Example: Multinomial distribution

Consider a sequence of n independent and identical experiments with r possible outcomes, each with probability q_i, $\sum_{i=1}^{r} q_i = 1$.
Let X_i be the number of experiments that yield outcome i, then:

$$p(n_1, \ldots, n_r) = \mathrm{P}_Z(X_1 = n_1, \ldots, X_r = n_r) = \frac{n!}{n_1! n_2! \cdots n_r!} q_1^{n_1} q_2^{n_2} \cdots q_r^{n_r}$$

This is due to independence, since a sequence has probability

$$q_1^{n_1} q_2^{n_2} \cdots q_r^{n_r} \quad (n_1 + n_2 + \ldots + n_r = n)$$

and the number of sequences that yield $(n_1, \ldots, n_r)$ is

$$\binom{n}{n_1}\binom{n - n_1}{n_2} \cdots \binom{n - \sum_{i=1}^{r-1} n_i}{n_r} = \frac{n!}{n_1! n_2! \cdots n_r!}$$

Multivariate Normal distribution

A random vector $X = (X_1, \ldots, X_n)$ with means $\mu = (\mu_1, \ldots, \mu_n)$ that has joint pdf

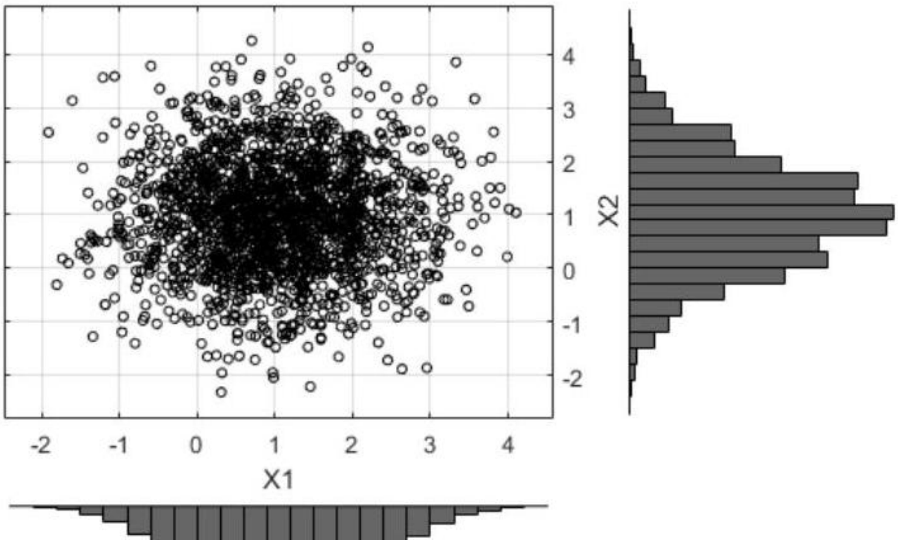
$$f_X = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} \exp(-\frac{1}{2}(x - \mu)^T \Sigma^{-1} (x - \mu))$$

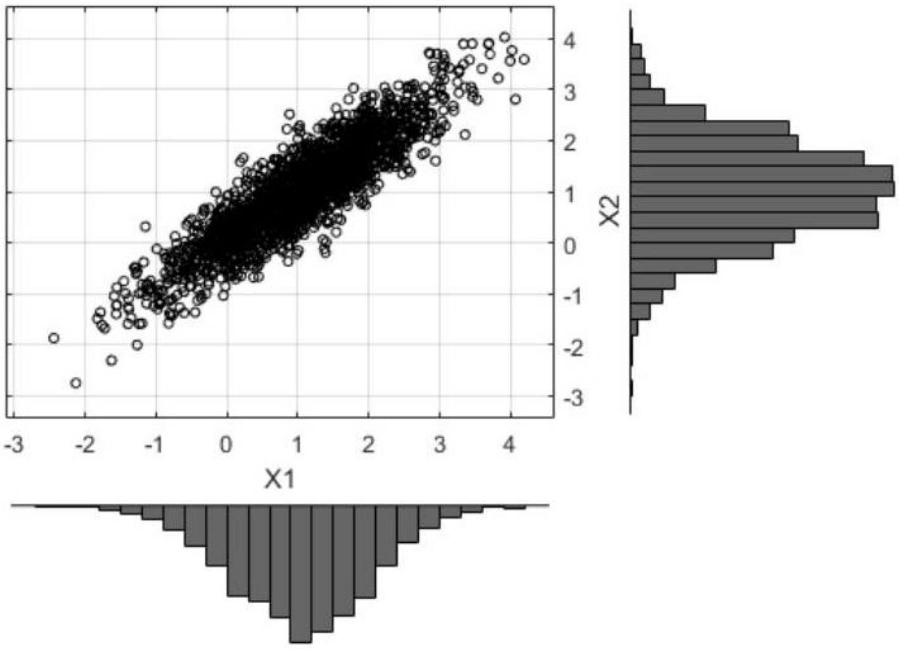
is said to have a multivariate Normal distribution, where
$\mu = (\mu_1, \ldots, \mu_n)$ is the vector of means of $X_1, \ldots, X_n$
and the covariance matrix $\Sigma = [\mathrm{Cov}(X_i, X_j); 1 \leq i, j \leq n]$, which must be positive definite for a pdf to exist.

Note that the r.vs. $X_1, \ldots, X_n$ need not be independent.

Example: Independent normal r.vs.

$$\mu = (1, 1), \mathrm{Var}(X_1) = \mathrm{Var}(X_2) = 1, \mathrm{Cov}(X_1, X_2) = \mathrm{Cov}(X_2, X_1) = 0$$





Example: $\mathrm{P}(X < Y)$

Let X, Y be independent exponential random variables with parameters λ, μ respectively. What is the probability that $X < Y$?

Solution 1: The first way to solve this is directly:

$$
\begin{aligned}
\mathrm{P}(X < Y) &= \int_{x<y} f(x,y)\mathrm{d}x \ \mathrm{d}y = \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{y} f(x,y)\mathrm{d}x \ \mathrm{d}y \\
&= \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{y} f_X(x) f_Y(y)\mathrm{d}x \ \mathrm{d}y \quad \text{(by independence)} \\
&= \int_{y=-\infty}^{\infty} F_X(y) f_Y(y)\mathrm{d}y = \int_{0}^{\infty} (1 - e^{-\lambda y})\mu e^{-\mu y} \ \mathrm{d}y \\
&= 1 - \frac{\mu}{\lambda + \mu} = \frac{\lambda}{\lambda + \mu}
\end{aligned}
$$

Example: $\mathrm{P}(X < Y)$ with conditional probabilities

Solution 2: The second way is more intuitive to some:

$$
\begin{aligned}
\mathrm{P}(X<Y) &= \int_{y=-\infty}^{\infty}\int_{X=-\infty}^{y} f(x,y)\mathrm{d}x \ \mathrm{d}y \\
&= \int_{y=-\infty}^{\infty}\int_{X=-\infty}^{y} f_{X|Y}(x \mid y) f_Y(y)\mathrm{d}x \ \mathrm{d}y \\
&= \int_{y=-\infty}^{\infty} F_{X|Y}(y \mid y) f_Y(y)\mathrm{d}y \\
&= \int_{0}^{\infty} (1-e^{-\lambda y})\mu e^{-\mu y} \ \mathrm{d}y \text{ (by independence)} \\
&= 1-\frac{\mu}{\lambda+\mu} = \frac{\lambda}{\lambda+\mu}.
\end{aligned}
$$

Example: modelling climate (continued)

- If today is cold ($\pi_{02}=1$), will it be hot in two days from now?

$$
\pi_0 R^2 = [\ 0 \quad 1\ ][\begin{array}{cc} 0.688 & 0.312 \\ 0.625 & 0.375 \end{array}] = [\ \underbrace{0.625}_{P(X_2=1)}\ \ 0.375]
$$

- What is the long-term probability of hot and cold days?
  $\lim_{n\to+\infty} \pi_0 R^n = [\ \pi_{01} \quad \pi_{02}\ ][\begin{array}{cc} 0.667 & 0.333 \\ 0.667 & 0.333 \end{array}] = [2/3, 1/3] = \pi_\infty$
  which holds in this example for any choice of vector π_0.

Example: climate modelling revisited

The daily temperatures DTMC is irreducible and aperiodic with

$$
\pi_\infty = \pi_0 \lim_{n\to\infty} R^n = (2/3, 1/3)
$$

To see that $\sum_j \pi_{\infty,j} = 1$ is necessary, note that
$\pi_\infty[\begin{array}{cc} 3/4 & 1/4 \\ 1/2 & 1/2 \end{array}] = \pi_\infty \Rightarrow \pi_\infty[\begin{array}{cc} -1/4 & 1/4 \\ 1/2 & -1/2 \end{array}] = (0,0)$ (singular)
Now, replacing an equation with $\sum_j \pi_{\infty,j} = 1$ we get instead

$$
\pi_\infty[\begin{array}{cc} -1/4 & 1 \\ 1/2 & 1 \end{array}] = (0,1) \Rightarrow \pi_\infty = (2/3, 1/3)
$$

Moreover, it is also $\pi^*_\infty = \pi_\infty = (2/3, 1/3)$.