Seminar 4

1. Fie $n \in \mathbb{N}^*$ și $d_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$,

$$d_1(x, y) = |x_1 - y_1| + \dots + |x_n - y_n| = \sum_{i=1}^{n} |x_i - y_i|.$$

$x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$

a) Arătați că d_1 este metrică pe $\mathbb{R}^n$.

Soluție. 1) $d_1(x,y) \geq 0 \ \forall x, y \in \mathbb{R}^n$ (evident)

2) $d_1(x,y) = 0 \Leftrightarrow \sum_{i=1}^{n} |x_i - y_i| = 0 \Leftrightarrow x_i - y_i = 0$
$\forall i = \overline{1,n} \Leftrightarrow x_i = y_i \ \forall i = \overline{1,n} \Leftrightarrow x = y \ \forall x, y \in \mathbb{R}^n.$

3) $d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i| = \sum_{i=1}^{n} |y_i - x_i| =$
$= d_1(y,x) \ \forall x, y \in \mathbb{R}^n.$

4) Fie $x, y, z \in \mathbb{R}^n$.

$$d_1(x,z) = \sum_{i=1}^{n} |x_i - z_i| = \sum_{i=1}^{n} |x_i - y_i + y_i - z_i| \leq$$

$$\leq \sum_{i=1}^{n} \left(|x_i - y_i| + |y_i - z_i| \right) = \sum_{i=1}^{n} |x_i - y_i| + \sum_{i=1}^{n} |y_i - z_i| =$$

$$= d_1(x,y) + d_1(y,z).$$

Prin urmare d_1 este metrică pe $\mathbb{R}^n$. $\square$

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b) Fie $(x^k)_k \subset \mathbb{R}^n$, $x^k = (x_1^k, x_2^k, \ldots, x_n^k)$ $\forall k \in \mathbb{N}$ și $x \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n)$. Arătați că $\lim\limits_{k\to\infty} x^k \overset{d_1}{=} x$ dacă și numai dacă $\lim\limits_{k\to\infty} x_i^k = x_i$ $\forall i = \overline{1,n}$.

<u>Soluție.</u> „$\Rightarrow$"

$$\lim_{k\to\infty} x^k \overset{d_1}{=} x \Rightarrow \lim_{k\to\infty} d_1(x^k, x) = 0 \Rightarrow$$

$\Rightarrow \forall \varepsilon > 0, \exists k_\varepsilon \in \mathbb{N}$ a.î. $\forall k \geq k_\varepsilon$, avem $d_1(x^k, x) < \varepsilon \Rightarrow \forall \varepsilon > 0, \exists k_\varepsilon \in \mathbb{N}$ a.î. $\forall k \geq k_\varepsilon$, avem

$$|x_1^k - x_1| + \ldots + |x_n^k - x_n| < \varepsilon.$$

Fie $i \in \{1, \ldots, n\}$ fixat arbitrar.

Pentru orice $\varepsilon > 0$ și orice $k \geq k_\varepsilon$ avem

$$|x_i^k - x_i| \leq |x_1^k - x_1| + \ldots + |x_n^k - x_n| < \varepsilon,$$

deci $\lim\limits_{k\to\infty} x_i^k = x_i$.

„$\Leftarrow$"

$\forall i = \overline{1,n}$ $\lim\limits_{k\to\infty} x_i^k = x_i \Rightarrow \forall i = \overline{1,n}, \forall \varepsilon > 0,$

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$\exists\, k_\varepsilon^i \in \mathbb{N}$ a.î. $\forall k \geq k_\varepsilon^i$, avem $|x_i^k - x_i| < \frac{\varepsilon}{n}$.

Alegem $k_\varepsilon = \max\{k_\varepsilon^1, k_\varepsilon^2, \ldots, k_\varepsilon^n\}$.

Prin urmare $\forall \varepsilon > 0$, $\forall k \geq k_\varepsilon$, avem $d_1(x^k, x) =$

$$= \sum_{i=1}^{n} |x_i^k - x_i| < n \cdot \frac{\varepsilon}{n} = \varepsilon.$$

Deci $\lim\limits_{k \to \infty} x^k \overset{d_1}{=} x$. $\square$

2. Fie $n \in \mathbb{N}^*$ și $d_2 \overset{\text{not.}}{=} d : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$,

$d(\underset{\substack{\| \\ (x_1, \ldots, x_n)}}{x}, \underset{\substack{\| \\ (y_1, \ldots, y_n)}}{y}) = \sqrt{\sum\limits_{i=1}^{n} (x_i - y_i)^2}$. Arătați că d_2 este metrică pe $\mathbb{R}^n$.

Soluție. 1) $d(x, y) \geq 0 \quad \forall x, y \in \mathbb{R}^n$ (evident).

2) $d(x, y) = 0 \Leftrightarrow \sqrt{\sum\limits_{i=1}^{n} (x_i - y_i)^2} = 0 \Leftrightarrow$

$\Leftrightarrow (x_i - y_i)^2 = 0 \ \forall i = \overline{1,n} \Leftrightarrow x_i = y_i \ \forall i = \overline{1,n} \Leftrightarrow$

$\Leftrightarrow x = y \ \forall x, y \in \mathbb{R}^n$.

3) $d(x, y) = \sqrt{\sum\limits_{i=1}^{n} (x_i - y_i)^2} = \sqrt{\sum\limits_{i=1}^{n} (y_i - x_i)^2} =$

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$= d(y, x) \ \forall\ x, y \in \mathbb{R}^n$.

4) Fie $x, y, z \in \mathbb{R}^n$. Arătăm că $d(x,z) \le d(x,y) + d(y,z)$.

Folosim inegalitatea Cauchy-Buniakowski-Schwarz (C.B.S.): $\forall\ n \in \mathbb{N}^*$, $\forall\ a_1, \dots, a_n \in \mathbb{R}$, $\forall\ b_1, \dots, b_n \in \mathbb{R}$, avem $\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$.

$$d(x,z) = \sqrt{\sum_{i=1}^{n}(x_i - z_i)^2} = \sqrt{\sum_{i=1}^{n}(x_i - y_i + y_i - z_i)^2} =$$

$$= \sqrt{\sum_{i=1}^{n}\left[(x_i - y_i)^2 + (y_i - z_i)^2 + 2(x_i - y_i)(y_i - z_i)\right]} =$$

$$= \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2 + \sum_{i=1}^{n}(y_i - z_i)^2 + 2\sum_{i=1}^{n}(x_i - y_i)(y_i - z_i)} \le$$

$$\underset{\uparrow}{\le} \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2 + \sum_{i=1}^{n}(y_i - z_i)^2 + 2\left(\sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}\right)\left(\sqrt{\sum_{i=1}^{n}(y_i - z_i)^2}\right)} =$$

$$\text{C.B.S.} \Rightarrow \left|\sum_{i=1}^{n}(x_i - y_i)(y_i - z_i)\right| \le \left(\sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}\right)\left(\sqrt{\sum_{i=1}^{n}(y_i - z_i)^2}\right)$$

$$= \sqrt{\left(\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2} + \sqrt{\sum_{i=1}^{n}(y_i-z_i)^2}\right)^2} =$$

$$= \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2} + \sqrt{\sum_{i=1}^{n}(y_i-z_i)^2} =$$

$$= d(x,y) + d(y,z).$$

Deci d este metrică pe $\mathbb{R}^n$. $\square$

3. Fie $n \in \mathbb{N}^*$, d_1 și d_2 ca mai sus. Arătați că există $a, b > 0$ a.î. $a\,d_1(x,y) \le d_2(x,y) \le b\,d_1(x,y)$ $\forall\, x, y \in \mathbb{R}^n$.

<u>Soluție.</u> Fie $x, y \in \mathbb{R}^n$.

$$d_2(x,y) = \sqrt{(x_1-y_1)^2 + \ldots + (x_n-y_n)^2} =$$

$$= \sqrt{|x_1-y_1|^2 + \ldots + |x_n-y_n|^2} \le \sqrt{\left(|x_1-y_1| + \ldots + |x_n-y_n|\right)^2} =$$

$$= |x_1-y_1| + \ldots + |x_n-y_n| = d_1(x,y).$$

Alegem $b = 1$.

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$$d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i| = \sum_{i=1}^{n} |x_i - y_i| \cdot 1 \leq$$

$$\underset{\text{C.B.S.}}{\leq} \left(\sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}\right)\left(\sqrt{\sum_{i=1}^{n} 1^2}\right) = d_2(x,y)\sqrt{n} \Rightarrow$$

$$\Rightarrow \frac{1}{\sqrt{n}} d_1(x,y) \leq d_2(x,y).$$

Alegem $a = \frac{1}{\sqrt{n}}$. $\square$

4. Fie $n \in \mathbb{N}^*$, d_1 ca mai sus și $d_\infty : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $d_\infty(x,y) = \max\{|x_i - y_i| \mid i = \overline{1,n}\}$. Arătați că există $a, b > 0$ a.î. $a d_1(x,y) \leq d_\infty(x,y) \leq b d_1(x,y)$ $\forall x, y \in \mathbb{R}^n$.

Soluție. $d_1(x,y) = |x_1 - y_1| + \ldots + |x_n - y_n| \leq$

$$\leq n \max\{|x_i - y_i| \mid i = \overline{1,n}\} = n\, d_\infty(x,y) \Rightarrow$$

$$\Rightarrow \frac{1}{n} d_1(x,y) \leq d_\infty(x,y).$$

Alegem $a = \frac{1}{n}$.

$$d_\infty(x,y) = \max\{|x_i - y_i| \mid i = \overline{1,n}\} \leq |x_1 - y_1| + \ldots + |x_n - y_n| =$$

$$= d_1(x,y).$$

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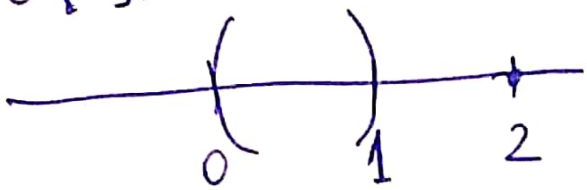
Alegem $b = 1$. $\square$

5. Faceți analiza topologică a mulțimii $A \subset \mathbb{R}$, unde:

a) $A = (0,1) \cup \{2\}$.

<u>Soluție.</u>



1) $\mathring{A} = ?$

$x \in \mathring{A} \Leftrightarrow \exists\, r > 0$ a.î. $(x-r, x+r) \subset A$.

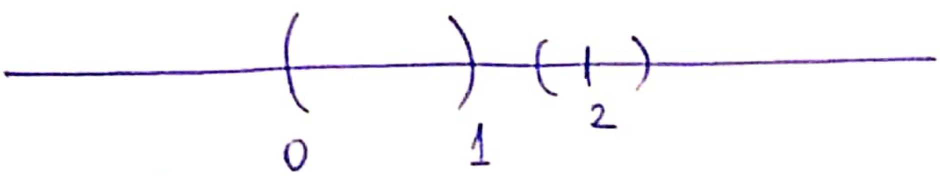
$\mathring{A} \subset A$

$$\left.\begin{array}{l}(0,1) \text{ deschisă} \\ (0,1) \subset A\end{array}\right\} \Rightarrow (0,1) \subset \mathring{A}.$$

Deci $(0,1) \subset \mathring{A} \subset (0,1) \cup \{2\}$.

Studiem dacă $2 \in \mathring{A}$.

$2 \in \mathring{A} \Leftrightarrow \exists\, r > 0$ a.î. $(2-r, 2+r) \subset A$.



Deci $2 \notin \mathring{A}$.

Așadar $\mathring{A} = (0,1)$.

2) $\overline{A} = ?$

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$x \in \overline{A} \Leftrightarrow \forall\, r>0$, avem $(x-r, x+r) \cap A \neq \emptyset$.

$A \subset \overline{A}$

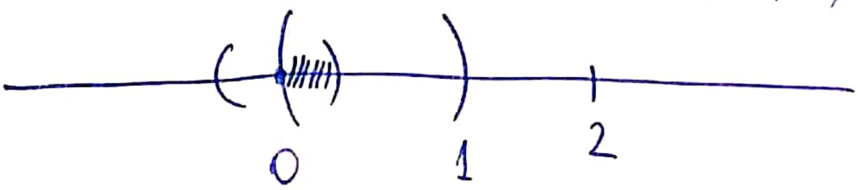
$$\left.\begin{array}{l} [0,1]\cup\{2\} \text{ închisă} \\ A \subset [0,1]\cup\{2\} \end{array}\right| \Rightarrow \overline{A} \subset [0,1]\cup\{2\}.$$

Deci $(0,1)\cup\{2\} \subset \overline{A} \subset [0,1]\cup\{2\}$.

Studiem dacă $0 \in \overline{A}$ și $1 \in \overline{A}$.

$0 \in \overline{A} \Leftrightarrow \forall\, r>0$, avem $\underbrace{(0-r, 0+r)}_{=(-r,r)} \cap \underbrace{A}_{=A} \neq \emptyset$.

$(-r, r) \cap A \neq \emptyset$.



Deci $0 \in \overline{A}$.

Analog $1 \in \overline{A}$.

Așadar $\overline{A} = [0,1]\cup\{2\}$.

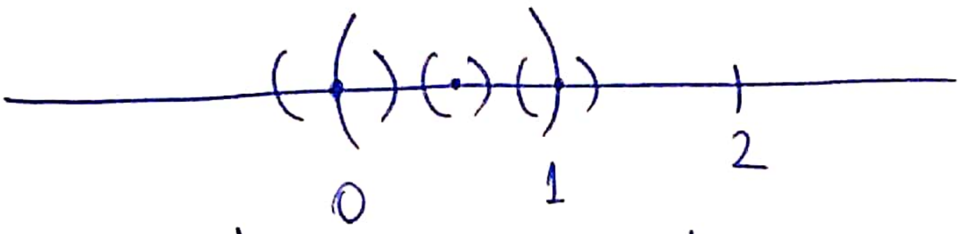
3) $A' = ?$

$x \in A' \Leftrightarrow \forall\, r>0$, avem $(x-r, x+r) \cap \left(A \setminus \{x\}\right) \neq \emptyset$.

$A' \subset \overline{A} = [0,1]\cup\{2\}$.

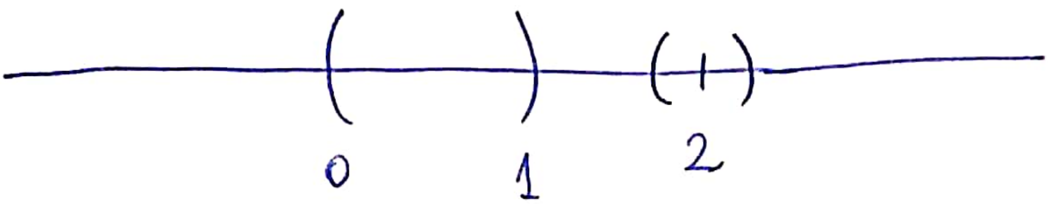
Fie $x \in [0,1]$.

$x \in A' \Leftrightarrow \forall\, r>0$, avem $(x-r, x+r) \cap \left(A \setminus \{x\}\right) \neq \emptyset$.



Deci $x \in A'$, i.e. $[0,1] \subset A'$.

$2 \in A' \Leftrightarrow \forall\, r>0$, avem $(2-r, 2+r) \cap (A \setminus \{2\}) \neq \emptyset$.



Deci $2 \notin A'$.

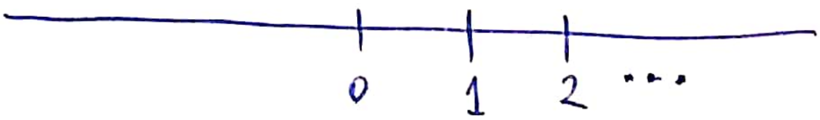
Așadar $A' = [0,1]$.

4) $Fr(A) = \partial A = \bar{A} \setminus \mathring{A} = \{0, 1, 2\}$.

5) $Izo(A) = {}'A = \bar{A} \setminus A' = \{2\}$. □

b) $A = \mathbb{N}$.

Soluție.



1) $\mathring{A} = \emptyset$, deoarece A nu conține niciun interval nedegenerat.

2) $A' = ?$

$x \in A' \Leftrightarrow \forall\, r>0$, avem $(x-r, x+r) \cap (A \setminus \{x\}) \neq \emptyset$.

Arătăm că $A' = \emptyset$.

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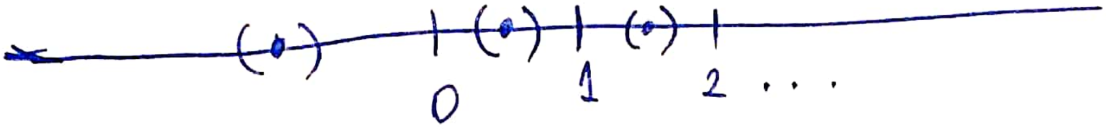
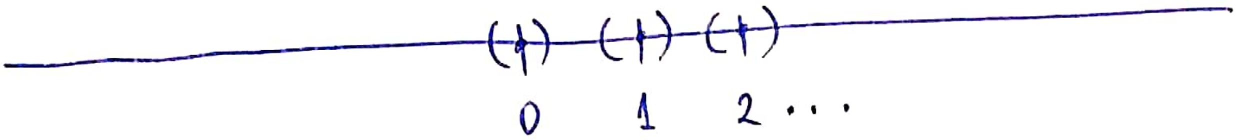
Fie $x \in \mathbb{R} \setminus \mathbb{N}$.



Alegem $r>0$ suficient de mic a.î. $(x-r, x+r) \cap A = \emptyset$.

Deci $x \notin A'$.

Fie $x \in \mathbb{N}$.



Alegem $r>0$ suficient de mic a.î. $(x-r, x+r) \cap A = \{x\}$.

Deci $x \notin A'$.

Așadar $A' = \emptyset$.

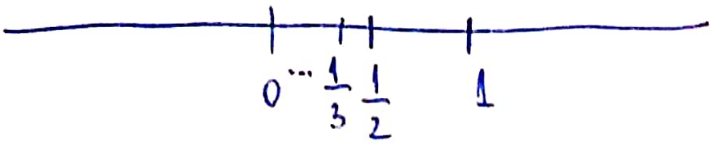
3) $\bar{A} = A \cup A' = \mathbb{N} \cup \emptyset = \mathbb{N}$.

4) $Fr(A) = \partial A = \bar{A} \setminus \mathring{A} = \mathbb{N} \setminus \emptyset = \mathbb{N}$.

5) $Izo(A) = {}'A = \bar{A} \setminus A' = \mathbb{N} \setminus \emptyset = \mathbb{N}$. □

c) $A = \{ \frac{1}{n} \mid n \in \mathbb{N}^* \} = \{ 1, \frac{1}{2}, \frac{1}{3}, \dots \}$.

Soluție.



1) $\mathring{A} = \emptyset$, deoarece A nu conține intervale nedegenerate.

2) $\overline{A}=?$

$A\subset\overline{A}$.

$x\in\overline{A}\iff\exists\,(x_m)_m\subset A$ a.î. $\lim\limits_{m\to\infty}x_m=x$.

Fie $(x_m)_m\subset A$ un șir convergent. Atunci limita sa poate fi un element din A sau 0 (deoarece $\lim\limits_{m\to\infty}\frac{1}{m}=0$)

(Dacă $x_m=\frac{1}{m}\ \forall m\in\mathbb{N}^*$, atunci $\lim\limits_{m\to\infty}x_m=0$, deci $0\in\overline{A}$)

Așadar $\overline{A}=\{0,1,\frac{1}{2},\frac{1}{3},\dots\}=\{0\}\cup\{\frac{1}{n}\mid n\in\mathbb{N}^*\}$.

3) $A'=?$

$A'\subset\overline{A}=\{0,1,\frac{1}{2},\dots\}$.

$x\in A'\iff\exists\,(x_m)_m\subset A\setminus\{x\}$ a.î. $\lim\limits_{m\to\infty}x_m=x$.

Dacă $x\in\{\frac{1}{n}\mid n\in\mathbb{N}^*\}$, atunci niciun șir $(x_m)_m\subset A\setminus\{x\}$ nu are limita x. Deci $x\notin A'$.

Dacă $x=0$, atunci șirul $(x_m)_m\subset A\setminus\{0\}$, $x_m=\frac{1}{m}\ \forall m\in\mathbb{N}^*$, are limita 0 ($\lim\limits_{m\to\infty}x_m=0$), deci $0\in A'$.

Așadar $A'=\{0\}$.

4) $\mathrm{Fr}(A)=\partial A=\overline{A}\setminus\mathring{A}=\{0,1,\frac{1}{2},\frac{1}{3},\dots\}\setminus\emptyset=\{0,1,\frac{1}{2},\frac{1}{3},\dots\}$.

5) $\mathrm{Izo}(A)={}'A=\overline{A}\setminus A'=\{0,1,\frac{1}{2},\frac{1}{3},\dots\}\setminus\{0\}=\{1,\frac{1}{2},\frac{1}{3},\dots\}$. $\square$

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