Probability Spaces

Sample Space Ω => set of all outcomes (mutually exclusive) of random experiment

Event $E \subseteq \Omega$ => any subset of sample space

Extreme Events => null event $\emptyset$ & universal event S

Elementary Event => singleton subsets of S

If $s \in E$ is experiment outcome, then E occurred

Null event $\emptyset$ never & universal event S always -occurs

For events $E_1, E_2, \ldots$:

E_1 or E_2 or ... => $\bigcup_i E_i$

E_1 and E_2 and ... => $\bigcap_i E_i$

$E_1, E_2, \ldots$ are **Mutually Exclusive** => $\forall i,j.\ E_i \cap E_j = \emptyset$ i.e. they're pairwise-disjoint

$E_1, E_2, \ldots$ are **Independent Events** =>

$P(\bigcap_{j=1}^n E_{i_j}) = \prod_{j=1}^n P(E_{i_j})$ for any finite subset $\{E_{i_1}, E_{i_2}, \ldots, E_{i_n}\}$

If events A, B are independent, then $\bar{A}, \bar{B}$ are also independent

σ-algebra $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ => family of subsets of Ω s.t.

nonempty: $\Omega \in \mathcal{F}$

closed under complements: $E \in \mathcal{F} \implies \bar{E} \in \mathcal{F}$

closed under countable union:

$E_1, E_2, \ldots \in \mathcal{F} \implies \bigcup_i E_i \in \mathcal{F}$

Immediate Basic Results:

$\emptyset \in \mathcal{F}$

closed under countable intersection:

$E_1, E_2, \ldots \in \mathcal{F} \implies \bigcap_i E_i \in \mathcal{F}$

$\{\emptyset, \Omega\}$ is smallest & $\mathcal{P}(\Omega)$ is largest σ-algebras

Generated σ-algebras:

$\sigma(\mathcal{G})$ => for family of subsets $\mathcal{G} \subseteq \mathcal{P}(\Omega)$, its smallest σ-algebra to contain $\mathcal{G}$ (exists & unique)

$\sigma(f)$ => for $f: \Omega \to E$ where $(E, \mathcal{E})$ is measurable space, $\sigma(f) = \{f^{-1}[F] \mid F \in \mathcal{E}\}$ i.e. all pre-images

trace σ-algebra of $B \in \mathcal{F}$ => $\mathcal{F}_B = \{B \cap A \mid A \in \mathcal{F}\}$

Probability Measure $P: \mathcal{F} \to [0,1]$ on $(\Omega, \mathcal{F})$

$\forall E \in \mathcal{F}, 0 \le P(E) \le 1$ i.e. between 0 and 1

$P(\Omega) = 1$ i.e. universal event S always occurs

σ-additive (countably additive) => $P(\bigcup_i E_i) = \sum_i P(E_i)$ for pairwise-disjoint events $E_1, E_2, \ldots \in \mathcal{F}$

Immediate Basic Results:

$P(\bar{E}) = 1 - P(E)$

$P(\emptyset) = 0$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Measurable Space $(\Omega, \mathcal{F})$ => sample space Ω with σ-algebra $\mathcal{F}$ on it

Probability Space $(\Omega, \mathcal{F}, P)$ => measurable space $(\Omega, \mathcal{F})$ with probability measure P on it

Conditional Probability

Conditional Probability => $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$ where $A, B \subseteq \Omega$ and $P(B) \neq 0$

The **conditional probability space** is $(B, \mathcal{F}_B, P(\cdot \mid B))$

Sample space $B \subseteq \Omega$

Trace σ-algebra $\mathcal{F}_B = \{B \cap A \mid A \in \mathcal{F}\}$

Probability measure $P(\cdot \mid B)$

If A, B are independent then $P(A \mid B) = P(A)$

A_1, A_2 are **Conditionally Independent** given B iff

$P(A_1 \cap A_2 \mid B) = P(A_1 \mid B) P(A_2 \mid B)$

Law Of Total Probability => for any events $\{B_1, B_2, \ldots\}$ which partition Ω, $P(A) = \sum_i P(A \mid B_i) P(B_i) = \sum_i P(A \cap B_i)$

Special Case =>

$P(A) = P(A \cap B) + P(A \cap \bar{B}) = P(A \mid B)P(B) + P(A \mid \bar{B})P(\bar{B})$

Bayes Theorem => $P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$

General Random Variables

Random Variable => measurable function $X: \Omega \to E$

$(\Omega, \mathcal{F}, P)$ is a probability space, $(E, \mathcal{E})$ is a measurable space

For every $B \in \mathcal{E}$ the pre-image of B under X is in $\mathcal{F}$,

i.e. $X^{-1}[B] = \{s \in \Omega \mid X(s) \in B\} \in \mathcal{F}$

i.e. $\sigma(X) \subseteq \mathcal{F}$ where $\sigma(X)$ is generated by function X

$g(X)(s) = (g \circ X)(s)$ is also random variable, for measurable function $g: E \to E$

Induced Probability $P_X(X \in B)$ => probability that X takes on value in $B \in \mathcal{E}$

$P_X(X \in B) = P(X^{-1}[B]) = P(\{s \in \Omega \mid X(s) \in B\})$

Also called **Pushforward Measure** of P onto $(E, \mathcal{E})$

induced by X => $(E, \mathcal{E}, P_X)$ is a probability space

Also called the **Probability Distribution** of X

Real Random Variables

Real Random Variable is RV who's co-domain is $E = \mathbb{R}$

Support $supp(X)$ => is range of X i.e. $supp(X) = X[\Omega]$

Simple RRV iff finite $supp(X)$

Discrete RRV iff countable $supp(X)$

Continuous RRV => uncountable $supp(X)$

Induced Probability => $P_X(X \le x) = P(\{s \in \Omega \mid X(s) \le x\})$

Cumulative Distribution Func. (CDF) $F_X(x) = P_X(X \le x)$

$F_X(x)$ is right-continuous => for any decreasing (x_n)

$\lim_{n\to\infty} x_n = x_L \implies \lim_{n\to\infty} F_X(x_n) = F_X(x_L)$

To check that function is valid CDF, must obey:

Monotonicity $\forall x_1, x_2 \in \mathbb{R}, x_1 < x_2 \implies F_X(x_1) \le F_X(x_2)$

$F_X(-\infty) = 0, F_X(\infty) = 1$

F_X is right-continuous

Simple Properties

$0 \le F_X(x) \le 1, \forall x \in \mathbb{R}$

$P_X(a < X \le b) = F_X(b) - F_X(a)$ for finite intervals $(a,b] \subseteq \mathbb{R}$

Moments of RRVs

Expectation $E[X]$ => the mean μ_X of distribution of X

Discrete => $E_X[g(X)] = \sum_x g(x) p_X(x)$

Continuous => $E_X[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$

Linearity => $E[\alpha g(X) + h(X) + \beta] = \alpha E[g(X)] + E[h(X)] + \beta$

Sum => for any $X_1, \ldots, X_n$

$E[\sum_{i=1}^n X_i] = \sum_{i=1}^n E[X_i]$ $E[\bar{X}] = E[\frac{\sum_{i=1}^n X_i}{n}] = \frac{\sum_{i=1}^n E[X_i]}{n}$

Independent Product => $E[\prod_{i=1}^n X_i] = \prod_{i=1}^n E[X_i]$

Independent and Identically Distributed (i.i.d.) =>

$E(\bar{X}) = \mu_X$

n-th Raw Moment $\mu'_n = E[X^n]$ => i.e. about zero

n-th Central Moment $\mu_n = E[(X - E[X])^n]$

Variance $\sigma_X^2 = Var(X) = E[(X - E(X))^2]$

$Var(X) = E[X^2] - (E[X])^2$

$Var(aX + b) = a^2 Var(X)$

Sum => for independent $X_1, \ldots, X_n$

$Var(\sum_{i=1}^n X_i) = \sum_{i=1}^n Var(X_i)$

$Var(\bar{X}) = Var(\frac{\sum_{i=1}^n X_i}{n^2}) = \frac{\sum_{i=1}^n Var(X_i)}{n}$

Independent and Identically Distributed (i.i.d.) =>

$Var(\bar{X}) = \frac{\sigma_X^2}{n}$

Standard Deviation $\sigma_X = sd(X) = \sqrt{Var(X)}$

n-th Standardized Moment $\tilde{\mu}_n = \frac{\mu_n}{\sigma^n} = \frac{E[(X - E[X])^n]}{(\sqrt{Var(X)})^n}$

Skewness $\gamma_1 = \tilde{\mu}_3 = \frac{E[(X-\mu)^3]}{\sigma^3}$ measures asymmetry

positive skew => distribution leans left

negative skew => distribution leans right

Moment Generating Function (MGF) $M_X(t) = E[e^{tX}]$

$E[X^n] = \frac{d^n M_X}{dt^n}\Big|_{t=0}$ if open interval around $t = 0$ exists

because $e^{tX} = 1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \ldots + \frac{t^nX^n}{n!} + \ldots$

so $M_X(t) = E[e^{tX}] = 1 + tE[X] + \frac{t^2E[X^2]}{2!} + \ldots + \frac{t^nE[X^n]}{n!} + \ldots$

Sum => for independent $X_1, \ldots, X_n$

let $X = \sum_{i=1}^n X_i$ => $M_X(t) = \prod_{i=1}^n M_{X_i}(t)$

Discrete Random Variables

Discrete RRV iff countable $supp(X)$

Let $supp(X) = \{x_1, x_2, \ldots\}$ be ordered s.t. $x_1 < x_2 < \ldots$

CDF F_X will be monotonic increasing step function,

i.e. $F_X(x_i) = F_X(x_{i-1}) + P_X(X = x_i)$

i.e. $P_X(X = x_i) = F_X(x_i) - F_X(x_{i-1})$

Probability Mass Function (PMF) $p(x) = P_X(X = x)$

$0 \le p(x) \le 1, \forall x \in \mathbb{R}$

$\sum_{x \in supp(X)} p(x) = 1$

$p(x_i) = F(x_i) - F(x_{i-1})$

$F(x_i) = \sum_{j=1}^i p(x_j)$

Notable Discrete Distributions

Bernoulli(p): Consider an experiment with **only two possible outcomes**, encoded as a random variable X taking values 1, with probability p; and 0, with probability (1-p). $X \sim Bernoulli(p)$.

Pmf: $p(x) = p^x(1-p)^{1-x}, x = 0, 1$. **Mean:** $\mu = p$. **Variance:** $\sigma^2 = p(1-p)$.

E.g. tossing a coin with probability p for heads: X =1 for heads;X =0 for tails.

Binomial(n, p): Consider n identical, independent Bernoulli(p) trials $X_1, \ldots, X_n$. Let $X = \sum_{i=1}^n X_i$ **be the total number of 1s observed in the n trials**. X is a random variable taking values in $\{0, 1, 2, \ldots, n\}$: $X \sim Binomial(n, p)$.

Pmf: $p(x) = \binom{n}{x} p^x (1-p)^{n-x}, x = 0, 1, 2, \ldots, n$. **Mean:** $\mu = np$. **Variance:** $\sigma^2 = np(1-p)$. **Skewness:** $\gamma_1 = \frac{1-2p}{\sqrt{np(1-p)}}$.

E.g. tossing a fair coin n times, X may be the number of heads obtained, p = 1/2.

Geometric(p): **[Memoryless]** Consider a potentially infinite sequence of independent Bernoulli(p) random variables $X_1, X_2, \ldots$. Suppose we define a quantity X by $X = \min\{i \mid i \ge 1, X_i = 1\}$ to be the **index of the first Bernoulli trial to result in a 1**. Then X is a random variable taking values in $supp(X) = \{1, 2, \ldots\}$: $X \sim Geometric(p)$.

Pmf: $p(x) = p(1-p)^{x-1}, x = 1, 2, \ldots$. **Mean:** $\mu = \frac{1}{p}$. **Variance:** $\sigma^2 = \frac{1-p}{p^2}$. **Skewness:** $\gamma_1 = \frac{2-p}{\sqrt{1-p}}$ and is always positive.

E.g. Tossing a coin: X is the number of tosses until the first head is obtained.

Poi(λ) (Poisson): Let X be a random variable on $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Define $p(x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0, 1, 2, \ldots, (i.e.\ x \in \mathbb{N})$, for some $\lambda > 0$, otherwise $p(x) = 0$. X is said to follow a Poisson distribution with parameter λ: $X \sim Poi(\lambda)$.

Mean: $\mu = \lambda$. **Variance:** $\sigma^2 = \lambda$. **Skewness:** $\gamma_1 = \frac{1}{\sqrt{\lambda}}$ is always positive but decreasing as λ grows.

Poisson random variables are concerned with the number of random events occurring per **unit** of time or space, if there is a constant rate of random events occurring across this unit.

E.g. the number of patients arriving at an emergency room in a hour.

If we have a **non-unit interval (or space) of length t**: λt can be used in the pmf instead of λ, so that $p(x) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}, x = 0, 1, 2, \ldots, X \sim Poi(\lambda t)$. Now we see λ as the rate at which random events occur and λt as the mean number of events in t.

Uniform $U(\{1,2,\ldots,n\})$: Let X be a random variable on $\{1,2,\ldots,n\}$ with pmf $p(x) = \frac{1}{n}, x = 1, 2, \ldots, n$. Then X is said to follow a discrete uniform distribution and $X \sim U(\{1,2,\ldots,n\})$.

Mean: $\mu = \frac{n+1}{2}$. **Variance:** $\sigma^2 = \frac{n^2-1}{12}$.

Distribution		pmf	μ	σ^2
Bernoulli(p)	$X \in \{0,1\}$	$p^x(1-p)^{1-x}$	p	$p(1-p)$
Binomial(n, p)	$X \in \{0,\ldots,n\}$	$\binom{n}{x}p^x(1-p)^{n-x}$	np	$np(1-p)$
Geometric(p)	$X \in \{1,2,\ldots\}$	$p(1-p)^{x-1}$	$\frac{1}{p}$	$\frac{(1-p)}{p^2}$
Alternative Geometric(p)	$Y = X - 1$, $X \sim$ Geometric(p)	$p(1-p)^y$	$\frac{(1-p)}{p}$	$\frac{(1-p)}{p^2}$
Poisson(λ)	$X \in \{0,1,\ldots\}$	$\frac{e^{-\lambda}\lambda^x}{x!}$	λ	λ
Uniform(1, n)	$X \in \{1,\ldots,n\}$	$\frac{1}{n}$	$\frac{n+1}{2}$	$\frac{n^2-1}{12}$

Continuous Random Variables

X is **(Absolutely) Continuous RRV** if $\exists f_X: \mathbb{R} \to \mathbb{R}$ such that $F_X(x) = \int_{u=-\infty}^{x} f_X(u) du$

f_X called **Probability Density Function (PDF)** of X

$P_X(a < X \le b) = P_X(X \le b) - P_X(X \le a)$

$P_X(a < X \le b) = F_X(b) - F_X(a) = \int_a^b f_X(x) dx$

$P_X(X = x) = P_X(\{x\}) = 0$ and

$P_X(X \in \{x_1, x_2, \ldots\}) = P_X(X = x_1) + P_X(X = x_2) + \ldots$ for countable sets

Properties of PDFs:

$F_X(x) = \int_{-\infty}^{x} f_X(t) dt$ & $f_X(x) = \frac{d}{dx} F_X(x)$

$f_X(x) \ge 0$ & $\int_{-\infty}^{\infty} f_X(x) dx = 1$

$f_X(x) \cdot h \approx P_X(X \in [x, x+h])$

Quantiles and Percentiles:

The **lower** and **upper quartiles** and **median** of sample of data are the points $(\frac{1}{4}, \frac{3}{4}, \frac{1}{2})$-way through the ordered dataset, respectively

α-quantile $Q_X(\alpha)$ => for continuous X and $0 \le \alpha \le 1$, the least number satisfying $P(X \le Q_X(\alpha)) = \alpha$

i.e. $Q_X(\alpha) = F_X^{-1}(\alpha)$

$Q(1/2)$ is median & **k-th percentile** is $Q(k/100)$

Notable Continuous Distributions

Uniform Distribution U(a, b): A continuous random variable X with range (a, b) has a uniform distribution on the interval (a, b) if its pdf is $f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & otherwise \end{cases}$ or equivalently its cdf is $F(x) = \begin{cases} 0 & x \le a \\ \frac{x-a}{b-a} & a < x < b \\ 1 & x \ge b \end{cases}$. We write $X \sim U(a, b)$.

To map $X \sim U(0,1)$ to $Y \sim U(a,b)$: $Y = a + (b-a)X$.

Mean: $E(X) = \frac{a+b}{2}$. **Variance:** $Var(X) = \frac{(b-a)^2}{12}$.

Exponential Distribution Exp(λ): Consider a random variable X with $supp(X) = [0, \infty)$ and pdf $f(x) = \lambda e^{-\lambda x}, x \ge 0$ for some $\lambda > 0$. Then X is a exponential (or negative exponential) random variable with rate parameter λ, and $X \sim Exp(\lambda)$.

Integration between 0 and x leads to the **cdf**:

$F(x) = 1 - e^{-\lambda x}, x \ge 0$.

Mean: $E(X) = \frac{1}{\lambda}$. **Variance:** $Var(X) = \frac{1}{\lambda^2}$.

Memoryless (Lack of memory) Property of the Exponential:

The **complementary cumulative distribution function** (or survival function, or tail distribution) is $P(X > x) = e^{-\lambda x}$.

Distribution of the **residual time** before the event occurs when X models time: $P(X > x + s \mid X > s) = P(X > x)$ for $x, s > 0$.

If we think of the exponential random variable as the time to an event, then knowledge that we have waited time s for the event tells us nothing about how much longer we will have to wait - the process has no memory.

Link with Poisson Distribution: If events in a random process occur according to a Poisson distribution with rate λ then the time between consecutive events has an exponential distribution with parameter λ.

Normal distribution $N(\mu, \sigma^2)$: A Normal (or Gaussian) random variable X with range $\mathbb{R}$ has pdf $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\{-\frac{(x-\mu)^2}{2\sigma^2}\}$, for some $\mu, \sigma \in \mathbb{R}, \sigma > 0$. X hash mean μ and variance σ^2, and $X \sim N(\mu, \sigma^2)$.

$\phi_X(t) = e^{it\mu - \frac{1}{2}\sigma^2t^2}, M_X(t) = e^{t\mu + \frac{1}{2}\sigma^2t^2}$

The cdf of X does not have an analytically tractable form for any (μ, σ), so we can only write: $F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} \exp\{-\frac{(t-\mu)^2}{2\sigma^2}\} dt$.

Setting $\mu = 0, \sigma = 1, Z \sim N(0,1)$ gives the special case of the **standard Normal** random variable, with simplified pdf $f(z) \equiv \phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$. Again for cdf we can only write: $F(z) \equiv \Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} dt$.

Linear Transformations of Normal Random Variables: To **standardise** any Normal random variable: $X \sim N(\mu, \sigma^2) \implies \frac{X-\mu}{\sigma} \sim N(0,1)$. Cdf: $F_X(x) = P\left(Z \le \frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right)$.

Using Table of Φ: The standard Normal pdf ϕ is **symmetric about 0**, so $\phi(-z) = \phi(z)$. For cdf Φ, this means $\Phi(z) = 1 - \Phi(-z)$. If $Z \sim N(0,1)$, $P(Z \ge z) = 1 - \Phi(z) = \Phi(-z)$.

Lognormal Distribution: Suppose $X \sim N(\mu, \sigma^2)$, and consider the transformation $Y = e^X$. It can be shown that the random variable Y has density $f_Y(y) = \frac{1}{\sigma y\sqrt{2\pi}} \exp\left[-\frac{(\log(y)-\mu)^2}{2\sigma^2}\right], y > 0$. Y is said to follow a lognormal distribution.

Distribution	rv	pdf	μ	σ^2
Exp(λ)	X	$\lambda e^{-\lambda x}$, $x > 0$	λ	λ^2
$N(\mu, \sigma^2)$	X	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $-\infty \le x \le +\infty$	μ	σ^2
Lognormal(μ, σ^2)	$Y = e^X$, $X \sim N(\mu, \sigma^2)$	$\frac{1}{\sigma y\sqrt{2\pi}} e^{-\frac{(\log(y)-\mu)^2}{2\sigma^2}}$, $y > 0$	$e^{\mu+\sigma^2/2}$	$e^{\sigma^2} - 1$
$U(a, b)$	X	$\frac{1}{b-a}$, $a \le x \le b$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$

Joint Random Variables

Product Probability Space $(\prod_i \Omega_i, \bigotimes_i \mathcal{F}_i, \bigtimes_i P_i)$

Let $(\Omega_1, \mathcal{F}_1, P_1), \ldots, (\Omega_n, \mathcal{F}_n, P_n)$ be probability spaces

$\prod_i \Omega_i$ is n-ary Cartesian product of $\Omega_1, \ldots, \Omega_n$

$\bigotimes_i \mathcal{F}_i$ is product σ-algebra of $\mathcal{F}_1, \ldots, \mathcal{F}_n$

$\bigotimes_i \mathcal{F}_i \triangleq \sigma(\{\prod_{i=1}^n E_i \mid E_1 \in \mathcal{F}_1, \ldots, E_n \in \mathcal{F}_n\})$

where $\sigma(\cdot)$ is the generated σ-algebra

$\bigtimes_i P_i$ is unique measure such that

$(\bigtimes_i P_i)(E_1, \ldots, E_n) = P_1(E_1) \cdots P_n(E_n)$

for every $E_1 \in \mathcal{F}_1, \ldots, E_n \in \mathcal{F}_n$

Multivariate Random Variable => $X: \Omega \to \mathbb{R}^n$

Consider RRVs $X_1, \ldots, X_n: \Omega \to \mathbb{R}$

NOTE: Usually take product space if their sample spaces differ

$X = (X_1, \ldots, X_n): \Omega \to \mathbb{R}^n$ is a **Random Vector**

$[X(s)]_i = X_i(s)$ => it's a random variable

from $(\Omega, \mathcal{F}, P)$ to $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$

$\mathcal{B}(\mathbb{R}^n)$ is **Borel σ-algebra** on $\mathbb{R}^n$

i.e. requirement that each X_i is also an RV

Induced Probability =>

$P_X(X_1 \le x_1, \ldots, X_n \le x_n)$

$= P(\{s \in \Omega \mid X_1(s) \le x_1 \wedge \cdots \wedge X_n(s) \le x_n\})$

RECALL: this pushforward measure is also called probability distribution of X => this measure on Borel σ-algebra is also called the

Joint Probability Distribution

Joint CDF $F_X(x_1, \ldots, x_n) = P_X(X_1 \le x_1, \ldots, X_n \le x_n)$

Recover **Marginal CDFs** for each X_i with

$F_{X_i}(x_i) = \lim_{j \ne i, x_j \to \infty} F_X(x_1, \ldots, x_n)$

e.g. for RVs X, Y => $F_X(x) = F(x, \infty)$ and $F_Y(y) = F(\infty, y)$

To check that function is valid CDF, must obey:

Monotonicity:

For every i, hold fixed all-but-the i-th component

Then for every x_{i_1}, x_{i_2} we have

$x_{i_1} < x_{i_2} \implies F_X(\ldots, x_{i_1}, \ldots) \le F_X(\ldots, x_{i_2}, \ldots)$

$0 \le F_X(x_1, \ldots, x_n) \le 1$

$F_X(\ldots, -\infty, \ldots) = 0, F_X(\infty, \ldots, \infty) = 1$

Interval Probabilities for bi-variate case $Z = (X, Y)$

$P_Z(x_1 < X \le x_2, Y \le y) = F(x_2, y) - F(x_1, y)$ hence

$P_Z(x_1 < X \le x_2, y_1 < Y \le y_2) = F(x_2, y_2) - F(x_1, y_2) - F(x_2, y_1) + F(x_1, y_1)$

Joint PMF $p(x_1, \ldots, x_n) = P_X(X_1 = x_1, \ldots, X_n = x_n)$

Recover **Marginal PMFs** for each X_i with

$p_{X_i}(x_i) = \sum_{x_1} \cdots \sum_{x_{i-1}} \sum_{x_{i+1}} \cdots \sum_{x_n} p(x_1, \ldots, x_n)$

e.g. for RVs X, Y => $p_X(x) = \sum_y p(x,y)$ & $p_Y(y) = \sum_x p(x,y)$

To check that function is valid PMF, must obey:

$0 \le p_X(x_1, \ldots, x_n) \le 1$ & $\sum_{x_1} \cdots \sum_{x_n} p_X(x_n, \ldots, x_1) = 1$

Multinomial Distribution => TODO: HEREEE!!!!! «« NOT DONE

$X_1, \ldots, X_n$ are **Jointly Continuous** if $\exists f: \mathbb{R}^n \to \mathbb{R}$ such

$F_X(x_1, \ldots, x_n) = \int_{t_n=-\infty}^{x_n} \cdots \int_{t_1=-\infty}^{x_1} f_X(t_1, \ldots, t_n)\, dt_1 \cdots dt_n$

f_X called **Joint PDF** of $X = (X_1, \ldots, X_n)$

$f_X(x_1, \ldots, x_n) = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} F_X(x_1, \ldots, x_n)$

To check that function is valid PMF, must obey:

$f_X(x_1, \ldots, x_n) \ge 0$ &

$\int_{t_n=-\infty}^{\infty} \cdots \int_{t_1=-\infty}^{\infty} f_X(t_1, \ldots, t_n) dt_1 \cdots dt_n = 1$

Recover **Marginal PDFs** for each X_i with

$f_{X_i}(x_i) = \int_{x_n=-\infty}^{\infty} \cdots \int_{x_{i+1}=-\infty}^{\infty} \int_{x_{i-1}=-\infty}^{\infty} \cdots \int_{x_1=-\infty}^{\infty} f_X(x_1, \ldots, x_n)\, dx_1 \cdots dx_n$

e.g. for RVs X, Y => $f_X(x) = \int_{y=-\infty}^{\infty} f(x,y) dy$ and $f_Y(y) = \int_{x=-\infty}^{\infty} f(x,y) dx$

X, Y are **Independent RRVs** if-and-only-if

General => $F(x,y) = F_X(x) F_Y(y)$

Discrete => $p(x,y) = p_X(x) p_Y(y)$

Continuous => $f(x,y) = f_X(x) f_Y(y)$

Joint Expectation $E[X] = E[X_1, \ldots, X_n]$

Discrete =>

$E_X[g(X)] = \sum_{x_n} \cdots \sum_{x_1} g(x_1, \ldots, x_n) p_X(x_1, \ldots, x_n)$

Continuous =>

$E_X[g(X)] = \int_{x_n=-\infty}^{\infty} \cdots \int_{x_1=-\infty}^{\infty} g(x_1, \ldots, x_n) f_X(x_1, \ldots, x_n) dx_1 \cdots dx_n$

Sum => $E[\sum_i g_i(X_i)] = \sum_i E_{X_i}[g_i(X_i)]$

Independent Product => $E[\prod_i g_i(X_i)] = \prod_i E_{X_i}[g_i(X_i)]$

Covariance =>

$\sigma_{XY} = Cov(X,Y) = E[(X - \mu_X)(Y - \mu_Y)] = E[XY] - \mu_X\mu_Y$

NOTE: $Var(X) = Cov(X,X)$

For independent RVs $E[XY] = E_X[X] E_Y[Y]$ so $\sigma_{XY} = 0$

Correlation => $\rho_{XY} = Cor(X,Y) = \frac{\sigma_{XY}}{\sigma_X\sigma_Y} = \frac{Cov(X,Y)}{sd(X)sd(Y)}$

Correlation is invariant to the scale of the X, Y

NOTE: $Var(X) = Cov(X,X)$

For independent RVs $\sigma_{XY} = \rho_{XY} = 0$

Multivariate Normal Distribution

Multivariate Normal distribution: A random vector $X = (X_1, \ldots, X_n)$ with means $\mu = (\mu_1, \ldots, \mu_n)$ that has joint pdf $f_X = \frac{1}{\sqrt{(2\pi)^n \det\Sigma}} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right)$ is said to have a multivariate Normal distribution where $\mu = (\mu_1, \ldots, \mu_n)$ is the vector of means of $X_1, \ldots, X_n$ and the covariance matrix $\Sigma = [Cov(X_i, X_j); 1 \le i, j \le n]$, which must be positive definite for a pdf to exist.

Conditional Distributions

Conditional Distribution $P_{Y|X}(B_Y \mid B_X) = \frac{P_{XY}(B_X, B_Y)}{P_X(B_X)}$

for any $B_X, B_Y \in \mathbb{R}$

Probability of Y falling inside B_Y given that we know X fell inside B_X

Conditional CDF $F_{Y|X}(y \mid x) = P(Y \le y \mid X = x)$

Discrete => $P(Y \le y \mid X = x) = \sum_{u=-\infty}^{y} p_{Y|X}(u \mid x)$

Continuous => $P(Y \le y \mid X = x) = \int_{u=-\infty}^{y} f_{Y|X}(u \mid x) du$

Interval Probabilities =>

$P(a < X \le b \mid Y = y) = F_{X|Y}(b \mid y) - F_{X|Y}(a \mid y)$

Law of Total Probability CDF =>

$F_X(x) = \int_{y=-\infty}^{\infty} F_{X|Y}(x \mid y) f_Y(y) dy$

Conditional PMF $p_{Y|X}(y \mid x) = \frac{p_{XY}(x,y)}{p_X(x)}$ if $p_X(x) > 0$

Bayes Theorem PMF => $p_{X|Y}(x \mid y) = \frac{p_{Y|X}(y \mid x) p_X(x)}{p_Y(y)}$

Law of Total Probability PMF =>

$p_X(x) = \sum_y p_{X|Y}(x \mid y) p_Y(y)$

Conditional PDF $f_{Y|X}(y \mid x) = \frac{f_{XY}(x,y)}{f_X(x)}$ if $p_X(x) > 0$

Bayes Theorem PDF => $f_{X|Y}(x \mid y) = \frac{f_{Y|X}(y \mid x) f_X(x)}{f_Y(y)}$

Law of Total Probability PDF =>

$f_X(x) = \int_{y=-\infty}^{\infty} f_{X|Y}(x \mid y) f_Y(y) dy$

Conditional Expectation $E_{Y|X}[Y \mid X = x]$

Discrete => $E_{Y|X}[Y \mid X = x] = \sum_y y\, p_{Y|X}(y \mid x)$

Continuous => $E_{Y|X}[Y \mid X = x] = \int_{y=-\infty}^{\infty} y f_{Y|X}(y \mid x) dy$

NOTE: $E_{Y|X}[Y \mid X = \cdot]$ is a function of x

Law of Total Expectation:

Define $E_{Y|X}[Y \mid X](s) = E_{Y|X}[Y \mid X = X(s)]$

Now $E_{Y|X}[Y \mid X]: \Omega \to \mathbb{R}$ is function of RRV X

Then we have $E_Y[Y] = E_X[E_{Y|X}[Y \mid X]]$

e.g. in continuous case => $= \int_x \int_y y f_{Y|X}(y \mid x) f_X(x) dy dx = \int_y \int_x y f(x,y) dx dy = \int_y y f_Y(y) dy$

Markov Chains

Discrete Time Markov Chains (DTMC) => generalization of sequences of i.i.d. RVs to support arbitrary (and possibly dependent) RVs

State Space J => elements $j \in J$ are states

$X_n, n \ge 0$ => takes values in J & models state at time n

Realization of $X_0, X_1, \ldots$ is called **Sample Path**

GOAL => calculate $P(X_n = j)$

i.e. probability that at time n system reaches state j

Markov property =>

$P(X_{n+1} = j_{n+1} \mid X_n = j_n, \ldots, X_1 = j_1, X_0 = j_0)$

$= P(X_{n+1} = j_{n+1} \mid X_n = j_n)$

i.e. choice of the next state depends on the current state only

How to specify DTMCs:

Initial Probability Vector => row-vector $\pi_0^T \in \mathbb{R}^n$

where $(\pi_0)_i = P(X_0 = i)$

Transition Probability Matrix => $R \in \mathbb{R}^{n \times n}$ where

$R_{ij} = P(X_{n+1} = j \mid X_n = i)$

Each transition probability R_{ij} is independent of the time n

Self-loops are allowed => e.g. $R_{jj} = 1$ means DTMC can never leave state j (e.g. a permanent fault)

R is a non-negative matrix who's rows sum to 1,

Also called **stochastic matrix**

$P(X_n = j \mid X_0 = i) = (R^n)_{ij}$ => $P(X_n = j) = (\pi_0 R^n)_j$

Classification Of DTMCs:

Irreducible DTMC => directed graph associated to R is strongly connected;

For any pair (i,j) there exist some sample path where, starting in state i, the DTMC eventually reaches state j

Periodic DTMC => time to return to a state is an integer multiple of a fixed period; otherwise it's **aperiodic DTMC**

Common Long-Term Behaviours Of DTMC:

Steady-State/Stationary Distribution => row-vector π_∞^* thats invariant under R,

i.e. $P(X_n = j) = (\pi_\infty^*)_j$ for all $n \ge 0, j \in J$

Limiting Distribution => row-vector π_∞ such that

$\pi_\infty = \lim_{n\to\infty} \pi_0 R^n$

If exits, its also a steady-state distribution => converse isn't true

Uniqueness and existence:

The two above are **not unique in general**

If irreducible and aperiodic DTMC:

Limiting & steady-state distributions both exist =>

unique and identical to each other

Elements of π_∞ all strictly positive

π_∞ is solution of $\pi_\infty R = \pi_\infty$ subject to $\sum_j (\pi_\infty)_j = 1$

i.e. left-eigenvector with left-eigenvalue $\lambda = 1$

If irreducible but not aperiodic DTMC:

limiting distribution π_∞ no longer exists

Steady-state distribution π_∞^* still exists =>

its the unique positive solution of $\pi_\infty^* R = \pi_\infty^*$ subject to $\sum_j (\pi_\infty^*)_j = 1$

Estimator Theory

Sample of Data $x = (x_1, \ldots, x_n)$ => may be seen as realisation of RVs $X = (X_1, \ldots, X_n)$ which are i.i.d

Assume that a single draw X_i follows a distribution $P(\cdot \mid \theta)$

where $\theta = (\theta_1, \ldots, \theta_r)$ are parameters we wish to estimate

I.i.d assumption implies => sampling is with replacement, i.e. any number of samples n may be collected

Statistic $T = T(X) = T(X_1, \ldots, X_n)$ => a function of random sample vector X

Since T is a random vector, its a random variable

If $T(X)$ used to approximate parameters in θ then, T is an **Estimator** for parameter θ

realisation $t(x)$ of estimator for particular data sample x is called **estimate** of θ

GOAL => study **sampling distribution**

$P(T \mid \theta) \equiv P(T(X) \mid \theta)$ for statistic T and its moments

Estimators for Mean => TODO: HEREEE!!!!!

Bias of Estimator $bias(T) = E[T \mid \theta] - \theta$

Requires to determine the expectation of $T(X)$ based on sampling distribution

If $bias(T) = 0$ we say T is **Unbiased**

Sample mean $\bar{X} = \frac{\sum_{i=1}^n X_i}{n}$ is unbiased estimate for the population mean μ i.e. $E[\bar{X}] = \mu$

Sample variance $\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2$ is known to be biased estimate of σ^2

Only if we know population mean μ then

$\frac{1}{n}\sum_{i=1}^n (X_i - \mu)^2$ is unbiased estimate of σ^2

Bessel's correction => $S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$ is an is unbiased estimate of σ^2, i.e. $E[S^2] = \sigma^2$

Efficiency of Estimators

Let $T \equiv T(X), H \equiv H(X)$ be unbiased estimators for a parameter θ

Assume we know the corresponding sampling distributions $P(T \mid \theta), P(H \mid \theta)$ => we can calculate their variances

T is **More Efficient** than H if:

1) $\forall\theta \quad Var(T \mid \theta) \le Var(H \mid \theta)$

2) $\exists\theta \quad Var(T \mid \theta) < Var(H \mid \theta)$

T is **Efficient** if its more efficient than any other possible estimator

e.g. $T = \bar{X}$ is more efficient than $H = X_1$ as estimator for μ (for $n \ge 2$)

Consistency of Estimators

Consistency allows us to recognize bad patterns as n grows large

T is a **Consistent Estimator** for parameter θ if

$\forall\varepsilon > 0, \quad P(|T(X) - \theta| > \varepsilon) \to 0$ as $n \to \infty$

i.e. all probability mass of the estimator *(seen as a random variable)* is asymptotically on the value θ

e.g. $T = \bar{X}$ is and $H = X_1$ isn't a consistent estimator for μ

Maximum Likelihood Estimation

1. The likelihood function, $L(\theta) = \prod_{i=1}^n f(x_i \mid \theta)$ is the product of the n pmf/pdf viewed as a function of a parameter θ.
2. Take the natural log of the likelihood to get the log-likelihood function $l(\hat{\theta}) = log(L(\theta))$ and collect terms involving θ.
3. Find the value of θ for which log-likelihood is maximised. This is typically done by finding $\hat{\theta}$ that solves $l'(\hat{\theta}) = \frac{d}{d\theta} log(L(\hat{\theta})) = 0$.
4. If the estimate $\hat{\theta}$ obtained in step 3 corresponds to a maximum $\frac{d^2}{d\theta^2} l(\hat{\theta}) < 0$, then $\hat{\theta}$ is confirmed as the maximum likelihood estimator (MLE) of θ.

Remarks:

- In large sample sizes, the MLE progressively becomes unbiased, efficient and consistent. This can be proved under mild technical assumptions.
- In small sample sizes there is no such guarantee and the quality of a MLE can vary. Bayesian parameter estimation is an area of statistics that deals with this problem.
- MLE generalizes to multi-parameter distributions. Yet this requires multivariate calculus and the maximization may give more than one answer if l has several peaks (local maxima).

Central Limit Theorem (CLT)

The CLT is a general result for sums of random variables. In statistics, it helps to study the distribution of the sample mean.

Let $X_1, X_2, \ldots, X_n$ be now n independent and identically distributed (i.i.d.) random variables from any probability distribution with mean μ and variance σ^2 both finite.

$\lim_{n\to\infty} \frac{S_n - n\mu}{\sqrt{n}\sigma} \sim N(0,1)$. Can also be written as:

$\lim_{n\to\infty} \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1), \bar{X} = \frac{S_n}{n} = \frac{\sum_{i=1}^n X_i}{n}$ is the sample mean.

Implications for the Sample Mean: The CLT thus implies that for large, but finite n, $\bar{X} \approx N(\mu, \frac{\sigma^2}{n})$

- A rule of thumb **for 'large n" is often n ≥ 30.**
- Amazingly, this result holds irrespective of the distribution of the $\{X_i\}$ (and including discrete random variables). CLT thus demonstrates that statistical regularity can arise even from the combination of highly-diverse random phenomena.
- If $X_i \sim N(\mu, \sigma^2), \forall i$, the result becomes exact even for finite n, since the sum of independent normal random variables is normally distributed.

Hypothesis Testing

The Null Hypothesis: there is no difference in user engagement between the two groups.

The Alternative Hypothesis: a difference exists, e.g., the new feature increases user engagement.

If the test group shows much higher engagement than the control group, then we will reject the null hypothesis and conclude that the evidence favours the alternative hypothesis.

Parametric tests: Suppose again we have a random i.i.d. sample $(X_1, \ldots, X_n)$ of a random variable X from an unknown distribution P.

Parametric tests typically assume the sample comes from a parametric family $P(\cdot \mid \theta)$ and test whether we could reasonably assume $\theta = \theta_0$ for some particular value θ_0.

Hypotheses: Formally, let H_0 be the null hypothesis and H_1 be the alternative hypothesis.

Two-sided test: $H_0: \theta = \theta_0$ versus $H_0: \theta \neq \theta_0$.

One-sided test: $H_0: \theta = \theta_0$ versus $H_1: \theta < \theta_0$; $H_0: \theta = \theta_0$ versus $H_1: \theta > \theta_0$.

Usually, the null hypothesis is formulated with an equality sign (=), while the alternative hypothesis uses one of $(\neq, <, >)$.

Rejection Region for a Test Statistic: To test the validity of H_0, we choose a **test statistic** T(X) of the data for which we can find the distribution under H_0.

The "art" of hypothesis testing is to define the test by identifying a rejection region $R \subseteq \mathbb{R}$ of low probability values of T under the assumption that H_0 is true, so that $P(T \in R \mid H_0) = \alpha$ for some small probability α (say 5%). We call α the **significance level** of the test. α is the overall probability of all rejection region.

Confidence level = 1- α.

Rule: A well-chosen rejection region will have relatively high probability under H_1, whilst retaining low probability under H_0.

We calculate the observed test statistic t(x) for our sample x:

1. If $t \in R$ we "**reject** the null hypothesis at the 100α% level".
2. If $t \notin R$ we "**retain** the null hypothesis at the 100α% level".

Dealing with unknown variance: Student's t distribution:

$T = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} \sim t_{n-1}$. $H_0: \mu = \mu_0$ **versus** $H_1: \mu \neq \mu_0$ **at the** α **level.**

Rejection region of observed test statistic $t = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}$ is then given by $R = \left(-\infty, -t_{n-1,\frac{\alpha}{2}}\right) \cup \left(t_{n-1,\frac{\alpha}{2}}, \infty\right)$.

Two-sample t-test: $T = \frac{(\bar{X} - \bar{Y}) - (\mu_x - \mu_y)}{\sqrt{s_P^2(\frac{1}{n_x} + \frac{1}{n_y})}} \sim t_{n_x+n_y-2}$.

$$(\bar{x} - \bar{y}) \pm t_{\frac{\alpha}{2}, n_x+n_y-2} \times s_p\sqrt{\frac{1}{n_x} + \frac{1}{n_y}}$$

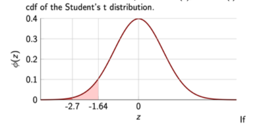
$$s_P^2 = \frac{\sum(x - \bar{x})^2 + \sum(y - \bar{y})^2}{n_x + n_y - 2} = \frac{(n_x - 1)s_x^2 + (n_y - 1)s_y^2}{n_x + n_y - 2}$$

p-Values: The **p-value** of the data is the probability of obtaining a test statistic at least as extreme as the one actually observed, assuming H_0 is correct. In other words, **the p-value is the maximum significance level at which we still reject the null hypothesis H_0 for that sample.** Thus, if we are given a fixed α, the null hypothesis H_0 is rejected if the p-value is less than or equal to α.

Rule: Smaller p-values suggest stronger evidence against H_0.

p-value in a One-sided Lower-Tailed Test ($H_1: \theta < \theta_0$):

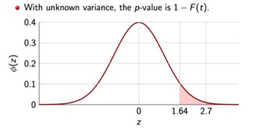
- With known variance, the p-value is $\Phi(z)$.
- With unknown variance, the p-value is $F(t)$, where $F(\cdot)$ is the cdf of the Student's t distribution.



If $z = -2.7$ and $H_1: \theta < \theta_0$ then $p = 0.003467$. The lighter shading is R for $\alpha = 0.05$.

p-value in a One-sided Upper-Tailed Test ($H_1: \theta > \theta_0$):

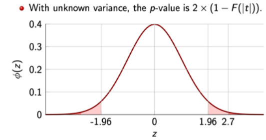
- With known variance, the p-value is $1 - \Phi(z)$.
- With unknown variance, the p-value is $1 - F(t)$.



If $z = 2.7$ and $H_1: \theta > \theta_0$ then $p = 0.003467$. The lighter shading is R for $\alpha = 0.05$.

p-value in a Two-sided Test:

- With known variance, the p-value is $2 \times (1 - \Phi(|z|))$.
- With unknown variance, the p-value is $2 \times (1 - F(|t|))$.



For example, if $z = 2.7$ then the **two-sided p-value** is $p = 0.006934$. The lighter shading is R for $\alpha = 0.05$.

p-value is the probability that a test statistic is at least as extreme as observed. Thus for fixed α, we reject H_0 if $p \le \alpha$.

Side	Tail	Var	P-Value
1	Low	σ^2	$p = \Phi(z)$
1	Low	S^2	$p = F(t)$ †
1	Up	σ^2	$p = 1 - \Phi(z)$
1	Up	S^2	$p = 1 - F(t)$ †
2	-	σ^2	$p = 2(1 - \Phi(\lvert z \rvert))$
2	-	S^2	$p = 2(1 - F(\lvert t \rvert))$†

† F is the CDF of the t-distribution.

Simulation

Discrete-Event Simulation (DES): A (discrete-event) simulation is a program that generates a random **sample path** through a state transition system, where time delays are associated with each state.

There is a single global "clock" – a virtual time. Not to be confused with elapsed real time.

State transitions are triggered by **events** which are ordered in time on a virtual timeline, an event "diary".

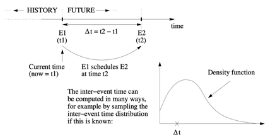
DES involves invoking events in time order; if an event is invoked ("occurs", "fires", "is triggered", …) at virtual time t the clock is updated to t and the code for the event:

- Updates the model state.
- Schedules zero or more new future events on the time line.

Note that the state is unchanged between events.

- In practice, DES is based on a few core design principles:
- The virtual time is a floating-point number (call it *now*)
- The state is defined by a set of program variables, which are typically discrete (e.g. booleans, integers, …)
- The timeline is a priority queue of *(Event, time)* pairs, ordered by virtual time – essentially an event *diary*.
- Events are implemented as objects, functions, procedures, methods, etc.
- A **scheduler** adds new *(Event, time)* pairs to the diary
- A **descheduler** similarly removes them from the diary.
- Additional measurement variables and code need to be added in order to accumulate performance measures (otherwise there will be no output!)

Evolving time:



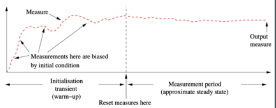
The times between events are r.v.s with an associated distribution – inter-event times are samples from that distribution. We need to be able to sample these distributions.

Designing a simulation model:

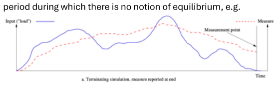
1. Identify the *entities* in the system that have to be modelled
2. Identify the *model states* (program state variables) – these specify where each entity is and what it is doing
3. Identify the *event types*, recalling that each state transition is triggered by an event (note that some events may be parameterisable, e.g. "arrival at location a")
4. For each *event*, specify i. how it changes the current state, ii. what new events need to be scheduled and what old events need to be cancelled (descheduled) when it fires
5. Add code to accumulate *measurements* whilst the simulation executes
6. Add code to *output results* when the program terminates, e.g. after T simulated time units, N occurrences of a specified event, etc.

Output Analysis

A **non-terminating simulation**, seeks to model a system at "equilibrium", a.k.a. "in steady state", where $p_s(t) \to p_s$ as $\to \infty$.



A **terminating simulation** models a system over a specified period during which there is no notion of equilibrium, e.g.



Equilibrium output analysis: We focus on non-terminating simulations.

Assume we are using simulation to estimate some "long-run" performance measure, e.g. mean population, mean response time, resource utilisation, etc.

The **initial state** is typically fixed (e.g. all queues empty), so the initial state probability distribution is different to the distribution after some time t >> 0, say, and measures take time to settle.

To avoid initialisation bias we must either:

1. Discard the initialisation transient by **resetting** the measured after some "warm-up" time has elapsed.
2. Measure for "long enough" to render any bias insignificant.

Confidence intervals: Discrete-event simulations are stochastic 随机的, so all outputs are random variables and each an **observation** of some measure, θ, of interest

If $X_i, 1 \le i \le n, n \ge 1$ are steady-state observations from a simulation then a **point estimate** for θ is the sample mean: $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

If we draw a large number of samples, $100(1-\alpha)\%$ of them will have the mean value that lies under this interval.

For any desired coverage probability level $1-\alpha$ we can define the $\boldsymbol{100(1-\alpha)\%}$ **confidence interval** for μ by $[\overline{X} - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}, \overline{X} + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}]$

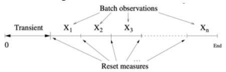
where z_α is the α-quantile of the standard normal.

For 95% confidence level: $\overline{X} \pm 1.96\frac{\sigma}{\sqrt{n}}$.

If n>1, a $\boldsymbol{100\alpha\%}$ **confidence interval** estimate for θ is $\overline{X} \pm t_{n-1,(1-\alpha)/2}\frac{S}{\sqrt{n}}$ where $t_{v,\alpha}$ is the α-quantile of the Student's t distribution with v degrees of freedom.

Confidence intervals in simulation: How do we apply confidence intervals to simulation outputs?

- Running n independent replications, possibly in parallel, guarantees independence of the X_i.
- Another approach is to run the model once, wait for it to warm up and reach (approximate) equilibrium, then divide the measurement time into batches, with X_i coming from batch i.



- If each X_i is itself a mean then this is called the **batch means** method.
- Yet the X_i may not be independent because the state at the end of one batch is the same as that at the start of the next!

If the X_i are **dependent**, then we have to take **covariances** into account to build an exact confidence interval.

The above confidence interval makes the assumption that $Var(\overline{X}) = \sigma^2/n$, and then uses S^2 as an estimate of σ^2.

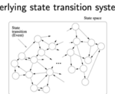
However, if the X_i are dependent, then it can be shown that $Var(\overline{X}) = \frac{\sigma^2}{n} + \frac{1}{n^2}[2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} Cov(X_i, X_j)$.

If covariances are positive S^2/n becomes an **under-estimate** of $Var(\overline{X})$ and the computed confidence intervals are narrower than they should be.

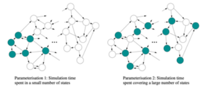
State-space coverage:

- Consider the underlying state transition system

s,



- Suppose the simulation is run for time T at (approximate) equilibrium and spends time T_s in state s
- T_s/T is then an estimate, $\hat{p}_s$, of (the unknown) state probability p_s
- Remark: Measures such as W could be computed directly, or in terms of $\hat{p}_s$ – the answer will be the same
- How good is the estimate $\hat{p}_s = T_s/T$? It clearly depends on T, but also on the distribution $p_s, s \in S$ itself:



Parameterisation 1: Simulation time spent in a small number of states

Parameterisation 2: Simulation time spent covering a large number of states

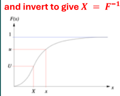
Filled states: largest subset $S' \subseteq S$ s.t. $inf\{p_i \mid i \in S'\} \ge sup\{p_j \mid j \in S - S'\}$ and $\sum_{i \in S'} p_i \le p_{max}$ for some p_{max}

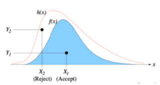
- For the same simulation time the C.I. for the left model (light load) should be narrower than the right (heavy load)

Distribution sampling

For r.v. X with support $supp(X)$, the objective is to define a sampling function: $U(0,1) \to supp(X)$ in terms of X's density/cdf (or pmf/cdf) function.

1. The Inverse Transform method: Suppose X is a continuous r.v. with cdf $F(x) = P(X \le x)$ and that we are trying to sample X, Let U ~ U(0,1). Because F(x) increases monotonically, we have: $P(F^{-1}(U) \le x) = P(U \le F(x)) = F(x)$, so **set** $\boldsymbol{U = F(X)}$ **and invert to give** $\boldsymbol{X = F^{-1}(U)}$.





(For Method 1) (For Method 2)

We can use the inverse transform method to sample a discrete r.v., X by inverting its cumulative distribution function, $F_X(x)$ (a"step function"). If $U \sim U(0,1)$, then the inverse transform methods returns $X = min\{x : F(x) \ge U\}$.

2. The Acceptance-Rejection (AR) Method: If F(x) cannot be explicitly inverted (e.g., normal cdf) we can sometimes work with the corresponding density function f(x). We choose a density function g(x) easy to sample from. Now we try to find a constant, c, so that **c g(x) = h(x) dominates f (x) for all x (i.e. c g(x)>f(x)):**

By construction, c is the area under h(x): $c = c\int_x^{\square} g(x)dx = \int_x^{\square} h(x)dx$.

To find c: We need to maximize $\frac{f(x)}{g(x)}$: Differentiate $\frac{f(x)}{g(x)}$, let $\frac{d}{dx}\frac{f(x)}{g(x)} = 0$ to find the maximum value of x, calculate c using the value of x ($c = max\frac{f(x)}{g(x)}$).

AR Algorithm:

1. Let X be a sample from the r.v. whose density function is g(x)
2. Generate a U(0,1) sample, U, and let $\boldsymbol{Y = Uh(X)}$
3. If $Y \le f(X)$,i.e.if $U \le \frac{f(X)}{h(X)} = \frac{f(X)}{cg(X)}$, where $U \sim (0,1)$, then **accept** X; otherwise **reject** it and start again.

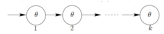
It's a "dart throwing" exercise (Monte Carlo simulation).

By construction, the samples X and Y define a point that lies under h(X); if (X,Y) lies under f(X) as well we accept X.

3. The Convolution Method: Some random variables are defined as the sum of two or more independent random variables.

We can **sample the individual distributions and sum the results.**

- Example: An Erlang(k, θ) random variable, X say, is defined as the sum of k independent exponentially-distributed random variables X_i, each with rate parameter θ



- Notice that

$$E[X] = E[X_1 + \ldots + X_k] = \frac{1}{\theta} + \frac{1}{\theta} + \cdots + \frac{1}{\theta} = \frac{k}{\theta}$$

- We can generate Erlang(k, θ) samples using the sampler for the exponential distribution: if $X_i \sim \exp(\theta)$ then

$$X = \sum_{i=1}^{k} X_i \sim \text{Erlang}(k, \theta)$$

- If $U_i \sim U(0,1)$ then X_i is sampled using $-\log U_i/\theta$
- We can save the more expensive log calculations in the summation by turning the sum into a product:

$$X = \sum_{i=1}^{k} -\frac{\log U_i}{\theta} = -\frac{1}{\theta}\log\prod_{i=1}^{k} U_i$$

4. The Compositio Method:

- The *Composition Method* applies to density functions that have the following structure (called a *mixture*):

$$f(x) = w_1 f_1(x) + w_2 f_2(x) + \ldots + w_n f_n(x)$$

where the $f_i(x)$ are density (or mass) functions and where $0 < w_i < 1, 1 \le i \le n, n > 0$ and $\sum_{i=1}^{n} w_i = 1$

- The sampling algorithm is straightforward:
  - Pick i w.p. w_i (discrete distribution sampling)
  - Sample the distribution whose density is f_i, e.g. using one of the above methods

This applies also to cdfs and mass functions with a similar algorithm.

Misc

Combination: order doesn't matter. Select item without regard to how they are arranged. $nCr = \frac{n!}{(n-r)!r!}$.

Permutation: order is important. Count number of ways you can arrange a set of items where the sequence or arrangement matters. $nPr = \frac{n!}{(n-r)!}$.

$$\left(\sum_{i=0}^{n} X_i\right)^2 = \sum_{i=0}^{n}\sum_{j=0}^{n} X_i X_j$$

$$\frac{\sum(x-\overline{x})^2}{n} = \frac{\sum x^2}{n} - \overline{x}^2$$

Taylor Series: $e^x = \sum_{i=0}^{\infty}\frac{x^n}{n!}$

Summations: $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$. $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$. $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$.

$\boldsymbol{f(x)}$	$\boldsymbol{\int f(x)\,dx}$
$\frac{1}{x}$	$ln\lvert x\rvert$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}ln\left\lvert\frac{x-a}{x+a}\right\rvert$ $(x > a)$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}ln\left\lvert\frac{a+x}{a-x}\right\rvert$ $(\lvert x\rvert < a)$

$\int uv'dx = uv - \int u'vdx$.

L(log)**A**(algebra)**T**(trigonometry)**E**(exponent)

$$\int\frac{f'(x)}{f(x)}dx = ln|f(x)|$$

$$\int f^n(x)f'(x)dx = \frac{1}{n+1}f^{n+1}(x) + C$$

$f(x)$	$f'(x)$
$ln\,x$	$\frac{1}{x}$
uv	$u'v + uv'$
$\frac{u}{v}$	$\frac{u'v - uv'}{v^2}$

$$x = rcos\theta, y = rsin\theta$$