Temă

4. Arătați că dacă $X \sim Exp(\lambda)$, atunci $[X]+1 \sim Geom(1-e^{-\lambda})$

$$\mathbb{P}([X]+1=1) = \mathbb{P}([X]=0) = \mathbb{P}(0 \le X < 1) = \int_0^1 \lambda e^{-\lambda x} dx =$$

$$= -\int_0^1 (e^{-\lambda x})' = -e^{-\lambda x}\Big/_0^1 = -e^{-\lambda} + e^0 = 1 - e^{-\lambda}$$

$$\mathbb{P}([X]+1=k) = \mathbb{P}([X]=k-1) = \mathbb{P}(k-1 \le x < k) = \int_{k-1}^{k} \lambda e^{-\lambda x} dx$$

$$= -e^{-\lambda x}\Big/_{k-1}^{k} = -e^{\lambda k} + e^{-\lambda(k-1)} = e^{-\lambda(k-1)}(-e^{-\lambda}+1)$$

$$= [1-(1-e^{-\lambda})]^{k-1} (\underbrace{1-e^{-\lambda}}_{p}) = (1-p)^{k-1} \cdot p, \text{ unde } p = 1-e^{-\lambda} \Rightarrow$$

$$\Rightarrow [X]+1 \sim Geom(1-e^{-\lambda})$$

5. Durata de viață a calculatoarelor din laboratorul 305 este o variabilă aleatoare continuă cu densitatea

$$f(x) = \begin{cases} k - \frac{x}{50}, & pt\ 0 \le x \le 10 \\ 0, & altfel \end{cases}$$

a) $k = ?$

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_0^{10} \left(k - \frac{x}{50}\right) dx + 0 = \int_0^{10} k\,dx - \frac{1}{50}\int_0^{10} x\,dx$$

$$= kx\Big/_0^{10} - \frac{x^2}{50 \cdot 2}\Big/_0^{10} = 10k - \frac{100}{100} = 10k - 1 \Rightarrow 10k = 2 \Rightarrow k = \frac{1}{5}$$

b) Care este probabilitatea ca un calculator să se strice în 5 ani? (i.e. $\mathbb{P}(X \le 5)$)

$$\mathbb{P}(X \le 5) = \int_0^5 \left(\frac{1}{5} - \frac{x}{10}\right) dx = \frac{x}{5}\Big/_0^5 - \frac{x^2}{50 \cdot 2}\Big/_0^5 = \underbrace{\frac{5}{5}}_{1} - \underbrace{\frac{25}{100}}_{\frac{1}{4}} = \frac{3}{4}$$

c) Care este durata medie de viață a unui calculator? Cu ce varianță?

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)\, dx = \int_0^{10} x\left(\frac{1}{5} - \frac{x}{50}\right) dx + 0 =$$

$$= \frac{1}{5}\int_0^{10} x\, dx - \frac{1}{50}\int_0^{10} x^2 dx = \frac{1}{5} \cdot \frac{x^2}{2}\Big|_0^{10} - \frac{1}{50} \cdot \frac{x^3}{3}\Big|_0^{10}$$

$$= \frac{10^{\cancel{2}}}{1\cancel{0}} - \frac{10^{\cancel{3}2}}{15\cancel{0}} = \frac{3}{10} - \frac{\cancel{100}^{20}}{\cancel{15}3} = \frac{10}{3} \simeq 3,(3)$$

$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_0^{10} x^2\left(\frac{1}{5} - \frac{x}{50}\right) dx + 0 =$$

$$= \frac{1}{5}\int_0^{10} x^2 dx - \frac{1}{50}\int_0^{10} x^3 dx = \frac{1}{5} \cdot \frac{x^3}{3}\Big|_0^{10} - \frac{1}{50} \cdot \frac{x^4}{4}\Big|_0^{10}$$

$$= \frac{\cancel{10^3}^{200}}{\cancel{15}3} - \frac{10^{\cancel{4}2}}{2\cancel{00}} = \frac{200}{3} - \frac{\cancel{100}^{3/50}}{\cancel{2}} = \frac{50}{3}$$

$$\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{50}{3}^{3} - \frac{100}{9} = \frac{50}{9} \simeq 5,(5)$$

2. Fie Z o variabilă aleatoare continuă normală standard, $Z \sim \mathcal{N}(0,1)$. Ce distribuție are $X := \sigma Z$? Argumentați.

$$X := \sigma Z \Rightarrow X \sim \mathcal{N}(0, \sigma^2) \text{ dacă } p_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{\frac{-x^2}{2\sigma^2}}$$

$$\mathbb{P}(X \le k) = \mathbb{P}(\sigma Z \le k) = \mathbb{P}\left(Z \le \frac{k}{\sigma}\right) = \mathbb{P}\left(-\infty < Z \le \frac{k}{\sigma}\right) =$$

$$= \int_{-\infty}^{\frac{k}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}} dz = \int_{-\infty}^{\frac{k}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2\sigma^2}{2\sigma^2}} dz \underset{\substack{x = \sigma z \Rightarrow z = \frac{x}{\sigma} \\ dx = \sigma dz \Rightarrow dz = \frac{dx}{\sigma}}}{=\!=\!=}$$

$$= \int_{-\infty \cdot \sigma}^{k \cdot \frac{\sigma}{\sigma}} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx = \int_{-\infty}^{k} p_X(x)\, dx \Rightarrow p_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{\frac{-x^2}{2\sigma^2}} \; (A) \Rightarrow$$

$$\Rightarrow X \sim \mathcal{N}(0, \sigma^2)$$

3. Dacă $X \sim Exp(\lambda)$ și $\alpha > 0$, ce distribuție are $Y := \alpha X$?
Dar $Z := X^2$? Argumentați

$$p_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{altfel} \end{cases}$$

$$\mathbb{P}(Y \leq k) = \mathbb{P}(\alpha X \leq k) = \mathbb{P}\left(X \leq \frac{k}{\alpha}\right) = \mathbb{P}\left(-\infty < X \leq \frac{k}{\alpha}\right)$$

$$= \int_{-\infty}^{\frac{k}{\alpha}} p_X(x)\,dx = \int_0^{\frac{k}{\alpha}} \lambda e^{-\lambda x}\,dx + 0 = \int_0^{\frac{k}{\alpha}} \lambda e^{-\frac{\lambda \alpha x}{\alpha}}\,dx$$

$$= \int_{0\cdot\alpha}^{k\cdot\frac{\alpha}{\alpha}} \frac{\lambda}{\alpha} e^{-\frac{\lambda y}{\alpha}}\,dy = \int_0^k \frac{\lambda}{\alpha} e^{-\frac{\lambda y}{\alpha}}\,dy \Rightarrow$$

$y = \alpha x \Rightarrow x = \frac{y}{\alpha}$

$dy = \alpha\,dx \Rightarrow dx = \frac{dy}{\alpha}$

$$\Rightarrow p_Y(y) = \begin{cases} \frac{\lambda}{\alpha} e^{-\frac{\lambda}{\alpha} y}, & y \geq 0 \\ 0, & \text{altfel} \end{cases} \Rightarrow Y \sim Exp\left(\frac{\lambda}{\alpha}\right)$$