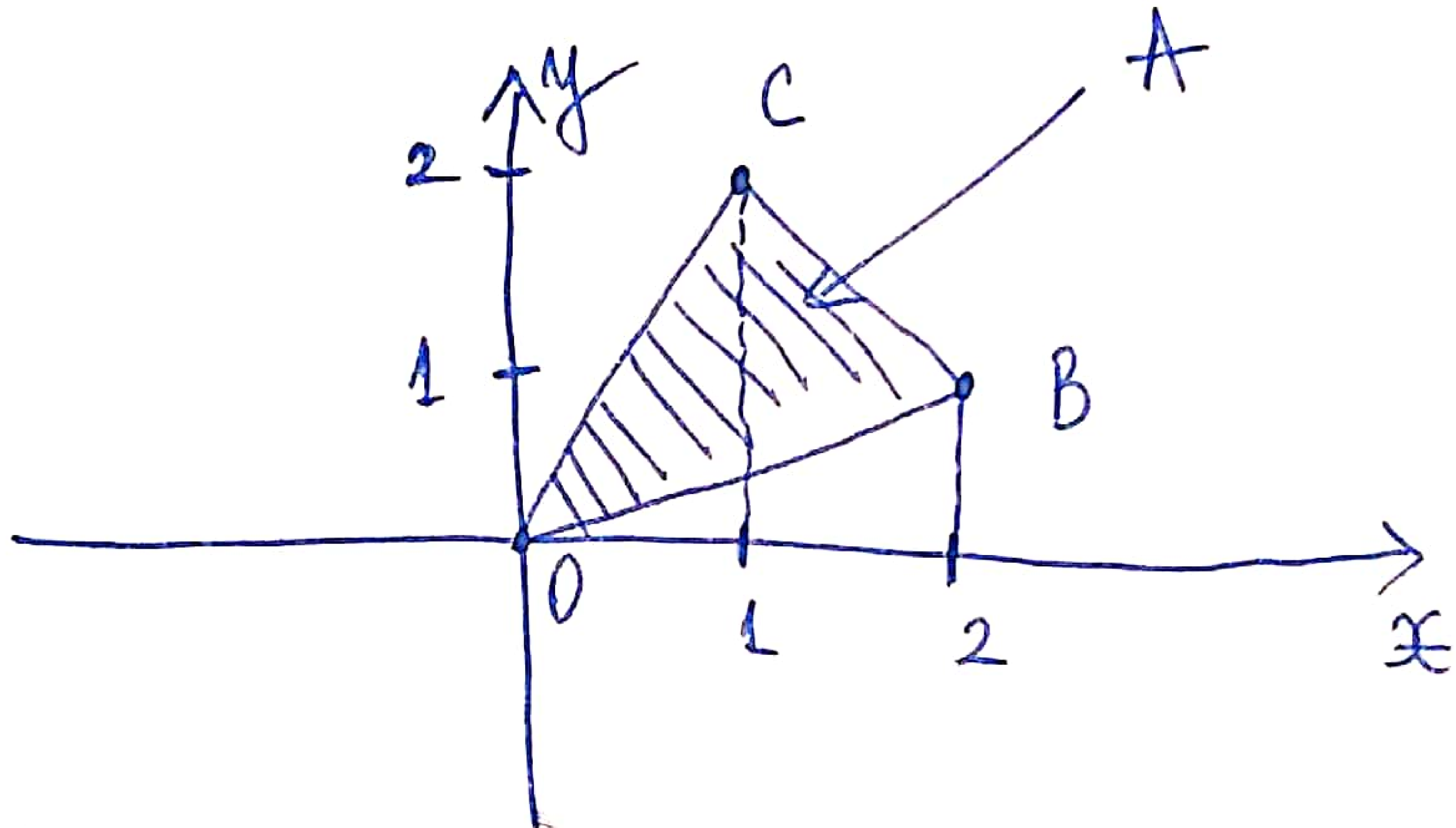
Seminar 14

1. Determinați:

a) $\iint_A x\,dx\,dy$, unde A este mulțimea plană mărginită de triunghiul OBC, $O(0,0)$, $B(2,1)$, $C(1,2)$.

Soluție.



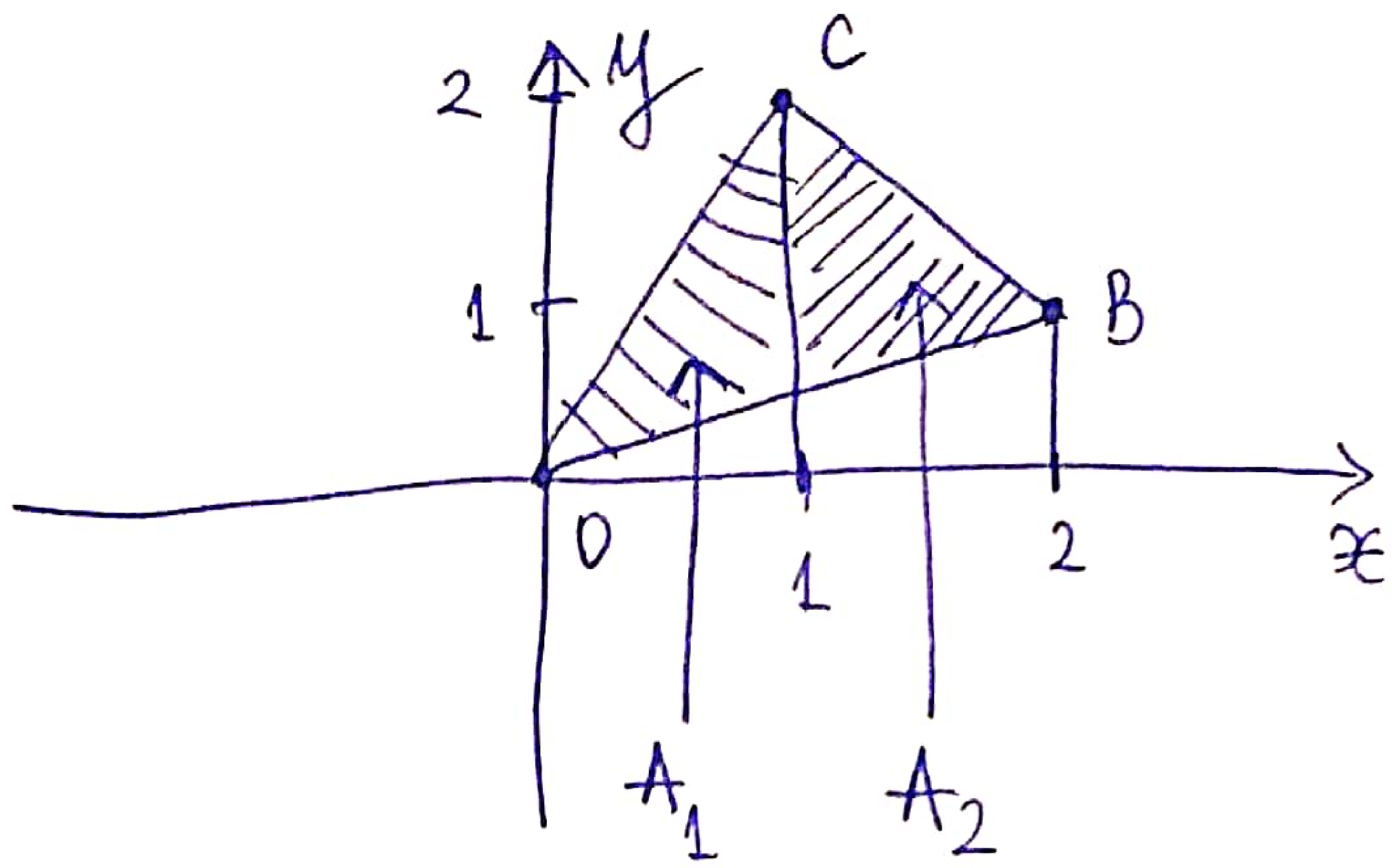
$$OB: \frac{y-y_0}{y_B-y_0}=\frac{x-x_0}{x_B-x_0} \Leftrightarrow \frac{y-0}{1-0}=\frac{x-0}{2-0} \Leftrightarrow y=\frac{x}{2}.$$

$$OC: \frac{y-y_0}{y_C-y_0}=\frac{x-x_0}{x_C-x_0} \Leftrightarrow \frac{y-0}{2-0}=\frac{x-0}{1-0} \Leftrightarrow \frac{y}{2}=x \Leftrightarrow$$

$$\Leftrightarrow y=2x.$$

$$BC: \frac{y-y_B}{y_C-y_B}=\frac{x-x_B}{x_C-x_B} \Leftrightarrow \frac{y-1}{2-1}=\frac{x-2}{1-2} \Leftrightarrow y-1=$$

$$=2-x \Leftrightarrow y=3-x.$$



$A = A_1 \cup A_2$, unde $A_1 = \{(x,y)\in\mathbb{R}^2 \mid x\in[0,1],$
$\frac{x}{2} \le y \le 2x\}$ și $A_2 = \{(x,y)\in\mathbb{R}^2 \mid x\in[1,2], \frac{x}{2}\le y \le 3-x\}$.

Fie $\alpha, \beta : [0,1] \to \mathbb{R}$, $\alpha(x) = \frac{x}{2}$, $\beta(x) = 2x$.

α, β continue.

A_1 mulțime măsurabilă Jordan și compactă.

Fie $\gamma, \delta : [1,2] \to \mathbb{R}$, $\gamma(x) = \frac{x}{2}$, $\delta(x) = 3-x$.

γ, δ continue.

A_2 mulțime măsurabilă Jordan și compactă.

$A = A_1 \cup A_2$.

$\mu(A_1 \cap A_2) = 0$.

Fie $f : A \to \mathbb{R}$, $f(x,y) = x$.

f continuă.

$$\iint_A f(x,y)\,dx\,dy = \iint_{A_1} f(x,y)\,dx\,dy + \iint_{A_2} f(x,y)\,dx\,dy.$$

$$\iint_{A_1} x\,dx\,dy = \int_0^1 \left(\int_{\frac{x}{2}}^{2x} x\,dy \right) dx = \int_0^1 xy \Big|_{y=\frac{x}{2}}^{y=2x} dx =$$

$$= \int_0^1 x\left(2x - \frac{x}{2}\right) dx = \int_0^1 \frac{3}{2} x^2\,dx = \frac{\cancel{3}}{2} \cdot \frac{x^3}{\cancel{3}} \Big|_{x=0}^{x=1} = \frac{1}{2}.$$

$$\iint_{A_2} x\,dx\,dy = \int_1^2 \left(\int_{\frac{x}{2}}^{3-x} x\,dy \right) dx = \int_1^2 xy \Big|_{y=\frac{x}{2}}^{y=3-x} dx =$$

$$= \int_1^2 x\left(3 - x - \frac{x}{2}\right) dx = \int_1^2 \left(3x - \frac{3}{2}x^2\right) dx =$$

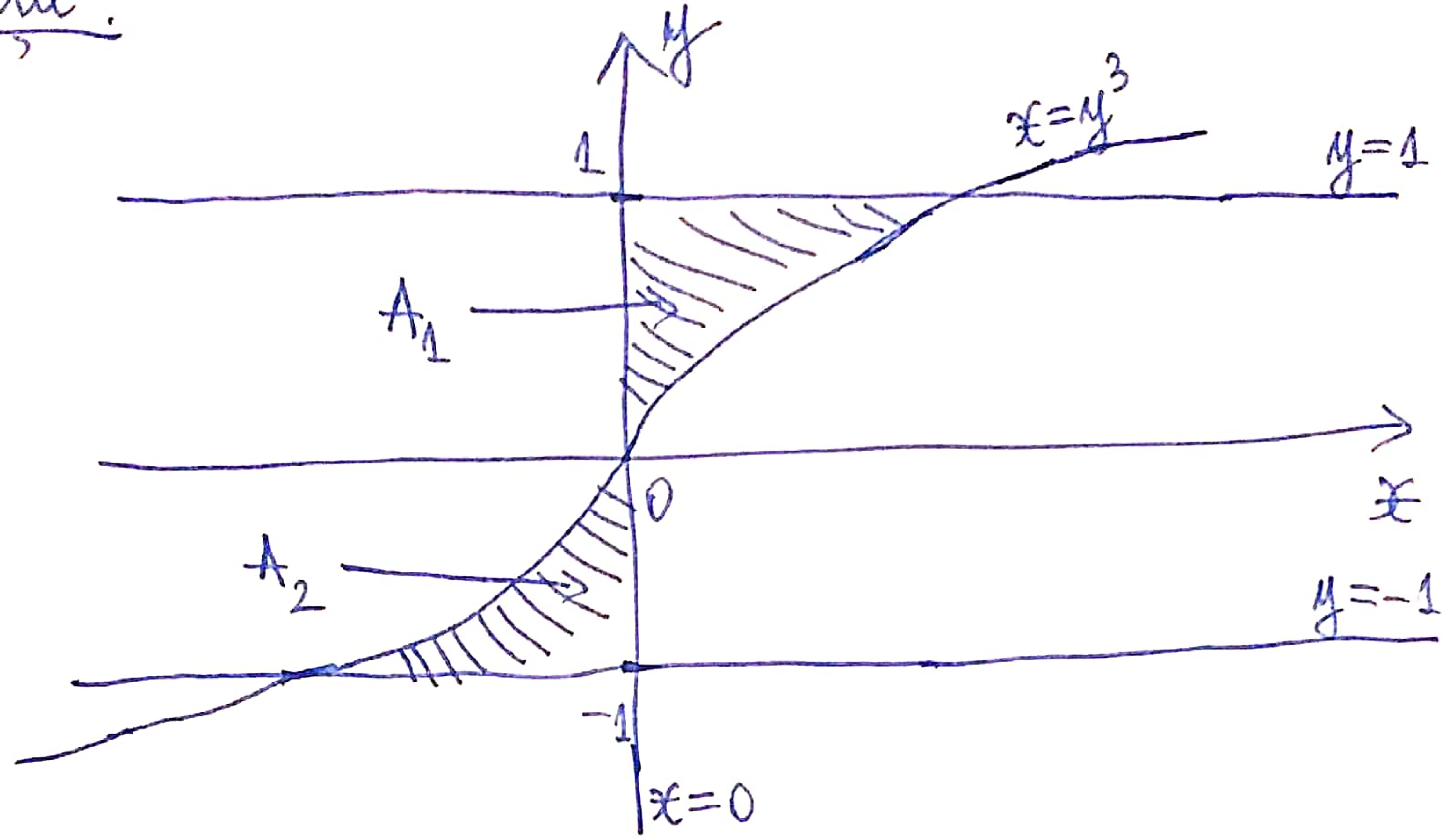
$$= 3\frac{x^2}{2}\Big|_{x=1}^{x=2} - \frac{\cancel{3}}{2}\,\frac{x^3}{\cancel{3}}\Big|_{x=1}^{x=2} = \frac{3}{2}(4-1) - \frac{1}{2}(8-1) =$$

$$= \frac{9}{2} - \frac{7}{2} = 1.$$

$$\iint_A x\,dx\,dy = \frac{1}{2} + 1 = \frac{3}{2}. \quad \square$$

b) $\iint_A e^{y^4}\,dx\,dy$, unde A este mulțimea plană mărginită de $x = y^3$, $y = 1$, $y = -1$, $x = 0$.

Soluție.



$A=A_1\cup A_2$, unde $A_1=\{(x,y)\in\mathbb{R}^2 \mid y\in[0,1],$
$0\le x\le y^3\}$, $A_2=\{(x,y)\in\mathbb{R}^2 \mid y\in[-1,0],\ y^3\le x\le 0\}$.

Fie $\varphi,\psi:[0,1]\to\mathbb{R}$, $\varphi(y)=0$, $\psi(y)=y^3$.

φ,ψ continue.

A_1 este mulțime măsurabilă Jordan și compactă.

Fie $\omega,\theta:[-1,0]\to\mathbb{R}$, $\omega(y)=y^3$, $\theta(y)=0$.

ω,θ continue.

A_2 este mulțime măsurabilă Jordan și compactă.

$\mu(A_1\cap A_2)=0$.

Fie $f:A\to\mathbb{R}$, $f(x,y)=e^{y^4}$.

f continuă.

$$\iint_A f(x,y)\,dx\,dy = \iint_{A_1} f(x,y)\,dx\,dy + \iint_{A_2} f(x,y)\,dx\,dy.$$

$$\iint_{A_1} e^{y^4}\,dx\,dy = \int_0^1 \left(\int_0^{y^3} e^{y^4}\,dx\right) dy = \int_0^1 e^{y^4} x \Big|_{x=0}^{x=y^3} dy =$$

$$= \int_0^1 e^{y^4} y^3\,dy = \frac{1}{4} e^{y^4} \Big|_{y=0}^{y=1} = \frac{1}{4}(e-1).$$

$$\iint_{A_2} e^{y^4}\,dx\,dy = \int_{-1}^0 \left(\int_{y^3}^0 e^{y^4}\,dx\right) dy = \int_{-1}^0 e^{y^4} x \Big|_{x=y^3}^{x=0} dy =$$

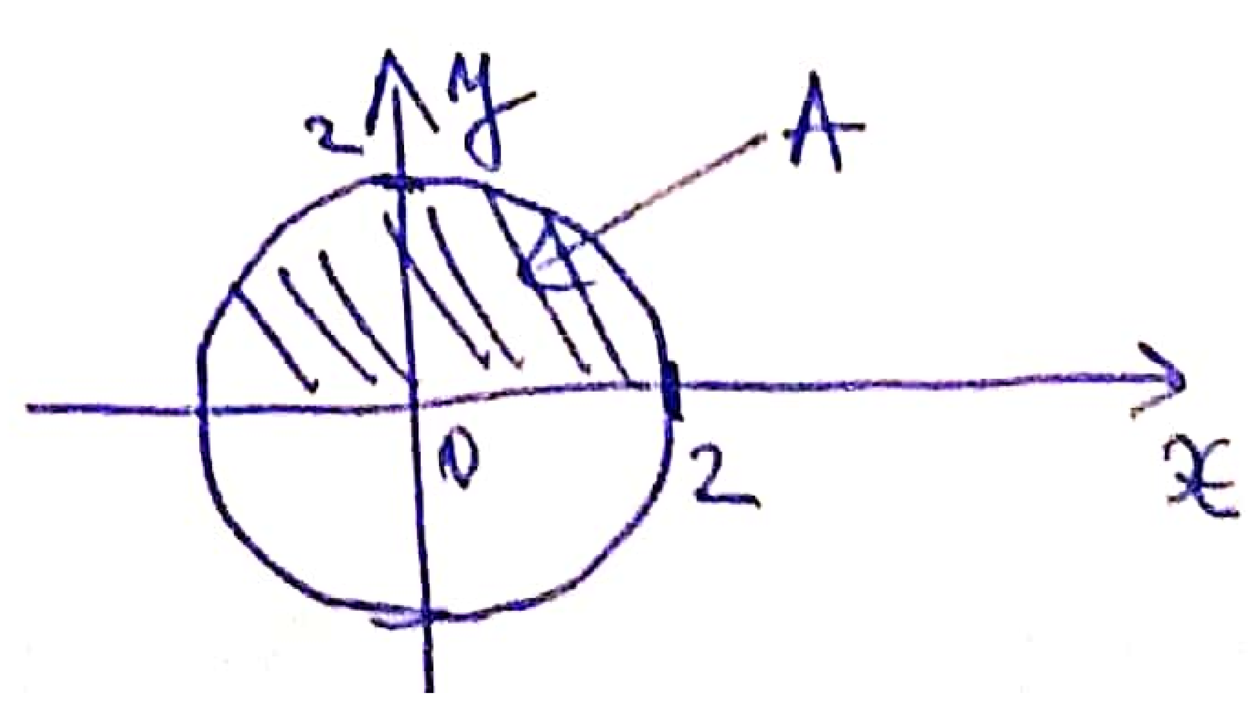
$$= \int_{-1}^0 e^{y^4}(-y^3)\,dy = -\frac{1}{4} e^{y^4} \Big|_{y=-1}^{y=0} = -\frac{1}{4}(1-e) = \frac{1}{4}(e-1).$$

$$\iint_A e^{y^4}\,dx\,dy = \frac{1}{4}(e-1) + \frac{1}{4}(e-1) = \frac{1}{2}(e-1). \;\square$$

[Observație. În exercițiile în care calculăm integralele prin schimbare de variabilă nu vom mai arăta că A este mulțime măsurabilă Jordan și că f este integrabilă Riemann.

c) $\iint_A e^{-x^2-y^2}\,dx\,dy$, unde $A = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 4, y \ge 0\}$.

Soluție.



Fie $f: A \to \mathbb{R}$, $f(x,y) = e^{-x^2-y^2}$.

S. V. $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, $r \in [0,\infty)$, $\theta \in [0, 2\pi]$.

$$(x,y) \in A \Leftrightarrow \begin{cases} x^2+y^2 \le 4 \\ y \ge 0 \end{cases} \Leftrightarrow \begin{cases} r^2 \le 4 \\ r\sin\theta \ge 0 \end{cases} \Leftrightarrow \begin{cases} r \in [0,2] \\ \sin\theta \ge 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} r \in [0,2] \\ \theta \in [0,\pi]. \end{cases}$$

Deci $B = [0,2] \times [0,\pi]$.

$$\iint_A f(x,y)\,dx\,dy = \iint_B r f(r\cos\theta, r\sin\theta)\,dr\,d\theta =$$

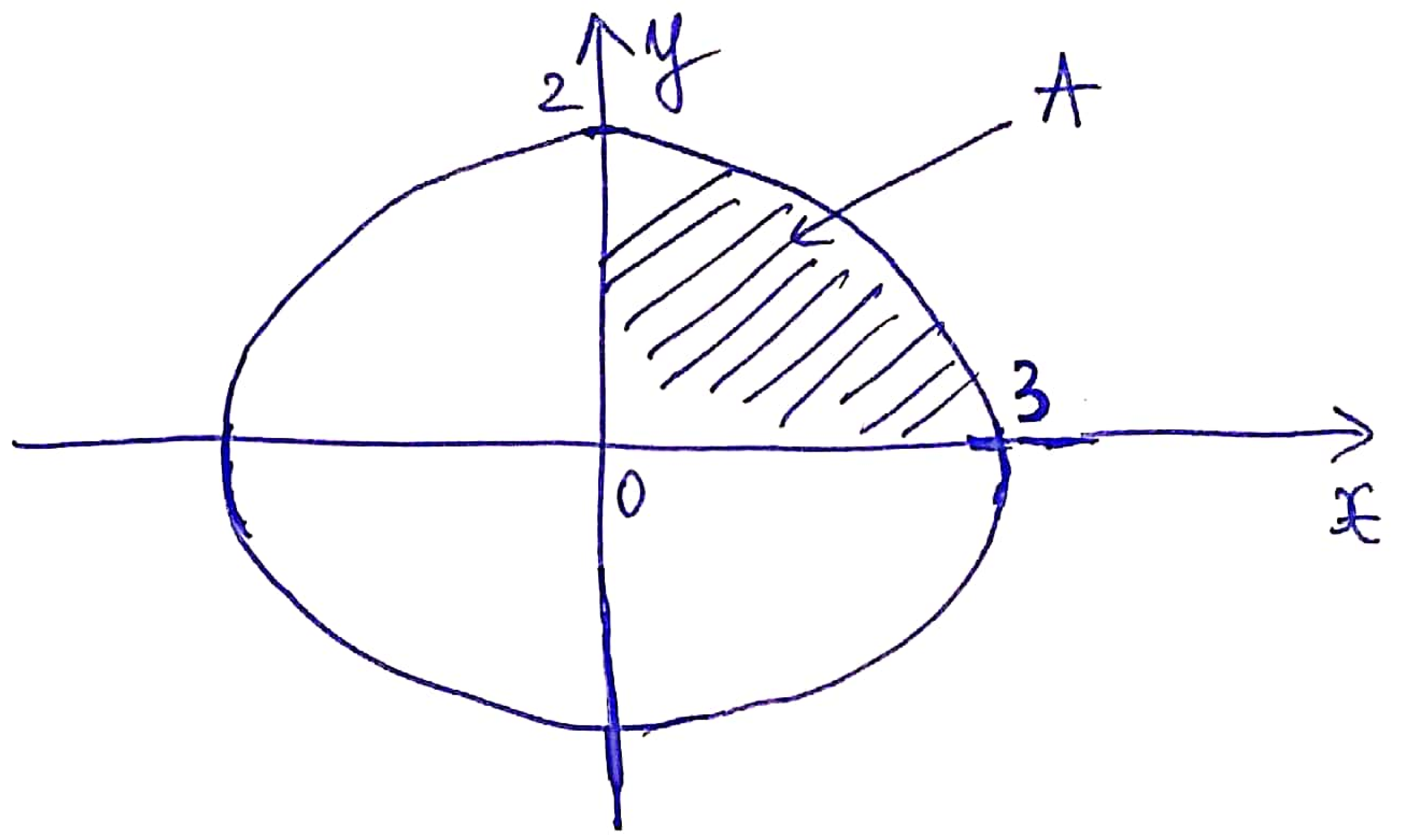
$$= \int_0^2 \left(\int_0^\pi r e^{-r^2} d\theta \right) dr = \int_0^2 r e^{-r^2} \theta \Big|_{\theta=0}^{\theta=\pi} dr =$$

$$= \int_0^2 r e^{-r^2} \pi\, dr = -\frac{\pi}{2} \int_0^2 (-2r) e^{-r^2} dr = -\frac{\pi}{2} \int_0^2 \left(e^{-r^2}\right)' dr =$$

$$= -\frac{\pi}{2} e^{-r^2} \Big|_{r=0}^{r=2} = -\frac{\pi}{2}\left(e^{-4} - 1\right) = \frac{\pi}{2}\left(1 - \frac{1}{e^4}\right). \square$$

d) $\iint_A \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}}\,dx\,dy$, unde $A = \{(x,y) \in \mathbb{R}^2 \mid \frac{x^2}{9} + \frac{y^2}{4} \le 1,\ x \ge 0,\ y \ge 0\}$.

Soluţie.



Fie $f: A \to \mathbb{R}$, $f(x,y) = \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}}$.

S.V. $\begin{cases} x = 3r\cos\theta \\ y = 2r\sin\theta \end{cases}$, $r \in [0, \infty)$, $\theta \in [0, 2\pi]$.

$(x,y) \in A \Leftrightarrow \begin{cases} \frac{x^2}{9} + \frac{y^2}{4} \le 1 \\ x \ge 0 \\ y \ge 0 \end{cases} \Leftrightarrow \begin{cases} r^2 \le 1 \\ 3r\cos\theta \ge 0 \\ 2r\sin\theta \ge 0 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} r \in [0,1] \\ \cos\theta \ge 0 \\ \sin\theta \ge 0 \end{cases} \Leftrightarrow \begin{cases} r \in [0,1] \\ \theta \in [0, \frac{\pi}{2}]. \end{cases}$

Deci $B = [0,1] \times [0, \frac{\pi}{2}]$.

$\iint_A f(x,y)\,dx\,dy = \iint_B 3\cdot 2\cdot r\, f(3r\cos\theta, 2r\sin\theta)\,dr\,d\theta =$

$= \iint_B 6r\sqrt{1-r^2}\,dr\,d\theta = \int_0^1 \left(\int_0^{\frac{\pi}{2}} 6r\sqrt{1-r^2}\,d\theta \right) dr =$

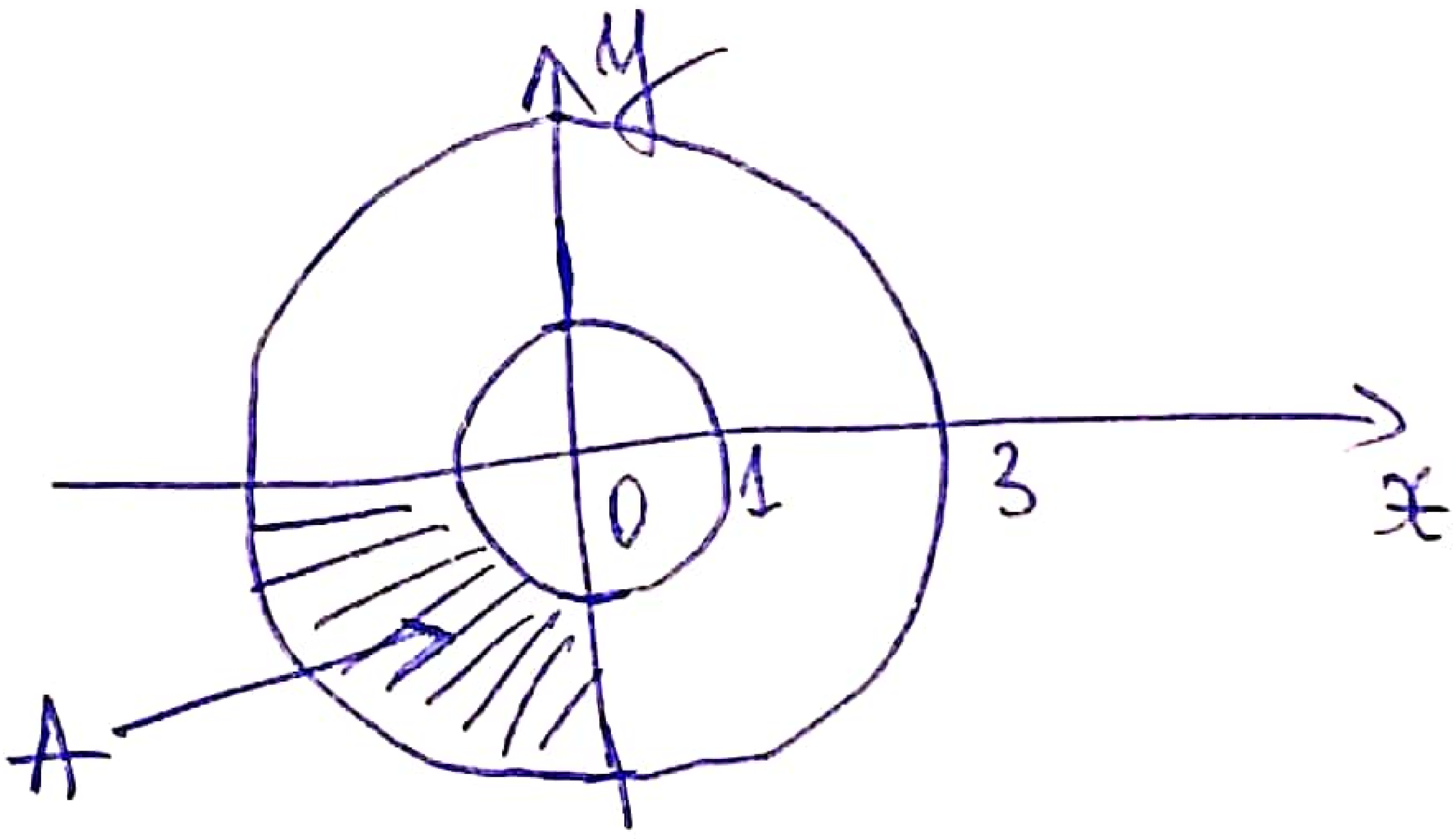
$$= \int_0^1 6r\sqrt{1-r^2}\,\theta\Big|_{\theta=0}^{\theta=\frac{\pi}{2}} dr = \frac{\pi}{2}\int_0^1 6r\sqrt{1-r^2}\,dr =$$

$$= -\frac{3\pi}{2}\int_0^1 (1-r^2)'(1-r^2)^{\frac{1}{2}}dr = -\frac{3\pi}{2}\left.\frac{(1-r^2)^{\frac{3}{2}}}{\frac{3}{2}}\right|_{r=0}^{r=1} =$$

$$= -\frac{3\pi}{2}\left(0 - \frac{1}{\frac{3}{2}}\right) = -\frac{\cancel{3}\pi}{\cancel{2}}\left(-\frac{\cancel{2}}{\cancel{3}}\right) = \pi. \quad \square$$

e) $\iint_A \sqrt{x^2+y^2}\,dx\,dy$, unde $A = \{(x,y)\in\mathbb{R}^2 \mid 1 \le x^2+y^2 \le 9,$ $x \le 0,\ y \le 0\}$.

Soluție.



Fie $f: A \to \mathbb{R}$, $f(x,y) = \sqrt{x^2+y^2}$.

S.V. $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, $r \in [0,\infty)$, $\theta \in [0, 2\pi]$.

$$(x,y)\in A \Leftrightarrow \begin{cases} 1 \le x^2+y^2 \le 9 \\ x \le 0 \\ y \le 0 \end{cases} \Leftrightarrow \begin{cases} 1 \le r^2 \le 9 \\ r\cos\theta \le 0 \\ r\sin\theta \le 0 \end{cases} \Leftrightarrow \begin{cases} r \in [1,3] \\ \cos\theta \le 0 \\ \sin\theta \le 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} r \in [1,3] \\ \theta \in \left[\pi, \frac{3\pi}{2}\right]. \end{cases}$$

Deci $B = [1,3] \times [\pi, \frac{3\pi}{2}]$.

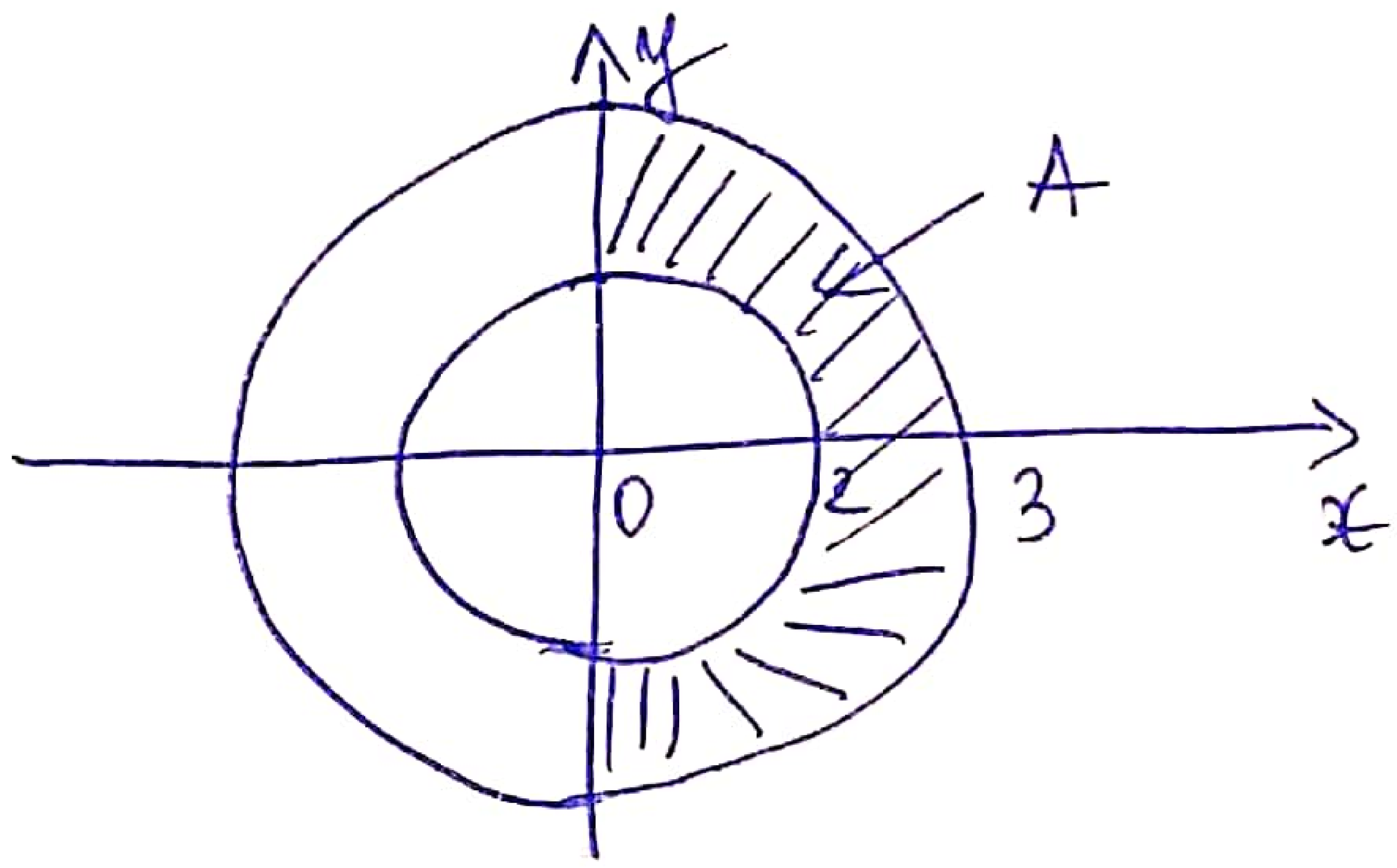
$$\iint_A f(x,y)\,dx\,dy = \iint_B r\, f(r\cos\theta, r\sin\theta)\,dr\,d\theta =$$

$$= \iint_B r\sqrt{r^2}\,dr\,d\theta = \int_1^3 \left(\int_\pi^{\frac{3\pi}{2}} r^2\,d\theta\right) dr = \int_1^3 r^2 \cdot \frac{\pi}{2}\,dr =$$

$$= \frac{\pi}{2} \cdot \frac{r^3}{3}\Big|_{r=1}^{r=3} = \frac{\pi}{6}(27-1) = \frac{\overset{13}{\cancel{26}}\pi}{\underset{3}{\cancel{6}}} = \frac{13\pi}{3}. \square$$

f) $\iint_A x\,dx\,dy$, unde $A = \{(x,y) \in \mathbb{R}^2 \mid 4 \le x^2+y^2 \le 9,\ x \ge 0\}$.

<u>Soluție.</u>



Fie $f: A \to \mathbb{R}$, $f(x,y) = x$.

S.V. $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, $r \in [0,\infty)$, $\theta \in [0, 2\pi]$.

$$(x,y) \in A \Leftrightarrow \begin{cases} 4 \le x^2+y^2 \le 9 \\ x \ge 0 \end{cases} \Leftrightarrow \begin{cases} 4 \le r^2 \le 9 \\ r\cos\theta \ge 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} r \in [2,3] \\ \theta \in [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi]. \end{cases}$$

Deci $B=[2,3]\times\left(\left[0,\frac{\pi}{2}\right]\cup\left[\frac{3\pi}{2},2\pi\right]\right)$.

$$\iint_A f(x,y)\,dx\,dy=\iint_B r\, f(r\cos\theta, r\sin\theta)\,dr\,d\theta=$$

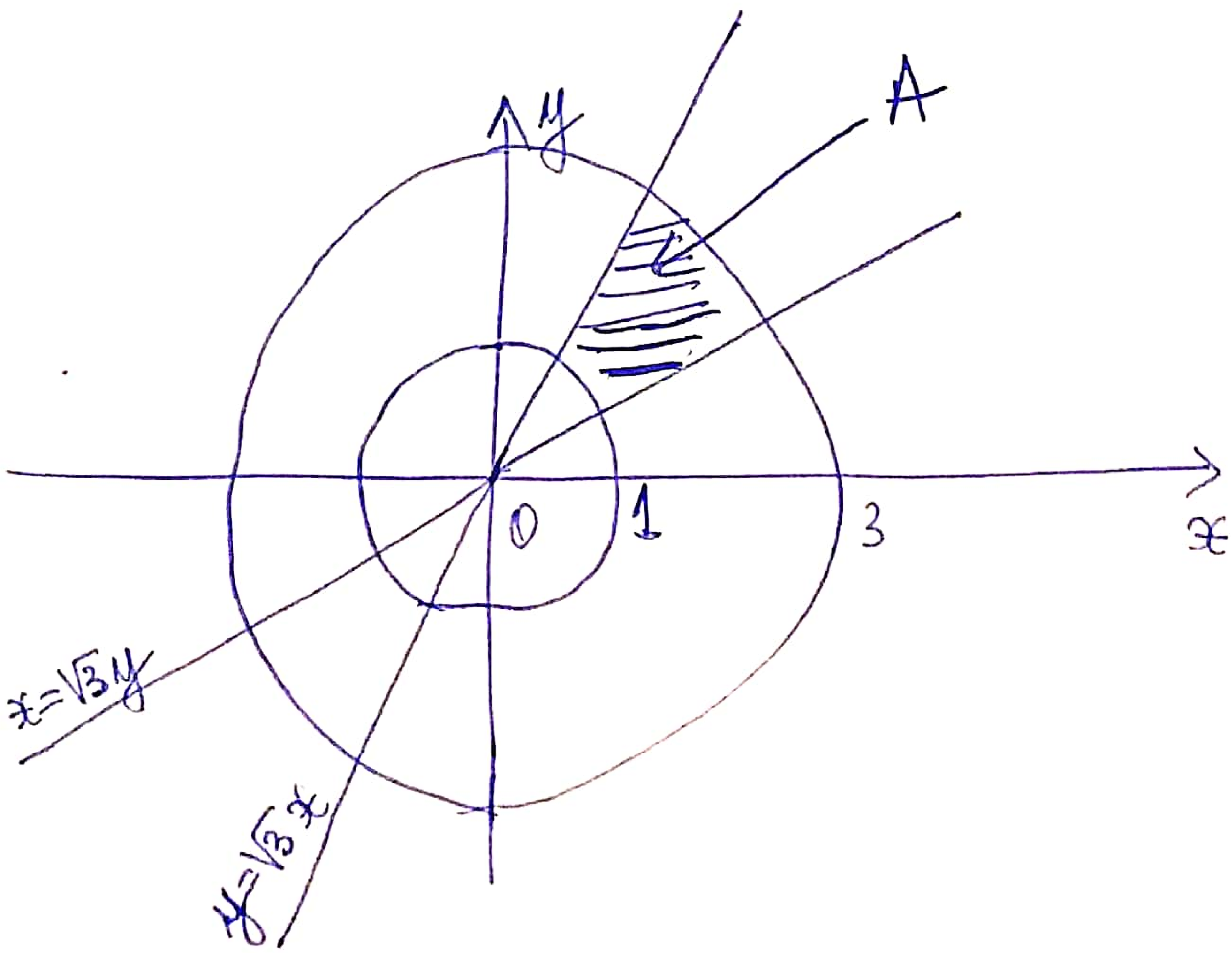
$$=\iint_B r\, r\cos\theta\,dr\,d\theta=\int_2^3\left(\int_0^{\frac{\pi}{2}} r^2\cos\theta\,d\theta+\int_{\frac{3\pi}{2}}^{2\pi} r^2\cos\theta\,d\theta\right)dr=$$

$$=\int_2^3\left(r^2\sin\theta\Big|_{\theta=0}^{\theta=\frac{\pi}{2}}+r^2\sin\theta\Big|_{\theta=\frac{3\pi}{2}}^{\theta=2\pi}\right)dr=$$

$$=\int_2^3\left(r^2+r^2\right)dr=\int_2^3 2r^2\,dr=2\,\frac{r^3}{3}\Big|_{r=2}^{r=3}=\frac{2}{3}(27-8)=\frac{38}{3}.\ \square$$

g) $\iint_A \operatorname{arctg}\frac{y}{x}\,dx\,dy$, unde $A=\{(x,y)\in\mathbb{R}^2 \mid 1\le x^2+y^2\le 9,$
$x\le\sqrt{3}\,y\le 3x\}$.

Soluție.



Fie $f: A \to \mathbb{R}$, $f(x,y) = \operatorname{arctg} \frac{y}{x}$.

S.V. $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, $r \in [0,\infty)$, $\theta \in [0, 2\pi]$.

$$(x,y) \in A \Leftrightarrow \begin{cases} 1 \le x^2 + y^2 \le 9 \\ x \le \sqrt{3}\, y \\ y \le \sqrt{3}\, x \end{cases} \Leftrightarrow \begin{cases} 1 \le r^2 \le 9 \\ r\cos\theta \le \sqrt{3}\, r\sin\theta \\ r\sin\theta \le \sqrt{3}\, r\cos\theta \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} r \in [1,3] \\ \operatorname{tg}\theta \ge \frac{1}{\sqrt{3}} \\ \operatorname{tg}\theta \le \sqrt{3} \end{cases} \Leftrightarrow \begin{cases} r \in [1,3] \\ \theta \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right] \end{cases}.$$

Deci $B = [1,3] \times \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$.

$$\iint_A f(x,y)\,dx\,dy = \iint_B r f(r\cos\theta, r\sin\theta)\,dr\,d\theta =$$

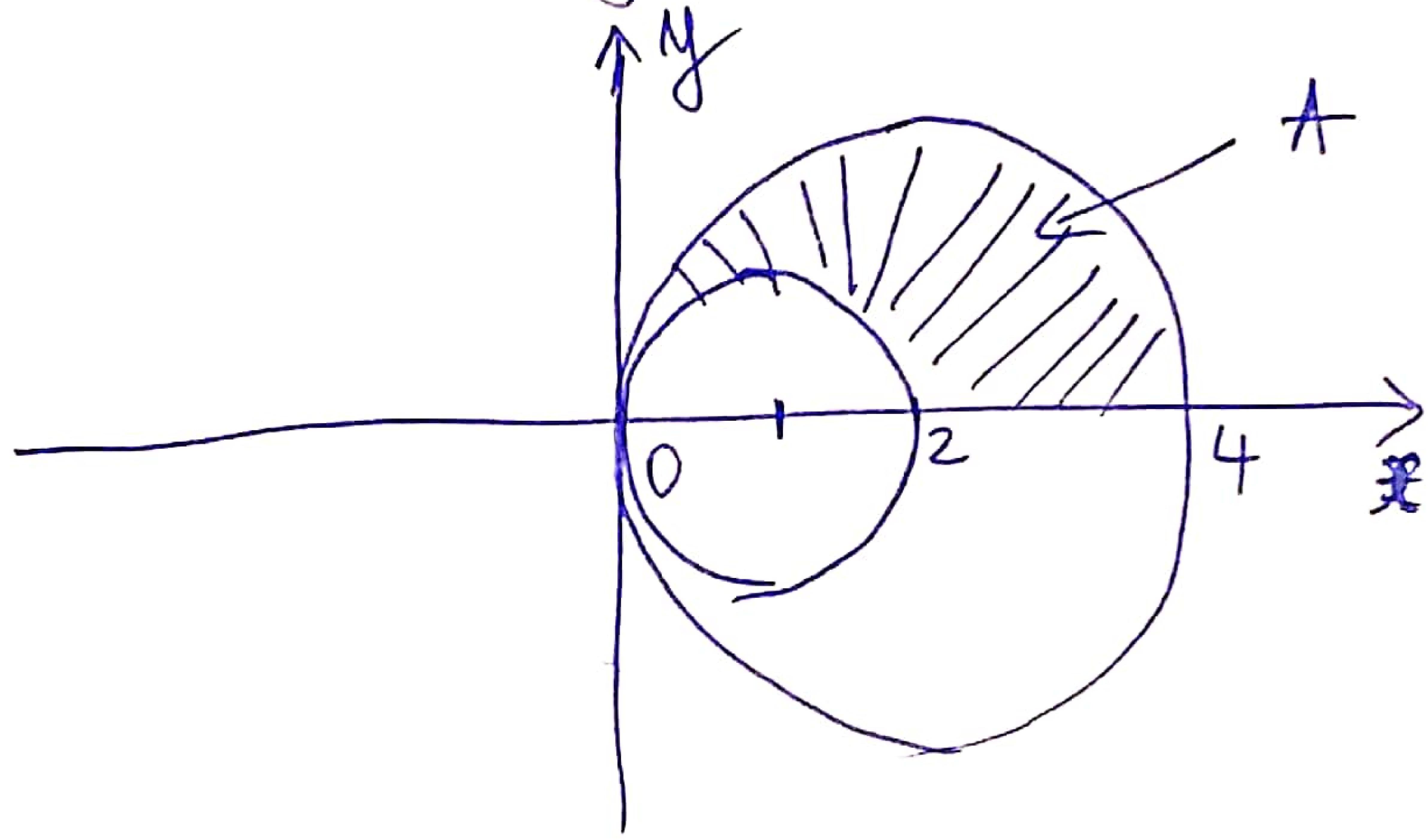
$$= \iint_B r \operatorname{arctg}(\operatorname{tg}\theta)\,dr\,d\theta = \int_1^3 \left(\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} r\theta\,d\theta \right) dr =$$

$$= \int_1^3 r\frac{\theta^2}{2} \Bigg|_{\theta=\frac{\pi}{6}}^{\theta=\frac{\pi}{3}} dr = \frac{1}{2}\int_1^3 r\left({}^{4)}\frac{\pi^2}{9} - \frac{\pi^2}{36} \right) dr =$$

$$= \frac{1}{2} \cdot \frac{\cancel{3}\pi^2}{\underset{12}{\cancel{36}}} \int_1^3 r\,dr = \frac{\pi^2}{24} \cdot \frac{r^2}{2} \Bigg|_{r=1}^{r=3} = \frac{\pi^2}{24} \cdot \frac{8}{2} = \frac{\pi^2}{6}. \square$$

h) $\iint_A \sqrt{x^2+y^2}\,dx\,dy$, unde $A=\{(x,y)\in\mathbb{R}^2 \mid 2x\le x^2+y^2\le$
$\le 4x,\ y\ge 0\}$.

Soluție. $2x\le x^2+y^2 \Leftrightarrow x^2+y^2-2x\ge 0 \Leftrightarrow (x-1)^2+y^2\ge 1$.
$x^2+y^2\le 4x \Leftrightarrow x^2-4x+y^2\le 0 \Leftrightarrow (x-2)^2+y^2\le 4$.



Fie $f: A\to\mathbb{R}$, $f(x,y)=\sqrt{x^2+y^2}$.

S.V. $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$, $r\in[0,\infty)$, $\theta\in[0,2\pi]$.

$(x,y)\in A \Leftrightarrow \begin{cases} 2x\le x^2+y^2 \\ x^2+y^2\le 4x \\ y\ge 0 \end{cases} \Leftrightarrow \begin{cases} 2r\cos\theta\le r^2 \\ r^2\le 4r\cos\theta \\ r\sin\theta\ge 0 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} 2\cos\theta\le r\le 4\cos\theta \\ \sin\theta\ge 0 \\ \cos\theta\ge 0 \end{cases} \Leftrightarrow \begin{cases} 2\cos\theta\le r\le 4\cos\theta \\ \theta\in\left[0,\frac{\pi}{2}\right]. \end{cases}$

Deci $B=\left\{(r,\theta)\in[0,\infty)\times[0,2\pi] \;\middle|\; \theta\in\left[0,\frac{\pi}{2}\right],\ 2\cos\theta \le r \le 4\cos\theta\right\}$.

$$\iint_A f(x,y)\,dx\,dy=\iint_B r\, f(r\cos\theta, r\sin\theta)\,dr\,d\theta =$$

$$=\iint_B r\sqrt{r^2}\,dr\,d\theta=\int_0^{\frac{\pi}{2}}\left(\int_{2\cos\theta}^{4\cos\theta} r^2\,dr\right)d\theta =$$

$$=\int_0^{\frac{\pi}{2}} \frac{r^3}{3}\Bigg|_{r=2\cos\theta}^{r=4\cos\theta} d\theta = \frac{1}{3}\int_0^{\frac{\pi}{2}}\left(64\cos^3\theta-8\cos^3\theta\right)d\theta =$$

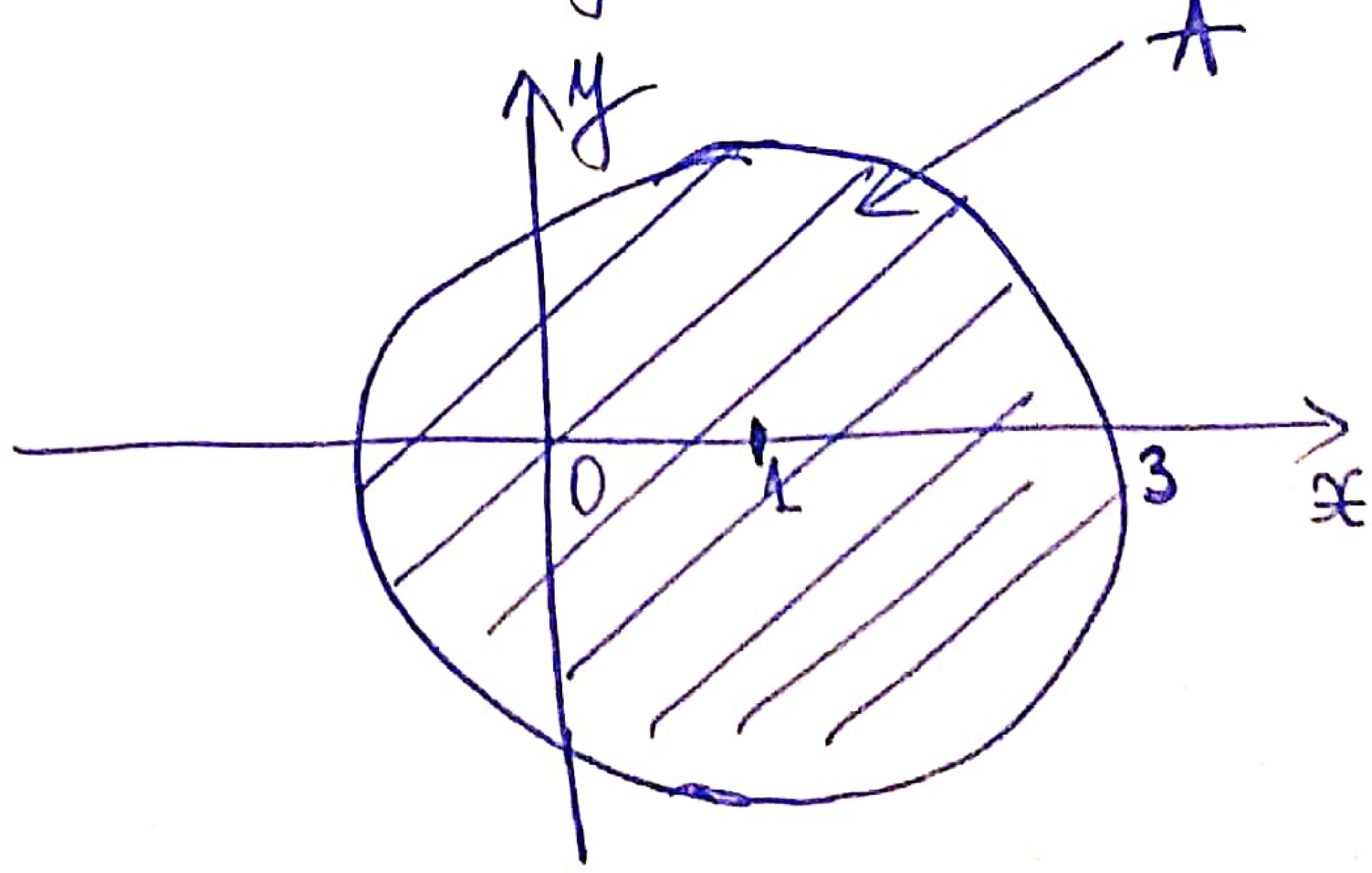
$$=\frac{56}{3}\int_0^{\frac{\pi}{2}}\cos^3\theta\,d\theta = \frac{56}{3}\int_0^{\frac{\pi}{2}}\cos\theta\left(1-\sin^2\theta\right)d\theta =$$

$$=\frac{56}{3}\left(\sin\theta\Bigg|_{\theta=0}^{\theta=\frac{\pi}{2}} - \frac{\sin^3\theta}{3}\Bigg|_{\theta=0}^{\theta=\frac{\pi}{2}}\right) =$$

$$=\frac{56}{3}\left(1-\frac{1}{3}\right)=\frac{56}{3}\cdot\frac{2}{3}=\frac{112}{9}. \ \square$$

i) $\iint_A y\,dx\,dy$, unde $A=\{(x,y)\in\mathbb{R}^2 \mid (x-1)^2+y^2\le 4\}$.

Soluție.



Fie $f: A \to \mathbb{R}$, $f(x,y) = y$.

S.V. $\begin{cases} x = 1 + r\cos\theta \\ y = r\sin\theta \end{cases}$, $r \in [0, \infty)$, $\theta \in [0, 2\pi]$.

$(x,y) \in A \Leftrightarrow (x-1)^2 + y^2 \leq 4 \Leftrightarrow r^2 \leq 4 \Leftrightarrow \begin{cases} r \in [0,2] \\ \theta \in [0, 2\pi]. \end{cases}$

Deci $B = [0,2] \times [0, 2\pi]$.

$$\iint_A f(x,y)\,dx\,dy = \iint_B r f(1 + r\cos\theta, r\sin\theta)\,dr\,d\theta =$$

$$= \iint_B r\, r\sin\theta\,dr\,d\theta = \int_0^2 \left(\int_0^{2\pi} r^2 \sin\theta\,d\theta \right) dr =$$

$$= \int_0^2 \left(-r^2\cos\theta\right)\Big|_{\theta=0}^{\theta=2\pi} dr = \int_0^2 0\,dr = 0. \quad \square$$