1. Исследовать функцию на условный экстремум:

$$U = 3 - 8x + 6y, \text{ если } x^2+y^2=36$$

$$L(\lambda, x, y) = 3 - 8x + 6y + \lambda(x^2+y^2-36)$$

$$\begin{cases} L'_x = -8 + \lambda\cdot 2x = 0 \\ L'_y = 6 + \lambda \cdot 2y = 0 \\ L'_\lambda = x^2+y^2-36 = 0 \end{cases} \quad \begin{cases} x = \dfrac{4}{\lambda} \\ y = -\dfrac{3}{\lambda} \\ \dfrac{16}{\lambda^2} + \dfrac{9}{\lambda^2} = 36 \end{cases} \quad \begin{cases} x = \dfrac{4}{\lambda} \\ y = -\dfrac{3}{\lambda} \\ \dfrac{25}{\lambda^2} = 36 \end{cases} \quad \lambda^2 = \frac{25}{36} = \pm\frac{5}{6}$$

$$x = \frac{4\cdot 6}{5} = \frac{24}{5}$$

$$y = -\frac{3\cdot 6}{5} = -\frac{18}{5}$$

Точки подозреваемые на экстремум:

$$\left(\frac{5}{6}; \frac{24}{5}; -\frac{18}{5}\right) \left(-\frac{5}{6}; -\frac{24}{5}; \frac{18}{6}\right)$$

$$L''_{xx} = 2\lambda \quad L''_{yy} = 2\lambda \quad L''_{\lambda\lambda} = 0$$

$$L''_{xy} = 0 \quad L''_{x\lambda} = 2x \quad L''_{y\lambda} = 2y$$

$$\begin{vmatrix} L''_{\lambda\lambda} & L''_{\lambda x} & L''_{\lambda y} \\ L''_{x\lambda} & L''_{xx} & L''_{xy} \\ L''_{y\lambda} & L''_{yx} & L''_{yy} \end{vmatrix} = \begin{vmatrix} 0 & 2x & 2y \\ 2x & 2\lambda & 0 \\ 2y & 0 & 2\lambda \end{vmatrix} = 0\cdot\begin{vmatrix} 2\lambda & 0 \\ 0 & 2\lambda \end{vmatrix} - 2x\cdot\begin{vmatrix} 2x & 0 \\ 2y & 2\lambda \end{vmatrix} + 2y\cdot\begin{vmatrix} 2x & 2\lambda \\ 2y & 0 \end{vmatrix} =$$

$$= 0 - 2x(2x\cdot 2\lambda - 0) + 2y(2x\cdot 0 - 2y\cdot 2\lambda) = -8x^2\lambda - 8y^2\lambda = -8\lambda(x^2+y^2)$$

Следовательно: $-8\cdot\frac{5}{6}\left(\left(\frac{24}{5}\right)^2 + \left(-\frac{18}{5}\right)^2\right) < 0$ – минимум $\left(\frac{5}{6}; \frac{24}{5}; -\frac{18}{5}\right)$

$\left(-\frac{5}{6}; -\frac{24}{5}; \frac{18}{6}\right)$ – максимум.

2. $U = 2x^2 + 12xy + 32y^2 + 15$, если $x^2+16y^2=64$

$$L(\lambda, x, y) = 2x^2+12xy+32y^2+15+\lambda(x^2+16y^2-64)$$

$$\begin{cases} L'_x = 4x + 12y + \quad + \lambda\cdot 2x = 0 \ (1) \\ L'_y = 12x + 64y + \lambda\cdot 32y = 0 \ (2) \\ L'_\lambda = x^2+16y^2-64 = 0 \ (3) \end{cases}$$

$$x(4+2\lambda) + 12y = 0$$

$$x = -\frac{12y}{4+2\lambda} =$$

$L''_{xx} = 4 + 2\lambda \qquad L''_{yy} = 64 + 32\lambda \qquad L''_{\lambda\lambda} = 0$

$L''_{xy} = 12 \qquad L''_{x\lambda} = 2x \qquad L''_{y\lambda} = 32y$

$$\Delta = \begin{vmatrix} L''_{\lambda\lambda} & L''_{\lambda x} & L''_{\lambda y} \\ L''_{x\lambda} & L''_{xx} & L''_{xy} \\ L''_{y\lambda} & L''_{yx} & L''_{yy} \end{vmatrix} = \begin{vmatrix} 0 & 2x & 32y \\ 2x & 4+2\lambda & 12 \\ 32y & 12 & 64+32\lambda \end{vmatrix} = 0 \cdot \begin{vmatrix} 4+2\lambda & 12 \\ 12 & 64+32\lambda \end{vmatrix} - 2x \cdot \begin{vmatrix} 2x & 12 \\ 32y & 64+32\lambda \end{vmatrix} +$$

$$+ 32y \begin{vmatrix} 2x & 4+2\lambda \\ 32y & 12 \end{vmatrix} = -2x\left(2x(64+32\lambda) - 32y \cdot 12\right) + 32y\left(24x - 32y(4+2\lambda)\right) =$$

$$= -2x\left(128x + 64\lambda x - 384y\right) + 32y\left(24x - 128y - 64y\lambda\right) =$$

$$= -256x^2 - 128x^2\lambda + 768xy + 768xy - 4096y^2 - 2048y^2\lambda =$$

$$= -256x^2 - 4096y^2 - 128x^2\lambda - 2048y^2\lambda + 1536xy =$$

$$= -256(x^2 - 16y^2) - 128\lambda(x^2 - 16y^2) + 1536xy = -(256 + 128\lambda)(x^2 - 16y^2) + 1536xy$$

Выражаем из (1) λ: Из (2):

$\lambda \cdot 2x = -4x - 12y \qquad \lambda \cdot 32y = -12x - 64y$

$\lambda = -\dfrac{4x + 12y}{2x} \qquad \lambda = -\dfrac{12x + 64y}{32y}$

Приравниваем:

$$\frac{4x + 12y}{2x} = \frac{12x + 64y}{32y}$$

$$32y(4x + 12y) = 2x(12x + 64y)$$

$$\cancel{128xy} + 384y^2 = 24x^2 + \cancel{128xy}$$

$$16y^2 = x^2$$

$$x = \pm\sqrt{16y} = \pm 4 \cdot y; \quad y = \pm\frac{x}{4}.$$

Подставляем y в (3):

$$x^2 + 16 \cdot \left(\pm\frac{x}{4}\right)^2 - 64 = 0$$

$$2x^2 = 64$$

$$x^2 = 32$$

$$x = \pm\sqrt{32} = \pm\sqrt{16 \cdot 2} = \pm 4\sqrt{2}$$

$$(\pm 4\sqrt{2})^2 + 16y^2 - 64 = 0$$

$$32 + 16y^2 - 64 = 0$$

$$y = \pm\sqrt{2}$$

Находим λ из (1)

$$4\cdot4\sqrt{2}+12\cdot\sqrt{2}+\lambda\cdot2\cdot4\sqrt{2}=0$$
$$\lambda\cdot8\sqrt{2}=-28\sqrt{2}$$
$$\lambda=-\frac{28}{8}=-\frac{7}{2}$$
$$\left(4\sqrt{2},\sqrt{2};-\frac{7}{2}\right)$$

$$4\cdot4\sqrt{2}+12\cdot(-\sqrt{2})+\lambda\cdot2\cdot4\sqrt{2}=0$$
$$16\sqrt{2}-12\sqrt{2}+\lambda\cdot8\sqrt{2}=0$$
$$\lambda\cdot8\sqrt{2}=-4\sqrt{2}$$
$$\lambda=-\frac{1}{2}$$
$$\left(4\sqrt{2};-\sqrt{2};-\frac{1}{2}\right)$$

$$4\cdot(-4\sqrt{2})+12\sqrt{2}+\lambda\cdot2\cdot4\sqrt{2}=0$$
$$-16\sqrt{2}+12\sqrt{2}+\lambda\cdot8\sqrt{2}=0$$
$$\lambda\cdot8\sqrt{2}=4\sqrt{2}$$
$$\lambda=\frac{1}{2}$$
$$\left(-4\sqrt{2},\sqrt{2},\frac{1}{2}\right)$$

$$4(-4\sqrt{2})+12\cdot(-\sqrt{2})+\lambda\cdot2\cdot4\sqrt{2}=0$$
$$-16\sqrt{2}-12\sqrt{2}+\lambda8\sqrt{2}=0$$
$$\lambda\cdot8\sqrt{2}=28\sqrt{2}$$
$$\lambda=\frac{28}{8}=\frac{7}{2}$$
$$\left(-4\sqrt{2},-\sqrt{2},\frac{7}{2}\right)$$

$$\Delta=-(256+128\lambda)(x^2-16y^2)+1536xy=-\left(256+128\cdot\left(-\frac{7}{2}\right)\right)(16\cdot2-16\cdot2)+$$
$$+1536\cdot4\sqrt{2}\cdot\sqrt{2}>0 - \text{максимум } \left(4\sqrt{2},\sqrt{2},-\frac{7}{2}\right)$$

для: $\left(-4\sqrt{2},\sqrt{2},\frac{1}{2}\right)=-\left(256+128\cdot\frac{1}{2}\right)\left((-4\sqrt{2})^2-16\cdot(\sqrt{2})^2\right)+1536(-4\sqrt{2})\cdot\sqrt{2}=$
$$=-(256+64)(16\cdot2-16\cdot2)+1536(-4)\cdot2<0 - \text{минимум.}$$

для: $\left(4\sqrt{2},-\sqrt{2};-\frac{1}{2}\right)$: $\Delta=-\left(256+128\left(-\frac{1}{2}\right)\right)\left((4\sqrt{2})^2-16\cdot(-\sqrt{2})^2\right)+1536(4\sqrt{2})(-\sqrt{2})=$
$$=-(256-64)(16\cdot2-16\cdot2)+1536\cdot4(-2)<0 - \text{минимум.}$$

для $\left(-4\sqrt{2},-\sqrt{2},\frac{7}{2}\right)$: $\Delta=-\left(256+128\cdot\frac{7}{2}\right)\left((-4\sqrt{2})^2-16(-\sqrt{2})^2\right)+1536\cdot(-4\sqrt{2})\cdot(-\sqrt{2})=$
$$=(-256+64\cdot7)(16\cdot2-16\cdot2)+1536\cdot4\cdot2>0 - \text{максимум.}$$

3. Найти производную функции $U = x^2 + y^2 + z^2$ по направлению вектора $\vec{c}(-9, 8, -12)$ в точке $M(8, -12, 9)$

$$|\vec{c}| = \sqrt{(-9)^2 + 8^2 + (-12)^2} = \sqrt{81 + 64 + 144} = 17$$

$$\vec{c}_0 = \frac{\vec{c}}{|\vec{c}|} = \left(-\frac{9}{17}, \frac{8}{17}; -\frac{12}{17}\right)$$

$$U'_x = 2x \qquad \operatorname{grad} U\Big|_{(8,-12,9)} = (2\cdot 8, 2\cdot(-12), 2\cdot 9) = (16, -24, 18)$$

$$U'_y = 2y$$

$$U'_z = 2z \qquad U'_{\vec{c}}\Big|_{(8,-12,9)} = \left(-\frac{9}{17}\cdot 16\right) + \frac{8}{17}\cdot(-24) - \frac{12}{17}\cdot 18 =$$

$$= \frac{-144 - 192 - 216}{17} = -\frac{552}{17}$$

4. Найти производную $U = e^{x^2+y^2+z^2}$ по направлению вектора $\vec{d} = (4, -13, -16)$ в точке $L(-16, 4, -13)$.

норма вектора: $|\vec{d}| = \sqrt{4^2 + (-13)^2 + (-16)^2} = \sqrt{16 + 169 + 256} = \sqrt{441} = 21$

нормированный вектор: $\vec{d}_0 = \frac{\vec{d}}{|\vec{d}|} = \left(\frac{4}{21}, -\frac{13}{21}, -\frac{16}{21}\right)$

$$U'_x = 2x\cdot e^{x^2+y^2+z^2}$$

$$U'_y = 2y\cdot e^{x^2+y^2+z^2} \qquad \operatorname{grad} U\Big|_{(-16,4,-13)} = \left(2\cdot(-16)e^{(-16)^2+4^2+(-13)^2},\ 8e^{(-16)^2+4^2+(-13)^2},\ -26e^{(-16)^2+4^2+(-13)^2}\right) =$$

$$U_z = 2z\cdot e^{x^2+y^2+z^2}$$

$$= \left(-32e^{441}, 8e^{441}, -26e^{441}\right)$$

$$U'_{\vec{d}}\Big|_{(-16,4,-13)} = \frac{4}{21}\left(-32e^{441}\right) - \frac{13}{21}\cdot 8e^{441} - \frac{16}{21}(-26)e^{441} =$$

$$= \left(-\frac{128}{21} - \frac{104}{21} + \frac{416}{21}\right)\cdot e^{441} = \frac{\text{[illegible]}}{21}e^{441}$$