Limite extreme

x = pct. limită dacă $\exists (x_{n_k})_n \subset \underbrace{(x_n)_n}_{\text{şir}}$ a.î. $\lim\limits_{k\to\infty} x_{n_k} = x$

$\mathcal{L} = \{x \in \overline{\mathbb{R}} \mid x$ pct. limită$\}$ subşir

$\overline{\lim}\, x_n = \limsup\limits_{n\to\infty} x_n$ cel mai mare pct. limită

$\underline{\lim}\, x_n = \liminf\limits_{n\to\infty} x_n$ — " — mic — " —

dacă $\overline{\lim}\, x_n = \underline{\lim}\, x_n$ atunci $\exists \lim\limits_{n\to\infty} x_n$

$$x_n = \frac{1}{2}\left(n-2-3\left[\frac{n-1}{3}\right]\right)\left(n-3-3\left[\frac{n-1}{3}\right]\right)$$

Idee: $\frac{n-1}{3} \Rightarrow 3$ subșiruri

$$x_{3n} = \frac{1}{2}\left(3n-2-3\left[\frac{3n-1}{3}\right]\right)\left(3n-3-3\left[\frac{3n-1}{3}\right]\right)$$

$$= \frac{1}{2}(3n-2-3(n-1))(3n-3-3(n-1))$$

$$= \frac{1}{2}(\underbrace{3n-2-3n+3}_{1})(\underbrace{3n-3-3n+3}_{0}) \xrightarrow{n\to\infty} 0$$

$$x_{3n+1} = \frac{1}{2}\left(3n+1-2-3\left[\frac{3n+1-1}{3}\right]\right)\left(3n+1-3-3\left[\frac{3n+1-1}{3}\right]\right)$$

$$= \frac{1}{2}(\underbrace{3n+1-2-3n}_{-1})(\underbrace{3n+1-3-3n}_{-2}) = 1 \xrightarrow{n\to\infty} 1$$

$$x_{3n+2} = \frac{1}{2}\left(3n+2-2-3\left[\frac{3n+1}{3}\right]\right)\left(3n+2-3-3\left[\frac{3n+1}{3}\right]\right)$$

$$= \frac{1}{2}(\underbrace{3n+2-2-3n}_{0})(\underbrace{3n+2-3-3n}_{-1}) \xrightarrow{n\to\infty} 0$$

Cum $\mathbb{N} = 3\mathbb{N} \cup (3\mathbb{N}+1) \cup (3\mathbb{N}+2)$

$\mathcal{L}((x_n)_n) = \{0, 1\}$.

$\overline{\lim}\, x_n = 1$

$\underline{\lim}\, x_n = 0$

Cum $\overline{\lim}\, x_n \neq \underline{\lim}\, x_n \Rightarrow \nexists \lim\limits_{n\to\infty} x_n$

$$x_n = \frac{n}{2n+1}\left(\cos \frac{n\pi}{\ldots}\right.$$

Idee: · știm cos ...
· chiar $\frac{n\pi}{2}$

$$x_{6n} = \frac{6n}{12n+1}\left(\cos \ldots\right.$$

Cum $\mathbb{N} = 6\mathbb{N} \cup (6\mathbb{N}+1) \cup \dots \cup (6\mathbb{N}+5)$

$L((x_n)) = \{0\}$

$\overline{\lim}\, x_n = 0$, $\underline{\lim}\, x_n = 0$ $\Rightarrow \exists \lim\limits_{n\to\infty} x_n = 0$

Limite extreme

$$x_n = \frac{n}{2n+1}\left(\sin\frac{n\pi}{3}\right)^n, \forall n \in \mathbb{N}$$

Idee: • știm sin periodic 2π $\Rightarrow$ 6 subșiruri

• avem $\frac{n\pi}{3}$

$$x_{6n} = \frac{6n}{12n+1}\Big(\underbrace{\sin\frac{6n\pi}{3}}_{0}\Big)^{6n} \xrightarrow{n\to\infty} 0$$

$$x_{6n+1} = \frac{6n+1}{12n+3}\left(\sin\frac{(6n+1)\pi}{3}\right)^{6n+1} = \frac{6n+1}{12n+3}\Big(\underbrace{\sin\frac{\pi}{3}}_{<1}\Big)^{6n+1} \xrightarrow{n\to\infty} 0$$

$$x_{6n+2} = \frac{6n+2}{12n+5}\left(\sin\frac{(6n+2)\pi}{3}\right)^{6n+2} = \frac{6n+2}{12n+5}\left(\sin\frac{2\pi}{3}\right)^{6n+2} \xrightarrow{n\to\infty} 0$$

$$x_{6n+3} = \frac{6n+3}{12n+7}\left(\sin\frac{(6n+3)\pi}{3}\right)^{6n+3} = \frac{6n+3}{12n+5}\Big(\underbrace{\sin\pi}_{0}\Big)^{6n+3} \xrightarrow{n\to\infty} 0$$

$$\sin\frac{(6n+1)\pi}{3} = \sin\left(\frac{6n\pi}{3}+\frac{\pi}{3}\right) = \sin\left(2n\pi + \frac{\pi}{3}\right)$$

Serii de numere reale

$S_n = \sum_{k=0}^{n} x_n = x_0 + x_1 + \dots + x_n$ ← suma parțială de rang n

$(x_n)_{n\in\mathbb{N}}$, $(S_n)_{n\in\mathbb{N}}$;

$((x_n)_{n\in\mathbb{N}}, (S_n)_{n\in\mathbb{N}}) \sim \sum_{n=0}^{\infty} x_n$ ← seria nr reale asociată șirului $(x_n)_{n\in\mathbb{N}}$.

$\sum_{n=0}^{\infty} x_n$ → convergentă dacă $(s_n)_{n\in\mathbb{N}}$ e convergent

divergentă e divergent

are suma în $\overline{\mathbb{R}}$ are limită în $\overline{\mathbb{R}}$; $\lim_{n\to\infty} s_n$ ← suma seriei

$$\sum_{n=0}^{\infty} x_n = \lim_{n\to\infty} s_n$$

$\sum_{n=0}^{\infty} x_n$ e abs convergentă dacă seria $\sum_{n=0}^{\infty} |x_n|$ e convergentă.

semiconvergentă dacă $\sum_{n=0}^{\infty} x_n$ e conv, dar nu e abs conv.

abs conv $\Longrightarrow$ conv.

$S_n = \sum_{k=0}^{n} x_k = \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}+\sqrt{k}} =$
$= \sum_{k=0}^{n} (\sqrt{k+1}-\sqrt{k}) =$
$= \sqrt{1}-\sqrt{0}+\sqrt{2}-\sqrt{1}+\ldots+\sqrt{n+1}-\sqrt{n} =$
$= \sqrt{n+1}-1$
$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sqrt{n+1}-1 = \infty$
$\Rightarrow \sum_{n=0}^{\infty} x_n$ e divergentă

... = suma parțială de rang n
... asociate șirului $(x_n)_{n\in\mathbb{N}}$
...nt
...nt
...ts în $\overline{\mathbb{R}}$; $\lim_{n\to\infty} s_n$ = suma seriei

abs conv $\Rightarrow$ conv

...nvergentă
... abs conv

Teoremă: $\sum_{n=0}^{\infty} x_n$ e convergentă $\Rightarrow \exists \lim_{n\to\infty} x_n = 0$.

Reciproc

Criteriu abs de divergență:

$(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ a.î. $\exists \lim_{n\to\infty} x_n \neq 0$ sau $\nexists \lim_{n\to\infty} x_n \Rightarrow \sum_{n=0}^{\infty} x_n$ e divergentă

Reciproc, $\boxed{NU}$, $x_n = \frac{1}{n} \to 0$, $\sum \frac{1}{n}$ divergentă

Studiem natura urm. serii:

$\sum_{n=1}^{\infty} n \sin\frac{1}{n}$; $\lim_{n\to\infty} n \sin\frac{1}{n} = \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1 \neq 0$

divergentă

$\sum_{n=1}^{\infty} \left(\frac{n+1}{n+2}\right)^n$; $\lim_{n\to\infty} \left(\frac{n+1}{n+2}\right)^n = \Big[\underbrace{\left(1-\frac{1}{n+2}\right)^{-(n+2)}}_{e}\Big]^{-\frac{n}{n+2}} = e^{-1} = \frac{1}{e} \neq 0$

div

$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$; $\lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} =$
$= \lim_{n\to\infty} (\sqrt{n+1}-\sqrt{n}) = 0$????

Serii de numere reale

Serii remarcabile

1. seria armonică $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$, $\alpha \in \mathbb{R}$ $\begin{cases} \to \text{conv}, & \alpha > 1 \\ \to \text{div}, & \alpha \leq 1 \end{cases}$

2. seria putere/geometrică: $\sum_{n=1}^{\infty} a^{n}$, $a \in \mathbb{R}$ $\begin{cases} \to \text{abs conv}, & a \in (-1, 1) \\ \to \text{divergentă}, & a \in (-\infty, -1] \cup [1, \infty) \end{cases}$

3. seria exponențială $\sum_{n=1}^{\infty} \frac{a^{n}}{n!}$, $a \in \mathbb{R} \longrightarrow$ abs convergentă, $\forall a \in \mathbb{R}$

Criterii de convergență pentru serii cu termeni pozitivi ($\mathbb{R}_+$)

1) Criteriul Raportului, fie $\sum_n x_n$, $x_n > 0$, $\exists \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \ell$

i) $\ell < 1 \Rightarrow \sum_n x_n$ convergentă

ii) $\ell = 1 \Rightarrow$ nu decide

iii) $\ell > 1 \Rightarrow \sum_n x_n$ divergentă

(*) a^n, $a!$, grade

Ex) $\sum_{n=1}^{\infty} \frac{\sqrt{n-1}}{n(n+1)} x^n$, $x > 0$, $x \neq 1$

fie $x_n = \frac{\sqrt{n-1}}{n(n+1)} x^n$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{(n+1)(n+2)} \cdot x^{n+1} \cdot \frac{n(n+1)}{\sqrt{n-1}} \cdot \frac{1}{x^n} = \lim_{n\to\infty} x \cdot \underbrace{\frac{\sqrt{n}}{\sqrt{n-1}}}_{\to 1} \cdot \underbrace{\frac{n}{n+2}}_{\to 1} = x$$

Conform Crit. Raportului

i) $x < 1 \Rightarrow \sum_n x_n$ convergentă

ii) $x > 1 \Rightarrow \sum_n x_n$ divergentă

iii) $x = 1 \Rightarrow$ nu decide (*) de urmărit

Concluzie: $x \in (0,1)$: Serie Conv.; $x \in (1, \infty)$: Serie DIV.

2) Crit. Radicalului. Fie seria $\sum_n x_n$, $x_n > 0$, $\exists \lim_{n\to\infty} \sqrt[n]{x_n} = l$ (*) a^n

i) $l < 1 \Rightarrow \sum_n x_n$ convergentă

ii) $l = 1 \to$ nu decide

iii) $l > 1 \Rightarrow \sum_n x_n$ divergentă

Ex. $\sum_{n=1}^{\infty} \left(\sqrt{(n+1)(n+x)} - n\right)^n$, $x > 0$

Fie $x_n = \left(\sqrt{(n+1)(n+x)} - n\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \sqrt[n]{\left(\sqrt{(n+1)(n+x)} - n\right)^n} = \lim_{n\to\infty} \sqrt{(n+1)(n+x)} - n = \lim_{n\to\infty} \frac{(n+1)(n+x) - n^2}{\sqrt{(n+1)(n+x)} + n}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2} + (x+1)n + x - \cancel{n^2}}{\underbrace{\sqrt{(n+1)(n+x)}}_{n^2+\dots} + n} = \frac{x+1}{2}$$

Conf. Crit. Radicalului:

i) $\frac{x+1}{2} < 1$ (i.e. $x < 1$) $\Rightarrow \sum_n x_n$ conv.

ii) $\frac{x+1}{2} > 1$ (i.e. $x > 1$) $\Rightarrow \sum_n x_n$ div.

iii) $\frac{x+1}{2} = 1$ (i.e. $x = 1$) $\Rightarrow$ nu decide

$$\lim_{n\to\infty} \sqrt[n]{x_n} = l$$

(*) $\quad a^n$

$$\ldots - n = \lim_{n\to\infty} \frac{(n+1)(n+x) - n^2}{\sqrt{(n+1)(n+x)} + n}$$

$x = 1 \Rightarrow$ Seria devine $\sum_{n=1}^{\infty} \left(\sqrt{(n+1)(n+1)} - n\right)^n = \sum_{n=1}^{\infty} 1^n$ divergentă (s. geom. $a=1$)

Concluzie: $x \in (0,1) \Rightarrow$ Seria conv.!

$x \in (1,\infty) \Rightarrow$ Seria Divergentă.

$\sum_{n=1}^{\infty} a^n$, $a \in \mathbb{R}$ — conv., $a \in (-1,1)$; div., $a \in \mathbb{R} \setminus (-1,1)$

$x = 1 \Rightarrow$ Seria devine $\sum_{n=1}^{\infty} \left(\underbrace{\sqrt{(n+1)(n+1)}}_{n+1} - n\right)^n = \sum_{n=1}^{\infty} 1^n$ divergentă (S geom. $a = 1$).

Concluzie: $x \in (0,1) \Rightarrow$ Seria conv!

$x \in [1, \infty) \Rightarrow$ Seria Divergentă.

$\sum_{n=1}^{\infty} a^n$, $a \in \mathbb{R}$ — $\to$ conv, $a \in (-1,1)$; $\to$ div, $a \in \mathbb{R} \setminus (-1,1)$

3). <u>Criteriul lui Raabe-Duhamel</u>

$\sum_{n=0}^{\infty} x_n \subseteq \mathbb{R}_+$ a.î. $\exists \lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = l \in \overline{\mathbb{R}}$

- $l < 1 \Rightarrow \sum x_n$ divergentă
- $l > 1 \Rightarrow \sum x_n$ convergentă

ex $\sum_{n=1}^{\infty} \frac{\sqrt{n-1}}{n(n+1)} \cdot x^n$, $x > 0$;

pt $x = 1$; $\sum_{n=1}^{\infty} \underbrace{\frac{\sqrt{n-1}}{n(n+1)}}_{a_n}$; RD. $\lim_{n\to\infty} n\left(\frac{\sqrt{n-1}}{n(n+1)} \cdot \frac{(n+1)(n+2)}{\sqrt{n}} - 1\right) =$

$= \lim_{n\to\infty} n\left(\frac{(n+2)\sqrt{n-1}}{n\sqrt{n}} - 1\right) = \lim_{n\to\infty} \cancel{n}\,\frac{(n+2)\sqrt{n-1} - n\sqrt{n}}{\cancel{n}\sqrt{n}} = \infty$

$\Rightarrow \sum x_n$ convergentă.

1) Criteriul Condensării a lui Cauchy

$(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$ desc cu $\lim\limits_{n\to\infty} x_n = 0$

$\Rightarrow \sum\limits_{n=0}^{\infty} x_n$ și $\sum\limits_{n=0}^{\infty} 2^n x_{2^n}$ au ac natură

ex $\sum\limits_{n=1}^{\infty} \frac{1}{n \ln n}$; $x_n = \frac{1}{n \ln n}$; desc, $\lim\limits_{n\to\infty} x_n = 0$

$\Rightarrow \boxed{\sum\limits_{n=1}^{\infty} x_n}$ și $\sum\limits_{n=1}^{\infty} 2^n x_{2^n}$ au ac nat

$$\sum_{n=1}^{\infty} 2^n x_{2^n} = \sum_{n=1}^{\infty} 2^n \frac{1}{2^n \cdot \ln 2^n} = \sum_{n=1}^{\infty} \frac{1}{\ln 2^n} = \sum_{n=1}^{\infty} \frac{1}{n \cdot \ln 2} = \frac{1}{\ln 2} \cdot \underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{} \leftarrow \text{serie armonică}, \alpha = 1$$

divergentă

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \begin{cases} \text{conv } \alpha > 1 \\ \text{div } \alpha \le 1 \end{cases}$$

$\Rightarrow \sum\limits_{n=1}^{\infty} x_n$ e divergentă.

5) Crit. de comparație cu inegalități: fie $\sum_n x_n$, $\sum_n y_n$, $x_n > 0$, $y_n > 0$, $\underline{x_n \leq y_n}$

i) $\sum_n y_n$ convergentă $\Rightarrow$ $\sum_n x_n$ convergentă

ii) $\sum_n x_n$ divergentă $\Rightarrow$ $\sum_n y_n$ divergentă

Ex) $\sum_{n=1}^{\infty} \frac{\sin(nx)}{2^n}$, $x \in \mathbb{R}$

$\sin nx \leq 1 \quad \Big| \cdot \frac{1}{2^n}$

$\underbrace{\frac{\sin nx}{2^n}}_{x_n} \leq \underbrace{\frac{1}{2^n}}_{y_n}$ (1)

$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ convergentă (s.g. $a \in (-1,1)$) (2)

$\frac{1}{2} \in (-1,1)$

Conf. CCI $\xRightarrow{1,2}$ $\sum_{n=1}^{\infty} x_n$ convergentă $\forall x \in \mathbb{R}$.

!!

$n! < n^n$

$e^x \geq x+1$

sin, cos mărg.

6) Crit. de comp. cu Limită: $\sum_n x_n$, $\sum_n y_n$, $x_n > 0$, $y_n > 0$. $\exists \lim_{n\to\infty} \frac{x_n}{y_n} = \ell$.

i) $\ell \in (0, \infty)$: $\sum_n x_n \sim \sum_n y_n$ (aceeași nat.)

ii) $\ell = 0$, $\sum_n y_n$ convergent $\Rightarrow \sum_n x_n$ conv.

iii) $\ell = \infty$, $\sum_n y_n$ divergent $\Rightarrow \sum_n x_n$ div.

ex $\sum_{n=1}^{\infty} \frac{a^n}{2^n + 5^n}$, $a > 0$ (S)

Fie $x_n = \frac{a^n}{2^n + 5^n}$, $y_n = \frac{a^n}{5^n}$.

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{\cancel{a^n}}{2^n + 5^n} \cdot \frac{5^n}{\cancel{a^n}} = \lim_{n\to\infty} \frac{\cancel{5^n}}{\cancel{5^n}\Big(\underbrace{\left(\frac{2}{5}\right)^n}_{\to 0} + 1\Big)} = 1 \in (0, \infty)$$

Conf CCL $\Rightarrow \sum_n x_n \sim \sum_n y_n$

$\sum_n y_n = \sum_n \left(\frac{a}{5}\right)^n$

$\frac{a}{5} \in (-1, 1)$

$a \in (-5, 5) \Rightarrow$ Serie convergentă (S geom. cu $a \in (-1,1)$)

$a \in \mathbb{R} \setminus (-5, 5) \Rightarrow$ Serie div.