Seminar 8 - 24.11.2021

Ex. 1: Se consideră $G = \left\{ \begin{pmatrix} m & n \\ \hat{0} & \hat{1} \end{pmatrix} \middle| \, m, n \in \mathbb{Z}_5,\ m \in \{\pm \hat{1}\} \right\}$.

Arătați că $(G, \cdot)$ este grup, „$\cdot$" înmulțirea matricelor.

Rez:

• Parte stabilă:

$$\begin{pmatrix} m & n \\ \hat{0} & \hat{1} \end{pmatrix} \begin{pmatrix} p & q \\ \hat{0} & \hat{1} \end{pmatrix} = \begin{pmatrix} \overset{\in \{\pm \hat{1}\}}{mp} & \overset{\in \mathbb{Z}_5}{mq+n} \\ \hat{0} & \hat{1} \end{pmatrix} \in G$$

$m, n, p, q \in \mathbb{Z}_5$, $m, p \in \{\pm \hat{1}\}$

• Asoc: OK

• Com:

$$\begin{pmatrix} p & q \\ \hat{0} & \hat{1} \end{pmatrix} \begin{pmatrix} m & n \\ \hat{0} & \hat{1} \end{pmatrix} = \begin{pmatrix} pm & pn+q \\ \hat{0} & \hat{1} \end{pmatrix}$$

„$\cdot$" com $\Leftrightarrow \begin{cases} mp = pm \ \checkmark \\ mq + n = pn + q \ ? \end{cases}$ $\forall m, n, p, q$

$m = \hat{1}$, $p = -\hat{1}$, $q = \hat{2}$, $n = \hat{3}$

$mq + n = \hat{2} + \hat{3} = \hat{0}$

$pn + q = -\hat{3} + \hat{2} = -\hat{1}$

„·" nu este com.

• Elem. neutru:

$$\begin{pmatrix} np & nq+m \\ \hat{0} & \hat{1} \end{pmatrix} = \begin{pmatrix} m & n \\ \hat{0} & \hat{1} \end{pmatrix} = \begin{pmatrix} pm & pn+q \\ \hat{0} & \hat{1} \end{pmatrix}$$

$$\begin{cases} mp = m \\ nq+n = n \\ pn+q = n \end{cases} \Rightarrow p = \hat{1} \quad q = \hat{0}$$

$$e = \begin{pmatrix} \hat{1} & \hat{0} \\ \hat{0} & \hat{1} \end{pmatrix}.$$

• Elem. sim.: $\begin{pmatrix} mp & mq+n \\ \hat{0} & \hat{1} \end{pmatrix} = \begin{pmatrix} \hat{1} & \hat{0} \\ \hat{0} & \hat{1} \end{pmatrix} = \begin{pmatrix} pm & pn+q \\ \hat{0} & \hat{1} \end{pmatrix}$

$$\begin{cases} mp = \hat{1} \\ mq+n = \hat{0} \\ pn+q = \hat{0} \end{cases} \Rightarrow p = m^{-1} \ (\text{în } \mathbb{Z}_5), \ m \in \{\pm\hat{1}\} \Rightarrow p = m$$

$$\overset{m=p}{\Rightarrow} \begin{cases} mq+n = \hat{0} \\ mn+q = \hat{0} \end{cases} \Rightarrow q = -mn$$

Obs: Puteți det. inversele luând cazuri: $m = \hat{1}$ sau $m = -\hat{1}$.

$$m \cdot (-mn) + n = -m^2 n + n = -n + n = \hat{0} \quad (\text{ok})$$

Ex. 2: Fie $G = [0,1)$. Pe G definim legea de compoziție

$x \circ y = \{x+y\}$, unde $\{a\}$ = partea fracționară a lui a.

Arătați că $(G, \circ)$ este grup abelian.

Rez:

- Parte stabilă: $x, y \in G \Rightarrow x \circ y = \{x+y\} \in [0,1) = G$.
- Asoc: $(x \circ y) \circ z = x \circ (y \circ z) \quad \forall x, y, z \in G$.

$$(x \circ y) \circ z = \{x+y\} \circ z = \{\{x+y\}+z\} \overset{\{a\} = a - [a]}{=\!=}$$

$$= \{x+y\} + z - [\{x+y\}+z] =$$

$$= x+y - [x+y] + z - [x+y-[x+y]+z] =$$

$$= x+y+z - [x+y] - [x+y+z - \underbrace{[x+y]}_{\in \mathbb{Z}}] \overset{[a+m] = [a]+m,\ m \in \mathbb{Z}}{=}$$

$$= x+y+z - \cancel{[x+y]} - [x+y+z] + \cancel{[x+y]}$$

$$= x+y+z - [x+y+z] = \{x+y+z\}$$

Analog se arată că $x \circ (y \circ z) = \{x+y+z\}$.

- Com : Evidentă

Obs: Dacă „$\circ$" comutativă, $x \circ (y \circ z) = (y \circ z) \circ x$.

- Elem. neutru : $\exists\, e \in G$ a.î. $x \circ e = e \circ x = x$, $\forall x \in G$

$\Rightarrow \{x+e\} = x \, (=\{x\}), \forall x \in [0,1)$.

$\Rightarrow e = 0$. $\quad (\{x\} = x)$

- Elem. sim: $\forall x \in G \ \exists\, y \in G$ a.î. $x \circ y = y \circ x = 0$

$\{x+y\} = 0$.

Dacă $x = 0$, $y = 0 \in G$.

Dacă $x \in (0,1)$, luăm $y = 1 - x \in (0,1)$.

$x \in [0,1)$
$-y \in (-1, 0] \Rightarrow 1-y \in (0,1]$.

$\Rightarrow (G, \cdot)$ grup com.

Ex. 3 : Det. toate morfismele de grupuri de la $(\mathbb{Z}, +)$ la $(\mathbb{Q}, +)$.

Rez: Fie $f : \mathbb{Z} \to \mathbb{Q}$ morfism de grupuri :

- $f(x+y) = f(x) + f(y)$
- $f(0) = 0$.

Fie $n \in \mathbb{N}^*$.

$$f(n) = f(\underbrace{1+1+\ldots+1}_{n \text{ ori}}) \overset{f \text{ morf.}}{=} \underbrace{f(1)+f(1)+\ldots+f(1)}_{n \text{ ori}} = nf(1)$$

Se obs. că dacă știm $f(1)$, știm $f(n)$, $n \in \mathbb{N}$.

$n \in \mathbb{N}^*$. Cum calculăm $f(-n)$?

$$0 = f(0) = f(n+(-n)) = f(n) + f(-n) = nf(1) + f(-n)$$

$\Rightarrow f(-n) = -nf(1)$.

Am obținut că $f(n) = nf(1)$, $\forall n \in \mathbb{Z}$.

Fie $f(1) = a \in \mathbb{Q}$. Morfismele de grupuri aditive de la $\mathbb{Z}$ la $\mathbb{Q}$ sunt: $f_a : \mathbb{Z} \to \mathbb{Q}$, $f(m) = a \cdot m, \forall m \in \mathbb{Z}$

Det. morfismele $f : (\mathbb{Q}, +) \to (\mathbb{Z}, +)$.

$\mathbb{Z} \subseteq \mathbb{Q}$ - Pentru $m \in \mathbb{Z}$ procedăm ca mai devreme

$\Rightarrow f(m) = m f(1), \forall m \in \mathbb{Z}$.

$\mathbb{Q} = \left\{ \frac{m}{n} \;\middle|\; m \in \mathbb{Z}, n \in \mathbb{N}^* \right\}$.

$m \in \mathbb{N}^*, n \in \mathbb{N}^*$

$f\left(\frac{m}{n}\right) = f\Big(\underbrace{\frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n}}_{m}\Big) = m f\left(\frac{1}{n}\right)$

$f\left(-\frac{m}{n}\right) = -m f\left(\frac{1}{n}\right) \quad \left(f(0) = f\left(\frac{m}{n}\right) + f\left(-\frac{m}{n}\right) \right)$.

Dacă aflăm $f\left(\frac{1}{n}\right)$, $n \in \mathbb{N}^*$, det. morfismul pt. orice $q \in \mathbb{Q}$.

$$f(1) = f\Big(\underbrace{\tfrac{1}{m} + \dots + \tfrac{1}{m}}_{m \text{ ori}}\Big) = \underbrace{f\left(\tfrac{1}{m}\right) + \dots + f\left(\tfrac{1}{m}\right)}_{m \text{ ori}} = m \cdot f\left(\tfrac{1}{m}\right)$$

$$\Rightarrow \underbrace{f\left(\tfrac{1}{m}\right)}_{\in \mathbb{Z}} = \frac{1}{m} \underbrace{f(1)}_{\in \mathbb{Z}} \in \mathbb{Z}$$

$$f\left(\tfrac{1}{m}\right) \in \mathbb{Z},\ \forall m \in \mathbb{N}^* \Rightarrow \frac{f(1)}{m} \in \mathbb{Z},\ \forall m \in \mathbb{N}^*$$

$\Rightarrow m \mid f(1),\ \forall m \in \mathbb{N}^*$ ($f(1)$ este multiplu al oricărui nr. natural)

$\Rightarrow f(1) = 0.$

Singurul morfism de grupuri $f : (\mathbb{Q}, +) \to (\mathbb{Z}, +)$ este morfismul nul, $f(q) = 0,\ \forall q \in \mathbb{Q}$.

Ex. 4: Fie $m, n \in \mathbb{N}$, $m, n \geq 2$. Determinați toate morfismele $f : (\mathbb{Z}_n, +) \longrightarrow (\mathbb{Z}_m, +)$.

Rez: Fie $f : \mathbb{Z}_n \to \mathbb{Z}_m$ morfism

- $f(\hat{x} + \hat{y}) = f(\underset{\in \hat{x}}{\hat{x}}) + \underset{\in \bar{x}}{f(\hat{y})}$, $\forall \hat{x}, \hat{y} \in \mathbb{Z}_n$
- $f(\hat{0}) = \bar{0}$

(*) f bine def $\Leftrightarrow f(\hat{x}) = f(\hat{y})$, $\forall x, y \in \mathbb{Z}$ cu $\hat{x} = \hat{y}$.

Dacă luăm în considerare doar primele 2 condiții, procedăm ca în ex. anterioare:

$$f(\hat{k}) = f(\underbrace{\hat{1} + \hat{1} + \ldots + \hat{1}}_{k \text{ ori}}) = \underbrace{f(\hat{1}) + f(\hat{1}) + \ldots + f(\hat{1})}_{k \text{ ori}} = \bar{k} \cdot \underset{\substack{\parallel \\ a \\ \in \mathbb{Z}_m}}{f(\hat{1})}.$$

$\hat{k} \in \{\hat{0}, \hat{1}, \ldots, \widehat{n-1}\}$

∇ Buna def: $\underset{(\hat{k} - \hat{l} = \hat{0})}{\hat{k} = \hat{l}} \Rightarrow f(\hat{k}) = f(\hat{l})$

$f(\hat{k}) = f(\hat{l}) \overset{f \text{ morf}}{\Longleftrightarrow} f(\widehat{k-l}) = f(\hat{0}) = \bar{0}$

$\underline{f(\widehat{k-l}) = \overline{0}}$

$(k-l)\cdot a = \overline{0}$, $\forall k, l \in \mathbb{Z}$, $\hat{k} = \hat{l}$ în $\mathbb{Z}_n$

$\in \mathbb{Z}_m$

$m \mid (k-l)\cdot a$

$\hat{k} = \hat{l}$ în $\mathbb{Z}_n \Rightarrow \widehat{k-l} = \hat{0}$ în $\mathbb{Z}_n \Rightarrow n \mid k-l$

$\Rightarrow k-l = n\cdot t,\ t \in \mathbb{Z}.$

$m \mid n\cdot t\cdot a$, $\forall t \in \mathbb{Z} \Rightarrow m \mid n\cdot a$

Fie $d = (m, n)$ (cmmdc)

$m = d m_1$, $n = d n_1$, $(m_1, n_1) = 1$

$$\left.\begin{array}{l} \underline{\underline{d}}\, m_1 \mid \underline{\underline{d}}\, n_1\cdot a \Rightarrow m_1 \mid n_1\cdot a \\ (m_1, n_1) = 1 \end{array}\right\} \Rightarrow m_1 \mid a.$$

$(4 \mid 2\cdot 6,\ \cancel{2}\cdot 2 \mid \cancel{2}\cdot 6 \Rightarrow 2 \mid 6)$

Morfismele $f : \mathbb{Z}_n \to \mathbb{Z}_m$ sunt de forma $f(\hat{k}) = \overline{ak}$ cu $\frac{m}{(m,n)} \mid a$.

Ex.: $f: \mathbb{Z}_{15} \to \mathbb{Z}_{18}$

$(15, 18) = 3$, $\frac{m}{d} = \frac{18}{3} = 6$.

Morfismele sunt: $f(\hat{k}) = \overline{ka}$ cu $a \in \{6, 12, 18\}$.

$f(\hat{k}) = \overline{6k}$

$f(\widehat{15k}) = \overline{6 \cdot 15k} = \overline{6 \cdot 3 \cdot 5k} = \overline{18 \cdot 5k} = \overline{0}$.

Obs: $f(\hat{k}) = \overline{3k}$

$f(\hat{0}) = \overline{0}$

$f(\widehat{15}) = \overline{45} = \overline{9} \neq \overline{0}$

f nu este bine def.