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Tutoriat 9

Ex 1 $\sum_{n\geq 1} \frac{2^n}{\sqrt[3]{n+1}\cdot\sqrt{n}}\cdot x^n, \ x>0$

Rezolvare:

Crit. Raportului:

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\cancel{2^{n+1}}\cdot\cancel{x^{n+1}}}{\sqrt[3]{n+2}\cdot\sqrt{n+1}}\cdot\frac{\sqrt[3]{n+1}\cdot\sqrt{n}}{\cancel{2^n}\cdot\cancel{x^n}} = \lim_{n\to\infty}\frac{2x\sqrt[3]{n+1}\cdot\sqrt{n}}{\sqrt[3]{n+2}\cdot\sqrt{n+1}} =$$

$$= \lim_{n\to\infty}\frac{2x\cdot(n+1)^{\frac{1}{3}}\cdot n^{\frac{1}{2}}}{(n+2)^{\frac{1}{3}}\cdot(n+1)^{\frac{1}{2}}} = \lim_{n\to\infty}\frac{2x\cdot\cancel{(n+1)^{\frac{2}{6}}}\cdot n^{\frac{3}{6}}}{(n+2)^{\frac{2}{6}}\cdot\cancel{(n+1)^{\frac{3}{6}}}} =$$

$$= \lim_{n\to\infty}\frac{2x\cdot n^{\frac{3}{6}}}{(n+2)^{\frac{2}{6}}\cdot(n+1)^{\frac{1}{6}}} = 2x\cdot\lim_{n\to\infty}\underbrace{\sqrt[6]{\frac{n^3}{(n+2)^2\cdot(n+1)}}}_{1} = 2x$$

i) $2x>1, \ x>\frac{1}{2} \to$ serie diverg.

ii) $2x<1, \ x<\frac{1}{2} \to$ serie converg.

iii) $2x=1, \ x=\frac{1}{2}$

$$a_n = \frac{\cancel{2^n}}{\sqrt[3]{n+1}\cdot\sqrt{n}}\cdot\frac{1}{\cancel{2^n}} = \frac{1}{\sqrt[3]{n+1}\cdot\sqrt{n}} = \frac{1}{(n+1)^{\frac{1}{3}}\cdot n^{\frac{1}{2}}} = \frac{1}{(n+1)^{\frac{2}{6}}\cdot n^{\frac{3}{6}}} =$$

$$a_n = \frac{1}{\sqrt[6]{n^3\cdot(n+1)^2}} > \frac{1}{\sqrt[6]{(n+1)^3(n+1)^2}} = \frac{1}{\sqrt[6]{(n+1)^5}} = \frac{1}{(n+1)^{\frac{5}{6}}} \quad \text{(Crit. Comp.)}$$

Seria arm. generalizată $\sum_{n\geq 1}\frac{1}{n^\alpha}$

$\to \alpha>1 \to$ seria conv

$\to \alpha<1 \to$ seria div

$\to \alpha=1 \to$ seria arm. clasică $\to$ serie div

$n\to n+1 \to \sum_{n\geq 1}\frac{1}{(n+1)^\alpha}, \ \alpha<1, \ \alpha=\frac{5}{6} \to$ seria div.

Integrala Curbilinie

Definiție!

Se num. drum în $\mathbb{R}^n$ orice funcție continuă $\gamma:[a;b]\subseteq\mathbb{R}\to\mathbb{R}^n$

$\gamma(t) = (\gamma_1(t), \gamma_2(t), \dots, \gamma_n(t))$

$\gamma_1, \gamma_2, \dots, \gamma_n : [a;b]\to\mathbb{R}$

$\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$

Observații!

$\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n) : [a;b] \longrightarrow \mathbb{R}^n$

a) γ drum în $\mathbb{R}^n \Leftrightarrow \gamma_1, \gamma_2, \dots, \gamma_n$ f. continue

b) γ drum de clasă $C^1 \Leftrightarrow \gamma_1, \gamma_2, \dots, \gamma_n$ f. derivabile și $\gamma_1', \gamma_2', \dots, \gamma_n'$ sunt f. continue

$$\gamma' = (\gamma_1', \gamma_2', \dots, \gamma_n')$$

$$\|\gamma'(t)\| = \sqrt{\gamma_1'(t)^2 + \dots + \gamma_n'(t)^2}$$

Integrala Curbilinie de I Tip

$f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ funcție continuă

$\gamma : [a,b] \subseteq \mathbb{R} \longrightarrow \mathbb{R}^n$ drum de clasă C^1

$$\int_\gamma f\, dl \ \underset{\text{definiție}}{=} \ \int_a^b f(\gamma(t)) \cdot \|\gamma'(t)\|\, dt$$

Ex2 Calculați $\int_\gamma xy\, dl$, unde $\gamma(t) = (t, t^2)$ $\forall\, t \in [0;1]$.

Rezolvare:

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ $\quad f(x,y) = xy$

f continuă pe $\mathbb{R}^2$

$\gamma : [0;1] \subseteq \mathbb{R} \longrightarrow \mathbb{R}^2$ $\quad \gamma(t) = (\underset{\gamma_1(t)}{t}, \underset{\gamma_2(t)}{t^2})$

$\left.\begin{array}{l} \gamma_1(t) = t \Rightarrow \gamma_1'(t) = 1 \quad \forall t \in [0;1] \\ \gamma_2(t) = t^2 \Rightarrow \gamma_2'(t) = 2t \quad \forall t \in [0;1] \\ \gamma_1', \gamma_2' \text{ f. continue pe } [0;1] \end{array}\right\} \Rightarrow \gamma$ drum de clasă C^1

$$I = \int_\gamma f\, dl = \int_0^1 f(\underset{x}{t}, \underset{y}{t^2}) \|\gamma'(t)\|\, dt = \int_0^1 t \cdot t^2 \cdot \|(1, 2t)\|\, dt$$

$$= \int_0^1 t^3 \cdot \sqrt{1+4t^2}\, dt = \frac{1}{8} \int_0^1 t^2 \cdot t \cdot \sqrt{1+4t^2} \cdot 8\, dt = \frac{1}{8} \int_0^1 t^2 \cdot \sqrt{\underbrace{1+4t^2}_{y}} \cdot 8t\,dt$$

$y = 1+4t^2 = 1 + 4 \cdot t^2$

$t^2 = \frac{(1+4t^2) - 1}{4} = \frac{y-1}{4}$

$y = 1+4t^2$

$dy = (1+4t^2)'\, dt = 8t\, dt$

$t = 0 \Rightarrow y = 1$

$t = 1 \Rightarrow y = 5$

$$dy = (1+4t^2)'\,dt = 8t\,dt \qquad t=1 \Rightarrow y=5$$

$$I = \frac{1}{8}\int_1^5 \frac{y-1}{4}\cdot\sqrt{y}\,dy = \frac{1}{32}\int_1^5 (y-1)\cdot\sqrt{y}\,dy = \frac{1}{32}\int_1^5 y\sqrt{y}\,dy - \frac{1}{32}\int_1^5 \sqrt{y}\,dy$$

$$= \frac{1}{32}\cdot\frac{y^{\frac{3}{2}+1}}{\frac{3}{2}+1}\Bigg|_1^5 - \frac{1}{32}\cdot\frac{y^{\frac{1}{2}+1}}{\frac{1}{2}+1}\Bigg|_1^5 = \frac{1}{80}\left(25\sqrt{5}-1\right) - \frac{1}{48}\left(5\sqrt{5}-1\right)$$

Ex3 Calculați $\int_C \frac{1}{y}\,dl$ unde $C \subseteq \mathbb{R}^2$ este curba descrisă de ecuația $x = \frac{y^3}{3}$ cu $y \in [1;2]$

Rezolvare:

$f : \mathbb{R}\times\mathbb{R}^* \longrightarrow \mathbb{R} \qquad f(x,y) = \frac{1}{y}$

f continuă pe $\mathbb{R}\times\mathbb{R}^*$

$\gamma : [a;b] \subseteq \mathbb{R} \longrightarrow \mathbb{R}^2 \qquad \mathrm{Im}\,\gamma = C$

$C: \ x = \frac{y^3}{3},\ y\in[1;2] \Rightarrow y = t \in [1,2] \Rightarrow x = \frac{t^3}{3}$

$\gamma : [1;2] \longrightarrow \mathbb{R}^2 \qquad \gamma(t) = \Big(\underbrace{\frac{t^3}{3}}_{\gamma_1(t)}, \underbrace{t}_{\gamma_2(t)}\Big)$

$\left.\begin{array}{l}\gamma_1(t)\in\mathbb{R}\ \forall t\in[1,2]\\ \gamma_2(t)\in\mathbb{R}^*\ \forall t\in[1,2]\end{array}\right\} \Rightarrow \gamma : [1;2] \to \mathbb{R}\times\mathbb{R}^*$

$\left.\begin{array}{l}\gamma_1(t) = \frac{t^3}{3} \Rightarrow \gamma_1'(t) = \frac{3t^2}{3} = t^2 \quad \forall t\in[1;2]\\ \gamma_2(t) = t \Rightarrow \gamma_2'(t) = 1 \quad \forall t\in[1,2]\\ \gamma_1', \gamma_2' \text{ f continue pe } [1;2]\end{array}\right\} \Rightarrow \gamma$ drum de clasă C^1

$$I = \int_C f\,dl = \int_\gamma f\,dl = \int_1^2 f\left(\frac{t^3}{3}, t\right)\|\gamma'(t)\|\,dt = \int_1^2 \frac{1}{t}\cdot\|(t^2,1)\|\,dt =$$

$$= \int_1^2 \frac{1}{t}\sqrt{t^4+1}\,dt = \int_1^2 \frac{t^3}{t^4}\sqrt{1+t^4}\,dt = \frac{1}{4}\int_1^2 \frac{1}{t^4}\cdot\sqrt{1+t^4}\cdot 4t^3\,dt \Rightarrow$$

$y = 1+t^4 \qquad t=1 \Rightarrow y=2$

$dy = 4t^3\,dt \qquad t=2 \Rightarrow y=17$

$$I = \frac{1}{4}\int_2^{17}\frac{1}{y-1}\cdot\sqrt{y}\cdot dy = \frac{1}{4}\int_{\sqrt{2}}^{\sqrt{17}}\frac{1}{x^2-1}\cdot\sqrt{x^2}\cdot 2x\,dx = \frac{1}{4}\int_{\sqrt{2}}^{\sqrt{17}}\frac{2x^2}{x^2-1}\,dx =$$

$$= \frac{1}{2}\int_{\sqrt{2}}^{\sqrt{17}}\frac{x^2}{x^2-1}\,dx$$

Notăm $x^2 = y$ $\qquad y=2 \Rightarrow x=\sqrt{2}$

$$- \frac{\cdot}{2} \int_{\sqrt{2}} \frac{\quad}{x^2-1}\, dx$$

Notăm $x^2 = y$ $\qquad y = 2 \Rightarrow x = \sqrt{2}$

$2x\,dx = dy \qquad y = 17 \Rightarrow x = \sqrt{17}$

$$I = \frac{1}{2}\int_{\sqrt{2}}^{\sqrt{17}} \frac{x^2-1+1}{x^2-1}\,dx = \frac{1}{2}\int_{\sqrt{2}}^{\sqrt{17}} \frac{x^2-1}{x^2-1}\,dx + \frac{1}{2}\int_{\sqrt{2}}^{\sqrt{17}} \frac{1}{x^2-1}\,dx =$$

$$= \frac{1}{2}\, x \Big|_{\sqrt{2}}^{\sqrt{17}} + \frac{1}{2}\cdot\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| \Big|_{\sqrt{2}}^{\sqrt{17}}$$

Integrala curbilinie de al II-lea tip

$$\int_\gamma w = \int_\gamma P_1\,dx_1 + P_2\,dx_2 + \dots + P_n\,dx_n$$

$w : D \subseteq \mathbb{R}^n \longrightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R})$

$w(x_1, \dots, x_n) = P_1(x_1, \dots, x_n)\,dx_1 + \dots + P_n(x_1, \dots, x_n)\,dx_n$

$P_1, P_2, \dots, P_n : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ f. continue

$\gamma : [a,b] \subseteq \mathbb{R} \longrightarrow D \subseteq \mathbb{R}^n$ drum de clasă C^1

$\gamma = (\gamma_1, \dots, \gamma_n)$

$$\int_\gamma w \overset{def}{=} \int_a^b \left[P_1(\gamma(t))\cdot\gamma_1'(t) + P_2(\gamma(t))\cdot\gamma_2'(t) + \dots + P_n(\gamma(t))\cdot\gamma_n'(t)\right]dt$$

Ex 4 Calculați $\int_C xy\,dx - y^2\,dy$ unde $C \subseteq \mathbb{R}^2$ este curbă deschisă de ecuație $y = x^3$, $x \in [0;1]$

Rezolvare:

$P, Q : \mathbb{R}^2 \longrightarrow \mathbb{R}$ $\qquad$ P, Q continue pe $\mathbb{R}^2$

$P(x,y) = xy$

$Q(x,y) = -y^2$

$\operatorname{Im}\gamma = C$

$C: y = x^3,\ x \in [0,1] \Rightarrow x = t \in [0;1] \Rightarrow y = t^3$

$\gamma : [0;1] \longrightarrow \mathbb{R}^2 \quad \gamma(t) = (t, t^3)$ drum în $\mathbb{R}^2$

$\gamma_1(t) \quad \gamma_2(t)$

$$\left.\begin{array}{l} \gamma_1(t) = t \Rightarrow \gamma_1'(t) = 1 \quad \forall\, t \in [0;1] \\ \gamma_2(t) = t^3 \Rightarrow \gamma_2'(t) = 3t^2 \quad \forall\, t \in [0;1] \\ \gamma_1', \gamma_2' \text{ f continue pe } [0;1] \end{array}\right\} \Rightarrow \gamma \text{ drum de clasă } C^1 \text{ în } \mathbb{R}^2$$

$$\int_C P\,dx + Q\,dy = \int_\gamma P\,dx + Q\,dy = \int_0^1 [P(t,t^3)\cdot\gamma_1'(t) + Q(t,t^3)\cdot\gamma_2'(t)]\,dt$$

$$= \int_0^1 (t^4 \cdot 1 + (-t^6)\cdot 3t^2)\,dt = \int_0^1 (t^4 - 3t^8)\,dt =$$

$$= \int_0^1 t^4\,dt - 3\int_0^1 t^8\,dt = \frac{t^5}{5}\Big|_0^1 - 3\cdot\left(\frac{t^9}{9}\Big|_0^1\right) = \frac{1}{5} - 0 - 3\cdot\frac{1}{9} + 0 =$$

$$= \frac{1}{5} - \frac{1}{3}$$

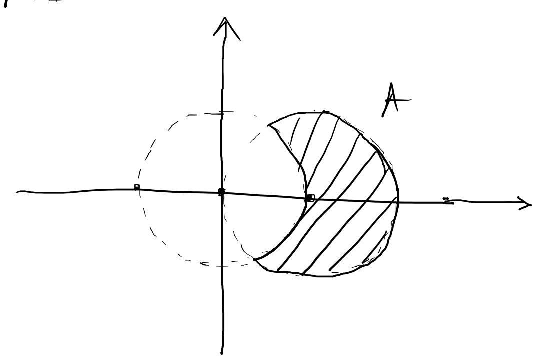
Ex 5 Calculați $\iint_A \frac{y^2}{x^2}\,dx\,dy$ unde $A = \{(x,y) \in \mathbb{R}^2 / 1 \le x^2 + y^2 \le 2x\}$

Rezolvare:

$$A = \begin{cases} 1 \le x^2 + y^2 \\ x^2 + y^2 \le 2x \end{cases}$$

$$x^2 + y^2 \le 2x \Rightarrow x^2 - 2x + y^2 \le 0 \,/_{+1} \Rightarrow x^2 - 2x + 1 + y^2 \le 1 \Rightarrow$$

$$\Rightarrow (x-1)^2 + y^2 \le 1$$



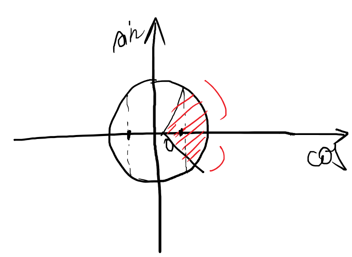
$$\varphi : \mathbb{R}^2 \to \mathbb{R}^2$$

$$\varphi(R,\theta) = (\underbrace{R\cos\theta}_{x}, \underbrace{R\sin\theta}_{y})$$

$$A = \varphi(B)$$

$$B: \begin{cases} 1 \le (R\cos\theta)^2 + (R\sin\theta)^2 \\ R^2\cos^2\theta + R^2\sin^2\theta \le 2R\cos\theta \\ R \ge 0 \\ \theta \in [0;2\pi] \text{ sau } \theta \in [-\pi;\pi] \end{cases} \Rightarrow B: \begin{cases} 1 \le R^2(\cos^2\theta + \sin^2\theta) \\ R^2(\cos^2\theta + \sin^2\theta) \le 2R\cos\theta \\ R \ge 0 \\ \theta \in [0;2\pi] \text{ sau } \theta \in [-\pi;\pi] \end{cases} \Rightarrow$$

$$\Rightarrow B: \begin{cases} 1 \le R^2 & (1) \\ R^2 \le 2R\cos\theta & (2) \\ R \ge 0 & (3) \\ \theta \in [0;2\pi] \text{ sau } \theta \in [-\pi;\pi] & (4) \end{cases}$$



$\theta \in [0; 2\pi]$ sau $\theta \in [-\pi; \pi]$ (4)

$$\left.\begin{array}{l}(3)\ R \geq 0 \\ (1)\ R^2 \geq 1\end{array}\right\} \Rightarrow R \geq 1$$

$$\left.\begin{array}{l}(2)\ R^2 \leq 2R\cos\theta \\ (3)\ R \geq 0\end{array}\right\} \Rightarrow R \leq 2\cos\theta$$

$\Rightarrow 1 \leq 2\cos\theta \Rightarrow \cos\theta \geq \frac{1}{2} \Rightarrow$

$\xRightarrow{(4)} \theta \in [0; \frac{\pi}{3}] \cup [\frac{5\pi}{3}; 2\pi]$

sau

$\boxed{\theta \in [-\frac{\pi}{3}; \frac{\pi}{3}]}$ IDEAL

Așadar $B = \begin{cases} R \geq 1 \\ R \leq 2\cos\theta \\ \theta \in [-\frac{\pi}{3}; \frac{\pi}{3}] \end{cases}$

$\Rightarrow B: \begin{cases} 1 \leq R \leq 2\cos\theta \\ -\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3} \end{cases}$ B simplă în raport cu axa OR

$$\iint_A \frac{y^2}{x^2}\,dx\,dy = \iint_{\varphi(B)} \frac{y^2}{x^2}\,dx\,dy = \iint_{\varphi(B)} \frac{R^2\sin^2\theta}{R^2\cos^2\theta}\,|R|\cdot dR\,d\theta =$$

$$= \iint_{\varphi(B)} tg^2\theta\, R\, dR\, d\theta = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(\int_1^{2\cos\theta} R\, tg^2\theta\, dR\right) d\theta = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} tg^2\theta \left.\frac{R^2}{2}\right|_1^{2\cos\theta} d\theta$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} tg^2\theta \left(\frac{4\cos^2\theta}{2} - \frac{1}{2}\right) d\theta = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} tg^2\theta \cdot 2\cos^2\theta\, d\theta - \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} tg^2\theta\, d\theta$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\sin^2\theta}{\cancel{\cos^2\theta}} \cdot 2\cancel{\cos^2\theta}\, d\theta - \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} tg^2\theta\, d\theta = \underbrace{\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 2\sin^2\theta\, d\theta}_{} - \underbrace{\frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} tg^2\theta\, d\theta}_{}$$

II $\rightarrow 2\sin^2\theta = 1 - \cos 2\theta$

I $\rightarrow$ adunăm și scădem 1

$$= \left.\left(\theta - \frac{\sin 2\theta}{2}\right)\right|_{-\frac{\pi}{3}}^{\frac{\pi}{3}} - \frac{1}{2}\left.\left(tg\,\theta - \theta\right)\right|_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$