Limite inf. & superioare.

inf → cel mai mare mai mic

sup → cel mai mic mai mare

$(x_m)_m \subset \overline{\mathbb{R}}$

$u_n = \sup\limits_{k \geq n} x_k$ $\qquad v_n \leq v_{n+1} \leq u_{n+1} \leq u_n$

$v_m = \inf\limits_{k \geq n} x_k$

Limita superioară $\limsup\limits_{n \to \infty} = \overline{\lim\limits_{n \to \infty}}\, x_n = \lim\limits_{n \to \infty} u_n = \inf\limits_{n \geq 1} u_n = \inf\limits_{n \geq 1}(\sup\limits_{k \geq n} x_k)$

Limita inferioară $\liminf\limits_{n \to \infty} = \underline{\lim\limits_{n \to \infty}}\, x_n = \sup\limits_{n \geq 1} v_n = \sup\limits_{n \geq 1}(\inf\limits_{k \geq n} x_k)$

(1) $x_n = \dfrac{n \cdot (-1)^n}{2n+1} + tg\dfrac{n\pi}{3}$ $\qquad \underline{\lim} = ?\ \overline{\lim} = ?$

Obs. că $(-1)^n = \begin{cases} 1, & n = 2k \\ -1, & n = 2k+1 \end{cases}$ $\qquad tg\dfrac{n\pi}{3} = \begin{cases} tg\, k\pi, & n = 3k \\ tg\left(k\pi + \frac{\pi}{3}\right), & n = 3k+1 \\ tg\left(k\pi + \frac{2\pi}{3}\right), & n = 3k+2 \end{cases} = \begin{cases} 0, & n = 3k \\ \sqrt{3}, & n = 3k+1 \\ -\sqrt{3}, & n = 3k+2 \end{cases}$

Alegem subșirurile $(x_{6k}), (x_{6k+1}), (x_{6k+2}), (x_{6k+3}), (x_{6k+4}), (x_{6k+5})$

$\lim\limits_{k \to \infty} x_{6k} = \lim\limits_{k \to \infty}\left(\dfrac{6k}{12k+1} + 0\right) = \dfrac{6}{12} = \dfrac{1}{2}$

$\lim\limits_{k \to \infty} x_{6k+1} = \lim\limits_{k \to \infty}\left(\dfrac{(6k+1)\cdot(-1)}{12k+3} + \sqrt{3}\right) = -\dfrac{6}{12} + \sqrt{3} = \sqrt{3} - \dfrac{1}{2}$

$\lim\limits_{k \to \infty} x_{6k+2} = \lim\limits_{k \to \infty}\left(\dfrac{6k+2}{12k+5} - \sqrt{3}\right) = \dfrac{1}{2} - \sqrt{3}$

$\lim\limits_{k \to \infty} x_{6k+3} = \lim\limits_{k \to \infty}\left(\dfrac{(6k+3)\cdot(-1)}{12k+7} + 0\right) = -\dfrac{1}{2}$

$\lim\limits_{k \to \infty} x_{6k+4} = \sqrt{3} + \dfrac{1}{2}$

$\lim\limits_{k \to \infty} x_{6k+5} = -\dfrac{1}{2} - \sqrt{3}$

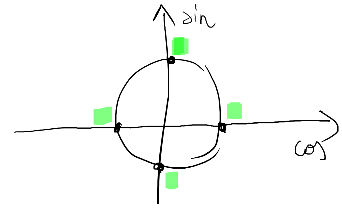
$L((x_n)_{n \in \mathbb{N}}) = \left\{\dfrac{1}{2}, \sqrt{3} - \dfrac{1}{2}, \dfrac{1}{2} - \sqrt{3}, -\dfrac{1}{2}, \dfrac{1}{2} + \sqrt{3}, -\dfrac{1}{2} - \sqrt{3}\right\}$

$\underline{\lim}\, x_n = \inf(L) = -\dfrac{1}{2} - \sqrt{3}$

$\overline{\lim}\, x_n = \sup(L) = \dfrac{1}{2} + \sqrt{3}$

Obs! $\underline{\lim} \neq \overline{\lim} \Rightarrow \nexists \lim\limits_{n \to \infty} x_n$ $\qquad \mathbb{R} \cup \{\pm\infty\} = \overline{\mathbb{R}}$

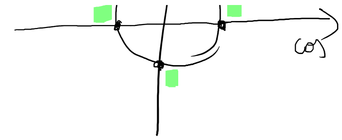
$\underline{\lim}\, x_n, \overline{\lim}\, x_n \in \mathbb{R} \Rightarrow$ șir mărginit



(2) $x_n = \left(1 + \dfrac{1}{n}\right)^{n \cdot (-1)^n} + \sin\dfrac{n\pi}{2}$ $\qquad \sin\dfrac{n\pi}{2} = \begin{cases} 0, & n = 4k \\ 1, & n = 4k+1 \\ 0, & n = 4k+2 \end{cases}$

$(-1)^n = \{ -1, n = 2k+1$

$$(-1)^n = \begin{cases} -1, & n=2k+1 \\ 1, & n=2k \end{cases} \qquad \sin\frac{n\pi}{2} = \begin{cases} 1, & n=4k+1 \\ 0, & n=4k+2 \\ -1, & n=4k+3 \end{cases}$$



$$\lim_{k\to\infty} x_{4k} = \lim_{k\to\infty} \left[\left(1+\frac{1}{4k}\right)^{4k} + 0\right] = e$$

$$\lim_{k\to\infty} x_{4k+1} = \lim_{k\to\infty} \left[\left(1+\frac{1}{4k+1}\right)^{-4k-1} + 1\right] = \frac{1}{e}+1$$

$$\lim_{k\to\infty} x_{4k+2} = e$$

$$\lim_{k\to\infty} x_{4k+3} = \frac{1}{e}-1$$

$$L\left((x_n)_{n\in\mathbb{N}}\right) = \left\{e, \frac{1}{e}+1, \frac{1}{e}-1\right\} \Rightarrow \begin{matrix} \underline{\lim}\, x_n = \frac{1}{e}-1 \\ \overline{\lim}\, x_n = e \end{matrix} \qquad \underline{\lim} \neq \overline{\lim} \Rightarrow \nexists \lim_{n\to\infty} x_n$$

(3) $x_n = \left\{1+\frac{1}{2}+\dots+\frac{1}{n}\right\}$, $n\in\mathbb{N}^*$ Este convergent?

$0 \le x_n < 1 \quad \forall n\in\mathbb{N}^* \Rightarrow x_n = $ mărginit

$$x_{n+1}-x_n = \left\{1+\frac{1}{2}+\dots+\frac{1}{n+1}\right\} - \left\{1+\frac{1}{2}+\dots+\frac{1}{n}\right\} = \cancel{1}+\cancel{\frac{1}{2}}+\dots+\cancel{\frac{1}{n}}+\frac{1}{n+1}-[\dots]-\left(\cancel{1}+\cancel{\frac{1}{2}}+\dots+\cancel{\frac{1}{n}}-[\dots]\right) =$$

$$= \frac{1}{n+1} + \left[1+\dots+\frac{1}{n}\right] - \left[1+\dots+\frac{1}{n+1}\right]$$

Pp că $(x_n)_{n\in\mathbb{N}^*}$ este convergent $\Rightarrow \exists \lim_{n\to\infty} x_n = l \in [0;1]$

$$\Rightarrow \lim_{n\to\infty}(x_{2n}-x_n) = 0 = l-l$$

$$x_{2n}-x_n = \left\{1+\dots+\frac{1}{2n}\right\} - \left\{1+\dots+\frac{1}{n}\right\} = 1+\dots+\frac{1}{2n} - [\dots] - \left(1+\dots+\frac{1}{n}\right) + [\dots] =$$

$$= \boxed{\frac{1}{n+1}+\dots+\frac{1}{2n}} - \underbrace{\left(\left[1+\dots+\frac{1}{2n}\right] - \left[1+\dots+\frac{1}{n}\right]\right)}_{y_n}$$

$$x_{2n}-x_n = \frac{1}{n+1}+\dots+\frac{1}{2n} - y_n$$

$$\left.\begin{matrix} y_n = \left(\frac{1}{n+1}+\dots+\frac{1}{2n}\right) + (x_n - x_{2n}) \\ y_n \in \mathbb{R} \quad \forall n\in\mathbb{N}^* \end{matrix}\right\} \Rightarrow \lim_{n\to\infty} y_n = \lim_{n\to\infty}\left(\frac{1}{n+1}+\dots+\frac{1}{2n} + x_n - x_{2n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{n+1}+\dots+\frac{1}{2n}\right) + \underbrace{\lim_{n\to\infty}(x_n - x_{2n})}_{=0}$$

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\dots+\frac{1}{2n}\right) = \lim_{n\to\infty}\left(1+\frac{1}{2}+\dots+\frac{1}{2n} - 1 - \frac{1}{2} - \dots - \frac{1}{n}\right) =$$

$$= \lim_{n\to\infty}\left[\underbrace{\left(1+\frac{1}{2}+\dots+\frac{1}{2n}\right) - \ln 2n} + \ln 2n - \left(\underbrace{1+\frac{1}{2}+\dots+\frac{1}{n} - \ln n} + \ln n\right)\right] =$$

$$= \lim_{n\to\infty} \left[\underbrace{\left(1+\frac{1}{2}+\dots+\frac{1}{2n}\right) - \ln 2n} + \ln 2n - \underbrace{\left(1+\frac{1}{2}+\dots+\frac{1}{n} - \ln n + \ln n\right)} \right] =$$

$$= \lim_{n\to\infty} \left[\underbrace{\left(1+\frac{1}{2}+\dots+\frac{1}{2n} - \ln 2n\right)}_{\gamma_{2n}} - \underbrace{\left(1+\frac{1}{2}+\dots+\frac{1}{n} - \ln n\right)}_{\gamma_n} + \ln 2n - \ln n \right] =$$

$$= \lim_{n\to\infty} \left[c - c + \ln 2n - \ln n \right] = \lim_{n\to\infty} (\ln 2n - \ln n) = \lim_{n\to\infty} \ln\frac{2n}{n} = \ln 2$$

$$\lim_{n\to\infty} y_n = \ln 2$$

Serii = perechi $(x_n)_{n\ge p}, (y_n)_{n\ge p}$ unde $y_n = \sum_{k=p}^{\infty} x_k = \sum_{k\ge p} x_k$

CRITERII

0. seria $\sum_{n\ge 1} x_n$ conv. $\Rightarrow x_n \to 0$

1. COMPARAȚIEI

* inegalități 1) $\sum b_n$ converg. și $a_n \le b_n \Rightarrow \sum a_n$ convg.

2) $\sum a_n$ diverg și $a_n \le b_n \Rightarrow \sum b_n$ divg.

* limită $\lim_{n\to\infty} \frac{a_n}{b_n} = l$

- $(0, +\infty)$ $\sum a_n \sim \sum b_n$
- 0 și $\sum b_n$ convg $\to \sum a_n$ convg
- ∞ $\sum b_n$ divg $\to \sum a_n$ divg

2. CONDENSĂRII

Dacă $a_n \searrow 0$ $\sum_{n\ge 1} a_n \sim \sum_{n\ge 1} 2^n \cdot a_{2^n}$

3. RAPORTULUI

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$

- $l < 1 \Rightarrow \sum a_n$ convg
- $l > 1 \Rightarrow \sum a_n$ divg

4. RĂDĂCINII / RADICALULUI

$\lim_{n\to\infty} \sqrt[n]{a_n} = l$

- $l < 1$ $\sum a_n$ convg
- $l > 1$ $\sum a_n$ divg.

5. RAABE - DUHAMEL

$a_n > 0$

$\lim_{n\to\infty} n \cdot \left(\frac{a_n}{a_{n+1}} - 1\right) = l$

- $l > 1$ $\sum a_n$ convg
- $l < 1$ $\sum a_n$ divg

6. O serie absolut convg. este convg.

Absolut convergența $\to$ seria modulelor pt serii cu termeni oarecare

ex. $\sum (-1)^n \cdot x_n$

$\sum |(-1)^n \cdot x_n|$

7. ABEL – DIRICLET

$a_n \searrow 0$

$\exists M$ a.î. $|\sum_{k=1}^{n} x_k| \le M \Rightarrow \sum_{n\ge1} a_n x_n$ conv.

8. LEIBNIZ

$a_n \ge 0$ $\quad \sum(-1)^n \cdot a_n$

a_n descrescător $a_n \to 0$ $\Big\} \Rightarrow \sum(-1)^n a_n$ conv.

(4) $\sum_{n=1}^{+\infty} \frac{a^n \cdot n!}{n^n}$, $a>0$

$x_n = \frac{a^n\, n!}{n^n}, n \in \mathbb{N}^*$ $\quad x_n > 0 \ \forall n \in \mathbb{N}^*$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{a^{n+1} \cdot (n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{a^n \cdot n!} = \lim_{n\to\infty} \frac{a \cdot n^n \cdot (n+1)}{(n+1)^{n+1}} = \lim_{n\to\infty} \frac{a \cdot n^n}{(n+1)^n} =$$

$$= a \cdot \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = a \cdot \lim_{n\to\infty} \left[\frac{\cancel{n}}{\cancel{n}(1+\frac{1}{n})}\right]^n = a \cdot \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n} = a \cdot \frac{1}{e} = \underbrace{\frac{a}{e}}_{\text{not } L}$$

I $a<e \Rightarrow L<1 \Rightarrow$ converg.

II $a>e \Rightarrow L>1 \Rightarrow$ diverg.

III $a=e \Rightarrow L=1$ $\qquad \frac{x_{n+1}}{x_n} = \frac{e}{(1+\frac{1}{n})^n}$

Știm $(1+\frac{1}{n})^n < e$ → ȘIRUL LUI EULER

$\underbrace{\quad}_{\frac{x_{n+1}}{x_n} > 1} \ \forall n \in \mathbb{N}^* \Rightarrow x_{n+1} > x_n \Rightarrow \exists \lim_{n\to\infty} x_n \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$

(x_n) cresc., $x_{n+1} > x_n > \ldots > x_1$

$x_n > x_1 \ \forall n \in \mathbb{N}^* \Rightarrow x_n > \frac{e \cdot 1!}{1^1} = e \ \forall n \in \mathbb{N}^*$ $\Big\} \Rightarrow$

$\Rightarrow \lim_{n\to\infty} x_n \ge e \Rightarrow \lim_{n\to\infty} x_n \ne 0 \Rightarrow$ seria $\sum x_n$ diverg. (C0) criteriul 0