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Gr. 212

Lucrare la algebră

① a) O relație de echivalență = o preordine, pe o mulțime A, care este și simetrică

Funcție injectivă = o funcție $f : A \to B$, astfel încât $\forall x_1, x_2 \in A$, $x_1 \neq x_2$ implică $f(x_1) \neq f(x_2)$

Subinel = o submulțime S a lui R, cu $(R, +, \cdot)$ inel, cu proprietatea că $\forall x, y \in S$, implică $x \cdot y$ și $x + y \in S$, iar cu operațiile induse S formează un inel

Liniar independentă = o listă de vectori $v = [v_1, v_2, \ldots, v_n]^t \in V^{n \times 1}$, V un K-spațiu vectorial care satisface: $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0$ implică $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$, cu $\alpha_1, \alpha_2, \ldots, \alpha_n$ scalari $\in K$.

(h) Submulțime a unei mulțimi ordonate care nu are infimum: $G \subseteq \mathbb{Z}$, $G = (-\infty, 0]$ în $(\mathbb{Z}, \leq)$

Element de ordin 2 în $\mathbb{R}^*$: -1 $((-1)^2 = 1)$

(2) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 2x-1, & x \in (-\infty, -1] \\ x-2, & x \in (-1, \infty) \end{cases}$

$g: \mathbb{R} \to (0, \infty)$, $g(x) = x^2 + x + 1$

a) f injectivă $\Leftrightarrow \forall x_1, x_2 \in \mathbb{R}, x_1 \neq x_2 \Rightarrow$
$\Rightarrow f(x_1) \neq f(x_2)$

Cazul $\underline{I}$: $x_1, x_2 \in (-\infty, -1]$, $x_1 \neq x_2$

$f(x_1) = 2x_1 - 1$ $\quad$ $2x_1 - 1 \neq 2x_2 - 1$

$f(x_2) = 2x_2 - 1$ $\quad$ $2x_1 \neq 2x_2$

$x_1 \neq x_2$ ✓

Cazul $\underline{II}$: $x_1 \in (-\infty, -1]$, $x_2 \in (-1, \infty)$

evident $x_1 \neq x_2$

$f(x_1) = 2x_1 - 1$ $\quad$ $2x_1 - 1 \neq x_2 - 2$

$f(x_2) = x_2 - 2$ $\quad$ $x_1 \neq \frac{x_2 - 1}{2}$

$-1 < x_2$

$-2 < x_2 - 1$ $\quad$ $x_1 \leq -1 < \frac{x_2 - 1}{2}$

$-1 < \frac{x_2 - 1}{2}$ $\quad$ $\Rightarrow x_1 \neq \frac{x_2 - 1}{2}$ ✓

Cazul $\underline{III}$: $x_1, x_2 \in (-1, \infty)$, $x_1 \neq x_2$

$f(x_1) = x_1 - 2$ $\quad$ $x_1 - 2 \neq x_2 - 2$

$f(x_2) = x_2 - 2$ $\quad$ $x_1 \neq x_2$ ✓

$\Rightarrow f$ injectivă

f surjectivă $\Leftrightarrow$ $\forall\, y \in \mathbb{R}$, $\exists\, x \in \mathbb{R}$ a.î.

$y = f(x)$ $\qquad f(x) = \begin{cases} 2x-1, & x \in (-\infty, -1] \\ x-2, & x \in (-1, \infty) \end{cases}$

$y = 2x - 1 \Rightarrow x = \dfrac{y+1}{2} \leq -1 \Rightarrow$

$\Rightarrow y \leq -3 \Rightarrow y \in (-\infty, -3]$

$y = x - 2 \Rightarrow x = y + 2 > -1 \Rightarrow y \in (-3, \infty)$

$y > -3$

$\Rightarrow \forall\, y \in \mathbb{R}$ ecuația $y = f(x)$ are loc

$\Rightarrow f$ surjectivă

g injectivă $\Leftrightarrow \forall\, x_1, x_2 \in \mathbb{R}$, $x_1 \neq x_2 \Rightarrow$

$\Rightarrow g(x_1) \neq g(x_2)$

Observăm: $g(1) = 1^2 + 1 + 1 = 3$

$g(-2) = (-2)^2 - 2 + 1 = 4 - 1 = 3$

$g(1) = g(-2)$

$1 \neq -2$

Deci $g \neq$ injectivă

g surjectivă $\Leftrightarrow \forall\, y \in (0, \infty)$, $\exists\, x \in \mathbb{R}$ a.î.

$y = g(x)$

Observăm g parabolă cu $a x^2$, $a=1>0 \Rightarrow$

$\Rightarrow$ parabolă cu vârful în jos (1)

$$g\left(-\frac{b}{2a}\right) = g\left(-\frac{1}{2}\right) = \frac{1}{4} - \frac{1}{2} + 1 = \frac{1-2+4}{4} =$$

$$= \frac{3}{4} \quad (2)$$

(1), (2) $\Rightarrow \forall\ y = g(x),\ y \geq \frac{3}{4}$

Observăm $y = \frac{1}{2} \in (0, \infty)$, $\nexists\, x$ a.î. $g(x) = \frac{1}{2}$

Deci $g \neq$ surjectivă

b) f bijectivă $\Rightarrow \exists\ f^{-1}(y): \mathbb{R} \to \mathbb{R} = h(y)$

$$h(y) = \begin{cases} \frac{y+1}{2}, & y \in (-\infty, -3] \\ y+2, & y \in (-3, \infty) \end{cases}$$

c)
$$\begin{tikzcd} \mathbb{R} \arrow[r, "f"] \arrow[dr, "g\circ f"'] & \mathbb{R} \arrow[d, "g"] \\ & (0,\infty) \end{tikzcd}$$

$$g \circ f(x) =$$

$$= g(f(x)) =$$

$$= \begin{cases} g(2x-1), & x \in (-\infty, -1] \\ g(x-2), & x \in (-1, \infty) \end{cases}$$

$$= \begin{cases} (2x-1)^2 + 2x - 1 + 1, & x \in (-\infty, -1] \\ (x-2)^2 + x - 2 + 1, & x \in (-1, \infty) \end{cases} =$$

$$= \begin{cases} 4x^2 - 2x + 1, & x \in (-\infty, -1] \\ x^2 - 3x + 3, & x \in (-1, \infty) \end{cases}$$

$$\begin{tikzcd} \mathbb{R} \arrow[r, "g"] \arrow[drr, "\cancel{f \circ g}"'] & (0, \infty) \neq & \mathbb{R} \arrow[d, "f"] \\ & & \mathbb{R} \end{tikzcd}$$

nu are loc $\left((0, \infty) \neq \mathbb{R} \right)$

d) $h_1 : [0, \infty) \to [0, 1]$

$h_1(x) = 1, \ \forall x \in [0, \infty)$

$$h_2(x) = \begin{cases} 0, & \forall x \in [0, \frac{1}{2}) \\ 1, & \forall x \in (\frac{1}{2}, \infty) \end{cases}$$

$h_2 : [0, \infty) \to [0, 1]$

$h_1 \circ g = h_2 \circ g$

$h_1(g(x)) = h_1(x^2 + x + 1) = 1 =$

$= h_2(\underbrace{x^2 + x + 1}_{> \frac{1}{2}, \ \forall x \in \mathbb{R}}) = h_2(g(x))$ ✓

(3) $S = \left\{ \begin{bmatrix} a & b \\ 3b & a \end{bmatrix} / a, b \in \mathbb{Z} \right\}$

a) $(S, +, \cdot) \leq (M_{2\times2}(\mathbb{Z}), +, \cdot)$

i) $0 \in S$ $\quad 0 = \begin{pmatrix} a & b \\ 3b & a \end{pmatrix}$ cu $a = 0 \in \mathbb{Z}$ ✓
$b = 0 \in \mathbb{Z}$

deci $0 \in S$

ii) $x, y \in S \Rightarrow x + y \in S$

$x = \begin{pmatrix} a_1 & b_1 \\ 3b_1 & a_1 \end{pmatrix} \quad y = \begin{pmatrix} a_2 & b_2 \\ 3b_2 & a_2 \end{pmatrix}$

$x + y = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ 3(b_1 + b_2) & a_1 + a_2 \end{pmatrix}$ $\quad \underset{\in \mathbb{Z}}{a_1} + \underset{\in \mathbb{Z}}{a_2} \in \mathbb{Z}$

$3(\underset{\in \mathbb{Z}}{b_1} + \underset{\in \mathbb{Z}}{b_2}) \in \mathbb{Z}$ ✓

deci $\forall x, y \in S \Rightarrow x + y \in S$

iii) $x \in S \Rightarrow -x \in S$

$x = \begin{pmatrix} a & b \\ 3b & a \end{pmatrix} \in S$

$-x = -\begin{pmatrix} a & b \\ 3b & a \end{pmatrix} = \begin{pmatrix} -a & b \\ -3b & -a \end{pmatrix}$

$-a \in \mathbb{Z}$

$-3b \in \mathbb{Z}$

deci $\forall x \in S \Rightarrow -x \in S$

iv) $x, y \in S \Rightarrow x y \in S$

$$x = \begin{pmatrix} a & b \\ 3b & a \end{pmatrix} \qquad y = \begin{pmatrix} e & f \\ 3f & e \end{pmatrix}$$

$$x \cdot y = \begin{pmatrix} a & b \\ 3b & a \end{pmatrix} \begin{pmatrix} e & f \\ 3f & e \end{pmatrix} = \begin{pmatrix} ae + 3bf & af + be \\ 3be + 3af & 3bf + ae \end{pmatrix}$$

$a, b, e, f \in \mathbb{Z} \Rightarrow ae + 3bf \in \mathbb{Z}$ ✓

$3(af + be) \in \mathbb{Z}$

deci $\forall x, y \in S \Rightarrow x y \in S$

i) → iv) $\Rightarrow$ S subinel al inelului $(M_{2\times 2}(\mathbb{Z}), +, \cdot)$

I_2 plus, $1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ 3b & b \end{pmatrix}$, cu $a = 1$, $b = 0$

apartine S, deci S - subinel cu unitate

(4) $S = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 - x_2 + 2x_3 = 0 \}$

a)

i) $0 \in S$

$0 = [0, 0, 0]$

$0 - 0 + 2 0 = 0 \Rightarrow 0 \in S$

ii) $x, y \in S \Rightarrow \alpha x + \beta y \in S$

$x \in S \Rightarrow x = (x_1, x_2, x_3)$

$y \in S \Rightarrow y = (y_1, y_2, y_3)$

$\alpha x + \beta y = (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2,$

$, \alpha x_3 + \beta y_3) \in S \Leftrightarrow$

$\Leftrightarrow \alpha x_1 + \beta y_1 - \alpha x_2 - \beta y_2 + 2 \alpha x_3 +$

$+ 2 \beta y_3 = 0$

$\alpha (x_1 - x_2 + 2x_3) + \beta (y_1 - y_2 + 2y_3) = 0$

$\alpha \cdot 0 + \beta \cdot 0 = 0$

$0 = 0$ ✓

Deci $S \leq_{\mathbb{R}} \mathbb{R}^3$

b) $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 - x_2 + 2x_3 = 0\}$

$x_1 - x_2 + 2x_3 = 0$ $\quad$ $x_3 = \alpha \quad , \alpha, \beta \in \mathbb{R}$

$x_1 = \beta$

$x_2 = 2\alpha + \beta$

$S = \{(\beta, 2\alpha + \beta, \alpha) \mid \alpha, \beta \in \mathbb{R}^3\} =$

$= \{\alpha(0, 2, 1) + \beta(1, 1, 0) \mid \alpha, \beta \in \mathbb{R}\} =$

$= \langle (0, 2, 1), (1, 1, 0) \rangle$

	S_1	S_2
e_1	0	1
e_2	2	1
e_3	1	0
S_2	0	1
e_2	2	0
e_3	1	0
S_2	0	1
e_2	0	0
S_1	1	0

$B_S = \{s_2, s_1\}$

$\dim B_S = 2$

	t_1	t_2
e_1	2	1
e_2	1	0
e_3	1	-1
t_2	2	1
e_2	1	0
e_3	-1	0
t_2	0	1
t_1	1	0
e_3	0	0

$B_T = \langle t_1, t_2 \rangle$

$\dim B_T = 2$

$S+T$	s_1	s_2	t_1	t_2
e_1	0	1	2	1
e_2	2	1	1	0
e_3	1	0	1	-1
s_2	0	1	2	1
e_2	2	0	-1	-1
e_3	1	0	1	0
s_2	0	1	2	1
e_2	0	0	-3	-1
s_1	1	0	1	0
s_2				0
t_2				1
s_1				0

(5) $f: \mathbb{R}^3 \to \mathbb{R}^2$, $f(x_1, x_2, x_3) =$

$= (x_1 - 2x_2 + x_3, 2x_1 - x_2 - x_3)$

a) $f \in Hom_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2) \Leftrightarrow$

$\Leftrightarrow f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$,

$\forall x, y \in \mathbb{R}^3$, $\alpha, \beta \in \mathbb{R}$

$f(\alpha x + \beta y) = f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2,$
$\alpha x_3 + \beta y_3) = (\alpha x_1 + \beta y_1 - 2\alpha x_2 - 2\beta y_2 +$
$+ \alpha x_3 + \beta y_3, 2\alpha x_1 + 2\beta y_1 - \alpha x_2 -$
$- \beta y_2 - \alpha x_3 - \beta y_3)$

$\alpha f(x) + \beta f(y) = \alpha(x_1 - 2x_2 + x_3,$
$2x_1 - x_2 - x_3) + \beta(y_1 - 2y_2 + y_3,$
$2y_1 - y_2 - y_3) = (\alpha x_1 + \beta y_1 - 2\alpha x_2 -$
$- 2\beta y_2 + \alpha x_3 + \beta y_3, 2\alpha x_1 + 2\beta y_1 -$
$- \alpha x_2 - \beta y_2 - \alpha x_3 - \beta y_3) = f(\alpha x + \beta y)$,
$\forall x, y \in \mathbb{R}^3$, $\alpha, \beta \in \mathbb{R}$ ✓

$\Rightarrow f \in Hom_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2$

h) $\text{Ker } f = \{ x \in \mathbb{R}^3 \ / \ f(x) = 0 \}$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_1 - x_2 - x_3 = 0 \end{cases} \qquad x_3 = \alpha \in \mathbb{R}$$

$$x_1 = 2x_2 - \alpha$$

$$4x_2 - 2\alpha - x_2 = \alpha$$

$$3x_2 = 3\alpha$$

$$x_2 = \alpha$$

$$x_1 = \alpha \Rightarrow \text{Ker } f = \{ [\alpha, \alpha, \alpha] \ / \ \alpha \in \mathbb{R}^3 \}$$

$$= \langle 1, 1, 1 \rangle$$

$$\dim = 1$$

$$\dim \text{Im } f = \dim \mathbb{R}^3 - \dim \text{Ker } f =$$

$$= 3 - 1 = 2$$