Definiții!

$f:[a,b] \longrightarrow \mathbb{R}$ mărg $\Delta: a = x_0 < x_1 < \dots < x_n = b$

Suma Riemann : $\sigma_\Delta(f, (\alpha_i)_{i=\overline{0,n-1}}) = \sum_{i=0}^{n-1} f(\alpha_i)(x_{i+1} - x_i)$

Suma Darboux Superioară

$$S_\Delta(f) = \sum_{i=0}^{n-1} M_i (x_{i+1} - x_i) \qquad M_i = \sup_{x \in [x_i, x_{i+1}]} f(x)$$

Suma Darboux Inferioară

$$s_\Delta(f) = \sum_{i=0}^{n-1} m_i (x_{i+1} - x_i), \quad m_i = \inf_{x \in [x_i, x_{i+1}]} f(x)$$

Integrala Definită $I = \int_a^b f(x)\,dx$

Integrala Sup. $\overline{\int_a^b} f = \inf_\Delta S_\Delta(f)$

Integrala Inf $\underline{\int_a^b} f = \sup_\Delta s_\Delta(f)$

Ex1 Studiați I. Riemann

a) $f:[0;1] \longrightarrow \mathbb{R}$, $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0,1] \\ 2, & x \in \mathbb{R} \setminus \mathbb{Q} \cap [0,1] \end{cases}$

Obs : f int. Riemann $\Leftrightarrow$ $\overline{\int_a^b} f = \underline{\int_a^b} f$

$\overline{\int_0^1} f(x)\,dx = \inf_\Delta S_\Delta(f)$

$\Delta: x_0 = 0 < x_1 < x_2 < \dots < x_n = 1$

$S_\Delta(f) = \sum_{i=0}^{n-1} M_i (x_{i+1} - x_i)$

$S_\Delta(f) = M_0(x_1 - x_0) + M_1(x_2 - x_1) + \dots + M_{n-1}(x_n - x_{n-1})$

$$\left.\begin{array}{l} M_0 = \sup_{t \in [x_0, x_1]} f(t) = 2 \\ M_1 = \sup_{t \in [x_1, x_2]} f(t) = 2 \\ \text{------} \\ M_{n-1} = \sup_{t \in [x_{n-1}, x_n]} f(t) = 2 \end{array}\right\} \Rightarrow$$

$S_\Delta(f) = 2(x_1 - x_0) + 2(x_2 - x_1) + \dots + 2(x_n - x_{n-1})$

$= 2(\cancel{x_1} - x_0 + \cancel{x_2} - \cancel{x_1} + \dots + x_n - \cancel{x_{n-1}})$

$= 2(x_n - x_0) = 2(1 - 0) = 2 \cdot 1 = 2$

$S_\Delta(f) = 2 \Rightarrow \overline{\int_0^1} f = 2$ (1)

$\underline{\int_0^1} f(x)\,dx = \sup_\Delta s_\Delta(f)$

$$s_\Delta(f) = m_0(x_1-x_0)+m_1(x_2-x_1)+\dots+m_{n-1}(x_n-x_{n-1})$$

$$\left.\begin{array}{l} m_0 = \inf\limits_{t\in[x_0,x_1]} f(t) = 1 \\ m_1 = \inf\limits_{t\in[x_1,x_2]} f(t) = 1 \\ \dots \\ m_{n-1} = 1 \end{array}\right\} \Rightarrow s_\Delta f = 1\cdot(\cancel{x_1}-x_0+\cancel{x_2}-\cancel{x_1}+\dots+x_n-\cancel{x_{n-1}}) = 1\cdot(x_n-x_0) = 1-0 = 1$$

$$\underline{\int_0^1} f = 1 \quad (2)$$

Din (1) și (2) $\Rightarrow$ f nu este int. Riemann

b) $f:[0,1]\to\mathbb{R}$ $\quad f(x)=\begin{cases}\dfrac{x^2}{x+1}, & x\in[0,1)\\ 2, & x=1\end{cases}$

f continuă pe $[0,1)$

$$\left.\begin{array}{l}\lim\limits_{\substack{x\to 1\\ x<1}} f(x) = \lim\limits_{\substack{x\to1\\ x<1}} \dfrac{x^2}{x+1} = \dfrac{1}{2} \\ f(1)=2\end{array}\right\} \Rightarrow f \text{ nu e cont. în } 1$$

D_f = m. punctelor de discont.

$D_f=\{1\}$ $\Rightarrow$ m. finită $\Rightarrow$ D_f m. neglijabilă Lebesgue (1)

$f(x)\geq 0$ $\forall x\in[0,1]$ $\Rightarrow$ f mărg. inferior

$f(x)\leq 2 \Leftrightarrow \dfrac{x^2}{x+1}\leq 2$ sau $2\leq 2$ (A) $\Leftrightarrow$ $\dfrac{x^2}{x+1}\leq 2$ $\quad\forall x\in[0,1) \Leftrightarrow x^2-2x-2\leq 0$ $\quad\forall x\in[0,1)$ $\Rightarrow$

$\Leftrightarrow x(x-2)-2\leq 0$ Adevărat $\forall x\in[0,1)$

$\Rightarrow f(x)\leq 2$ $\forall x\in[0,1]$ $\Rightarrow$ f mărg. superior

f mărg. (2)

Din (1) și (2) $\Rightarrow$ f int. Riemann

c) $f:[0,1]\to\mathbb{R}$ $\quad f(x)=\begin{cases}\dfrac{1}{2^n}, & \exists n\in\mathbb{N}^* \text{ a.î. } x=\dfrac{1}{n}\\ 0, & x\neq\dfrac{1}{n}\ \forall n\in\mathbb{N}^*\end{cases}$

$$\left.\begin{array}{l} f(x) \geq 0 \quad \forall x \in [0,1] \Rightarrow f \text{ m. inf.} \\ f(x) \leq \frac{1}{2} \quad \forall x \in [0,1] \Rightarrow f \text{ m. sup.} \end{array}\right\} \Rightarrow f \text{ mărg. } (1)$$

$$\left.\begin{array}{l} f \text{ continuă pe } [0;1] \setminus \{\frac{1}{n}, n \in \mathbb{N}^*\} \Rightarrow D_f \subseteq \{\frac{1}{n} \mid n \in \mathbb{N}^*\} \\ \{\frac{1}{n} \mid n \in \mathbb{N}^*\} \text{ numărabilă} \Rightarrow \text{neglijabilă Lebesgue} \end{array}\right\} \Rightarrow$$

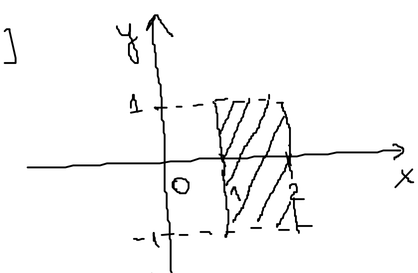
D_f neglijabilă Lebesgue (2)

Din (1) și (2) $\Rightarrow f$ int. Riemann

Integrale Multiple : $\iint_D f(x,y)\,dx\,dy$

* D = interval precizat.

<u>EX2</u> Calculați $\iint_D (y^2+2x^2)\,dx\,dy \quad D=[1,2]\times[-1,1]$



Rezolvare:

$f: D \longrightarrow \mathbb{R} \quad f(x,y) = y^2+2x^2$

$$\iint_{[1,2]\times[-1,1]} f(x,y)\,dx\,dy = \int_{-1}^{1}\left(\int_{1}^{2} f(x,y)\,dx\right)dy = \int_{1}^{2}\left(\int_{-1}^{1} f(x,y)\,dy\right)dx =$$

$$= \int_{-1}^{1}\left(\int_{1}^{2}(y^2+2x^2)\,dx\right)dy = \int_{-1}^{1}\left(\int_{1}^{2} y^2\,dx + 2\int_{1}^{2} x^2\,dx\right)dy =$$

$$= \int_{-1}^{1}\left(y^2 x\Big|_1^2 + 2\cdot\frac{x^3}{3}\Big|_1^2\right)dy = \int_{-1}^{1}\left(2y^2-y^2+2\cdot\frac{8}{3}-2\cdot\frac{1}{3}\right)dy =$$

$$= \int_{-1}^{1}\left(y^2+\frac{14}{3}\right)dy = \int_{-1}^{1} y^2\,dy + \frac{14}{3}\int_{-1}^{1} 1\,dy = \frac{y^3}{3}\Big|_{-1}^{1} + \frac{14}{3}y\Big|_{-1}^{1} = \frac{1}{3}+\frac{1}{3}+\frac{14}{3}+\frac{14}{3} =$$

$$= \frac{30}{3} = 10$$

* D nu este precizat ca interval ci ca mulțime

<u>EX3</u> Calculați $\iint_D (xy^2)\,dx\,dy$, unde $D = \{(x,y)\in\mathbb{R}^2 \mid y \geq x^2,\ y \leq x\}$

Rezolvare:

$f : D \to \mathbb{R} \quad f(x,y) = xy^2$

f continuă pe D

$D = \{(x,y) \in \mathbb{R}^2 \mid y \geq x^2, y \leq x\}$

D mulțimea din $\mathbb{R}^2$ cuprinsă între curbele de ecuație $y = x^2$ și $y = x$

$$\begin{cases} y \geq x^2 \\ y \leq x \end{cases} \Rightarrow x^2 \leq y \leq x \Rightarrow x^2 \leq x \Rightarrow x^2 - x \leq 0 \Rightarrow x(x-1) \leq 0 \Rightarrow x \in [0;1]$$

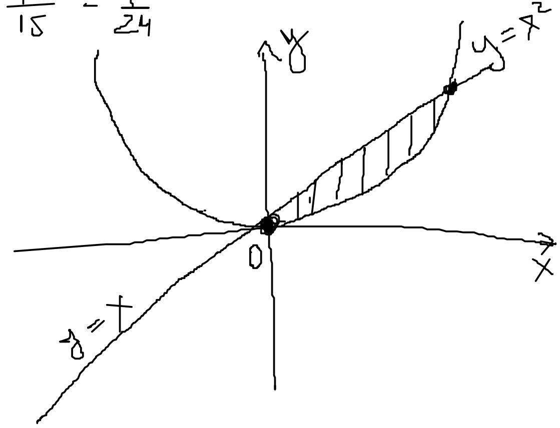
$$\left.\begin{cases} x \in [0,1] \\ y \in [x^2, x] \end{cases} \Rightarrow D \text{ simplă în rap. cu axa } Oy \Rightarrow \begin{array}{l} D \in \mathcal{J}(\mathbb{R}) \\ D = \text{închisă} \end{array}\right\} \Rightarrow$$

$\Rightarrow f$ int. Riemann

$$\iint_D f(x,y)\,dx\,dy = \int_0^1 \left(\int_{x^2}^{x} f(x,y)\,dy \right) dx = \int_0^1 \left(\int_{x^2}^{x} xy^2\,dy \right) dx =$$

$$= \int_0^1 \left(\frac{xy^3}{3} \Big|_{x^2}^{x} \right) dx = \int_0^1 \left(\frac{x \cdot x^3}{3} - \frac{x \cdot x^6}{3} \right) dx = \frac{1}{3} \int_0^1 x^4\,dx -$$

$$- \frac{1}{3} \int_0^1 x^7\,dx = \frac{1}{3} \cdot \frac{x^5}{5} \Big|_0^1 - \frac{1}{3} \frac{x^8}{8} \Big|_0^1 = \frac{1}{15} - \frac{1}{24}$$



* Schimbare de Variabilă / Coordonate Polare

pt $A = \{(x,y) \in \mathbb{R}^2 \;/\; (x-a)^2 + (y-b)^2 \leq r^2\}$

$(x-a)^2 + (y-b)^2 = r^2 \to \mathcal{C}((a,b), r)$

$f : A \to \mathbb{R} \quad f(x,y)$

$\varphi : \mathbb{R}^2 \to \mathbb{R}^2 \quad \varphi(R, \theta) = (\underbrace{R\cos\theta}_{x}, \underbrace{R\sin\theta}_{y})$

Condiții generale $R \geq 0 \quad \theta \in [0, 2\pi] \;/\; [-\pi, \pi]$

$$B: \begin{cases} (R\cos\theta - a)^2 + (R\sin\theta - b)^2 \leq r^2 \\ R \geq 0 \\ \theta \in [0; 2\pi) \;/\; \theta \in [-\pi, \pi] \end{cases} \to \text{se scot } R \text{ și } \theta \text{ intervale}$$

$$I = \iint_A f(x,y)\,dx\,dy = \iint_B f(R\cos\theta, R\sin\theta)\,|R|\,dR\,d\theta$$

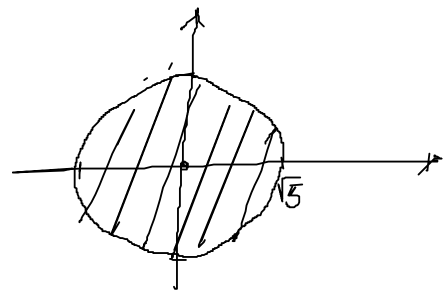
$R \in [p,q]$

$A = \varphi(B) \longrightarrow B$

$(x,y) \longrightarrow (R\cos\theta, R\sin\theta)$ $\qquad \theta \in [m,n]$

$dx\,dy \longrightarrow |R|\,dR\,d\theta$

Ex4. $\iint_A e^{x^2+y^2}\,dx\,dy \qquad A = \{(x,y) / x^2+y^2 \le 5\}$



Rezolvare:

$$\left.\begin{array}{l} x^2+y^2=5 \Rightarrow a=b=0 \\ r=\sqrt{5} \end{array}\right\} \Rightarrow \mathcal{C}((0,0),\sqrt{5})$$

$f : A \to \mathbb{R} \quad f(x,y) = e^{x^2+y^2}$

$\varphi : \mathbb{R}^2 \Longrightarrow \mathbb{R}^2 \quad \varphi(R,\theta) = (\underbrace{R\cos\theta}_{x}, \underbrace{R\sin\theta}_{y})$

$R \ge 0, \quad \theta \in [0, 2\pi]$

$A: x^2+y^2 \le 5$

$$B : \begin{cases} (R\cos\theta)^2 + (R\sin\theta)^2 \le 5 \\ R \ge 0 \\ \theta \in [0,2\pi] \end{cases} \Leftrightarrow \begin{cases} R^2 \le 5 \\ R \ge 0 \\ \theta \in [0,2\pi] \end{cases} \Rightarrow \begin{cases} R \in [0,\sqrt{5}] \\ \theta \in [0,2\pi] \end{cases}$$

$B = [0,\sqrt{5}] \times [0, 2\pi]$

$$\iint_{A=\varphi(B)} f(x,y)\,dx\,dy = \iint_B f(R\cos\theta, R\sin\theta)\,|R|\,dR\,d\theta = \iint_{[0;\sqrt{5}]\times[0;2\pi]} e^{R^2\cos^2\theta + R^2\sin^2\theta}\,|R|\,dR\,d\theta =$$

$$= \int_0^{2\pi}\left(\int_0^{\sqrt{5}} e^{R^2}|R|\,dR\right)d\theta = \int_0^{2\pi}\left(\frac{1}{2}e^5 - \frac{1}{2}e^0\right)d\theta = \int_0^{2\pi}\frac{1}{2}(e^5-1)\,d\theta = \frac{1}{2}(e^5-1)\theta\Big|_0^{2\pi}$$

$$= \pi \cdot (e^5 - 1)$$