$a=1$, seria devine $\sum_{n=2}^{\infty} \frac{1}{\sqrt[n]{n!}}$

Știm că $\underbrace{\frac{1}{n}}_{D} \le \frac{1}{\sqrt[n]{n!}}$ (3)

$\sum_n \frac{1}{n}$ divergentă (serie armonică generalizată $a=1$) (5)

$\xRightarrow{cci} \sum_n x_n$ divergentă când $a=1$

Concluzie: $\sum_n x_n$
- → Convergentă $a<1$
- → Divergentă $a \ge 1$

$\sum a^n$ (C) $a \in (-1,1)$

$n! \le n^n$

$\sum x_n$

$\frac{x_{n+1}}{x_n} = l$

$x_n \le y_n$

$\sum y_n$ C → $\sum x_n$ C

$\sum x_n$ D → $\sum y_n$ D.

$\sum \frac{1}{n^\alpha}$

$\alpha \le 1$ D

$\alpha > 1$ C

$\sum_{n=2}^{\infty} \frac{a^n}{\sqrt[n]{n!}}$, $a>0$

Fie $x_n = \frac{a^n}{\sqrt[n]{n!}}$

$\sqrt[n]{n!} \ge 1 \Rightarrow \frac{1}{\sqrt[n]{n!}} \le 1 \Rightarrow \frac{a^n}{\sqrt[n]{n!}} \le a^n$ (1)

$\sum_n a^n$ convergentă pe $a<1$ (2)

$\xRightarrow{C.C.I} \sum_n x_n$ Conv. când $a<1$

$n! \le n^n \Rightarrow \sqrt[n]{n!} \le n \Rightarrow \frac{1}{\sqrt[n]{n!}} \ge \frac{1}{n} \Rightarrow \frac{a^n}{\sqrt[n]{n!}} \ge \frac{a^n}{n}$ (3)

Fie $y_n = \frac{a^n}{n}$

$\lim_{n\to\infty} \frac{y_{n+1}}{y_n} = \lim_{n\to\infty} \frac{a^{\cancel{n+1}}}{n+1} \cdot \frac{n}{\cancel{a^n}} = a$

dacă $a>1 \Rightarrow \sum_n y_n$ divergentă (4)

Obs: $a<1 \Rightarrow \sum y_n$ C

$a>1 \Rightarrow \sum y_n$ D

$\underbrace{\frac{a^n}{n}}_{D} \le \frac{a^n}{\sqrt[n]{n!}}$

$\xRightarrow{cci} \sum_n x_n$ divergentă dacă $a>1$

Criterii de convergență pt serii cu termeni oarecare ($\mathbb{R}$)

① Crit lui Leibniz $\sum_n x_n$, $(x_n)_n \subset \mathbb{R}$, $(x_n)_n \searrow$

$$\lim_{n\to\infty} x_n = 0$$

Atunci $\sum_n (-1)^n x_n$ convergentă

exemplu $\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n}$

fie $x_n = \frac{1}{\ln n}$

AC $\longrightarrow$ C

$$\ln n \le \ln(n+1)$$

$$\frac{1}{\ln n} \ge \frac{1}{\ln(n+1)}$$

$$x_n \ge x_{n+1} \longrightarrow (x_n)_n \searrow$$

$$\left.\begin{array}{l}\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{\ln n} = 0\end{array}\right| \Rightarrow \sum_{n=2}^{\infty} (-1)^n x_n \text{ Convergentă}$$

$$\sum_n \left|(-1)^n \cdot \frac{1}{\ln n}\right| = \sum_n \frac{1}{\ln n}$$

$$y_n = \frac{1}{n}.$$

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{n}{\ln n} = \infty$$

$\sum_n \frac{1}{n}$ div (s. arm. gen. $\alpha = 1$) $\Rightarrow$

$\Rightarrow \sum_n |x_n|$ Div.

$$\sum_{n=1}^{\infty} \frac{\sin n \cdot \sin n^2}{\sqrt{n}}; \quad x_n = \frac{1}{\sqrt{n}} \to \text{monoton (desc) și } 0 < x_n \leq 1, \forall n \in \mathbb{N}$$

$$y_n = \sin n \cdot \sin n^2$$

$$\sum_{n=0}^{\infty} y_n = \sum_{n=0}^{\infty} \sin n \cdot \sin n^2 = \sum \frac{1}{2}\left[\cos(n-n^2) - \cos(n+n^2)\right] =$$

$$= \frac{1}{2} \cdot \underbrace{\left[1 - \cos(n+n^2)\right]}_{\leq 2} \leq 1$$

$$\cos x \geq -1 \mid (-1)$$

$$-\cos x \leq +1 \mid +1$$

$$\overset{\text{Abel}}{\Longrightarrow} \sum_{n=1}^{\infty} \frac{\sin n \cdot \sin n^2}{\sqrt{n}} \text{ e conv.}$$

② Criteriul lui Abel:

$(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}$ a.î. $\left| \begin{array}{l} x_{n+1} \leq x_n, \forall n \in \mathbb{N}, \lim\limits_{n \to \infty} x_n = 0 \\ \exists M \in \mathbb{R} \text{ a.î. } \left\{ \begin{array}{l} |y_1 + \dots + y_n| \leq M \\ \left(\sum\limits_{k=1}^{n} y_k\right)_{n \in \mathbb{N}} \leq M \end{array} \right. \end{array} \right| \Longrightarrow \sum\limits_{n=0}^{\infty} x_n y_n$ e convergent

Ex. $\sum\limits_{n=1}^{\infty} \frac{\cos n \cdot \cos \frac{1}{n}}{n}$; $\quad x_n = \frac{1}{n} \to$ desc

$\lim\limits_{n \to \infty} x_n = 0$

$y_n = \cos n \cdot \cos \frac{1}{n}$

$(*) \sum\limits_{k=1}^{n} \underbrace{\cos k}_{\leq 1} \cdot \underbrace{\cos \frac{1}{k}}_{\leq 1} \leq \underline{n}$.

$\Rightarrow \sum\limits_{n=1}^{\infty} x_n y_n = \sum\limits_{n=1}^{\infty} \frac{\cos n \cos \frac{1}{n}}{n}$ conv

Relucăm: $y_n - \frac{1}{n}$; $\sum y_n$ conv

$x_n = \cos n \cdot \cos \frac{1}{n}$

$-1 \leq \cos n \cdot \cos \frac{1}{n} \leq 1 \to$

$\Rightarrow x_n$ mărginit; $\quad \underline{x_{n+1} - x_n} = ?$

③ Criteriul lui Dirichlet

$(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}, \left| \begin{array}{l} (x_n)_{n \in \mathbb{N}} \text{ monoton și mărginit} \\ \sum\limits_{n=0}^{\infty} y_n \text{ e conv} \end{array} \right| \Rightarrow \sum\limits_{n=0}^{\infty} x_n y_n$ e conv.

Algoritm pt studiul convergenței unei serii

1. Verificăm dacă seria are termeni pozitivi
2. Încercăm studiul conv. cu definiția
   $\lim_{n\to\infty} S_n = x_1 + x_2 + \dots + x_n$
3. Dacă nu putem calcula $\lim_{n\to\infty} S_n$,
   aplicăm criterii
   → RAPORT | RADICAL.
   → RAABE-DUHAMEL $n\left(\frac{x_n}{x_{n+1}} - 1\right)$
4. Încercăm inegalități cunoscute | criterii cu limite.

I. $a<1 \Rightarrow \sum x_n$ e div

II. $a>1 \Rightarrow \sum x_n$ e conv.

III. $a=1$

Şirul $(x_n)_{n\in\mathbb{N}}$ devine:

$$x_n = \frac{n!}{2\cdot 3 \cdots (n+1)} = \frac{n!}{(n+1)!} = \frac{1}{n+1} \text{ — div}$$

$y_n = \frac{1}{n}$ (Criteriul de comp. cu limită)

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{n}{n+1} = 1 \in (0,\infty)$$

$$\sum x_n \sim \sum y_n$$

$\boxed{Ex}$: $x_n = \frac{n!}{(a+1)\cdots(a+n)}$, $a>-1$; $x_n>0, \forall n\in\mathbb{N}$

Aplicăm criteriul RAPORTULUI şi calculăm:

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{(n+1)!}{(a+1)\cdots(a+n)(a+n+1)} \cdot \frac{(a+1)\cdots(a+n)}{n!} =$$

$$= \lim_{n\to\infty} \frac{n+1}{a+n+1} = 1$$

Criteriul raportului NU se pronunţă. Aplicăm criteriul Raabe-Duhamel

$$\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{n!}{(a+1)\cdots(a+n)} \cdot \frac{(a+1)\cdots(a+n)(a+n+1)}{(n+1)!} - 1\right) =$$

$$= \lim_{n\to\infty} n\left(\frac{a+n+1}{n+1} - 1\right) = \lim_{n\to\infty} n\cdot\frac{a+n+1-n-1}{n+1} = \lim_{n\to\infty} \frac{a\cdot n}{n+1} = a.$$

Discuţie: I $a<1 \Rightarrow \sum_{n=1}^{\infty} x_n$ e divergent

II $a>1$

$(f_n)_n$ - şir de funcţii , $\sum_{n=1}^{\infty} f_n(x)$ - serie de funcţii

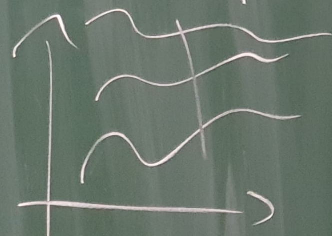
$f_n: X \to \mathbb{R}$

1. Convergenţă simplă: $f_n \xrightarrow[n\to\infty]{s} f$, $\forall x \in X$, avem $\lim_{n\to\infty} f_n(x) = f(x)$.

$f: X \to \mathbb{R}$.

2. Convergenţă uniformă: $f_n \xrightarrow[n\to\infty]{u} f$

$$\lim_{n\to\infty} \sup |f_n(x) - f(x)| = 0$$



Cum rezolv?

Pas 1: $\lim_{n\to\infty} f_n(x) = f$

Pas 2: Găsim A a.î. $\exists \lim_{n\to\infty} f_n(x) = f$, $f: A \to \mathbb{R}$.

Pas 3: $\lim_{n\to\infty} \sup |f_n(x) - f(x)| = \ell$ $\begin{cases} 1.\ \text{Tabel de var.} \\ 2.\ \text{Mărginiri} \end{cases}$

$\to \ell = 0$, $f_n \xrightarrow[n\to\infty]{u} f$

$\ell \neq 0$, $f_n \not\xrightarrow[n\to\infty]{u} f$

$\boxed{Ex1}$ Să se studieze conv simplă și uniformă pt $f_n: [0,+\infty) \to \mathbb{R}$, $f_n(x) = \sqrt{x+\frac{1}{n}}$, $\forall x \in [0,\infty)$, $\forall n \in \mathbb{N}^*$

I. Studiem conv simplă

Fixez $x \in [0,+\infty)$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \sqrt{x+\frac{1}{n}} = \sqrt{x}$$

Definim $f: A \to \mathbb{R}$, $f(x) = \sqrt{x}$

$$f_n \xrightarrow[{[0,+\infty)}]{s} f$$

II. Studiem conv uniformă

Vrem să calc $\sup_{x\in A} |f_n(x) - f(x)|$.

Fie $n \in \mathbb{N}^*$

$$\sup_{x\in A} |f_n(x) - f(x)| = \sup_{x\in A} \left|\sqrt{x+\frac{1}{n}} - \sqrt{x}\right| \le \frac{1}{\sqrt{n}}$$

$A = [0,+\infty)$

$$\sqrt{x+\frac{1}{n}} - \sqrt{x} = \frac{x+\frac{1}{n}-x}{\sqrt{x+\frac{1}{n}}+\sqrt{x}} = \frac{\frac{1}{n}}{\sqrt{x+\frac{1}{n}}+\sqrt{x}} \le \frac{\frac{1}{n}}{\sqrt{\frac{1}{n}}} = \frac{1}{\sqrt{n}}$$

$$0 \le \sup_{x\in A} |f_n(x) - f(x)| \le \frac{1}{\sqrt{n}}$$

$$\lim_{n\to\infty} \sup \ldots = 0 \Rightarrow f_n \xrightarrow[{[0,\infty)}]{u} f \ \checkmark$$

$\ldots(x) = \sqrt{x+\frac{1}{n}},\ \forall x \in [0,\infty),\ \forall n \in \mathbb{N}^*$

$\frac{\frac{1}{n}}{\sqrt{x+\frac{1}{n}}+\sqrt{x}} \le \frac{\frac{1}{n}}{\sqrt{\frac{1}{n}}} = \frac{1}{\sqrt{n}}$

$= \frac{1}{\sqrt{n}}$

$f_n \xrightarrow[{[0,\infty)}]{u} f$ ✓

$\boxed{\text{Ex 2}}$ $f_n(x) : [0,1] \to \mathbb{R},\ f_n(x) = x^{7n}(1-x^{3n})$

CS: fixez $x \in [0,1]$

$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^{7n}(1-x^{3n}) = l$

$x = 0 \Rightarrow l = 0$

$x = 1 \Rightarrow l = 0$

$x \in (0,1) \Rightarrow l = 0$

$\Rightarrow l = 0$

Definim $f : A \to \mathbb{R},\ f(x) = 0,\ A = [0,1]$

$f_n \xrightarrow[{[0,1]}]{s} f$

CU: $\lim_{n\to\infty} \sup |f_n(x) - f(x)| = \lim_{n\to\infty} \sup |\underbrace{x^{7n}(1-x^{3n}) - 0}_{g_n}|$

$g_n'(x) = n x^{7n-1}(7 - 10x^{3n})$

$g_n'(x) = 0 \Rightarrow \begin{cases} x = 0 \\ x = \sqrt[3n]{7/10} \end{cases}$

x	$[0$		$\sqrt[3n]{7/10}$		$1]$
$g_n'(x)$	0	$+$	0	$-$	$-$
$g_n(x)$	0	$\nearrow$	$\frac{3}{10}\left(\frac{7}{10}\right)^{7/3}$ max	$\searrow$	0

$g_n(0) = 0$

$g_n(\sqrt[3n]{7/10}) = \frac{3}{10}\left(\frac{7}{10}\right)^{7/3}$

$g_n(1) = 0$

$\lim_{n\to\infty} \sup |g_n(x)| = \lim_{n\to\infty} \frac{3}{10}\left(\frac{7}{10}\right)^{7/3} = \frac{3}{10}\cdot\left(\frac{7}{10}\right)^{7/3} \neq 0$

$f_n \overset{u}{\underset{[0,1]}{\not\to}} f$