Discrete Mathematics MT17: Problem Sheet 5

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December 2017

1 5.1

(i) Let $X = \{B_0, B_2, ..., B_{n-1}\}$ be a chain. As $X \subseteq P(\{1, 2, 3, 4, 5, 6\})$ we have that $0 \leq |B_i| \leq 6$. Assuming $|X| \geq 7$ there would be at least two different elements of X with the same cardinality (by the pigeonhole principle). But, under these conditions these two elements would not be comparable, contradicting the fact that X is a chain. Thus $|X| \leq 6$.
(ii) On the same lines, we have that $|B_i| = i$ for $i \in \{0, 1, ..., 6\}$, so every set B_i is build as $B_i = B_{i-1} \cup b_i$ with $B_0 = \emptyset$ and the b_i's being unique elements from $\{1, 2, 3, 4, 5, 6\}$. There is a natural bijection between these and permutations on $1, 2, 3, 4, 5, 6$, so the total number of chains is $6! = 720$.
(iii) The required antichain is $Y = \{S \subseteq \{1, 2, 3, 4, 5, 6\} : |S| = 3\}$. Its maximality and uniqueness are established by Sperner's Theorem.

2 5.2

(i) The relation is a preorder.
(ii) The relation is a partial order. Every pair of elements have a lub, constructible as $z_i = max(x_i, y_i)$.
(iii) The relation is a partial order. Some pairs of elements don't have a lub, such as $(x, y) = (19, 20)$.
(iv) The relation is a linear order. Every pair of elements has a lub, given by the "larger element" in the pair.

3 5.3

(i) $glb((1, 3), (1, 3) = (1, 1)$
(ii) $glb((2, 3), (3, 2) = (1, 6)$
(iii) $glb((6, 2), (3, 3) = (3, 3)$

4 5.4

Let $S = \{a + b : a \in A \wedge b \in B\}$.
We'll use the alternative definition of the lub found in the lecture notes:

(i) We'll prove that $x \leq lubA + lubB$ for every $x \in S$. As $x \in S$ we know that $x = a + b$ for some $a \in A$, $b \in B$. But, by definition, $a \leq lubA$ and $b \leq lubB$, therefore $x = a + b \leq lubA + lubB$.

(ii) We'll prove that for every $y < lubA + lubB$ there is a number $x \in S$ such that $y < x$. Let $d = \frac{lubA + lubB - y}{2}$ and $a' = lubA - d, b' = lubB - d$. Note that $a' + b' = y$. As $a' < lubA$ and $b' < lubB$ there are two numbers a'' and b'' such that $a' < a'' < lubA$ and $b' < b'' < lubB$. Adding up these two relations we get that $y < a'' + b'' < lubA + lubB$. Finally, by setting $x = a'' + b'' \in S$ we get the conclusion.

5 5.5

Proof by double inclusion:

(i) We'll show that $S \cap T = \emptyset$ implies that $S \oplus T = S \cup T$. By definition $S \oplus T = (S \setminus T) \cup (T \setminus S) = S \cup T$ (as the sets are disjoint).

(ii) We'll show that $S \oplus T = S \cup T$ implies $S \cap T = \emptyset$. Assume for a contradiction that $S \cap T \neq \emptyset$, and pick some $a \in S \cap T$ (same as saying $a \in S$ and $a \in T$). We know that $S \oplus T = S \cup T$, so it must be that $a \in S \oplus T$. $a \in S \oplus T$ is equivalent to $a \in (S \setminus T) \cup (T \setminus S)$, or to saying that either $a \in S \wedge a \notin T$ or $a \notin S \wedge a \in T$, none of which are actually true, as we already know that both $a \in S$ and $a \in T$ hold true. This is a contradiction, so it must be that $S \cap T = \emptyset$.

6 5.6

Variations on a well established method:

Let $x = k \cdot n$ be number divisible by n, for some non-negative integer k. The condition that x has exactly m digits can be written as $10^{m-1} \leq x < 10^m$ or $10^{m-1} - 1 < x \leq 10^m - 1$ or $\frac{10^{m-1}-1}{n} < k \leq \frac{10^m - 1}{n}$. By the subtract rule the number of such integers k is given by $\left\lfloor \frac{10^m-1}{n} \right\rfloor - \left\lfloor \frac{10^{m-1}-1}{n} \right\rfloor$.

The number of 4 digit numbers which are divisible by both a and b can be found by substituting $n = lcm(a, b)$ in the above-derived formula. Let $f(n) = \left\lfloor \frac{10^4-1}{n} \right\rfloor - \left\lfloor \frac{10^{4-1}-1}{n} \right\rfloor$. For $a = 6$ and $b = 15$ we get that $n = 30$ and a total of $f(30) = 333 - 33 = 300$ such numbers.

In order to count those numbers that are divisible by at least one of 6 and 15 we will use the inclusion-exclusion principle together with our function f. The answer is $f(6) + f(15) - f(30) = 1500 + 600 - 300 = 1800$.

Similarly, the number of numbers that are divisible by at least one of 6, 10 and 15 is given by $f(6) + f(10) + f(15) - f(lcm(6, 10)) - f(lcm(6, 15)) - f(lcm(10, 15)) + f(lcm(6, 10, 15)) = f(6) + f(10) + f(15) - f(30) - f(30) - f(30) + f(30) = f(6) + f(10) + f(15) - 2 \cdot f(30) = 1500 + 900 + 15 - 2 \cdot 300 = 2400$.

7 5.7

$f(n) = \sum_{i=0}^{i=k} a_i n^i$

(i) For two different integers n and m we can write $f(n) - f(m) = \sum_{i=0}^{i=k} a_i(n^i - m^i) = \sum_{i=1}^{i=k} a_i(n^i - m^i) = \sum_{i=0}^{i=k} a_i(n-m) \sum_{l=0}^{l=i} n^l m^{i-l} = (n-m) \sum_{i=0}^{i=k} \sum_{l=0}^{l=i} a_i n^l m^{i-l}$

(ii) Let x be an integer such that $f(x) = 1$. By writing (i) for $n = x$ and $m \in \{0, 3\}$ we get that both $x - 0$ and $x - 3$ divide $f(x) - 0 = 1$, or x and $x - 3$ both divide 1, which implies that $x = 1$ and $x - 3 = -1$. These two can't both hold true at the same time, contradicting the existence of x, meaning that such an x can not exist.

8 5.8

The classic Catalan sequence:

(i) $C_0 = 1, C_1 = 1, C_2 = 2, C_3 = 5, C_4 = 14$

(ii) It trivially holds for $n = 0$. Assume it holds for some $n \geq 0$, let's prove it works for $n + 1$.

Inductive hypothesis: $C_n = \frac{1}{n+1}\binom{2n}{n}$

$C_{n+1} = \frac{4n+2}{n+2} C_n = \frac{4n+2}{n+2} \frac{1}{n+1}\binom{2n}{n} = \frac{(4n+2)(2n)!}{(n+1)(n+2)(n!)^2} = \frac{1}{n+2} \frac{2(2n+1)(2n)!}{(n+1)(n!)^2} = \frac{1}{n+2} \frac{(2n+2)(2n+1)(2n)!}{(n+1)^2(n!)^2} = \frac{1}{n+2} \frac{(2n+2)!}{((n+1)!)^2} = \frac{1}{n+2}\binom{2n+2}{n+1}$, which establishes the inductive step.

(iii) This fact is quite trivial once (ii) has been established: Every factor of C_n should also be a factor of $(n + 1)(n!)^2 C_n = (2n)!$. As this number is a product of $2n$ terms at most equal to $2n$, all its prime factors must also be at most equal to $2n$, or, by noting that $2n$ is even and non-prime, at most $2n-1$.

(iv) It trivially holds for $n = 4$. Assume it holds for some $n \geq 4$, let's prove it works for $n + 1$.

Inductive hypothesis: $C_n \geq 2n$

Multiplying both sides by $\frac{4n+2}{n+2}$:

$C_{n+1} \geq \frac{2n(4n+2)}{n+2}$

All we have left to do is show that:

$\frac{2n(4n+2)}{n+2} \geq 2(n+1)$

Or $n(4n+2) \geq (n+1)(n+2)$

Or $3n^2 - n - 2 >= 0$

This expression becomes $42 \geq 0$ when $n = 4$ and the left hand side can only further increase, so it holds for all $n \geq 4$, thus establishing the inductive step.

(v) First, look at C_0, C_1, C_2, C_3 and establish that, indeed, only C_2 and C_3 are prime amongst them. Now, assume $n \geq 4$ to show that these are, indeed, the only Catalan primes. From (iii) we know that all prime factors of C_n shall be $< 2n$ and from (iv) we know that $C_n \geq 2n$, meaning that C_n's largest prime factor is strictly smaller than itself, showing C_n's compositeness.

(vi) We know that $bn^{n+\frac{1}{2}}exp(-n) \leq n! \leq cn^{n+\frac{1}{2}}exp(-n)$ for some positive constants b and c.

For later use, take inverses: $\frac{1}{bn^{n+\frac{1}{2}}exp(-n))} \geq \frac{1}{n!} \geq \frac{1}{cn^{n+\frac{1}{2}}exp(-n)}$

And then rewrite, flipping the order: $\frac{1}{cn^{n+\frac{1}{2}}exp(-n)} \leq \frac{1}{n!} \leq \frac{1}{bn^{n+\frac{1}{2}}exp(-n))}$

Write the original fact for $2n$: $b(2n)^{2n+\frac{1}{2}}exp(-2n) \leq (2n)! \leq c(2n)^{2n+\frac{1}{2}}exp(-2n)$

Now, square the final version of the original fact and multiply it by the fact for $2n$ to yield:

$\frac{b(2n)^{2n+\frac{1}{2}}exp(-2n)}{(cn^{n+\frac{1}{2}}exp(-n))^2} \leq \frac{(2n)!}{(n!)^2} \leq \frac{c(2n)^{2n+\frac{1}{2}}exp(-2n)}{(bn^{n+\frac{1}{2}}exp(-n))^2}$

Or,

$\frac{b(2n)^{2n+\frac{1}{2}}}{(cn^{n+\frac{1}{2}})^2} \leq \frac{(2n)!}{(n!)^2} \leq \frac{c(2n)^{2n+\frac{1}{2}}}{(bn^{n+\frac{1}{2}})^2}$

This simplifies to:

$\frac{b4^n\sqrt{2}}{c^2\sqrt{n}} \leq \frac{(2n)!}{(n!)^2} \leq \frac{c4^n\sqrt{2}}{b^2\sqrt{n}}$

Divide everything by $n+1$:

$\frac{b4^n\sqrt{2}}{c^2(n+1)\sqrt{n}} \leq C_n \leq \frac{c4^n\sqrt{2}}{b^2(n+1)\sqrt{n}}$

We may now just as well drop the left inequality, as it's no longer needed:

$C_n \leq \frac{c4^n\sqrt{2}}{b^2(n+1)\sqrt{n}} \leq \frac{c4^n\sqrt{2}}{b^2n\sqrt{n}} = \frac{c\sqrt{2}}{b^2}4^n n^{-\frac{3}{2}}$, for all $n \geq 1$, establishing that $C_n \in O(4^n n^{-\frac{3}{2}})$, as required.