \dfrac{|x^5| y^2}{x^8+y^4} = |x| \cdot \dfrac{x^4 y^2}{x^8+y^4} \leq \dfrac{|x|}{2} \xrightarrow{x\to 0} 0$

$l = 0 \Rightarrow f$ cont pe $\mathbb{R}^2$

[$m_g \leq m_a$]

$\sqrt{x^8 y^4} \leq \dfrac{x^8+y^4}{2}$

$\dfrac{x^4 y^2}{x^8+y^4} \leq \dfrac{1}{2}$

(V2) $= \left(\frac{x^8}{x^8+y^4}\right)^{\frac{5}{8}} \cdot \left(\frac{y^4}{x^8+y^4}\right)^{\frac{2}{4}} \cdot (x^8+y^4)^{\frac{5}{8}+\frac{2}{4}-1}$

$\underbrace{\left(\frac{x^8}{x^8+y^4}\right)}_{\leq 1}$, $\underbrace{\left(\frac{y^4}{x^8+y^4}\right)}_{\leq 1}$

$\leq (x^8+y^4)^{\frac{5}{8}+\frac{2}{4}-1} = (x^8+y^4)^{\frac{1}{8}} \xrightarrow[y\to 0]{x\to 0} 0$

$l = 0 \Rightarrow f$ cont pe $\mathbb{R}^2$

$= \frac{x^{8\cdot\frac{5}{8}}}{\cancel{n^{\frac{5}{8}}}} \cdot \frac{y^{4\cdot\frac{2}{4}}}{\cancel{n^{\frac{2}{4}}}} \cdot n^{\frac{5}{8}+\frac{2}{4}} \cdot n^{-1}$

(1) $f(x,y) = \begin{cases} \frac{x^5 \cdot y \ldots}{x^8 + \ldots} \\ 0 \end{cases}$

f continuă ...

Studiem c...

$l = \lim\limits_{\substack{x\to 0\\ y\to 0}} |f(x,y) - \ldots$

$|f(x,y)| \ldots$

(V1)

$= \ldots$

4) $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=\begin{cases}\dfrac{x^2y^3}{x^2+y^4}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

f e continuă pe $\mathbb{R}^2\setminus\{(0,0)\}$

$$\lim_{(x,y)\to(0,0)} |f(x,y)-f(0,0)| = \lim_{(x,y)\to(0,0)} \left|\frac{x^2y^3}{x^2+y^4}\right| \overset{\left[\frac{0}{0}\right]}{=} ?$$

Luăm șirurile $x_n=\frac{1}{n}$, $y_n=\frac{1}{n}$;

$$\lim_{n\to\infty} \frac{\frac{1}{n^2}\cdot\frac{1}{n^3}}{\frac{1}{n^2}+\frac{1}{n^4}} = \lim_{n\to\infty} \frac{1}{n^3}\cdot\frac{n^4}{n^2+1} = \lim_{n\to\infty}\frac{1}{n(n^2+1)} = 0$$

$$0 \le |f(x,y)-f(0,0)| = \left|\frac{x^2y^3}{x^2+y^4}\right| \le \left|\frac{x^2y^3}{x^2}\right| = |y^3| \xrightarrow{(x,y)\to(0,0)} 0$$

$\Rightarrow \lim_{(x,y)\to(0,0)}|f(x,y)-f(0,0)|=0$

$\Rightarrow f$ e cont. în $(0,0)$.

$f=\{\ 0, (x,y)=(0,0)$

5) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

f e cont pe $\mathbb{R}^2 \setminus \{(0,0)\}$.

Luăm $x_n = \frac{1}{n}$, $y_n = 0$;

$$\lim_{n\to\infty} f\left(\frac{1}{n}, 0\right) = \lim_{n\to\infty} 0 = \underline{0}$$

Luăm $x_n = 0$, $y_n = \frac{1}{n}$

$$\lim_{n\to\infty} f\left(0, \frac{1}{n}\right) = \underline{0}$$

Luăm $x_n = \frac{1}{n}$, $y_n = \frac{1}{n}$

$$\lim_{n\to\infty} f\left(\frac{1}{n}, \frac{1}{n}\right) = \lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{2}{n^2}} = \underline{\frac{1}{2}} \neq 0$$

$\Rightarrow f$ nu e cont în $(0,0)$.

$x^2+y^2 \neq 0$

(6) $f(x,y) = \begin{cases} \dfrac{(x^2-y^2)^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

f. continuă pe $\mathbb{R}^2 \setminus \{(0,0)\}$ (op. cu funcții elementare)

Studiem cont. în $(0,0)$.

Fie $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$$|f(x,y) - f(0,0)| = \left| \frac{(x^2-y^2)^2}{x^2+y^2} - 0 \right| = \frac{(x^2-y^2)^2}{x^2+y^2} = (x^2-y^2) \cdot \underbrace{\frac{x^2-y^2}{x^2+y^2}}_{\leq 1}$$

$$\leq (x^2-y^2) \cdot 1 \xrightarrow[y \to 0]{x \to 0} 0$$

f. continuă pe $\mathbb{R}^2$

(7) Să se studieze conv simplă și conv uniformă

$f_n : \mathbb{R} \to \mathbb{R}, \ f_n(x) = \frac{2nx}{n^2+x^2}, \ \forall x \in \mathbb{R}, \forall n \in \mathbb{N}^*$

* Studiem convergența simplă

$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{2nx}{n^2+x^2} = 0, \ x \neq 0, \quad f_n(0) = 0$

Luăm funcția $f : A \to \mathbb{R}, \ f(x) = 0$, unde $A = \mathbb{R}$

Spunem că $f_n \xrightarrow[\mathbb{R}]{s} f$

* Studiem convergența uniformă: $f_n(n)$

Fixăm $n \in \mathbb{N}^*$

$\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = \sup_{x\in\mathbb{R}} \left|\frac{2nx}{n^2+x^2}\right| \underset{x=n}{\geq} \frac{2n^2}{2n^2} = 1$

Cum $\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| \geq 1 > 0 \Rightarrow \lim_{n\to\infty} \sup_{x\in\mathbb{R}} |f_n(x) - f(x)| \neq 0 \Rightarrow f_n \overset{u}{\underset{\mathbb{R}}{\not\to}} f$

$0 \leq \sup_{x\in\mathbb{R}} |f_n(x) - f(x)| \leq$ ceva ce tinde la 0

$1 <$

(8) $f(x,y) = \begin{cases} \frac{(x^2-y^2)^2}{x^2+y^2}, & (x,y) \neq \\ 0, & (x,y) = (0,0) \end{cases}$

f continuă pe $\mathbb{R}^2 \setminus \{(0,0)\}$

Studiem cont. în $(0,0)$.

Fie $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$|f(x,y) - f(0,0)| = \left| \frac{(x^2}{x} \right.$

$\leq$

f continuă în