$\boxed{3.2.35.}$ $f: V \to W$ aplicație liniară

$X \subseteq V$

$f(\langle X \rangle) = \langle f(X) \rangle$

$f(\langle X \rangle) = \{ f(\alpha_1 x_1 + \dots + \alpha_n x_n) \mid n \in \mathbb{N}, x_1, \dots, x_n \in X, \alpha_1, \dots, \alpha_n \in K \}$

$f(X) = \{ f(x_i) \mid x_i \in X \}$

$\Rightarrow \langle f(X) \rangle = \{ \alpha_1 f(x_1) + \dots + \alpha_n f(x_n) \mid x_1, \dots, x_n \in X, \alpha_1, \dots, \alpha_n \in K, n \in \mathbb{N} \}$

$P(n): f(\alpha_1 x_1 + \dots + \alpha_n x_n) = \alpha_1 f(x_1) + \dots + \alpha_n f(x_n)$

$P(1): f(\alpha_1 x_1) = \alpha_1 f(x_1)$ (Definiția aplicației liniare)

fie $P(k)$ Adevărată $\Rightarrow f(\alpha_1 x_1 + \dots + \alpha_k x_k) = \alpha_1 f(x_1) + \dots + \alpha_k f(x_k)$

$P(k+1): f(\alpha_1 x_1 + \dots + \alpha_k x_k + \alpha_{k+1} x_{k+1}) = f(\alpha_1 x_1 + \dots + \alpha_k x_k) + f(\alpha_{k+1} x_{k+1}) =$

$= \alpha_1 f(x_1) + \dots + \alpha_k f(x_k) + \alpha_{k+1} f(x_{k+1})$

$\overset{PIM}{\Longrightarrow}$ $P(n)$ Adevărată $\Rightarrow f(\langle X \rangle) = \langle f(X) \rangle$

3.2.38. $f: V \to W$, $v = [v_1, \dots, v_n]^t \in V^{n \times 1}$

$$f(v) = [f(v_1), \dots, f(v_n)]^t \in W^{n \times 1}$$

a) f injectivă, v lin. ind. $\Rightarrow \alpha_1 v_1 + \dots + \alpha_n v_n = 0 \Leftrightarrow \alpha_1 = \dots = \alpha_n = 0$

$$f(\alpha_1 v_1 + \dots + \alpha_n v_n) = f(\alpha_1 v_1) + f(\alpha_2 v_2 + \dots + \alpha_n v_n) = \dots =$$

$$= f(\alpha_1 v_1) + \dots + f(\alpha_n v_n) = \alpha_1 f(v_1) + \dots + \alpha_n f(v_n)$$

$f: V \to W$ morfism de grupuri $\to f(0) = 0$, f injectivă $\Rightarrow f(0) = 0$ val. unică

$\Rightarrow$ dacă $\alpha_1 v_1 + \dots + \alpha_n v_n = 0 \Rightarrow f(\alpha_1 v_1 + \dots + \alpha_n v_n) = 0$

$\Rightarrow \alpha_1 f(v_1) + \dots + \alpha_n f(v_n) = 0 \Leftrightarrow \alpha_1 = \dots = \alpha_n = 0 \Rightarrow$ liniar independentă

b) f surjectivă, $\langle v \rangle = V \Rightarrow \underline{\langle f(v) \rangle = W}$

f surjectivă $\Rightarrow f(V) = W \Rightarrow f(\langle v \rangle) = W$

din 3.2.35 $\Rightarrow f(\langle v \rangle) = \langle f(v) \rangle \Rightarrow \langle f(v) \rangle = W$ c.c.t.d.

c) f bijectivă și v bază a lui $V \Rightarrow \underline{f(v) \text{ bază a lui } W}$

v bază $\Rightarrow v$ liniar independentă $\overset{a)}{\Rightarrow} f(v)$ liniar independentă (1)

$\Rightarrow \langle v \rangle = V \overset{b)}{\Rightarrow} \langle f(v) \rangle = W$ (2)

$\overset{(1)(2)}{\Longrightarrow} f(v)$ e bază a lui W