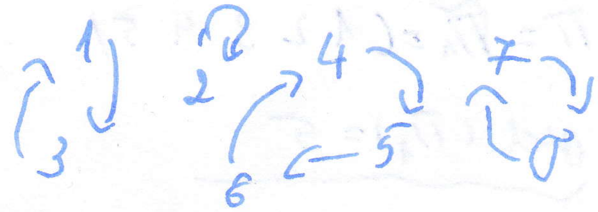
2.1.74. $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 2 & 1 & 5 & 6 & 4 & 8 & 7 \end{pmatrix} \in S_8$

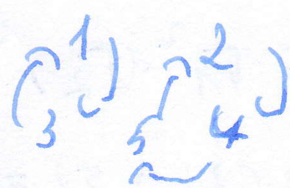
$\sigma = (1\ 3)(4\ 5\ 6)(7\ 8)$



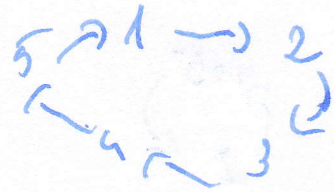
2.1.75. $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix} \in S_5$

$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix} \in S_5$

a) $\sigma = (1\ 3)(2\ 4\ 5)$



$\pi = (1\ 2\ 3\ 4\ 5)$



b) $\sigma \cdot \pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 5 & 2 & 3 \end{pmatrix}$

$\pi \cdot \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 2 & 1 & 3 \end{pmatrix}$

$\sigma^{-1} = \begin{pmatrix} 3 & 4 & 1 & 5 & 2 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 2 & 4 \end{pmatrix}$

$\pi^{-1} = \begin{pmatrix} 2 & 3 & 4 & 5 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}$

c) $\mathrm{ord}(\sigma) =$ c.m.m.m.c $(2, 3) = 6$

$\sigma = \sigma_1 \sigma_2 = (1\ 3)(2\ 4\ 5)$ cicli disjuncți

$\mathrm{ord}(\sigma_1) = 2 \quad \mathrm{ord}(\sigma_2) = 3 \quad (2, 3) = 1$

$\operatorname{ord}(\pi) = 5$

$\pi = \pi_1 = (1\ 2\ 3\ 4\ 5)$

$\operatorname{ord}(\pi_1) = 5$

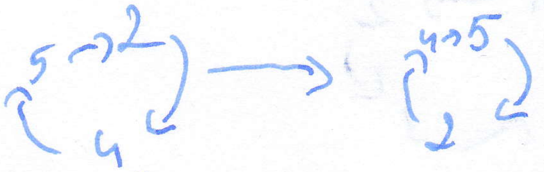
$\langle \sigma \rangle = \{\sigma, \sigma^2, \sigma^3, \sigma^4, \sigma^5, e\}$

$\sigma = \sigma_1 \cdot \sigma_2 = (1\ 3)(2\ 4\ 5) = \sigma_2 \cdot \sigma_1$

$$\begin{cases} \operatorname{ord}(\sigma_1) = 2 \Rightarrow \sigma_1^2 = e \\ \operatorname{ord}(\sigma_2) = 3 \Rightarrow \sigma_2^3 = e. \end{cases}$$

$$\sigma^2 = (\sigma_1 \cdot \sigma_2)^2 = \sigma_1^2 \cdot (\sigma_2)^2 = (\sigma_2)^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 3 & 2 & 4 \end{pmatrix}$$

$$(2\ 4\ 5)(2\ 4\ 5) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 5 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 5 & 2 \end{pmatrix}$$



$\sigma(2) = 4$ $\quad$ $\sigma(2) = 5$

$\sigma(4) = 5$ $\quad$ $\sigma(4) = 2$

$\sigma(5) = 2$ $\quad$ $\sigma(5) = 4.$

$$\sigma^3 = (\sigma_1)^3 \cdot (\sigma_2)^3 = \sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 4 & 5 \end{pmatrix}$$

$$\sigma^4 = (\sigma_1)^4 (\sigma_2)^4 = \sigma_2 = (2\ 4\ 5) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 5 & 2 \end{pmatrix}$$

$(\sigma_2)^4 = \sigma_2$

$$\sigma^5 = \sigma_1^5 \cdot \sigma_2^5 = \sigma_1 \cdot \sigma_2^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 4 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 3 & 2 & 4 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 2 & 4 \end{pmatrix}$$

d) $\varepsilon(\overline{\sigma}) = (-1)^{m(\overline{\sigma})} = (-1)^5 = (-1)$

$m(\overline{\sigma}) = 2 + 2 + 1 = 5$

$\varepsilon(\overline{\pi}) = (-1)^{m(\overline{\pi})} = (-1)^4 = 1.$

$m(\overline{\pi}) = 1 + 1 + 1 + 1 = 4$

2.2.39. Să se rez în $\mathbb{Z}_6$

a) $\hat{4}x + \hat{5} = \hat{1} \,/ + \hat{1} \Rightarrow \hat{4}\hat{x} = \hat{2}$ gcd$(4, 6) \neq 1$

$\Downarrow$

fac tabelul de valori pt $\hat{4}x$

x	$\hat{0}$	$\hat{1}$	$\hat{2}$	$\hat{3}$	$\hat{4}$	$\hat{5}$
$\hat{4}x$	$\hat{0}$	$\hat{4}$	$\hat{2}$	$\hat{0}$	$\hat{4}$	$\hat{2}$

$\Rightarrow x \in \{\hat{2}, \hat{5}\}$

b) $\hat{5}x + \hat{3} = \hat{1} \,/ + \hat{3} \Rightarrow \hat{5}x = \hat{4}$ gcd$(5, 6) = 1 \Rightarrow \hat{5}$ admite simetric în rap cu $\cdot$ în $\mathbb{Z}_6$

$\hat{5}^{-1} = \hat{5} \Rightarrow \hat{5}x = \hat{4} \,/ \cdot \hat{5} \Rightarrow x \cdot \hat{5} \cdot \hat{5} = \hat{4} \cdot \hat{5} \Rightarrow x = \hat{2}$

2.2.40. a) $\mathbb{Z}+i\mathbb{Z}=\{a+ib \mid a,b\in\mathbb{Z}\}$ subinel $(\mathbb{C},+,\cdot)$

(1) $\mathbb{Z}+i\mathbb{Z}\subseteq\mathbb{C}$ (A)

(2) $0\in\mathbb{Z}+i\mathbb{Z}$

$$\left.\begin{array}{l} 0\in\mathbb{Z}+i\mathbb{Z} \\ 0=0+i\cdot 0,\ 0\in\mathbb{Z}\end{array}\right| \Rightarrow 0\in\mathbb{Z}+i\mathbb{Z}$$

(3) $z_1,z_2\in\mathbb{Z}+i\mathbb{Z}\Rightarrow z_1+z_2\in\mathbb{Z}+i\mathbb{Z}$

$$\left.\begin{array}{l} z_1=a+ib \\ z_2=c+id \\ a,b,c,d\in\mathbb{Z}\end{array}\right| \Rightarrow z_1+z_2=(a+c)+i(b+d)=\left.\begin{array}{l}u+iv \\ u,v\in\mathbb{Z}\end{array}\right|\Rightarrow$$

$z_1+z_2\in\mathbb{Z}+i\mathbb{Z}$

(4) $z\in\mathbb{Z}+i\mathbb{Z}\Rightarrow -z\in\mathbb{Z}+i\mathbb{Z}$

$$\begin{array}{l} z=a+ib\Rightarrow \\ a,b\in\mathbb{Z}\end{array}\ \left.\begin{array}{l} -z=-a-ib=(-a)+i\cdot(-b) \\ \Rightarrow -a,-b\in\mathbb{Z}\end{array}\right|\Rightarrow -z\in\mathbb{Z}+i\mathbb{Z}$$

(5) $z_1,z_2\in\mathbb{Z}+i\mathbb{Z}\Rightarrow z_1\cdot z_2\in\mathbb{Z}+i\mathbb{Z}$

$$\left.\begin{array}{l} z_1=a+ib \\ z_2=c+id \\ a,b,c,d\in\mathbb{Z}\end{array}\right| \Rightarrow z_1\cdot z_2=ac+iad+ibc+i^2bd=$$

$$=\underbrace{(ac-bd)}_{u\in\mathbb{Z}}+i\,(\underbrace{ad+bc}_{v\in\mathbb{Z}})=u+iv$$

Din (1),(2),(3),(4),(5) $\Rightarrow(\mathbb{Z}+i\mathbb{Z},+,\cdot)$ subinel $(\mathbb{C},+,\cdot)$

b) $R = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \middle/ a, b \in \mathbb{R} \right\}$ subinel$(M_{2\times 2}(\mathbb{R}), +, \cdot)$

(1) $R \subseteq M_{2\times 2}$ (A)

(2) $O_2 \in R \Leftarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in R \Leftarrow \begin{matrix} a = 0 \\ b = 0 \end{matrix}$

(3) $A, B \in R \Rightarrow A + B \in R$

$$\left. \begin{matrix} A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \\ B = \begin{pmatrix} c & d \\ -d & c \end{pmatrix} \end{matrix} \right| \Rightarrow A + B = \begin{pmatrix} \overset{\mathbb{R}}{\downarrow} & \overset{\mathbb{R}}{\downarrow} \\ a+c & b+d \\ -(b+d) & a+c \end{pmatrix} \in R$$

(4) $A \in R \Rightarrow -A \in R$

$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \Rightarrow -A = -\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} -a & -b \\ b & -a \end{pmatrix} = \begin{pmatrix} u & v \\ -v & u \end{pmatrix} \in R$$

$$\left. \begin{matrix} u = -a \\ v = -b \end{matrix} \right] \in \mathbb{R}$$

(5) $A, B \in R \Rightarrow A \cdot B \in R.$

$$\left. \begin{matrix} A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \\ B = \begin{pmatrix} c & d \\ -d & c \end{pmatrix} \end{matrix} \right| \Rightarrow A \cdot B = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \begin{pmatrix} \overbrace{ac - bd}^{u \in \mathbb{R}} & \overbrace{ad + bc}^{v \in \mathbb{R}} \\ -(ad+bc) & ac - bd \end{pmatrix}$$

$$= \begin{pmatrix} u & v \\ -v & u \end{pmatrix} \Rightarrow A \cdot B \in R.$$

Din (1), (2), (3), (4), (5) $\Rightarrow (R, +, \cdot)$ subinel. al lui $(M_{2\times 2}(\mathbb{R}), +, \cdot)$

c) $R \cong \mathbb{Z}+i\mathbb{Z}$ (- inelele sunt izomorfe)

Fie $f: \mathbb{Z}+i\mathbb{Z} \to R$, $f(z) = \begin{pmatrix} Re(z) & Im(z) \\ -Im(z) & Re(z) \end{pmatrix}$

Vom considera $z = a+ib$, $a,b \in \mathbb{Z} \Rightarrow f(z) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \Rightarrow$

$f(a+ib) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$

(1) $f(z_1+z_2) = f(z_1)+f(z_2)$

$M_S = f(z_1+z_2) = f((a+c)+i(b+d)) = \begin{pmatrix} a+c & b+d \\ -(b+d) & a+c \end{pmatrix}$

$z_1 = a+ib$

$z_2 = c+id$

$M_d = f(z_1)+f(z_2) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} + \begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ -(b+d) & a+c \end{pmatrix}$ $\Rightarrow$

$M_S = M_d$. (1)

(2) $f(z_1 \cdot z_2) = f(z_1) \cdot f(z_2)$

$M_S\ f(z_1 \cdot z_2) \overset{(a)5}{=\!=} f((ac-bd)+i(ad+bc)) = \begin{pmatrix} ac-bd & ad+bc \\ -(ad+bc) & ac-bd \end{pmatrix}$

$M_D\ f(z_1) \cdot f(z_2) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} c & d \\ -d & c \end{pmatrix} \overset{b{-}5}{=\!=} \begin{pmatrix} ac-bd & ad+bc \\ -(ad+bc) & ac-bd \end{pmatrix}$

$M_S = M_D$ (2)

(2) + (1) $\Rightarrow$ f morfism.

Dem că f bijectivă.

(1) inj. $\forall z_1, z_2 \in \mathbb{Z}+i\mathbb{Z}$, $f(z_1) = f(z_2) \Rightarrow z_1 = z_2$.

$f(z_1) = f(z_2) \Leftrightarrow \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} c & d \\ -d & c \end{pmatrix} \Rightarrow \begin{cases} a = c \\ b = d \end{cases}$, dar

$z_1 = a+ib = c+id = z_2 \Rightarrow$ f inj.

(2) f surj

$\forall A \in R \Rightarrow \exists z_1 \in \mathbb{Z}+i\mathbb{Z}$ a.î. $f(z_1)=A$

Fie $A=\begin{pmatrix} c & d \\ -d & c \end{pmatrix} \Rightarrow f(z_1)=\begin{pmatrix} c & d \\ -d & c \end{pmatrix} \Leftrightarrow \begin{pmatrix} a & b \\ -b & a \end{pmatrix}=\begin{pmatrix} c & d \\ -d & c \end{pmatrix} \Rightarrow$

$z_1=a+ib$

$\begin{cases} a=c \\ b=d \end{cases} \Rightarrow \exists z_1=c+id, \forall A=\begin{pmatrix} c & d \\ -d & c \end{pmatrix} \in R.$

Din $(1)+(2) \Rightarrow$ f bij + f morfism $\Big| \Rightarrow$ f izomorfism $\Rightarrow \mathbb{Z}+i\mathbb{Z} \simeq R$.

(c) Sunt $\mathbb{Z}+i\mathbb{Z}$ și/sau R dom. de integritate.

Pp. că $\exists z_1, z_2 \in \mathbb{Z}+i\mathbb{Z}$, $z_1 \neq 0$ ~~și~~ sau $z_2 \neq 0$ | a.î. $z_1 \cdot z_2 = 0$.

$z_1 \cdot z_2 = (ac-bd)+i(ad+bc)=0+i\cdot 0$

Rezolv sistemul $\begin{cases} ac-bd=0 \\ ad+bc=0 \end{cases} \Rightarrow M=\left(\begin{array}{cc|c} a & -b & 0 \\ b & a & 0 \end{array}\right)$

Fie z_1 arbitrar fixat

$\Downarrow$

necunoscutele sunt c și d

$\det M = a^2+b^2 \neq 0, \forall a,b \in \mathbb{Z} \xrightarrow{\text{Cramer}}$ sistemul are soluție unică (1)

Dar sistemul este unul omogen (termenii liberi $=0$) (2)

Din (1) și (2) $\Rightarrow$ soluția unică e cea banală (0,0)

$\Rightarrow c=0, d=0 \Rightarrow z_2=0$ (absurd)

Pp. fiind falsă $\Rightarrow \nexists$ divizori ai lui 0 $\Rightarrow \mathbb{Z}+i\mathbb{Z}$ domeniu de integritate $\xrightarrow[\text{izomorf}]{\text{e}}$ R domeniu de int.