Tutoriat 6

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Grup factor. Teorema fundamentală de izomorfism

Relații de echivalență pe un grup

G grup, $H \leq G$, $x, y \in G$

a) Spunem că x este congruent la stânga modulo H cu y

$(x \equiv_s y \pmod H) \iff x^{-1}y \in H$

b) Similar, x congruent la dreapta mod. H cu y:

$$x \equiv_d y \pmod H \iff xy^{-1} \in H$$

Obs: „$\equiv_s$", „$\equiv_d$" sunt relații de echivalență

Notații pt. mulțimile factor:

$$\left(G/H\right)_s \overset{not}{=} G/_{\equiv_s \pmod H} \qquad \left(G/H\right)_d \overset{not}{=} G/_{\equiv_d \pmod H}$$

$$\hat{x}^s = \{ y \in G \mid x^{-1}y \in H \} = \{ y \in G \mid y \in xH \} = \{ xh \mid h \in H \} = xH$$

Similar, $\hat{x}^d = Hx$

Prop-Def: G grup, $H \leq G$. Atunci $\left|\left(G/H\right)_s\right| = \left|\left(G/H\right)_d\right| \overset{not}{=} |G:H|$ și se numește indicele lui H în G

Ex: $n \in \mathbb{N}$, $n \geq 2$, $|\mathbb{Z} : n\mathbb{Z}| = n$

$n\mathbb{Z} \leq \mathbb{Z}$, $x \overset{n}{\equiv} y$, $|\mathbb{Z}/_{\equiv}| = n$

$\hat{x} \in \mathbb{Z}/_{\equiv}$, $\hat{x} = \{ y \in \mathbb{Z} \mid x - y \,\vdots\, n \}$

Def: G grup, $H \leq G$

Spunem că H are indice
- finit dacă $|G : H| \in \mathbb{N}$
- infinit dacă $|G : H| = \infty$

Teorema Lagrange

Fie G grup finit (are un nr. finit de elemente) și $H \leq G$. Atunci
$|G| = |H| \cdot |G : H|$ (indice)

Cu alte cuvinte, $|H|$ îl divide $|G|$ (= ordinul lui H (nr. de elem) îl divide pe cel al lui G)

Corolar

Fie G grup finit, $K \leq H \leq G$. Atunci
$|G : K| = |G : H| \cdot |H : K|$

Subgrupuri normale: $H \leq G$. Spunem că H e subgrup normal ($H \trianglelefteq G$) dacă $\forall x \in G$, $xhx^{-1} \in H$, $\forall h \in H$ ($\Leftrightarrow x^{-1}Hx \subseteq H$)

Propoziție. $H \leq G$. Sunt echivalente:

1) $H \trianglelefteq G$
2) $xHx^{-1} = H$, $\forall x \in G$
3) $xH = Hx$, $\forall x \in G$
4) $(G/H)_s = (G/H)_d$

Obs: G grup comutativ, atunci oricare subgrup al său e normal.

Grup factor

Fie G grup, $H \trianglelefteq G$. Notăm $G/H \overset{not}{=} (G/H)_s = (G/H)_d$

$\hat{x} = xH$, $\hat{y} = yH$ și definim:

$$\hat{x} \cdot \hat{y} := \widehat{xy} \in G/H$$

Spunem că $(G/H, \cdot)$ s.n. GRUPUL FACTOR AL LUI G PRIN H

În plus, $\pi : G \longrightarrow G/H$, $\pi(x) = \hat{x}$ morfism surjectiv de grupuri

Ex: 1) $H = \{1\} \trianglelefteq G$. Atunci $G/_{\{1\}} \simeq G$, deoarece

$G/_{\{1\}} = \{ \{x\} \mid x \in G \}$.

2) $G = (\mathbb{Z}, +)$, $n \in \mathbb{N}$, $n \geq 2$, $H = n\mathbb{Z} \trianglelefteq \mathbb{Z}$

$$\mathbb{Z}/_{n\mathbb{Z}} \overset{\text{not}}{=} \mathbb{Z}_n = \{ \hat{0}, \hat{1}, \ldots, \widehat{n-1} \}$$

$\hat{a}, \hat{b} \in \mathbb{Z}_n$, $\hat{a} + \hat{b} := \widehat{a+b} \in \mathbb{Z}_n$

PROPRIETATEA DE UNIVERSALITATE A GRUPULUI FACTOR (PUGF)

G grup, $H \trianglelefteq G$, $\pi : G \to G/_H$, $\pi(g) = \hat{g}$

$$\begin{tikzcd} G \arrow[r, "\pi"] \arrow[d, "\varphi"'] & G/_H \arrow[dl, dashed, "\exists!\, \bar{\varphi}"] \\ G' & \end{tikzcd}$$

Atunci $\forall G'$ grup și $\forall \varphi : G \to G'$ morfism, $Ker(\varphi) \supseteq H$, $\exists!\, \bar{\varphi} : G/_H \to G'$ morfism a.î. $\bar{\varphi} \circ \pi = \varphi$

a) $\bar{\varphi}$ surj $\Leftrightarrow$ φ surj.

b) $\bar{\varphi}$ inj $\Leftrightarrow$ $Ker(\varphi) = H$

TEOREMA FUNDAMENTALĂ DE IZOMORFISM (TFI)

Fie $f : G \to G'$ morfism de grupuri. Atunci:

$\tilde{f} : G/_{Ker(f)} \xrightarrow{\sim} Im\, f$, $\tilde{f}(\hat{x}) = f(x)$ izo.

Ex: $\mathbb{R} \leq (\mathbb{C}, +)$. Arătați: $\mathbb{C}/_{\mathbb{R}} \simeq (\mathbb{R}, +)$

PUGF:

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\pi"] \arrow[d, "f"'] & \mathbb{C}/_{\mathbb{R}} \arrow[dl, dashed, "\bar{f}"] \\ \mathbb{R} & \end{tikzcd}$$

$f : \mathbb{C} \to \mathbb{R}$, $f(a + bi) = b$

f surj

$Ker(f) = ?$

$a + bi \in Ker(f) \Leftrightarrow f(a+bi) = 0 \Leftrightarrow b = 0$ ...

$Ker(f) = \{ a \mid a \in \mathbb{R} \} = \mathbb{R}$

PUGF $\Rightarrow$ $\exists!\, \bar{f} : \mathbb{C}/_{\mathbb{R}} \xrightarrow{\sim} \mathbb{R}$, $\bar{f}(\widehat{a + bi}) = b$ izo

TFI $f(a+bi) = b$ surj. $Ker(f) = \mathbb{R}$ $\overset{\text{TFI}}{\Longrightarrow}$ $\tilde{f} : \mathbb{C}/_{\mathbb{R}} \xrightarrow{\sim} Im(f) = \mathbb{R}$, $\tilde{f}(\widehat{a+bi}) = b$ izo

Exerciții

1) Aflați subgrupurile normale și grupurile factor ale lui $(\mathbb{Z}_{12}, +)$

SOL :

$H \leq \mathbb{Z}_{12}$: $x, y \in H$, $x - y \in H$

$\{\hat{0}\}, \mathbb{Z}_{12} \leq \mathbb{Z}_{12}$

Bonus : $\mathcal{L}(\mathbb{Z}_n) = \{ d\mathbb{Z}/_{n\mathbb{Z}} \mid d \mid n \}$ (grupurile factor ale lui $\mathbb{Z}_n$)

Căutăm $d \mid 12$

$d \in \{1, 2, 3, 4, 6, 12\}$

$\{0\} \leq \mathbb{Z}_{12} \Rightarrow \mathbb{Z}_{12}/_{\{0\}} \simeq \mathbb{Z}_{12}$

$d = 1 \Rightarrow H = \mathbb{Z}_{12} \leq \mathbb{Z}_{12} \Rightarrow \mathbb{Z}_{12}/\mathbb{Z}_{12} \simeq \{0\}$

$d = 2 \Leftrightarrow \langle \hat{2} \rangle = \{\hat{0}, \hat{2}, \hat{4}, \hat{6}, \hat{8}, \hat{10}\} \leq \mathbb{Z}_{12}$

$(x, y \in \langle \hat{2} \rangle,\ x - y \in \langle \hat{2} \rangle)$

$|\langle \hat{2} \rangle| = 6$, $|\mathbb{Z}_{12}| = 12$

T. Lagrange : $\underset{12}{|\mathbb{Z}_{12}|} = \underset{6}{|\langle \hat{2} \rangle|} \cdot |\mathbb{Z}_{12}/\langle 2 \rangle| \Rightarrow |\mathbb{Z}_{12}/\langle 2 \rangle| = 2$

$\boxed{\mathbb{Z}_{12}/\langle 2 \rangle \simeq \mathbb{Z}_2}$

(Mai târziu: un grup finit este izomorf fie cu $\mathbb{Z}_n$, fie cu S_n)

$\boxed{d = 3} \Leftrightarrow \langle \hat{3} \rangle = \{\hat{0}, \hat{3}, \hat{6}, \hat{9}\} \leq \mathbb{Z}_{12}$

$\langle \hat{3} \rangle = 3\mathbb{Z}/_{12\mathbb{Z}}$

$\mathbb{Z}_{12}/\langle 3 \rangle = \dfrac{\mathbb{Z}/_{12\mathbb{Z}}}{3\mathbb{Z}/_{12\mathbb{Z}}} \simeq \mathbb{Z}/_{3\mathbb{Z}} = \mathbb{Z}_3$

$\boxed{d = 4} \Leftrightarrow \langle \hat{4} \rangle = \{\hat{0}, \hat{4}, \hat{8}\}$

$\mathbb{Z}_{12}/\langle 4 \rangle = \dfrac{\mathbb{Z}/_{12\mathbb{Z}}}{4\mathbb{Z}/_{12\mathbb{Z}}} \simeq \mathbb{Z}/_{4\mathbb{Z}} = \mathbb{Z}_4$

Similar pt $d = 6$

$(\mathbb{Z},+)$, $12\mathbb{Z} \trianglelefteq \mathbb{Z}$, $y \in \mathbb{Z}$ (Mă bazez pe intuiție)

$8 \equiv y \Leftrightarrow -8+y \in 12\mathbb{Z} \Leftrightarrow -8+y \in \{12m \mid m \in \mathbb{Z}\}$

$\Leftrightarrow y \in \{12m+8 \mid m \in \mathbb{Z}\}$

$\Leftrightarrow y = 12\mathbb{Z}+8 \Longleftrightarrow \hat{y} = \hat{8}$

Ce înseamnă factorizarea printr-un subgrup?

$\mathbb{Z},+$ $\quad y = nx + r \xrightarrow[\text{la } \mathbb{Z}_n]{\text{factorizare}} \hat{y} = \hat{r}$

Produs ce conține n devine $\hat{0}$

2)) Arătați că $f : \mathbb{Z} \xrightarrow{\sim} \mathbb{Z}\times\mathbb{Z}/_{\langle(2,3)\rangle}$, $f(m) := \widehat{(m,m)}$,

$(\forall)\, m \in \mathbb{Z}$ este izomorfism de grupuri.

- f bijectivă

- f inj $\Leftrightarrow \mathrm{Ker}(f) = \{0\}$ $\quad \left(\mathrm{Ker}(f) = \{ f^{-1}(\widehat{0,0}) \mid x \in \mathbb{Z} \text{ a.î. } f(x) = \widehat{(0,0)} \} \right.$

Fie $m \in \mathrm{Ker}(f) \Rightarrow f(m) = \widehat{(0,0)} \Leftrightarrow \widehat{(m,m)} = \widehat{(0,0)}$

$\Leftrightarrow \widehat{(m,m)} - \widehat{(0,0)} \in \langle(2,3)\rangle = \{t(2,3) \mid t \in \mathbb{Z}\}$

$\Leftrightarrow (m,m) \in \langle 2,3 \rangle$

$\Leftrightarrow \exists t \in \mathbb{Z}$ a.î.

$\begin{cases} m = 2t \\ m = 3t \end{cases} \Rightarrow t = 0 \Rightarrow m = 0$

$\Leftrightarrow \mathrm{Ker}(f) = \{0\}$

$\boxed{\begin{array}{l} (\mathbb{Z},+) \text{ el. neu} = 0 \\ (\mathbb{Z}_n,+) \text{ el. neu} = \hat{0} \end{array}}$

- f surj

Fie $\widehat{(a,b)} \in \mathbb{Z}\times\mathbb{Z}/_{\langle(2,3)\rangle}$

Vrem $m \in \mathbb{Z}$ a.î. $f(m) = \widehat{(a,b)} = \widehat{(m,m)}$

$\Leftrightarrow \widehat{(a,b)} - \widehat{(m,m)} \in \langle(2,3)\rangle = \{t(2,3) \mid t \in \mathbb{Z}\} = \{(2t,3t)\}$

$\Leftrightarrow (a-m, b-m) \in \langle(2,3)\rangle \Leftrightarrow \exists t \in \mathbb{Z}$ a.î. $\begin{cases} a-m = 2t \\ b-m = 3t \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} a = 2t+m \\ b = 3t+m \end{cases} \Big| \overset{\ominus}{\Rightarrow} b-a = t \quad \Rightarrow \underline{m = 3a-2b \in \mathbb{Z}} \Rightarrow f$ surj.

• f morfism

$f(0)=(\hat{0},0)$

$f(m+n) = f(m)+f(n) = (\widehat{m+n}, m+n)$

Deci f izo

Subgrupul generat de o mulțime

G grup, $X \subseteq G$. $\langle X\rangle = \{x_1^{\alpha_1}x_2^{\alpha_2}\dots x_n^{\alpha_n} \mid x_i\in X, \alpha_i\in\mathbb{Z}\} \le G$

$(G,\cdot)$

$(G,+) \Rightarrow \langle X\rangle = \{\alpha_1 x_1+\alpha_2 x_2+\dots+\alpha_n x_n \mid x_i\in X, \alpha_i\in\mathbb{Z}\}$

Ex: $(\mathbb{Z},+)$, $\langle 35\rangle = \{35t \mid t\in\mathbb{Z}\} = 35\mathbb{Z} \le \mathbb{Z}$

$(\mathbb{Q},\cdot)$, $\langle 2\rangle = \{2^t \mid t\in\mathbb{Z}\} \le \mathbb{Q}$

3) $\mathbb{Z} \trianglelefteq (\mathbb{R},+)$, $U=\{z\in\mathbb{C} \mid |z|=1\} \le (\mathbb{C}^*,\cdot)$

Arătați că: $\mathbb{R}/\mathbb{Z} \simeq U$

SOL: TUGF

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\pi"] \arrow[d, "\varphi"'] & \mathbb{R}/\mathbb{Z} \arrow[dl, dashed, "\exists!\,\bar{\varphi}"] \\ U & \end{tikzcd}$$

Vreau $\varphi:\mathbb{R}\to U$ morfism surjectiv

$\varphi(r) = \cos(2\pi r) + i\sin(2\pi r)$

surjectivă; morfism (1)

$\mathrm{Ker}(\varphi) = ?$

$\mathrm{Ker}(\varphi) = \{\varphi^{-1}(\underset{\text{el. neu}}{1}) \mid x\in\mathbb{R} \text{ a.î. } \varphi(x)=1\}$

$f(r)=1 \Leftrightarrow \cos(2\pi r)+i\sin(2\pi r) = 1+0i \Leftrightarrow \begin{cases}\cos(2\pi r)=1\\ \sin(2\pi r)=0\end{cases}, \forall r\in\mathbb{Z}$

Deci $\mathrm{Ker}(\varphi)=\mathbb{Z}$ (2)

Din (1), (2) $\overset{\text{TUGF}}{\Longrightarrow} \exists!\ \bar{\varphi}: \mathbb{R}/\mathbb{Z} \xrightarrow{\sim} U$, $\bar{\varphi}(\hat{r}) = \varphi(r)$ izo

Ce e $\mathbb{R}/\mathbb{Z}$? $x, y \in (\mathbb{R}, +)$

$x \equiv y \iff \underbrace{x-y}_{} \in \mathbb{Z} \iff x \in \{m+y \mid m \in \mathbb{Z}\}$

$x \in \mathbb{Z} + y$

$\Downarrow$

$\hat{x} = \hat{y}$

$x = \sqrt{3} \iff x - \sqrt{3} \in \mathbb{Z}$

$\sqrt{3} - x \in \mathbb{Z}$

$\sqrt{3} \in \mathbb{Z} + x$

$x = [x] + \{x\}$

$\widehat{\sqrt{3}} = \widehat{\{x\}} = \hat{y}$

(calcul haotic pe intuiție)

G grup, $H \trianglelefteq G$, $\hat{x}, \hat{y} \in G/H$

* $\hat{x} = \hat{y} \iff x - y \in H$