Basic identities of matrix/vector ops

$(A+B)^T=A^T+B^T$ $(AB)^T=B^TA^T$ $(A^{-1})^T=(A^T)^{-1}$
$(AB)^{-1}=B^{-1}A^{-1}$

For $A\in\mathbb{R}^{m\times n}$, A_{ij} is the i-th **ROW** then j-th **COLUMN**
$(A^T)_{ij}=A_{ji}$ $(AB)_{ij}=A_{i*}\cdot B_{*j}=\sum_k A_{ik}B_{kj}$
$(Ax)_i=A_{i*}\cdot x=\sum_j A_{ij}x_j$ $x^Ty=x\cdot y=\sum_i x_iy_i$
$x^TAx=\sum_i\sum_j A_{ij}x_ix_j$ $xe_k^T=[0|\dots|x|\dots|0]$
$e_kx^T=[0^T;\dots;x^T;\dots;0^T]$

Scalar-multiplication + addition distributes over:
column-blocks => $\lambda A+B=\lambda[A_1|\dots|A_c]+[B_1|\dots|B_c]=[\lambda A_1+B_1|\dots|\lambda A_c+B_c]$
row-blocks => $\lambda A+B=\lambda[A_1;\dots;A_r]+[B_1;\dots;B_r]=[\lambda A_1+B_1;\dots;\lambda A_r+B_r]$

Matrix-multiplication distributes over:
column-blocks => $AB=A[B_1|\dots|B_p]=[AB_1|\dots|AB_p]$
row-blocks => $AB=[A_1;\dots;A_p]B=[A_1B;\dots;A_pB]$
outer-product sum => $AB=[A_1|\dots|A_p][B_1;\dots;B_p]=\sum_{i=1}^p A_iB_i$
e.g. For $A=[a_1|\dots|a_n]$, $B=[b_1;\dots;b_n]$ => $AB=\sum_i a_ib_i$

Projection: definition & properties

A **projection** $\pi:V\to V$ is a **endomorphism** such that $\pi\circ\pi=\pi$, i.e. it leaves its image unchanged (its idempotent)
A **square matrix** P such that $P^2=P$ is called a **projection matrix**
It is called an **orthogonal projection matrix** if $P^2=P=P^\dagger$ (conjugate-transpose)
Eigenvalues of a **projection matrix** must be 0 or 1
Because $\pi:V\to V$ is a **linear map**, its **image space** $U=\text{im}(\pi)$ and **null space** $W=\ker(\pi)$ are **subspaces** of V
$\pi|_U$ is the **identity operator** on U
The linear map $\pi^*=I_V-\pi$ is a projection with $W=\text{im}(\pi^*)=\ker(\pi)$ and $U=\ker(\pi^*)=\text{im}(\pi)$, i.e. they are swapped
π is a projection **along** W **onto** U
π^* is a projection **along** U **onto** W
π^* is the **identity operator** on W
V can be decomposed as $V=U\oplus W$ meaning every vector $x\in V$ can be uniquely written as $x=u+w$
$u\in U$ and $u=\pi(x)$
$w\in W$ and $w=x-\pi(x)=(I_V-\pi)(x)=\pi^*(x)$
An **orthogonal projection** further satisfies $U\perp W$, i.e. the **image** and **kernel** of π are **orthogonal subspaces**
infact they are eachother's **orthogonal compliments**, i.e. $U^\perp=W$, $W^\perp=U$ *(because finite-dimensional vectorspaces)*
so we have $\pi(x)\cdot y=\pi(x)\cdot\pi(y)=x\cdot\pi(y)$
or equivalently, $\pi(x)\cdot(y-\pi(y))=(x-\pi(x))\cdot\pi(y)=0$

By Cauchy–Schwarz inequality we have $\|\pi(x)\|\le\|x\|$
The orthogonal projection onto the line containing vector u is $\text{proj}_u=\hat u\hat u^T$, i.e. $\text{proj}_u(v)=\frac{u\cdot v}{u\cdot u}u$; $\hat u=\frac{u}{\|u\|}$
A special case of $\pi(x)\cdot(y-\pi(y))=0$ is $u\cdot(v-\text{proj}_u v)=0$ since $\text{proj}_u(u)=u$

If $U\subseteq\mathbb{R}^n$ is a k-dimensional subspace with **orthonormal basis (ONB)** $\langle u_1,\dots,u_k\rangle\in\mathbb{R}^m$
Let $U=[u_1|\dots|u_k]\in\mathbb{R}^{m\times k}$ matrix
Orthogonal projection onto U is $\pi_U=UU^T$
Can be rewritten as $\pi_U(v)=\sum_i(u_i\cdot v)u_i$
If $\langle u_1,\dots,u_k\rangle$ is **not orthonormal**, then "normalizing factor" $(U^TU)^{-1}$ is added => $\pi_U=U(U^TU)^{-1}U^T$
For **line subspaces** $U=\text{span}\{u\}$, we have $(U^TU)^{-1}=(u^Tu)^{-1}=1/(u\cdot u)=1/\|u\|^2$

Gram-Schmidt (GS) to gen. ONB from lin. ind. vectors

Gram-Schmidt is **iterative** projection => we use ***current*** j**-dim subspace**, to get ***next*** $(j+1)$**-dim subspace**
Assume **orthonormal basis (ONB)** $\langle q_1,\dots,q_j\rangle\in\mathbb{R}^m$ for j-dim subspace $U_j\subset\mathbb{R}^m$
Let $Q_j=[q_1|\dots|q_j]\in\mathbb{R}^{m\times j}$ be the matrix
$P_j=Q_jQ_j^T$ is orthogonal projection **onto** U_j
$P_{\perp j}=I_m-Q_jQ_j^T$ is orthogonal projection **onto** $(U_j)^\perp$ *(orthogonal compliment)*
Uniquely decompose next $U_j\not\ni a_{j+1}=v_{j+1}+u_{j+1}$
$v_{j+1}=P_j(a_{j+1})\in U_j$ => discard it!!
$u_{j+1}=P_{\perp j}(a_{j+1})\in(U_j)^\perp$ => we're after this!!
Let $q_{j+1}=\hat u_{j+1}$ => we have **next ONB** $\langle q_1,\dots,q_{j+1}\rangle$ for U_{j+1} => start next iteration
$u_{j+1}=(I_m-Q_jQ_j^T)a_{j+1}=a_{j+1}-Q_jc_j$ where
$c_j=[q_1\cdot a_{j+1},\dots,q_j\cdot a_{j+1}]^T$

Notice: $Q_jc_j=\sum_{i=1}^j(q_i\cdot a_{j+1})q_i=\sum_{i=1}^j\text{proj}_{q_i}(a_{j+1})$, so rewrite as
$u_{j+1}=a_{j+1}-\sum_{i=1}^j(q_i\cdot a_{j+1})q_i=a_{j+1}-\sum_{i=1}^j\text{proj}_{q_i}(a_{j+1})$
$\underbrace{a_1,\dots,a_n\in\mathbb{R}^m}_{n}$ $m\ge n$ $U_n=\text{span}\{a_1,\dots,a_n\}$
We apply Gram-Schmidt to build **ONB** $\langle q_1,\dots,q_n\rangle\in\mathbb{R}^m$ for $U_n\subset\mathbb{R}^m$
$j=1$ => $u_1=a_1$ and $q_1=\hat u_1$, i.e. ***start of iteration***
$j=2$ => $u_2=a_2-(q_1\cdot a_2)q_1$ and $q_2=\hat u_2$, ***etc...***
Linear independence ***guarantees*** that $a_{j+1}\notin U_j$
For exams: compute $u_{j+1}=a_{j+1}-Q_jc_j$
1) Gather $Q_j=[q_1|\dots|q_j]\in\mathbb{R}^{m\times j}$
2) Compute $c_j=[q_1\cdot a_{j+1},\dots,q_j\cdot a_{j+1}]^T\in\mathbb{R}^j$
3) Compute $Q_jc_j\in\mathbb{R}^m$, and subtract from a_{j+1}

Properties: dot-product & norm

$x^Ty=y^Tx=x\cdot y=\sum_i x_iy_i$ $x\cdot y=\|a\|\|b\|\cos\hat{xy}$
$x\cdot y=y\cdot x$ $x\cdot(y+z)=x\cdot y+x\cdot z$ $\alpha x\cdot y=\alpha(x\cdot y)$
$x\cdot x=\|x\|^2=0\iff x=0$
for $x\ne 0$, we have $x\cdot y=x\cdot z\implies x\cdot(y-z)=0$
$|x\cdot y|\le\|x\|\|y\|$ **(Cauchy-Schwartz inequality)**
$\|u+v\|^2+\|u-v\|^2=2\|u\|^2+2\|v\|^2$ **(parallelogram law)**
$\|u+v\|\le\|u\|+\|v\|$ **(triangle inequality)**
$u\perp v\iff\|u+v\|^2=\|u\|^2+\|v\|^2$ **(pythagorean theorem)**
$\|c\|^2=\|a\|^2+\|b\|^2-2\|a\|\|b\|\cos\hat{ba}$ **(law of cosines)**

Transformation matrix & linear maps

For linear map $f:\mathbb{R}^n\to\mathbb{R}^m$, ordered bases $\langle b_1,\dots,b_n\rangle\in\mathbb{R}^n$ and $\langle c_1,\dots,c_m\rangle\in\mathbb{R}^m$
$A=F_{CB}\in\mathbb{R}^{m\times n}$ is the **transformation-matrix** of f w.r.t to bases B and C
$f(b_j)=\sum_{i=1}^m A_{ij}c_i$ -> each b_j basis gets mapped to a linear combination of $\sum_i a_ic_i$ bases
If f^{-1} exists *(i.e. its bijective and $m=n$)* then $(F_{CB})^{-1}=F^{-1}{}_{BC}$ *(where $F^{-1}{}_{BC}$ is the* ***transformation-matrix*** *of f^{-1})*

The transformation matrix of the identity map is called change-in-basis matrix
The identity matrix I_m represents $\text{id}_{\mathbb{R}^m}$ w.r.t. the standard basis $E_m=\langle e_1,\dots,e_m\rangle$ => i.e. $I_m=I_{EE}$
If $B=\langle b_1,\dots,b_m\rangle$ is a basis of $\mathbb{R}^m$, then $I_{EB}=[b_1|\dots|b_m]$ is the transformation matrix from B to E
$I_{BE}=(I_{EB})^{-1}$, so => $F_{CB}=I_{CE}F_{EE}I_{EB}$

Dot-product uniquely determines a vector w.r.t. to basis
If $a_j=x\cdot b_j$; $x=\sum_i a_jb_j$, we call a the coordinate-vector of x w.r.t. to B
Rank-nullity theorem:
$\dim(\text{im}(f))+\dim(\ker(f))=\text{rk}(A)+\dim(\ker(A))=n$
f is injective/monomorphism **iff** $\ker(f)=\{0\}$ **iff** A is full-rank

Orthogonality concepts

$u\perp v\iff u\cdot v=0$, i.e. u and v are orthogonal
u and v are orthonormal **iff** $u\perp v$, $\|u\|=1=\|v\|$
$A\in\mathbb{R}^{n\times n}$ is orthogonal **iff** $A^{-1}=A^T$
Columns of $A=[a_1|\dots|a_n]$ are orthonormal basis (ONB) $C=\langle a_1,\dots,a_n\rangle\in\mathbb{R}^n$, so $A=I_{EC}$ is change-in-basis matrix
Orthogonal transformations preserve lengths/angles/distances => $\|Ax\|_2=\|x\|_2$, $A\hat{x}A y=\hat{xy}$
Therefore can be seen as a succession of reflections and planar rotations
$\det(A)=1$ or $\det(A)=-1$, and all **eigenvalues** of A are s.t. $|\lambda|=1$

$A\in\mathbb{R}^{m\times n}$ is semi-orthogonal **iff** $A^TA=I$ or $AA^T=I$
If $n>m$ then **all** m **rows** are orthonormal vectors
If $m>n$ then **all** n **columns** are orthonormal vectors
$U\perp V\subset\mathbb{R}^n\iff u\cdot v=0$ for all $u\in U, v\in V$, i.e. they are **orthogonal subspaces**
Orthogonal compliment of $U\subset\mathbb{R}^n$ is the subspace
$U^\perp=\{x\in\mathbb{R}^n\mid\forall y\in\mathbb{R}^n:x\perp y\}$
$=\{x\in\mathbb{R}^n\mid\forall y\in\mathbb{R}^n:\|x\|\le\|x+y\|\}$
$\mathbb{R}^n=U\oplus U^\perp$ and $(U^\perp)^\perp=U$
$U\perp V\iff U^\perp=V$ and vice-versa
$Y\subseteq X\implies X^\perp\subseteq Y^\perp$ and $X\cap X^\perp=\{0\}$
Any $x\in\mathbb{R}^n$ can be uniquely decomposed into $x=x_i+x_k$, where $x_i\in U$ and $x_k\in U^\perp$
For matrix $A\in\mathbb{R}^{m\times n}$ and for row-space $R(A)$, column-space $C(A)$ and null space $\ker(A)$
$R(A)^\perp=\ker(A)$ and $C(A)^\perp=\ker(A^T)$
Any $b\in\mathbb{R}^m$ can be uniquely decomposed into
$b=b_i+b_k$, where $b_i\in C(A)$ and $b_k\in\ker(A^T)$
$b=b_i+b_k$, where $b_i\in R(A)$ and $b_k\in\ker(A)$

Vector norms (beyond euclidean)

vector norms are such that: $\|x\|=0\iff x=0$, $|\lambda x|=|\lambda|\|x\|$, $\|x+y\|\le\|x\|+\|y\|$
ℓ_p norms: $\|x\|_p=(\sum_{i=1}^n|x_i|^p)^{1/p}$
$p=1$: $\|x\|_1=\sum_{i=1}^n|x_i|$
$p=2$: $\|x\|_2=\sqrt{\sum_{i=1}^n x_i^2}=\sqrt{x\cdot x}$
$p=\infty$: $\|x\|_\infty=\lim_{p\to\infty}\|x\|_p=\max_{1\le i\le n}|x_i|$
Any two norms in $\mathbb{R}^n$ are equivalent, meaning there exist $r>0$; $s>0$ such that: $\forall x\in\mathbb{R}^n, r\|x\|_a\le\|x\|_b\le s\|x\|_a$
Equivalence of ℓ_1,ℓ_2 and ℓ_∞ => $\|x\|_\infty\le\|x\|_2\le\|x\|_1$, $\|x\|_1\le\sqrt{n}\|x\|_2$, $\|x\|_2\le\sqrt{n}\|x\|_\infty$
Induce **metric** $d(x,y)=\|y-x\|$, has additional properties:
Translation invariance: $d(x+w,y+w)=d(x,y)$
Scaling: $d(\lambda x,\lambda y)=|\lambda|d(x,y)$

Matrix norms

Matrix norms are such that: $\|A\|=0\iff A=0$, $|\lambda A|=|\lambda|\|A\|$, $\|A+B\|\le\|A\|+\|B\|$
Matrices $\mathbb{F}^{m\times n}$ are a vector space so **matrix norms** ***are vector norms***, all results apply
Sub-multiplicative matrix norm (assumed by default) is also such that $\|AB\|\le\|A\|\|B\|$
Common matrix norms, for some $A\in\mathbb{R}^{m\times n}$:
$\|A\|_1=\max_j\|A_{*j}\|_1$
$\|A\|_2=\sigma_1(A)$, i.e. largest singular value of A *(square-root of largest eigenvalue of A^TA or AA^T)*
$\|A\|_\infty=\max_i\|A_{i*}\|_1$, note that $\|A\|_1=\|A^T\|_\infty$
Frobenius norm: $\|A\|_F=\sqrt{\sum_{i=1}^m\sum_{j=1}^n|A_{ij}|^2}$

A matrix norm $\|\cdot\|$ on $\mathbb{R}^{m\times n}$ is **consistent** with the vector norms $\|\cdot\|_a$ on $\mathbb{R}^n$ and $\|\cdot\|_b$ on $\mathbb{R}^m$ if for all $A\in\mathbb{R}^{m\times n}$ and $x\in\mathbb{R}^n$ => $\|Ax\|_b\le\|A\|\|x\|_a$
If $a=b$, $\|\cdot\|$ is **compatible** with $\|\cdot\|_a$
Frobenius norm is **consistent** with ℓ_2 norm => $\|Av\|_2\le\|A\|_F\|v\|_2$
For a vector norm $\|\cdot\|$ on $\mathbb{R}^n$, the **subordinate matrix norm** $\|\cdot\|$ on $\mathbb{R}^{m\times n}$ is
$\|A\|=\max\{\|Ax\|:x\in\mathbb{R}^n,\|x\|=1\}$
$=\max\{\frac{\|Ax\|}{\|x\|}:x\in\mathbb{R}^n,x\ne0\}$
$=\max\{\|Ax\|:x\in\mathbb{R}^n,\|x\|\le1\}$
Vector norms are **compatible** with their **subordinate matrix norms**
For $p=1,2,\infty$ **matrix norm** $\|\cdot\|_p$ is **subordinate** to the vector norm $\|\cdot\|_p$ *(and thus* ***compatible*** *with)*

Properties of matrices

Consider $A\in\mathbb{R}^{m\times n}$
If $Ax=x$ for all x then $A=I$
For square A, the **trace** of A is the **sum** if its **diagonals**, i.e. $\text{tr}(A)$

A is symmetric **iff** $A=A^T$, A is Hermitian, **iff** $A=A^\dagger$, i.e. its equal to its conjugate-transpose
AA^T and A^TA are **symmetric** *(and* ***positive semi-definite****)*
For real matrices, **Hermitian/symmetric** are equivalent conditions
Every eigenvalue λ_j of **Hermitian** matrices is real
geometric multiplicity of λ_j = geometric multiplicity of λ_j
eigenvectors x_1,x_2 associated to distinct eigenvalues λ_1,λ_2 are **orthogonal**, i.e. $x_1\perp x_2$

A is triangular **iff** all entries above **(*lower-triangular*)** or below **(*upper-triangular*)** the main diagonal are **zero**
Determinant => $|A|=\prod_i a_{ii}$, i.e. the product of diagonal elements

A is diagonal **iff** $A_{ij}=0, i\ne j$, i.e. if all off-diagonal entries are zero
Written as $\text{diag}_{m\times n}(a_1,\dots,a_p)$, $p=\min(m,n)$, where $a=[a_1,\dots,a_p]^T\in\mathbb{R}^p$ diagonal entries of A
For $x\in\mathbb{R}^n$ $Ax=\text{diag}_{m\times n}(a_1,\dots,a_p)[x_1\dots x_n]^T=[a_1x_1\dots a_px_p\ 0\dots0]^T\in\mathbb{R}^m$ *(if $p=m$ those tail-zeros don't exist)*
$\text{diag}_{m\times n}(a)+\text{diag}_{m\times n}(b)=\text{diag}_{m\times n}(a+b)$
Consider $\text{diag}_{n\times k}(c_1,\dots,c_q)$, $q=\min(n,k)$, then
$\text{diag}_{m\times n}(a_1,\dots,a_p)\text{diag}_{n\times k}(c_1,\dots,c_q)$
$=\text{diag}_{m\times k}(a_1c_1,\dots,a_rc_r,0,\dots,0)=\text{diag}(s)$
Where $r=\min(p,q)=\min(m,n,k)$, and $s\in\mathbb{R}^S$, $S=\min(m,k)$
Inverse of square-diagonals =>
$\text{diag}(a_1,\dots,a_n)^{-1}=\text{diag}(a_1^{-1},\dots,a_n^{-1})$, i.e. diagonals **cannot be zero** *(division by zero undefined)*

Determinant of square-diagonals =>
$|\text{diag}(a_1,\dots,a_n)|=\prod_i a_i$ *(since they are technically triangular matrices)*

The **(column) rank** of A is number of linearly independent columns, i.e. $\text{rk}(A)$
I.e. its the **number of pivots** in **row-echelon-form**
I.e. its the **dimension** of the **column-space** $\text{rk}(A)=\dim(C(A))$
I.e. its the **dimension** of the **image-space** $\text{rk}(A)=\dim(\text{im}(f_A))$ of linear map $f_A(x)=Ax$
The **(row) rank** of A is number of linearly independent rows
The **row/column ranks** are **always the same**, hence $\text{rk}(A)=\dim(C(A))=\dim(R(A))=\dim(C(A^T))=\text{rk}(A^T)$
A is full-rank **iff** $\text{rk}(A)=\min(n,m)$, i.e. its as linearly independent as possible

Two matrices $A,\tilde A\in\mathbb{R}^{m\times n}$ are **equivalent** if there exist two invertible matrices $P\in\mathbb{R}^{m\times m}$ and $Q\in\mathbb{R}^{n\times n}$ such that $A=P\tilde AQ^{-1}$
Two matrices $A,\tilde A\in\mathbb{R}^{m\times n}$ are **similar** if there exists an invertible matrix $P\in\mathbb{R}^{n\times n}$ such that $A=P\tilde AP^{-1}$
Similar matrices are equivalent, with $Q=P$
A is diagonalisable **iff** A is similar to some diagonal matrix D

Properties of determinants

Consider $A\in\mathbb{R}^{n\times n}$, then $A_{ij}'\in\mathbb{R}^{(n-1)\times(n-1)}$ the (i,j)**-minor matrix** of A, obtained by deleting i**-th row** and j**-th column** from A
Then we define **determinant** of A, i.e. $\det(A)=|A|$, as
$\det(A)=\sum_{k=1}^n(-1)^{i+k}A_{ik}\det(A_{ik}')$, i.e. expansion along i**-th row** *(for any i)
$\det(A)=\sum_{k=1}^n(-1)^{k+j}A_{kj}\det(A_{kj}')$, i.e. expansion along j**-th column** *(for any j)*
When $\det(A)=0$ we call A a **singular matrix**
Common determinants
For $n=1$, $\det(A)=A_{11}$
For $n=2$, $\det(A)=A_{11}A_{22}-A_{12}A_{21}$
$\det(I_n)=1$
Multi-linearity in columns/rows: if $A=[a_1|\dots|a_j|\dots|a_n]=[a_1|\dots|\lambda x_j+\mu y_j|\dots|a_n]$ then
$\det(A)=\lambda\det([a_1|\dots|x_j|\dots|a_n])+\mu\det([a_1|\dots|y_j|\dots|a_n])$
And the exact same linearity property for **rows**
Immediately leads to: $|A|=|A^T|$, $|\lambda A|=\lambda^n|A|$, and $|AB|=|BA|=|A||B|$ *(for any $B\in\mathbb{R}^{n\times n}$)*
Alternating: if any **two columns** of A are **equal** *(or any* ***two rows*** *of A are* ***equal****)*, then $|A|=0$ (its singular)
Immediately from this (and **multi-linearity**) => if **columns** *(or* ***rows****)* are linearly-dependent *(some are linear combinations of others)* then $|A|=0$
Stated in other terms => $rk(A)<n\iff|A|=0\iff$ $\text{RREF}(A)\ne I_n\iff|A|=0$ *(reduced row-echelon-form)*
$\iff C(A)\ne\mathbb{R}^n\iff|A|=0$ *(column-space)*
For more equivalence to the above, see invertible matrix theorem
Interaction with **EROs/ECOs**:
Swapping rows/columns **flips the sign**
Scaling a row/column by $\lambda\ne0$ **will scale the determinant** by λ *(by multi-linearity)*
Remember **to scale** by λ^{-1} **to maintain equality**, i.e. $\det(A)=\lambda^{-1}\det([a_1|\dots|\lambda a_j|\dots|a_n])$
Invariant under **addition** of rows/columns
Link to invertable matrices => $|A^{-1}|=|A|^{-1}$ which means A is invertible $\iff|A|\ne0$, i.e. **singular matrices are not invertible**
For block-matrices:
$\det\begin{pmatrix}A&B\\0&D\end{pmatrix}=\det(A)\det(B)=\det\begin{pmatrix}A&0\\C&D\end{pmatrix}$
$\det\begin{pmatrix}A&B\\C&D\end{pmatrix}=\det(A)\det(D-CA^{-1}B)$
$=\det(D)\det(A-BD^{-1}C)$ if A or D are invertible, *respectively*
Sylvester's determinant theorem: $\det(I_m+AB)=\det(I_n+BA)$
Matrix determinant lemma:
$\det(A+uv^T)=(1+v^TA^{-1}u)\det(A)$
$\det(A+UV^T)=\det(I_m+V^TA^{-1}U)\det(A)$
$\det(A+UWV^T)=\det(W^{-1}+V^TA^{-1}U)\det(W)\det(A)$

Tricks for computing determinant

If block-triangular matrix then apply $\det\begin{pmatrix}A&B\\0&D\end{pmatrix}=\det(A)\det(B)$
If *close* to triangular matrix apply **EROs/ECOs** to get it there, then its just product of diagonals
If **Cholesky/LU/QR** is possible and cheap then do it, then apply $|AB|=|A||B|$
If all else fails, try to find row/column with **MOST** zeros
Perform minimal **EROs/ECOs** to get that row/column to be **all-but-one** zeros
Don't forget to keep track of sign-flipping & scaling-factors
Do Laplace expansion along *that* row/column => notice **all-but-one** minor matrix determinants **go to zero**

Representing EROs/ECOs as transformation matrices

For $A\in\mathbb{R}^{m\times n}$, suppose a sequence of:
EROs transform $A\rightsquigarrow_{EROs}A'$ => there is matrix R s.t. $RA=A'$
ECOs transform $A\rightsquigarrow_{ECOs}A'$ => there is matrix C s.t. $AC=A'$
Both transform $A\rightsquigarrow_{EROs+ECOs}A'$ => there are matrices R,C s.t. $RAC=A'$

FORWARD: to compute these **transformation matrices**:
Start with $[I_m|A|I_n]$, i.e. A and identity matrices
For every **ERO** on A, do the same to **LHS** (i.e. I_m)
For every **ECO** on A, do the same to **RHS** (i.e. I_n)
Once done, you should get $[I_m|A|I_n]\rightsquigarrow[R|A'|C]$ with $RAC=A'$

If the sequences of **EROs** and **ECOs** were $R_1,\dots,R_\lambda$ and $C_1,\dots,C_\mu$ *respectively*
$R=R_\lambda\cdots R_1$ and $C=C_1\cdots C_\mu$ so $(R_\lambda\cdots R_1)A(C_1\cdots C_\mu)=A'$
$R^{-1}=R_1^{-1}\cdots R_\lambda^{-1}$ and $C^{-1}=C_\mu^{-1}\cdots C_1^{-1}$, where R_i^{-1},C_j^{-1} are **inverse EROs/ECOs** *respectively*

BACKWARD: once $R_1,\dots,R_\lambda$ and $C_1,\dots,C_\mu$ for which $RAC=A'$ are **known**, starting with $[I_m|A|I_n]$
For $i=1\to\lambda$ perform R_i on A, perform $R_{\lambda-i+1}^{-1}$ on **LHS** (i.e. I_m)
For $j=1\to\mu$ perform C_j on A, perform $C_{\mu-j+1}^{-1}$ on **RHS** (i.e. I_n)
You should get $[I_m|A|I_n]\rightsquigarrow[R^{-1}|A'|C^{-1}]$ with $A=R^{-1}A'C^{-1}$

You can mix-and-match the **forward/backward** modes i.e. inverse operations in inverse order for one, and operations in normal order for the other
e.g. you can do $[I_m|A|I_n]\rightsquigarrow[R^{-1}|A'|C]$ to get $AC=R^{-1}A'$ => useful for LU factorization

Eigen-values/vectors

Consider $A\in\mathbb{R}^{n\times n}$, non-zero $x\in\mathbb{C}^n$ is an **eigenvector** with **eigenvalue** $\lambda\in\mathbb{C}$ for A if $Ax=\lambda x$
If $Ax=\lambda x$ then $A(kx)=\lambda(kx)$ for $k\ne0$, i.e. kx is also an **eigenvector**
A has at most n distinct **eigenvalues**
The set of all eigenvectors associated with eigenvalue λ is called **eigenspace** E_λ of A
$E_\lambda=\ker(A-\lambda I)$
The geometric multiplicity of λ is $\dim(E_\lambda)=\dim(\ker(A-\lambda I))$
The **spectrum** $Sp(A)=\{\lambda_1,\dots,\lambda_n\}$ of A is the set of all eigenvalues of A
The characteristic polynomial of A is $P(\lambda)=|A-\lambda I|=\sum_{i=0}^n a_i\lambda^i$
$a_0=|A|$, $a_{n-1}=(-1)^{n-1}\text{tr}(A)$, $a_n=(-1)^n$
$\lambda\in\mathbb{C}$ is eigenvalue of A **iff** λ is a root of $P(\lambda)$
The algebraic multiplicity of λ is **the number of times it is repeated as root of** $P(\lambda)$
$1\le$ geometric multiplicity of λ $\le$ algebraic multiplicity of λ
Let $\lambda_1,\dots,\lambda_n\in\mathbb{C}$ be *(potentially non-distinct)* **eigenvalues** of A, with $x_1,\dots,x_n\in\mathbb{C}^n$ their **eigenvectors**
$\text{tr}(A)=\sum_i\lambda_i$ and $\det(A)=\prod_j\lambda_j$
A is diagonalisable **iff** there exist a basis of $\mathbb{R}^n$ consisting of $x_1,\dots,x_n$
A is diagonalisable **iff** $r_j=g_j$, where r_j = geometric multiplicity of λ_j and g_j = geometric multiplicity of λ_j
Eigenvalues of A^k are $\lambda_1,\dots,\lambda_n$
Let $P=[x_1|\dots|x_n]$, then $AP=[Ax_1|\dots|Ax_n]=[\lambda_1x_1|\dots|\lambda_nx_n]=[x_1|\dots|x_n]\text{diag}(\lambda_1,\dots,\lambda_n)=PD$
If P^{-1} exists then
$A=PDP^{-1}$, i.e. A is diagonalisable
$P=I_{EB}$ is **change-in-basis** matrix for basis $B=\langle x_1,\dots,x_n\rangle$ of eigenvectors
If $A=F_{EE}$ is transformation-matrix of linear map f, then $F_{EE}=I_{EB}F_{BB}I_{BE}$
Spectral theorem: if A is Hermitian then P^{-1} exists:
If x_i,x_j associated to different eigenvalues then $x_i\perp x_j$
If associated to same eigenvalue λ then **eigenspace** E_λ has spanning-set $\{x_{\lambda_i},\dots\}$
$x_1,\dots,x_n$ are linearly independent => apply Gram-Schmidt $q_{\lambda_i},\dots\leftarrow x_{\lambda_i},\dots$
Then $\{q_{\lambda_i},\dots\}$ is orthonormal basis (ONB) of E_λ
$Q=\langle q_1,\dots,q_n\rangle$ is an ONB of $\mathbb{R}^n$ => $Q=[q_1|\dots|q_n]$ is orthogonal matrix i.e. $Q^{-1}=Q^T$
$q_1,\dots,q_n$ are still eigenvectors of A => $A=QDQ^T$ ***(spectral decomposition)***
$A=QDQ^T$ can be interpreted as scaling in direction of its eigenvectors:
1) Perform a succession of reflections/planar rotations to change coordinate-system
2) Apply scaling by λ_j to each dimension q_j
Undo those reflections/planar rotations

Extension to $\mathbb{C}^n$

Standard inner product: $\langle x,y\rangle=x^\dagger y=\sum_i\overline{x_i}y_i$
Conjugate-symmetric: $\langle x,y\rangle=\overline{\langle y,x\rangle}$
Standard *(induced)* norm: $\|x\|=\sqrt{\langle x,y\rangle}=\sqrt{x^\dagger y}$
We can diagonalise real matrices in $\mathbb{C}$ which lets us diagonalise more matrices than before

Least Square Method

If we are solving $Ax=b$ and $b\notin C(A)$, *i.e. no solution*, then **Least Square Method** is:
Finding x which **minimizes** $\|Ax-b\|_2$
Recall for $A\in\mathbb{R}^{m\times n}$ we have unique decomposition for any $b\in\mathbb{R}^m$ $b=b_i+b_k$
where $b_i\in C(A)$ and $b_k\in\ker(A^T)$
$\|Ax-b\|_2$ is minimized $\iff\|Ax-b_i\|_2=0\iff Ax=b_i$

$A^TAx=A^Tb$ is the **normal equation** which gives solution to least square problem:
$\|Ax-b\|_2$ is minimized $\iff Ax=b_i\iff A^TAx=A^Tb$

Linear Regression

Let $y=f(t)=\sum_{j=1}^n s_jf_j(t)$ be a **mathematical model**, where f_j are **basis functions** and s_j are **parameters**
Let (t_i,y_i), $1\le i\le m$, $m\gg n$ be a set of **observations**, and $t,y\in\mathbb{R}^m$ are vectors representing those **observations**
$f_j(t)=[f_j(t_1),\dots,f_j(t_m)]^T$ **is transformed vector**
$A=[f_1(t)|\dots|f_n(t)]\in\mathbb{R}^{m\times n}$ is a matrix of columns
$z=[s_1,\dots,s_n]^T$ is vector of parameters
Then we get equation $Az=y$ => **minimizing** $\|Az-y\|_2$ is the solution to Linear Regression
So applying **LSM** to $Az=y$ is *precisely* what **Linear Regression** is
We can use normal equations for this => $\|Az-y\|_2$ is minimized $\iff A^TAz=A^Ty$
Solution to **normal equations** unique **iff** A is full-rank, i.e. it has linearly-independent columns

Positive (semi-)definite matrices

Consider symmetric $A\in\mathbb{R}^{n\times n}$, i.e. $A=A^T$
A is positive-definite **iff** $x^TAx>0$ for all $x\ne0$
A is positive-definite **iff** all its eigenvalues are **strictly positive**
A is positive-definite => all its diagonals are **strictly positive**
A is positive-definite => $\max(A_{ij},A_{jj})>|A_{ij}|$, i.e. **strictly larger coefficient** on the diagonals
A is positive-definite => all **upper-left submatrices** are *also* positive-definite
Sylvester's criterion: A is positive-definite **iff** all **upper-left submatrices** have strictly positive determinant

A is positive semi-definite **iff** $x^TAx\ge0$ for all x
A is positive semi-definite **iff** all its eigenvalues are **non-negative**
A is positive semi-definite => all its diagonals are **non-negative**
A is positive semi-definite => $\max(A_{ii},A_{jj})\ge|A_{ij}|$, i.e. **no coefficient larger** than on the diagonals
A is positive semi-definite => all **upper-left submatrices** are *also* positive semi-definite
A is positive semi-definite => it has a Cholesky Decomposition

For any $M\in\mathbb{R}^{m\times n}$, MM^T and M^TM are symmetric and **positive semi-definite**

Singular Value Decomposition (SVD) & Singular Values

Singular Value Decomposition of $A\in\mathbb{R}^{m\times n}$ is *any decomposition* of the form $A=USV^T$, where
Orthogonal $U=[u_1|\dots|u_m]\in\mathbb{R}^{m\times m}$ and
$V=[v_1|\dots|v_n]\in\mathbb{R}^{n\times n}$
$S=\text{diag}_{m\times n}(\sigma_1,\dots,\sigma_p)$ where $p=\min(m,n)$ and $\sigma_1\ge\dots\ge\sigma_p\ge0$
$\sigma_1,\dots,\sigma_p$ are **singular values** of A
(Positive) **singular values** are *(positive)* **square-roots of eigenvalues** of AA^T or A^TA
i.e. $\sigma_1^2,\dots,\sigma_p^2$ are **eigenvalues** of AA^T or A^TA
$\|A\|_2=\sigma_1$ *(link to matrix norms)*
Let $r=\text{rk}(A)$, then number of strictly positive **singular values** is r
i.e. $\sigma_1\ge\dots\ge\sigma_r>0$ and $\sigma_{r+1}=\dots=\sigma_p=0$
$A=\sum_{i=1}^r\sigma_iu_iv_i^T$

SVD is *similar* to spectral decomposition, except it always exists
If $n\le m$ then work with $A^TA\in\mathbb{R}^{n\times n}$:
Obtain eigenvalues $\sigma_1^2\ge\dots\ge\sigma_n^2\ge0$ of A^TA
Obtain **orthonormal** eigenvectors $v_1,\dots,v_n\in\mathbb{R}^n$ of A^TA (apply **normalization** e.g. **Gram-Schmidt**!!!! to **eigenspaces** E_{σ_i})
$V=[v_1|\dots|v_n]\in\mathbb{R}^{n\times n}$ is orthogonal so $V^T=V^{-1}$
$r=\text{rk}(A)$ = no. of strictly +ve σ_j
Let $u_i=\frac{1}{\sigma_i}Av_i$ then $u_1,\dots,u_r\in\mathbb{R}^m$ are **orthonormal** *(therefore linearly independent)*
The orthogonal compliment of $\text{span}\{u_1,\dots,u_r\}$ =>
$\text{span}\{u_1,\dots,u_r\}^\perp=\text{span}\{u_{r+1},\dots,u_m\}$
Solve for unit-vector u_{r+1} s.t. it is orthogonal to $u_1,\dots,u_r$
Then solve for unit-vector u_{r+2} s.t. it is orthogonal to $u_1,\dots,u_{r+1}$
And so on...
$U=[u_1|\dots|u_m]\in\mathbb{R}^{m\times m}$ is orthogonal so $U^T=U^{-1}$
$S=\text{diag}_{m\times n}(\sigma_1,\dots,\sigma_n)$, **AND DONE!!!**
If $m<n$ then let $B=A^T$
apply above method to B => $B=A^T=USV^T$
$A=B^T=VS^TU^T$

Tricks: Computing orthonormal vector-set extensions

You have **orthonormal** vectors $u_1,\dots,u_r\in\mathbb{R}^m$ => need to **extend** to **orthonormal** vectors $u_1,\dots,u_m\in\mathbb{R}^m$
Special case => two 3D vectors => use **cross-product** => $a\times b\perp a,b$

Extension via standard basis $I_m=[e_1|\dots|e_m]$ using (tweaked) GS:
Choose candidate vector: just work through $e_1,\dots,e_m$ sequentially starting from e_1 => denote the current candidate e_k
Orthogonalize: Starting from $j=r$ going to $j=m$ with each iteration => with current orthonormal vectors $u_1,\dots,u_j$
Compute
$w_{j+1}=e_k-\sum_{i=1}^j(e_k\cdot u_i)u_i=e_k-\sum_{i=1}^j(u_i)_ku_i$
$=e_k-U_jc_j$
Where $U_j=[u_1|\dots|u_j]$ and $c_j=[(u_1)_k,\dots,(u_j)_k]^T$
NOTE: $e_k\cdot u_i=(u_i)_k$, i.e. k-th component of u_i
If $w_{j+1}=0$ then $e_k\in\text{span}\{u_1,\dots,u_j\}$ => discard w_{j+1}, choose next candidate e_{k+1}, try this step again
Normalize: $w_{j+1}\ne0$ so compute unit vector $u_{j+1}=\hat w_{j+1}$
Repeat: keep repeating the above steps, now with **new** orthonormal vectors $u_1,\dots,u_{j+1}$

SVD Application: Principal Component Analysis (PCA)

Assume $A_{\text{uncentered}}\in\mathbb{R}^{m\times n}$ represent m **samples** of n**-dimensional data** *(with $m\ge n$)*
Data centering: subtract mean of each column from that column's elements
Let the **resulting matrix** be $A\in\mathbb{R}^{m\times n}$, who's **columns** have **mean zero**
PCA is done on **centered** data-matrices like A:
SVD exists i.e. $A=USV^T$ and $r=\text{rk}(A)$
Let $A=[r_1;\dots;r_m]$ be rows $r_1,\dots,r_m\in\mathbb{R}^n$ => each row *corresponds to* a sample
Let $A=[c_1|\dots|c_n]$ be columns $c_1,\dots,c_n\in\mathbb{R}^m$ => each column *corresponds to* one dimension of the data
Let $X_1,\dots,X_n$ be **random variables** where each X_i *corresponds* to column c_i
i.e. each X_i *corresponds* to i-th component of data
i.e. random vector $X=[X_1,\dots,X_n]^T$ models the data $r_1,\dots,r_m$
Co-variance matrix of X is $\text{Cov}(A)=\frac{1}{m-1}A^TA$ =>
$(A^TA)_{ij}=(A^TA)_{ji}=\text{Cov}(X_i,X_j)$

$v_1,\dots,v_r$ *(columns of V)* are **principal axes** of A
Let $w\in\mathbb{R}^n$ be some unit-vector => let $\alpha_j=r_j\cdot w$ be the **projection/coordinate** of sample r_j onto w
Variance (Bessel's correction) of $\alpha_1,\dots,\alpha_m$ is
$\text{Var}_w=\frac{1}{m-1}\sum_j\alpha_j^2=\frac{1}{m-1}w^T\left(\sum_j r_j^Tr_j\right)w$
$=\frac{1}{m-1}w^TA^TAw$
First (principal) axis defined =>
$w_{(1)}=\arg\max_{\|w\|=1}w^TA^TAw$
$=\arg\max_{\|w\|=1}(m-1)\text{Var}_w=v_1$
i.e. $w_{(1)}$ the direction that maximizes variance Var_w,
i.e. maximizes variance of **projections on line** $\mathbb{R}w_{(1)}$

$\sigma_1u_1,\dots,\sigma_ru_r$ *(columns of US)* are **principal components/scores** of A
Recall: $A=\sum_{i=1}^r\sigma_iu_iv_i^T$ with $\sigma_1\ge\dots\ge\sigma_r>0$, so that relates principal axes and principal components
Data compression: If $\sigma_1\gg\sigma_2$ then **compress** A by projecting in direction of principal component =>
$A\approx\sigma_1u_1v_1^T$

Cholesky Decomposition

Consider **positive (semi-)definite** $A\in\mathbb{R}^{n\times n}$
Cholesky Decomposition is $A=LL^T$ where L is lower-triangular
For positive semi-definite => **always exists**, but **non-unique**
For positive-definite => **always *uniquely* exists** s.t. diagonals of L are positive

Finding a Cholesky Decomposition:
Compute LL^T and solve $A=LL^T$ by matching terms
For square roots always pick positive
If there is **exact solution** then **positive-definite**
If there are **free variables** at the end, then **positive semi-definite**
i.e. the decomposition is a **solution-set** parameterized on **free variables**
e.g. $\begin{bmatrix}1&1&1\\1&1&1\\1&1&2\end{bmatrix}=LL^T$ where $L=\begin{bmatrix}1&0&0\\1&0&0\\1&c&\sqrt{1-c^2}\end{bmatrix}$, $c\in[0,1]$

If $A=LL^T$ you can use forward/backward substitution to **solve equations**
For $Ax=b$ => let $y=L^Tx$
Solve $Ly=b$ by forward substitution to **find** y
Solve $L^Tx=y$ by backward substitution to **find** x
For $n=3$ => $L=\begin{bmatrix}l_{11}&0&0\\l_{21}&l_{22}&0\\l_{31}&l_{32}&l_{33}\end{bmatrix}$,
$LL^T=\begin{bmatrix}l_{11}^2&l_{11}l_{21}&l_{11}l_{31}\\l_{11}l_{21}&l_{21}^2+l_{22}^2&l_{21}l_{31}+l_{22}l_{32}\\l_{11}l_{31}&l_{21}l_{31}+l_{22}l_{32}&l_{31}^2+l_{32}^2+l_{33}^2\end{bmatrix}$

Forward/backward substitution

Forward substitution: for lower-triangular
$L=\begin{bmatrix}\ell_{1,1}&&0\\\vdots&\ddots&\\\ell_{n,1}&\dots&\ell_{n,n}\end{bmatrix}$
For $Lx=b$, just **solve** the first row
$\ell_{1,1}x_1=b_1\implies x_1=\frac{b_1}{\ell_{1,1}}$ and **substitute down**
Then **solve** the second row
$\ell_{2,1}x_1+\ell_{2,2}x_2=b_2\implies x_2=\frac{b_2-\ell_{2,1}x_1}{\ell_{2,2}}$ and **substitute down**
...and so on until all x_j are solved
Backward substitution: for upper-triangular
$U=\begin{bmatrix}u_{1,1}&\dots&u_{1,n}\\&\ddots&\vdots\\0&&u_{n,n}\end{bmatrix}$
For $Ux=b$, just **solve** the last row
$u_{n,n}x_n=b_n\implies x_n=\frac{b_n}{u_{n,n}}$ and **substitute up**
Then **solve** the second-to-last row
$u_{n-1,n-1}x_{n-1}+u_{n-1,n}x_n=b_{n-1}$
$\implies x_{n-1}=\frac{b_{n-1}-u_{n-1,n-1}x_{n-1}}{u_{n-1,n}}$ and **substitute up**
...and so on until all x_j are solved

Thin QR Decomposition w/ Gram-Schmidt (GS)

Consider **full-rank** $A=[a_1|\dots|a_n]\in\mathbb{R}^{m\times n}$ $(m\ge n)$, i.e. $a_1,\dots,a_n\in\mathbb{R}^m$ are linearly independent
Apply GS $q_1,\dots,q_n\leftarrow\text{GS}(a_1,\dots,a_n)$ to build **ONB** $\langle q_1,\dots,q_n\rangle\in\mathbb{R}^m$ for $C(A)$
For exams: more efficient to compute as $u_{j+1}=a_{j+1}-Q_jc_j$
1) Gather $Q_j=[q_1|\dots|q_j]\in\mathbb{R}^{m\times j}$ **all-at-once**
2) Compute $c_j=[q_1\cdot a_{j+1},\dots,q_j\cdot a_{j+1}]^T\in\mathbb{R}^j$ **all-at-once**
3) Compute $Q_jc_j\in\mathbb{R}^m$, and subtract from a_{j+1} **all-at-once**
Can now rewrite $a_j=\sum_{i=1}^j(q_i\cdot a_j)q_i=Q_jc_j$

Choose $Q = Q_n = [q_1 | \dots | q_n] \in \mathbb{R}^{m\times n}$, notice its semi-orthogonal since $Q^TQ = I_n$

Notice => $a_j = Q_jc_j = Q[q_1\cdot a_j, \dots, q_j\cdot a_j, 0, \dots, 0]^T = Qr_j$

Let $R = [r_1 | \dots | r_n] \in \mathbb{R}^{n\times n}$ =>

$A = QR = Q\begin{bmatrix} q_1^Ta_1 & \cdots & q_1^Ta_n \\ & \ddots & \vdots \\ 0 & & q_n^Ta_n \end{bmatrix}$, notice its upper-triangular

Full QR Decomposition

Consider **full-rank** $A = [a_1 | \dots | a_n] \in \mathbb{R}^{m\times n}$ $(m \geq n)$, i.e. $a_1, \dots, a_n \in \mathbb{R}^m$ are linearly independent

Apply QR decomposition to obtain:

ONB $\langle q_1, \dots, q_n\rangle \in \mathbb{R}^m$ for $C(A)$

Semi-orthogonal $Q_1 = [q_1 | \dots | q_n] \in \mathbb{R}^{m\times n}$ and upper-triangular $R_1 \in \mathbb{R}^{n\times n}$, where $A = Q_1R_1$

Compute basis extension to obtain remaining $q_{n+1}, \dots, q_m \in \mathbb{R}^m$, where $\langle q_1, \dots, q_m\rangle$ is ONB for $\mathbb{R}^m$

Notice $\langle q_{n+1}, \dots, q_m\rangle$ is ONB for $C(A)^\perp = \ker(A^T)$

Let $Q_2 = [q_{n+1} | \dots | q_m] \in \mathbb{R}^{m\times(m-n)}$, let $Q = [Q_1 | Q_2] \in \mathbb{R}^{m\times m}$, let $R = [R_1; 0_{m-n}] \in \mathbb{R}^{m\times n}$

Then **full QR decomposition** is

$A = QR = [Q_1 | Q_2]\begin{bmatrix} R_1 \\ 0_{m-n} \end{bmatrix} = Q_1R_1$

Q is **orthogonal**, i.e. $Q^{-1} = Q^T$, so its a basis transformation

$\mathrm{proj}_{C(A)} = Q_1Q_1^T$, $\mathrm{proj}_{C(A)^\perp} = Q_2Q_2^T$ are orthogonal projections **onto** $C(A)$, $C(A)^\perp = \ker(A^T)$ *respectively*

Notice: $QQ^T = I_m = Q_1Q_1^T + Q_2Q_2^T$

Generalizable to $A \in \mathbb{C}^{m\times n}$ by changing transpose to conjugate-transpose

Lines and hyperplanes in $\mathbb{E}^n (=\mathbb{R}^n)$

Consider **standard Euclidean space** $\mathbb{E}^n (=\mathbb{R}^n)$ with standard basis $(e_1, \dots, e_n) \in \mathbb{R}^n$ with standard origin $0 \in \mathbb{R}^n$

A **line** $L = \mathbb{R}n + c$ is *characterized* by direction $n \in \mathbb{R}^n$ $(n \neq 0)$ and offset from origin $c \in L$

It is customary that:

$\hat n$ is a **unit vector**, i.e. $\|n\| = \|\hat n\| = 1$

$c \in L$ is **closest point to origin**, i.e. $c \perp n$

If $c \neq \lambda n$ => L **not** vector-subspace of $\mathbb{R}^n$ i.e. $0 \notin L$, i.e. L doesn't go through the origin

L is affine-subspace of $\mathbb{R}^n$

If $c = \lambda n$, i.e. $L = \mathbb{R}n$ => L is vector-subspace of $\mathbb{R}^n$ i.e. $0 \in L$, i.e. L goes through the origin

L has $\dim(L) = 1$ and orthonormal basis (ONB) $\{\hat n\}$

A **hyperplane** $P = (\mathbb{R}n)^\perp + c = \{x + c \mid x \in \mathbb{R}^n, x \perp n\} = \{x \in \mathbb{R}^n \mid x\cdot n = c\cdot n\}$ is *characterized* by normal $n \in \mathbb{R}^n$ $(n \neq 0)$ and offset from origin $c \in P$

It *represents* an $(n-1)$**-dimensional slice** of the n**-dimensional space**

It is customary that:

n is a **unit vector**, i.e. $\|n\| = \|\hat n\| = 1$

$c \in P$ is **closest point to origin**, i.e. $c = \lambda n$

With those => $P = \{x \in \mathbb{R}^n \mid x\cdot n = \lambda\}$

If $c\cdot n \neq 0$ => P **not** vector-subspace of $\mathbb{R}^n$ i.e. $0 \notin P$, i.e. P doesn't go through the origin

P is affine-subspace of $\mathbb{R}^n$

If $c\cdot n = 0$, i.e. $P = (\mathbb{R}n)^\perp$ => P is vector-subspace of $\mathbb{R}^n$ i.e. $0 \in P$, i.e. P goes through the origin

P has $\dim(P) = n-1$

Notice $L = \mathbb{R}n$ and $P = (\mathbb{R}n)^\perp$ are orthogonal complements, so:

$\mathrm{proj}_L = \hat n\hat n^T$ is orthogonal projection **onto** L *(along* P*)*

$\mathrm{proj}_P = \mathrm{id}_{\mathbb{R}^n} - \mathrm{proj}_L = I_n - \hat n\hat n^T$ is orthogonal projection **onto** P *(along* L*)*

$L = \mathrm{im}(\mathrm{proj}_L) = \ker(\mathrm{proj}_P)$ and $P = \ker(\mathrm{proj}_L) = \mathrm{im}(\mathrm{proj}_P)$

$\mathbb{R}^n = \mathbb{R}n \oplus (\mathbb{R}n)^\perp$, i.e. all vectors $v \in \mathbb{R}^n$ uniquely decomposed into $v = v_L + v_P$

Householder Maps: reflections

Two points $x, y \in \mathbb{E}^n$ are **reflections** w.r.t hyperplane $P = (\mathbb{R}n)^\perp + c$ if:

1) The translation $\vec{xy} = y - x$ is *parallel* to normal n, i.e. $\vec{xy} = \lambda n$

2) Midpoint $m = 1/2(x+y) \in P$ *lies* on P, i.e. $m\cdot n = c\cdot n$

Suppose $P_u = (\mathbb{R}u)^\perp$ goes through the origin with unit normal $u \in \mathbb{R}^n$

Householder matrix $H_u = I_n - 2uu^T$ is reflection w.r.t. hyperplane P_u

Recall: let $L_u = \mathbb{R}u$

$\mathrm{proj}_{L_u} = uu^T$ and $\mathrm{proj}_{P_u} = I_n - uu^T$ =>

$H_u = \mathrm{proj}_{P_u} - \mathrm{proj}_{L_u}$

Visualize as preserving component in P_u, then flipping component in L_u

H_u is involutory, orthogonal and symmetric, i.e. $H_u = H_u^{-1} = H_u^T$

Modified Gram-Schmidt

Go check Classical GM first, as this is just an alternative computation method

Let $P_{\perp q_j} = I_m - q_jq_j^T$ be *projector* onto hyperplane $(\mathbb{R}q_j)^\perp$, i.e. orthogonal compliment of line $\mathbb{R}q_j$

Notice: $P_{\perp j} = I_m - Q_jQ_j^T = \prod_{i=1}^{j}(I_m - q_iq_i^T) = \prod_{i=1}^{j} P_{\perp q_i}$

Re-state: $u_{j+1} = (I_m - Q_jQ_j^T)a_{j+1}$ =>

$u_{j+1} = \left(\prod_{i=1}^{j} P_{\perp q_i}\right)a_{j+1} = (P_{\perp q_j} \cdots P_{\perp q_1})a_{j+1}$

Projectors $P_{\perp q_1}, \dots, P_{\perp q_j}$ are iteratively applied to a_{j+1} removing its components along q_1, then along q_2, *and so on...*

Let $u_k^{(j)} = \left(\prod_{i=1}^{j} P_{\perp q_i}\right)a_k$, i.e. a_k *without* its components along $q_1, \dots, q_j$

Notice: $u_j = u_j^{(j-1)}$, thus $q_j = \hat u_j = u_j^{(j-1)}/r_{jj}$ where $r_{jj} = \|u_j^{(j-1)}\|$

Iterative step:

$u_k^{(j)} = (P_{\perp q_j})u_k^{(j-1)} = u_k^{(j-1)} - (q_j\cdot u_k^{(j-1)})q_j$

i.e. each **iteration** j of MGS computes $P_{\perp q_j}$ *(and projectors under it)* **in one go**

At *start* of iteration $j \in 1..n$ we have ONB $q_1, \dots, q_{j-1} \in \mathbb{R}^m$ and residual $u_j^{(j-1)}, \dots, u_n^{(j-1)} \in \mathbb{R}^m$

Compute $r_{jj} = \|u_j^{(j-1)}\|$ => $q_j = u_j^{(j-1)}/r_{jj}$

For each $k \in (j+1)..n$, compute $r_{jk} = q_j\cdot u_k^{(j-1)}$ => $u_k^{(j)} = u_k^{(j-1)} - r_{jk}q_j$

Next ONB $(q_1, \dots, q_j)$ and **next residual** $u_{j+1}^{(j)}, \dots, u_n^{(j)}$

NOTE: for $j = 1$ => $q_1, \dots, q_{j-1} = \emptyset$, i.e. none yet

By *end* of iteration $j = n$, we have **ONB** $(q_1, \dots, q_n) \in \mathbb{R}^m$

$A = [a_1 | \dots | a_n] = [q_1 | \dots | q_n]\begin{bmatrix} r_{11} & \cdots & r_{1n} \\ & \ddots & \vdots \\ 0 & & r_{nn} \end{bmatrix} = QR$

corresponds to thin QR decomposition

Where $A \in \mathbb{R}^{m\times n}$ is full-rank, $Q \in \mathbb{R}^{m\times n}$ is semi-orthogonal, and $R \in \mathbb{R}^{n\times n}$ is upper-triangular

Classical vs. Modified Gram-Schmidt

These algorithms both compute thin QR decomposition

Classical Gram-Schmidt
```
for j = 1 to n do
    u_j = a_j
    for i = 1 to j − 1 do
        r_ij = q_i^* a_j
        u_j = u_j − r_ij q_i
    end for
    r_jj = ||u_j||_2
    q_j = u_j / r_jj
end for
```

Modified Gram-Schmidt
```
for j = 1 to n do
    u_j = a_j
end for
for j = 1 to n do
    r_jj = ||u_j||_2
    q_j = u_j / r_jj
    for k = j + 1 to n do
        r_jk = q_j^* u_k
        u_k = u_k − r_jk q_j
    end for
end for
```

Computes at j-th step:

Classical GS => j-th column of Q and the j-th column of R

Modified GS => j-th column of Q and the j-th row of R

Both have **flop (floating-point operation)** count of $O(2mn^2)$

NOTE: **Householder method** has $2(mn^2 - n^3/3)$ **flop** count, but better numerical properties

Recall: $Q^\dagger Q = I_n$ => check for loss of orthogonality with $\|I_n - Q^\dagger Q\| = \mathrm{loss}$

Classical GS => $\|I_n - Q^\dagger Q\| \approx \mathrm{Cond}(A)^2\epsilon_{mach}$

Modified GS => $\|I_n - Q^\dagger Q\| \approx \mathrm{Cond}(A)\epsilon_{mach}$

NOTE: **Householder method** has $\|I_n - Q^\dagger Q\| \approx \epsilon_{mach}$

Multivariate Calculus

Consider $f: \mathbb{R}^n \to \mathbb{R}$

When clear write i**-th component** of input as i instead of x_i

Level curve w.r.t. to $c \in \mathbb{R}$ is all points s.t. $f(x) = c$

Projecting level curves onto $\mathbb{R}^n$ gives f's **contour-map**

n_k**-th order partial derivative** w.r.t i_k of ..., of n_1**-th order partial derivative** w.r.t i_1 of f is:

$\frac{\partial^{n_k+\cdots+n_1} f}{\partial x_{i_k}^{n_k}\cdots\partial x_{i_1}^{n_1}} = \partial_{i_k}^{n_k}\cdots\partial_{i_1}^{n_1} f = f_{i_1\cdots i_k}^{(n_1,\dots,n_k)}$

Its an N**-th order partial derivative** where $N = \sum_k n_k$

$\nabla f = [\partial_1 f, \dots, \partial_n f]^T$ is gradient of f => $(\nabla f)_i = \frac{\partial f}{\partial x_i}$

$\nabla^T f = (\nabla f)^T$ is transpose of ∇f, i.e. $\nabla^T f$ is row vector

$D_u f(x) = \lim_{\delta\to 0}\frac{f(x+\delta u) - f(x)}{\delta}$ is **directional-derivative** of f

It is rate-of-change in direction u, where $u \in \mathbb{R}^n$ is unit-vector

$D_u f(x) = \nabla f(x)\cdot u = \|\nabla f(x)\|\|u\|\cos(\theta)$ => $D_u f(x)$ **maximized** when $\cos\theta = 1$

i.e. when x, u are parallel => hence $\nabla f(x)$ is direction of **max.** rate-of-change

f has **local minimum** at x_{loc} if there's radius $r > 0$ s.t. $\forall x \in B[r; x_{loc}]$ we have $f(x_{loc}) \leq f(x)$

f has **global minimum** x_{glob} if $\forall x \in \mathbb{R}^n$ we have $f(x_{glob}) \leq f(x)$

A **local minimum** satisfies optimality conditions:

$\nabla f(x) = 0$, e.g. for $n=1$ its $f'(x) = 0$

$\nabla^2 f(x)$ is positive-definite, e.g. for $n=1$ its $f''(x) > 0$

$H(f) = \nabla^2 f = J(\nabla f)^T$ is **Hessian** => $H(f)_{ij} = \frac{\partial^2 f}{\partial x_i\partial x_j}$

Interpret $F: \mathbb{R}^n \to \mathbb{R}^m$ as m functions $F_i: \mathbb{R}^n \to \mathbb{R}$ *(one per output-component)*

$J(F) = [\nabla^T F_1; \dots; \nabla^T F_m]$ is **Jacobian** => $J(F)_{ij} = \frac{\partial F_i}{\partial x_j}$

Conditioning

A **problem** is some $f: X \to Y$ where X, Y are normed vector-spaces

A problem **instance** is f with fixed input $x \in X$, shortened to **just** "problem" (with $x \in X$ implied)

δx is **small perturbation** of x => $\delta f = f(x + \delta x) - f(x)$

A problem (instance) is:

Well-conditioned if all small δx lead to small δf, i.e. if κ is small (e.g. 1, 10, 10^2)

Ill-conditioned if some small δx lead to large δf, i.e. if κ is large (e.g. 10^6, 10^{16})

***Absolute* condition number** $\mathrm{cond}(x) = \hat\kappa(x) = \hat\kappa$ of f at x is

$\hat\kappa = \lim_{\delta\to 0}\sup_{\|\delta x\|\leq\delta}\frac{\|\delta f\|}{\|\delta x\|}$

=> for most problems simplified to $\hat\kappa = \sup_{\delta x}\frac{\|\delta f\|}{\|\delta x\|}$

If Jacobian $J_f(x)$ exists then $\hat\kappa = \|J_f(x)\|$, where matrix norm $\|\cdot\|$ induced by norms on X and Y

***Relative* condition number** $\kappa(x) = \kappa$ of f at x is

$\kappa = \lim_{\delta\to 0}\sup_{\|\delta x\|\leq\delta}\left(\frac{\|\delta f\|}{\|f(x)\|} / \frac{\|\delta x\|}{\|x\|}\right)$

=> for most problems simplified to

$\kappa = \sup_{\delta x}\left(\frac{\|\delta f\|}{\|f(x)\|} / \frac{\|\delta x\|}{\|x\|}\right)$

If Jacobian $J_f(x)$ exists then $\kappa = \frac{\|J_f(x)\|}{\|f(x)\|/\|x\|}$

More important than $\hat\kappa$ for *numerical analysis*

***Matrix* condition number** $\mathrm{Cond}(A) = \kappa(A) = \|A\|\|A^{-1}\|$

=> comes up so often that has its own name

$A \in \mathbb{C}^{m\times m}$ is well-conditioned if $\kappa(A)$ is small, ill-conditioned if **large**

$\kappa(A) = \kappa(A^{-1})$; $\kappa(A) = \kappa(\gamma A)$; $\|\cdot\| = \|\cdot\|_2 \implies \kappa(A) = \frac{\sigma_1}{\sigma_m}$

For $A \in \mathbb{C}^{m\times n}$, the problem $f_A(x) = Ax$ has $\kappa = \|A\|\frac{\|x\|}{\|Ax\|}$ => if A^{-1} exists then $\kappa \leq \mathrm{Cond}(A)$

If $Ax = b$, problem of finding x given b is just $f_{A^{-1}}(b) = A^{-1}b$ => $\kappa = \|A^{-1}\|\frac{\|b\|}{\|x\|} \leq \mathrm{Cond}(A)$

For $b \in \mathbb{C}^m$, the problem $f_b(A) = A^{-1}b$ *(i.e. finding* x *in* $Ax = b$*)* has $\kappa = \|A\|\|A^{-1}\| = \mathrm{Cond}(A)$

Stability

Given a problem $f: X \to Y$, an **algorithm** for f is $\tilde f: X \to Y$

Input $x \in X$ is first rounded to $\mathrm{fl}(x)$, i.e. $\tilde f(x) = \tilde f(\mathrm{fl}(x))$

Absolute error => $\|\tilde f(x) - f(x)\|$

relative error => $\frac{\|\tilde f(x) - f(x)\|}{\|f(x)\|}$

$\tilde f$ is **accurate** if $\forall x \in X$, $\frac{\|\tilde f(x) - f(x)\|}{\|f(x)\|} = O(\epsilon_{mach})$

$\tilde f$ is **stable** if $\forall x \in X$, $\exists\tilde x \in X$ s.t. $\frac{\|\tilde f(x) - f(\tilde x)\|}{\|f(\tilde x)\|} = O(\epsilon_{mach})$ and $\frac{\|\tilde x - x\|}{\|x\|} = O(\epsilon_{mach})$

i.e. nearly the right answer to nearly the right question

outer-product is stable

$\tilde f$ is **backwards stable** if $\forall x \in X$, $\exists\tilde x \in X$ s.t. $\tilde f(x) = f(\tilde x)$ and $\frac{\|\tilde x - x\|}{\|x\|} = O(\epsilon_{mach})$

i.e. exactly the right answer to nearly the right question, a **subset of stability**

$\oplus, \ominus, \otimes, \oslash$, **inner-product**, back-substitution w/ triangular systems, are backwards stable

If **backwards stable** $\tilde f$ and f has condition number $\kappa(x)$ then relative error $\frac{\|\tilde f(x) - f(x)\|}{\|f(x)\|} = O(\kappa(x)\epsilon_{mach})$

Accuracy, stability, backwards stability are **norm-independent** for fin-dim X, Y

Big-O meaning for numerical analysis

In complexity analysis $f(n) = O(g(n))$ as $n \to \infty$

But in numerical analysis $f(\epsilon) = O(g(\epsilon))$ as $\epsilon \to 0$, i.e. $\limsup_{\epsilon\to 0}\|f(\epsilon)\| / \|g(\epsilon)\| < \infty$

i.e. $\exists C, \delta > 0$ s.t. $\forall\epsilon$, we have $0 < \|\epsilon\| < \delta \implies \|f(\epsilon)\| \leq C\|g(\epsilon)\|$

$O(g)$ is set of functions $\{f : \limsup_{\epsilon\to 0}\|f(\epsilon)\| / \|g(\epsilon)\| < \infty\}$

Smallness partial order $O(g_1) \preceq O(g_2)$ defined by set-inclusion $O(g_1) \subseteq O(g_2)$

i.e. as $\epsilon \to 0$, $g_1(\epsilon)$ goes to zero ***faster*** than $g_2(\epsilon)$

Roughly same hierarchy as complexity analysis but **flipped** *(some don't fit the pattern)*

e.g. $\dots, O(\epsilon^3) < O(\epsilon^2) < O(\epsilon) < O(1)$

Maximum:

$O(\max(|g_1|, |g_2|)) = O(g_2) \iff O(g_1) \preceq O(g_2)$

e.g. $O(\max(\epsilon^k, \epsilon)) = O(\epsilon)$

Using functions $f_1, \dots, f_n$, let $\Phi(f_1, \dots, f_n)$ be formula **defining some function**

Then $\Phi(O(g_1), \dots, O(g_n))$ is the class of functions $\{\Phi(f_1, \dots, f_n) : f_1 \in O(g_1), \dots, f_n \in O(g_n)\}$

e.g. $\epsilon^{O(1)} = \{\epsilon^{f(\epsilon)} : f \in O(1)\}$

General case:

$\Phi_1(O(f_1), \dots, O(f_m)) = \Phi_2(O(g_1), \dots, O(g_n))$ means $\Phi_1(O(f_1), \dots, O(f_m)) \subseteq \Phi_2(O(g_1), \dots, O(g_n))$

e.g. $\epsilon^{O(1)} = O(k^\epsilon)$ means $\{\epsilon^{f(\epsilon)} : f \in O(1)\} \subseteq O(k^\epsilon)$; *not necessarily true*

Special case: $f = \Phi(O(g_1), \dots, O(g_n))$ means $f \in \Phi(O(g_1), \dots, O(g_n))$

e.g. $(\epsilon + 1)^2 = \epsilon^2 + O(\epsilon)$ means $\epsilon \mapsto (\epsilon+1)^2 \in \{\epsilon^2 + f(\epsilon) : f \in O(\epsilon)\}$; *not necessarily true*

Let $f_1 = O(g_1)$, $f_2 = O(g_2)$ and let $k \neq 0$ be a constant

$f_1f_2 = O(g_1g_2)$; $f\cdot O(g) = O(fg)$; $O(|k|\cdot g) = O(g)$

$f_1 + f_2 = O(\max(|g_1|, |g_2|))$

=> **if** $g_1 = g = g_2$ **then** $f_1 + f_2 = O(g)$

Floating-point numbers

Consider **base/radix** $\beta \geq 2$ *(typically 2)* and **precision** $t \geq 1$ *(24 or 53 for IEEE single/double precisions)*

Floating-point numbers are discrete subset

$F = \left\{(-1)^s\left(m/\beta^t\right)\beta^e \mid 1 \leq m \leq \beta^t,\ s \in \mathbb{B}, m, e \in \mathbb{Z}\right\}$

s is **sign-bit**, m/β^t is **mantissa**, e is **exponent** *(8-bit for single, 11-bit for double)*

Equivalently, can restrict to $\beta^{t-1} \leq m \leq \beta^t - 1$ for unique m and e

$F \subset \mathbb{R}$ is idealized *(ignores over/underflow)*, so is countably infinite and self-similar *(i.e.* $F = \beta F$*)*

For all $x \in \mathbb{R}$ there exists $\mathrm{fl}(x) \in F$ s.t. $|x - \mathrm{fl}(x)| \leq \epsilon_{mach}|x|$

Equivalently $\mathrm{fl}(x) = x(1+\delta)$, $|\delta| \leq \epsilon_{mach}$

Machine epsilon $\epsilon_{machine} = \epsilon_{mach} = \frac{1}{2}\beta^{1-t}$ is maximum relative gap between FPs

Half the gap between 1 and next largest FP

$2^{-24} \approx 5.96\times 10^{-8}$ and $2^{-53} \approx 10^{-16}$ for single/double

FP arithmetic: let $*, \circledast$ be real and floating counterparts of arithmetic operation

For $x, y \in F$ we have $x \circledast y = \mathrm{fl}(x * y) = (x * y)(1+\epsilon)$, $|\delta| \leq \epsilon_{mach}$

Holds for ***any*** arithmetic operation $\circledast = \oplus, \ominus, \otimes, \oslash$

Complex floats implemented pairs of real floats, so above applies to complex ops as-well

Caveat: $\epsilon_{mach} = \frac{1}{2}\beta^{1-t}$ must be scaled by factors on the order of $2^{3/2}, 2^{5/2}$ for $\otimes, \oslash$ *respectively*

$(x_1 \oplus \dots \oplus x_n) \approx (x_1 + \dots + x_n) + \sum_{i=1}^{n} x_i\left(\sum_{j=i}^{n}\delta_j\right)$; $|\delta_j| \leq \epsilon_{mach}$

$(x_1 \otimes \dots \otimes x_n) \approx (x_1 \times \dots \times x_n)(1+\epsilon)$, $\epsilon \leq 1.06(n-1)\epsilon_{mach}$

$\mathrm{fl}(\sum x_iy_i) = \sum x_iy_i(1+\epsilon_i)$ where $1+\epsilon_i = (1+\delta_i)\times(1+\eta_i)\cdots(1+\eta_n)$ and $|\delta_j|, |\eta_i| \leq \epsilon_{mach}$

$1+\epsilon_i \approx 1+\delta_i+(\eta_i+\dots+\eta_n)$

$|\mathrm{fl}(x^Ty) - x^Ty| \leq \sum|x_iy_i||\epsilon_i|$

Assuming $n\epsilon_{mach} \leq 0.1$ =>

$|\mathrm{fl}(x^Ty) - x^Ty| \leq \phi(n)\epsilon_{mach}|x|^T|y|$, where $|x|_i = |x_i|$ is vector and $\phi(n)$ is small function of n

Summing a series is more stable if terms added in order of increasing magnitude

For **FP matrices**, let $|M|_{ij} = |M_{ij}|$, i.e. matrix $|M|$ of absolute values of M

$\mathrm{fl}(\lambda A) = \lambda A + E$; $|E|_{ij} \leq |\lambda A|_{ij}\epsilon_{mach}$

$\mathrm{fl}(A + B) = (A + B) + E$; $|E|_{ij} \leq |A + B|_{ij}\epsilon_{mach}$

$\mathrm{fl}(AB) = AB + E$; $|E|_{ij} \leq n\epsilon_{mach}(|A||B|)_{ij} + O\left(\epsilon_{mach}^2\right)$

Taylor series about $a \in \mathbb{R}$ is

$f(x) = \sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k + O\left((x-a)^{n+1}\right)$ as $x \to a$

Need $a = 0$ => $f(x) = \sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}x^k + O\left(x^{n+1}\right)$ as $x \to 0$

e.g. $(1+\epsilon)^p = \sum_{k=0}^{n}\binom{p}{k}\epsilon^k + O\left(\epsilon^{n+1}\right) = \sum_{k=0}^{n}\frac{p!}{k!(p-k)!}\epsilon^k + O\left(\epsilon^{n+1}\right)$ as $\epsilon \to 0$

Elementary Matrices

Identity $I_n = [e_1 | \dots | e_n] = [e_1; \dots; e_n]$ has elementary vectors $e_1, \dots, e_n$ for rows/columns

Row/column switching: permutation matrix P_{ij} obtained by switching e_i and e_j in I_n *(same for rows/columns)*

Applying P_{ij} **from left** will swap rows, **from right** will swap columns

$P_{ij} = P_{ij}^T = P_{ij}^{-1}$, i.e. applying twice will **undo** it

Row/column scaling: $D_i(\lambda)$ obtained by scaling e_i by λ in I_n *(same for rows/columns)*

Applying P_{ij} **from left** will scale rows, **from right** will scale columns

$D_i(\lambda) = \mathrm{diag}(1, \dots, \lambda, \dots, 1)$ so all **diagonal** properties apply, e.g. $D_i(\lambda)^{-1} = D_i(\lambda^{-1})$

Row addition: $L_{ij}(\lambda) = I_n + \lambda e_ie_j^T$ performs $R_i \leftarrow R_i + \lambda R_j$ when applying **from left**

$\lambda e_ie_j^T$ is zeros except for λ **in** (i,j)**-th entry**

$L_{ij}(\lambda)^{-1} = L_{ij}(-\lambda)$ both triangular matrices

LU factorization w/ Gaussian elimination

Recall: you can represent **EROs** and **ECOs** as transformation matrices R, C *respectively*

LU **factorization** => finds $A = LU$ where L, U are lower/upper triangular *respectively*

Naive Gaussian Elimination performs $[I_m \mid A \mid I_n] \rightsquigarrow [R^{-1} \mid U \mid I_n]$ to get $AI_n = R^{-1}U$ using only row addition

R^{-1}, i.e. **inverse EROs** in reversed order, is **lower-triangular** so $L = R^{-1}$

Algorithm 1 Gaussian elimination
```
U = A, L = I
for k = 1 to m − 1 do
    for j = k + 1 to m do
        ℓ_{j,k} = u_{j,k} / u_{k,k}
        u_{j,k:m} = u_{j,k:m} − ℓ_{j,k} u_{k,k:m}
    end for
end for
```

The **pivot element** is simply diagonal entry $u_{kk}^{(k-1)}$; fails if $u_{kk}^{(k-1)} \approx 0$

$\tilde L\tilde U = A + \delta A$, $\frac{\|\delta A\|}{\|L\|\cdot\|U\|} = O(\epsilon_{mach})$; only **backwards stable** if $\|L\|\cdot\|U\| \approx \|A\|$

Work required: $\sim\frac{2}{3}m^3$ flops $\sim O\left(m^3\right)$

Solving $Ax = LUx$ is $\sim\frac{2}{3}m^3$ flops *(back substitution is* $O(m^2)$*)*

NOTE: Householder triangularisation requires $\sim\frac{4}{3}m^3$

Partial pivoting computes $PA = LU$ where P is a permutation matrix => $PP^T = I$, i.e. its orthogonal

For each column j, finds largest entry and row-swaps to make it new pivot => P_j

Then performs normal elimination on that column => L_j

Result is $L_{m-1}P_{m-1}\dots L_2P_2L_1P_1A = U$, where $L_{m-1}P_{m-1}\dots L_2P_2L_1P_1 = L'_{m-1}\dots L'_1P_{m-1}\dots P_1$

Setting $L = \left(L'_{m-1}\dots L'_1\right)^{-1}$, $P = P_{m-1}\dots P_1$ gives $PA = LU$

Algorithm 2 Gaussian elimination with partial pivoting
```
U = A, L = I, P = I
for k = 1 to m − 1 do
    i = argmax_{i≥k} |u_{i,k}|
    u_{k,k:m} ↔ u_{i,k:m}
    ℓ_{k,1:k−1} ↔ ℓ_{i,1:k−1}
    p_{k,:} ↔ p_{i,:}
    for j = k + 1 to m do
        ℓ_{j,k} = u_{j,k} / u_{k,k}
        u_{j,k:m} = u_{j,k:m} − ℓ_{j,k} u_{k,k:m}
    end for
end for
```

Work required: $\sim\frac{2}{3}m^3$ flops $\sim O\left(m^3\right)$; results in $L_{ij} \leq 1$ so $\|L\| = O(1)$

Stability depends on **growth-factor** $\rho = \frac{\max_{i,j}|u_{i,j}|}{\max_{i,j}|a_{i,j}|}$

=> for partial pivoting $\rho \leq 2^{m-1}$

$\|U\| = O(\rho\|A\|)$ => $\tilde L\tilde U = \tilde PA + \delta A$, $\frac{\|\delta A\|}{\|A\|} = O(\rho\epsilon_{machine})$

=> only **backwards stable** if $\rho = O(1)$

Full pivoting is $PAQ = LU$ finds largest entry in **bottom-right submatrix**

Makes it **pivot** with row/column swaps before normal elimination

Very expensive $O(m^3)$ search-ops, **partial pivoting** only needs $O(m^2)$

Metric spaces & limits

Metrics obey these axioms

$d(x,x) = 0$; $x \neq y \implies d(x,y) > 0$; $d(x,y) = d(y,x)$; $d(x,z) \leq d(x,y) + d(y,z)$

For metric spaces, **mix-and-match** these infinite/finite limit definitions:

$\lim_{x\to+\infty} f(x) = +\infty \iff \forall r \in \mathbb{R}, \exists N \in \mathbb{N}, \forall x > N: f(x) > r$

$\lim_{x\to p} f(x) = L \iff \forall\varepsilon > 0, \exists\delta > 0, \forall x \in X: 0 < d_X(x,p) < \delta \implies d_Y(f(x), L) < \varepsilon$

Cauchy sequences, i.e. $\forall\varepsilon > 0, \exists N \in \mathbb{N}, \forall m, n \geq N: d(a_m, a_n) < \varepsilon$, converge in **complete spaces**

You can manipulate matrix limits much **like in real analysis**, e.g. $\lim_{n\to\infty}(A^nB + C) = (\lim_{n\to\infty}A^n)B + C$

Turn **metric limit** $\lim_{n\to\infty}x_n = L$ into **real limit** $\lim_{n\to\infty}d(x_n, L) = 0$ to leverage real analysis

Bounded **monotone sequences** converge in $\mathbb{R}$

Sandwich theorem for limits in $\mathbb{R}$ => pick easy upper/lower bounds

$\lim_{n\to\infty}r^n = 0 \iff |r| < 1$ and

$\lim_{n\to\infty}\sum_{i=0}^{n}ar^i = \frac{a}{1-r} \iff |r| < 1$

Iterative Techniques

Systems of Equations

Let $A, R, G \in \mathbb{R}^{n\times n}$ where G^{-1} exists => **splitting**

$A = G + R$ helps iteration

$Ax = b$ rewritten as $x = Mx + c$ where $M = -G^{-1}R$; $c = G^{-1}b$

Define $f(x) = Mx + c$ and sequence $x^{(k+1)} = f(x^{(k)}) = Mx^{(k)} + c$ with starting point $x^{(0)}$

Limit of $\langle x_k\rangle$ is fixed point of f => unique fixed point of f is **solution** to $Ax = b$

If $\|\cdot\|$ is consistent norm and $\|M\| < 1$ then $\langle x_k\rangle$ converges for any $x^{(0)}$ *(because Cauchy-completeness)*

We want to find $\|M\| < 1$ and easy to compute M; c

Stopping criterion usually the relative residual $\frac{\|b - Ax^{(k)}\|}{\|b\|} \leq \epsilon$

Assume A's **diagonal is non-zero** *(w.l.o.g. permute/change basis if isn't)* then $A = D + L + U$, where D is **diagonal** of A, L, U are strict **lower/upper triangular** parts of A

Jacobi Method:

$G = D$; $R = L + U$ => $M = -D^{-1}(L+U)$; $c = D^{-1}b$

$x_i^{(k+1)} = \frac{1}{A_{ii}}\left(b_i - \sum_{j\neq i}A_{ij}x_j^{(k)}\right)$ => $x_i^{(k+1)}$ only needs b_i; $x^{(k)}$; A_{i*} => row-wise parallelization

Gauss-Seidel (G-S) Method:

$G = D + L$; $R = U$ => $M = -(D+L)^{-1}U$; $c = (D+L)^{-1}b$

$x_i^{(k+1)} = \frac{1}{A_{ii}}\left(b_i - \sum_{j=1}^{i-1}A_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n}A_{ij}x_j^{(k)}\right)$

Computing $x_i^{(k+1)}$ needs b_i; $x^{(k)}$; A_{i*} and $x_j^{(k+1)}$ for $j < i$ => lower storage requirements

Successive over-relaxation (SOR):

$G = \omega^{-1}D + L$; $R = (1-\omega^{-1})D + U$ =>

$M = -(\omega^{-1}D + L)^{-1}((1-\omega^{-1})D + U)$; $c = -(\omega^{-1}D + L)^{-1}b$

$x_i^{(k+1)} = \frac{\omega}{A_{ii}}\left(b_i - \sum_{j=1}^{i-1}A_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n}A_{ij}x_j^{(k)}\right) + (1-\omega)x_i^{(k)}$ for **relaxation factor** $\omega > 1$

If A is **strictly row diagonally dominant** then Jacobi/Gauss-Seidel methods converge; A is **strictly row diagonally dominant** if $|A_{ii}| > \sum_{j\neq i}|A_{ij}|$

If A is positive-definite then **G-S** and **SOR** $(\omega \in (0,2))$ converge

Eigenvalue Problems

If A is diagonalizable then eigen-decomposition is $A = X\Lambda X^{-1}$

Dominant λ_1; x_1 are such that $|\lambda_1|$ is strictly largest for which $Ax = \lambda x$

Rayleigh quotient for Hermitian $A = A^\dagger$ is $R_A(x) = \frac{x^\dagger Ax}{x^\dagger x}$

Eigenvectors are stationary points of R_A

$R_A(x)$ is closest to being like eigenvalue of x, i.e. $R_A(x) = \mathrm{argmin}_\alpha\|Ax - \alpha x\|_2$

$R_A(x) - R_A(v) = O(\|x - v\|^2)$ as $x \to v$ where v is eigenvector

Power iteration: define sequence $b^{(k+1)} = \frac{Ab^{(k)}}{\|Ab^{(k)}\|}$ with initial $b^{(0)}$ s.t. $\|b^{(0)}\| = 1$

Assume **dominant** λ_1; x_1 exist for A, and that $\mathrm{proj}_{x_1}\left(b^{(0)}\right) \neq 0$

Under above assumptions,

$\mu_k = R_A\left(b^{(k)}\right) = \frac{b^{(k)\dagger}Ab^{(k)}}{b^{(k)\dagger}b^{(k)}}$ converges to **dominant** λ_1

$\langle b_k\rangle$ converges to some **dominant** x_1 associated with λ_1 => $\|Ab^{(k)}\|$ converges to $|\lambda_1|$

If $\mathrm{proj}_{x_1}\left(b^{(0)}\right) = 0$ then $\langle b_k\rangle$; $\langle\mu_k\rangle$ converge to second **dominant** λ_2; x_2 instead

If **no dominant** λ *(i.e. multiple eigenvalues of maximum* $|\lambda|$*)* then $\langle b_k\rangle$ will converge to linear combination of their corresponding eigenvectors

Slow convergence if **dominant** λ_1 not "very dominant"

$\|b^{(k)} - \alpha_kx_1\| = O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right)$ for **phase factor** $\alpha_k \in \{-1, 1\}$ it may alternate if $\lambda_1 < 0$

$\alpha_k = \frac{(\lambda_1)^kc_1}{|\lambda_1|^k|c_1|}$ where $c_1 = x_1^\dagger b^{(0)}$ and assuming $b^{(k)}$; x_1 are normalized

$(A - \sigma I)$ has **eigenvalues** $\lambda - \sigma$

=> power-iteration on $(A - \sigma I)$ has $\frac{\lambda_2 - \sigma}{\lambda_1 - \sigma}$

Eigenvector guess => estimated eigenvalue

Inverse (power-)iteration: perform power iteration on $(A - \sigma I)^{-1}$ to get $\lambda_{1,\sigma}$ **closest** to σ

$(A - \sigma I)^{-1}$ has eigenvalues $(\lambda - \sigma)^{-1}$ so power iteration will yield **largest** $(\lambda_{1,\sigma} - \sigma)^{-1}$

i.e. will yield **smallest** $\lambda_{1,\sigma} - \sigma$, i.e. will yield $\lambda_{1,\sigma}$ **closest** to σ

$\|b^{(k)} - \alpha_kx_{1,\sigma}\| = O\left(\left|\frac{\lambda_{1,\sigma} - \sigma}{\lambda_{2,\sigma} - \sigma}\right|^k\right)$ where $x_{1,\sigma}$ corresponds to $\lambda_{1,\sigma}$ and $\lambda_{2,\sigma}$ is 2nd-closest to σ

Efficiently compute eigenvectors for **known eigenvalues** σ

Eigenvalue guess => estimated eigenvector

Algorithm 3 Inverse iteration
```
for k = 1, 2, 3, ... do
    x̂^(k) = (A − σI)^{−1} x^(k−1)
    x^(k) = x̂^(k) / max(x̂^(k))
    λ^(k) = (x^(k))^T A x^(k)
end for
```

Can reduce matrix inversion $O(m^3)$ to $O(m^2)$ by pre-factorization

Nonlinear Systems of Equations

Recall that $\nabla f(x)$ is direction of **max.** rate-of-change $|\nabla f(x)|$

Idea: Search for stationary point by **gradient descent**: $x^{(k+1)} = x^{(k)} - \alpha\nabla f(x^{(k)})$ for step length α

If A is positive-definite, solving $Ax = b$ and $\min_x f(x) = \frac{1}{2}x^TAx - x^Tb$ are equivalent

Get iterative methods $x^{(k+1)} = x^{(k)} - \alpha^{(k)}p^{(k)}$ for step length $\alpha^{(k)}$ and directions $p^{(k)}$

Conjugate gradient (CG) method: if $A \in \mathbb{R}^{n\times n}$ symmetric then $\langle u, v\rangle_A = u^TAv$ is an inner-product

GC chooses $p^{(k)}$ that are conjugate w.r.t. A, i.e. $\langle p^{(i)}, p^{(j)}\rangle_A = 0$ for $i \neq j$

And chooses $\alpha^{(k)}$ s.t. **residuals** $r^{(k)} = -\nabla f(x^{(k)}) = b - Ax^{(k)}$ are orthogonal

$k = 0$ => $p^{(0)} = -\nabla f(x^{(0)}) = r^{(0)}$

$k \geq 1$ => $p^{(k)} = r^{(k)} - \sum_{i<k}\frac{\langle p^{(i)}, r^{(k)}\rangle_A}{\langle p^{(i)}, p^{(i)}\rangle_A}p^{(i)}$

$\alpha^{(k)} = \mathrm{argmin}_\alpha f(x^{(k)} + \alpha^{(k)}p^{(k)}) = \frac{p^{(k)}\cdot r^{(k)}}{\langle p^{(k)}, p^{(k)}\rangle_A}$

Without rounding errors, **CG** converges in $\leq n$ iterations

Similar to to Gram-Schmidt *(but different inner-product)*

$\langle p^{(0)}, \dots, p^{(n-1)}\rangle$ and $\langle r^{(0)}, \dots, r^{(n-1)}\rangle$ are bases for $\mathbb{R}^n$

QR Algorithm to find Schur decomposition $A = QUQ^\dagger$

Any $A \in \mathbb{C}^{m\times m}$ has **Schur decomposition** $A = QUQ^\dagger$

Q is unitary, i.e. $Q^\dagger = Q^{-1}$ and upper-triangular U

Diagonal of U contains **eigenvalues** of A

Algorithm 1 Basic QR iteration
```
for k = 1, 2, 3, ... do
    A^(k−1) = Q^(k−1) R^(k−1)
    A^(k) = R^(k−1) Q^(k−1)
end for
```

For $A \in \mathbb{R}^{m\times m}$ each iteration $A^{(k)} = Q^{(k)}R^{(k)}$ produces orthogonal $Q^{(k)T} = Q^{(k)-1}$

So $A^{(k+1)} = R^{(k)}Q^{(k)} = (Q^{(k)T}Q^{(k)})R^{(k)}Q^{(k)} = Q^{(k)T}A^{(k)}Q^{(k)}$ means $A^{(k+1)}$ is **similar** to $A^{(k)}$

Setting $A^{(0)} = A$ we get $A^{(k)} = (\tilde Q^{(k)})^TA\tilde Q^{(k)}$ where $\tilde Q^{(k)} = Q^{(0)}\dots Q^{(k-1)}$

Under certain conditions **QR algorithm** converges to **Schur decomposition**

We can **apply shift** $\mu^{(k)}$ at iteration k

=> $A^{(k)} - \mu^{(k)}I = Q^{(k)}R^{(k)}$; $A^{(k+1)} = R^{(k)}Q^{(k)} + \mu^{(k)}I$

If **shifts** are good eigenvalue estimates then last column of $\tilde Q^{(k)}$ converges quickly to an **eigenvector**

Estimate $\mu^{(k)}$ with Rayleigh quotient =>

$\mu^{(k)} = (A_k)_{mm} = (\tilde q_m^{(k)})^TA\tilde q_m^{(k)}$ where $\tilde q_m^{(k)}$ is m-th column of $\tilde Q^{(k)}$