**FMI, Info, Anul I**
**Logică matematică şi computaţională**

# Seminar 7

**(S7.1) (Metoda reducerii la absurd)**
Să se arate că pentru orice mulţime de formule $\Gamma$ şi orice formule $\varphi, \psi$,

$$\Gamma \cup \{\neg\psi\} \vdash \neg(\varphi \to \varphi) \quad \Rightarrow \quad \Gamma \vdash \psi.$$

**Demonstraţie:** Avem

| | | | |
|---|---|---|---|
| (1) | $\Gamma \cup \{\neg\psi\}$ | $\vdash \neg(\varphi \to \varphi)$ | Ipoteză |
| (2) | $\Gamma$ | $\vdash \neg\psi \to \neg(\varphi \to \varphi)$ | Teorema deducţiei |
| (3) | $\Gamma$ | $\vdash (\neg\psi \to \neg(\varphi \to \varphi)) \to ((\varphi \to \varphi) \to \psi)$ | (A3) şi Propoziţia 1.37.(i) |
| (4) | $\Gamma$ | $\vdash (\varphi \to \varphi) \to \psi$ | (MP): (2), (3) |
| (5) | $\Gamma$ | $\vdash \varphi \to \varphi$ | Propoziţiile 1.45 şi 1.39.(ii) |
| (6) | $\Gamma$ | $\vdash \psi$ | (MP): (4), (5). |

□

**(S7.2)** Să se arate că pentru orice formule $\varphi, \psi$,

(i) $\{\psi, \neg\psi\} \vdash \varphi$;

(ii) $\vdash \neg\psi \to (\psi \to \varphi)$;

(iii) $\vdash \neg\neg\varphi \to \varphi$;

(iv) $\vdash \varphi \to \neg\neg\varphi$.

**Demonstraţie:** Demonstrăm (i):

| | | | |
|---|---|---|---|
| (1) | | $\vdash \neg\psi \to (\neg\varphi \to \neg\psi)$ | (A1) |
| (2) | $\{\neg\psi\}$ | $\vdash \neg\varphi \to \neg\psi$ | Teorema deducţiei |
| (3) | $\{\neg\psi\}$ | $\vdash (\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$ | (A3) şi Propoziţia 1.37.(i) |
| (4) | $\{\neg\psi\}$ | $\vdash \psi \to \varphi$ | (MP): (2), (3) |
| (5) | $\{\psi, \neg\psi\}$ | $\vdash \varphi$ | Teorema deducţiei. |

Punctul (ii) se obţine din (i) aplicând de două ori Teorema deducţiei:

| | | | |
|---|---|---|---|
| (1) | $\{\psi, \neg\psi\}$ | $\vdash \varphi$ | se aplică (i) |
| (2) | $\{\neg\psi\}$ | $\vdash \psi \to \varphi$ | Teorema deducţiei |
| (3) | | $\vdash \neg\psi \to (\psi \to \chi)$ | Teorema deducţiei. |

Demonstrăm în continuare (iii).

| | | | |
|---|---|---|---|
| (1) | $\{\neg\varphi, \neg\neg\varphi\}$ | $\vdash \neg(\varphi \to \varphi)$ | se aplică (i) |
| (2) | $\{\neg\neg\varphi\}$ | $\vdash \varphi$ | (1) şi (S7.1) |
| (3) | | $\vdash \neg\neg\varphi \to \varphi.$ | Teorema deducţiei. |

Demonstrăm (iv):

| | | |
|---|---|---|
| (1) | $\vdash \neg\neg\neg\varphi \to \neg\varphi$ | se aplică (iii) cu $\varphi := \neg\varphi$ |
| (2) | $\vdash (\neg\neg\neg\varphi \to \neg\varphi) \to (\varphi \to \neg\neg\varphi)$ | (A3) |
| (3) | $\vdash \varphi \to \neg\neg\varphi$ | (MP): (1), (2). |

□

**(S7.3) ("Reciproca" axiomei 3)**
Să se arate că pentru orice formule $\varphi, \psi$,

$$\vdash (\varphi \to \psi) \to (\neg\psi \to \neg\varphi).$$

**Demonstraţie:**

| | | | |
|---|---|---|---|
| (1) | $\{\varphi \to \psi, \neg\psi, \neg\neg\varphi\}$ | $\vdash \varphi \to \psi$ | Propoziţia 1.37.(ii) |
| (2) | $\{\varphi \to \psi, \neg\psi, \neg\neg\varphi\}$ | $\vdash \neg\psi$ | Propoziţia 1.37.(ii) |
| (3) | $\{\varphi \to \psi, \neg\psi, \neg\neg\varphi\}$ | $\vdash \neg\neg\varphi$ | Propoziţia 1.37.(ii) |
| (4) | $\{\varphi \to \psi, \neg\psi, \neg\neg\varphi\}$ | $\vdash \neg\neg\varphi \to \varphi$ | (S7.2).(iii) şi Propoziţia 1.39.(ii) |
| (5) | $\{\varphi \to \psi, \neg\psi, \neg\neg\varphi\}$ | $\vdash \varphi$ | (MP): (3), (4) |
| (6) | $\{\varphi \to \psi, \neg\psi, \neg\neg\varphi\}$ | $\vdash \psi$ | (MP): (1), (5) |
| (7) | $\{\varphi \to \psi, \neg\psi, \neg\neg\varphi\}$ | $\vdash \neg\psi \to (\psi \to \neg(\varphi \to \varphi))$ | (S7.2).(ii) şi Propoziţia 1.39.(ii) |
| (8) | $\{\varphi \to \psi, \neg\psi, \neg\neg\varphi\}$ | $\vdash \psi \to \neg(\varphi \to \varphi)$ | (MP): (2), (7) |
| (9) | $\{\varphi \to \psi, \neg\psi, \neg\neg\varphi\}$ | $\vdash \neg(\varphi \to \varphi)$ | (MP): (6), (8) |
| (10) | $\{\varphi \to \psi, \neg\psi\}$ | $\vdash \neg\varphi$ | (9) şi (S7.1) |
| (11) | $\{\varphi \to \psi\}$ | $\vdash \neg\psi \to \neg\varphi$ | Teorema deducţiei |
| (12) | | $\vdash (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$ | Teorema deducţiei. |

□

**(S7.4)** Să se arate că pentru orice formule $\varphi, \psi$,

$$\{\psi, \neg\varphi\} \vdash \neg(\psi \to \varphi).$$

**Demonstraţie:** Avem

| | | | |
|---|---|---|---|
| (1) | $\{\psi, \neg\varphi, \neg\neg(\psi \to \varphi)\}$ | $\vdash \psi$ | Propoziţia 1.37.(ii) |
| (2) | $\{\psi, \neg\varphi, \neg\neg(\psi \to \varphi)\}$ | $\vdash \neg\varphi$ | Propoziţia 1.37.(ii) |
| (3) | $\{\psi, \neg\varphi, \neg\neg(\psi \to \varphi)\}$ | $\vdash \neg\neg(\psi \to \varphi)$ | Propoziţia 1.37.(ii) |
| (4) | $\{\psi, \neg\varphi, \neg\neg(\psi \to \varphi)\}$ | $\vdash \neg\neg(\psi \to \varphi) \to (\psi \to \varphi)$ | (S7.2).(iii) şi Prop. 1.39.(ii) |
| (5) | $\{\psi, \neg\varphi, \neg\neg(\psi \to \varphi)\}$ | $\vdash \psi \to \varphi$ | (MP): (3), (4) |
| (6) | $\{\psi, \neg\varphi, \neg\neg(\psi \to \varphi)\}$ | $\vdash \varphi$ | (MP): (1), (5) |
| (7) | $\{\psi, \neg\varphi, \neg\neg(\psi \to \varphi)\}$ | $\vdash \neg\varphi \to (\varphi \to \neg(\varphi \to \varphi))$ | (S7.2).(ii) şi Prop. 1.39.(ii) |
| (8) | $\{\psi, \neg\varphi, \neg\neg(\psi \to \varphi)\}$ | $\vdash \varphi \to \neg(\varphi \to \varphi)$ | (MP): (2), (7) |
| (9) | $\{\psi, \neg\varphi, \neg\neg(\psi \to \varphi)\}$ | $\vdash \neg(\varphi \to \varphi)$ | (MP): (6), (8) |
| (10) | $\{\psi, \neg\varphi\}$ | $\vdash \neg(\psi \to \varphi)$ | (9) şi (S7.1). |

$\square$