Lab 4 - notite

Wednesday, March 22, 2023 12:33 AM

Rezolvarea sistemelor lineare

Sisteme liniare triunghiulare → *backward/forward substitution*

$$
\begin{cases}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots \quad \cdots \quad \cdots + a_{1n}x_n = b_1 \\
\qquad a_{22}x_2 + a_{23}x_3 + \cdots \quad \cdots \quad \cdots + a_{2n}x_n = b_2 \\
\qquad\qquad \ddots \\
\qquad\qquad a_{ii}x_i + \boxed{a_{i\,i+1}x_{i+1} + \cdots \quad \cdots + a_{in}x_n} = b_i \\
\qquad\qquad\qquad \ddots \\
\qquad\qquad\qquad\qquad a_{n-1\,n-1}x_{n-1} + a_{n-1\,n}x_n = b_{n-1} \\
\qquad\qquad\qquad\qquad\qquad a_{nn}x_n = b_n
\end{cases}
$$

Sisteme lineare compatibile determinate

$$
\begin{cases}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
\vdots \\
a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \\
\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n
\end{cases}
\Leftrightarrow
\begin{bmatrix}
a_{11} & a_{12} & \dots & a_{1n} \\
a_{21} & a_{22} & \dots & a_{2n} \\
\vdots & \vdots & \dots & \vdots \\
a_{i1} & a_{i2} & \dots & a_{in} \\
\vdots & \vdots & \vdots & \vdots \\
a_{n1} & a_{n2} & \dots & a_{nn}
\end{bmatrix}
\cdot
\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix}
$$

$$A \text{ inversabilă } (\det A \neq 0),\ A \cdot x = b \ \Rightarrow\ x = A^{-1} \cdot b.$$

Eliminarea gaussiană naivă

$$
\begin{bmatrix}
a_{11} & a_{12} & \dots & a_{1k} & \dots & a_{1j} & \dots & a_{1n} \\
0 & a_{22} & \dots & a_{2k} & \dots & a_{2j} & \dots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots & \dots & \vdots & \dots & \vdots \\
0 & a_{k2} & \dots & a_{kk} & \dots & a_{kj} & \dots & a_{kn} \\
\vdots & \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\
0 & a_{i2} & \dots & a_{ik} & \dots & a_{ij} & \dots & a_{in} \\
\vdots & \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\
0 & a_{n2} & \dots & a_{nk} & \dots & a_{nj} & \dots & a_{nn}
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix}
$$

$$
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1k} & \cdots & a_{1j} & \cdots & a_{1n} \\
0 & a_{22} & \cdots & a_{2k} & \cdots & a_{2j} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & a_{kk} & \cdots & a_{kj} & \cdots & a_{kn} \\
\vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & a_{ik} & \cdots & a_{ij} & \cdots & a_{in} \\
\vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & a_{nk} & \cdots & a_{nj} & \cdots & a_{nn}
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix}
$$

$$
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1k} & \cdots & a_{1j} & \cdots & a_{1n} \\
0 & a_{22} & \cdots & a_{2k} & \cdots & a_{2j} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & a_{kk} & \cdots & a_{kj} & \cdots & a_{kn} \\
\vdots & \vdots & \cdots & \vdots & \ddots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & a_{jj} & \cdots & a_{jn} \\
\vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & a_{nn}
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix}
$$

Exemplu (în care eliminarea gaussiană naivă nu funcționează):

$$
\begin{cases} \varepsilon x_1 + x_2 = 1 \\ x_1 + x_2 = 2 \end{cases}
\Rightarrow
\begin{cases} \varepsilon x_1 + x_2 = 1 \\ (1 - \varepsilon^{-1}) x_2 = 2 - \varepsilon^{-1} \end{cases}
\Rightarrow x_2 = \frac{2 - \varepsilon^{-1}}{1 - \varepsilon^{-1}} \approx 1,\ x_1 = \frac{1 - x_2}{\varepsilon} \approx 0
$$

$$
\begin{cases} x_1 + x_2 = 2 \\ \varepsilon x_1 + x_2 = 1 \end{cases}
\Rightarrow
\begin{cases} x_1 + x_2 = 2 \\ (1 - \varepsilon) x_2 = 1 - 2\varepsilon \end{cases}
\Rightarrow x_2 = \frac{1 - 2\varepsilon}{1 - \varepsilon} \approx 1,\ \ x_1 = 2 - x_2 \approx 1
$$

Descompunerea L,U,P și alte descompuneri

$$L = \begin{bmatrix} 1 & & & & \\ \ell_{21} & 1 & & \text{\Huge 0} & \\ \ell_{31} & \ell_{32} & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \ell_{n1} & \ell_{n2} & \ell_{n3} & \cdots & 1 \end{bmatrix} \qquad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ & u_{22} & u_{23} & \cdots & u_{2n} \\ & & u_{33} & \cdots & u_{3n} \\ & \text{\Huge 0} & & \ddots & \vdots \\ & & & & u_{nn} \end{bmatrix}$$

$P = I_n$ cu liniile permutate

★ Descompunerea L,U,P: $P \cdot A = L \cdot U$

Rezolvarea sistemului $A \cdot x = b$ se reduce la rez. a două sisteme triunghiulare:

$$A \cdot x = b \Leftrightarrow (P \cdot A) \cdot x = P \cdot b \Leftrightarrow L \cdot U \cdot x = P \cdot b \Leftrightarrow L \cdot y = P \cdot b, U \cdot x = y$$

! Descompunerea *L,U,P* a matricei A se poate folosi pentru b-uri diferite în rezolvarea sistemului $A \cdot x = b \Rightarrow$ economie de memorie și timp de execuție.

★ Dacă A este **simetrică pozitiv definită**, atunci putem obține descompunerea Cholesky ($L = R^*, U = R, P = I_n$):

$$A = R^* \cdot R, \text{ unde } R \text{ este triunghiulară superior.}$$

Temă: De scris funcții (din curs) pt.:

1) descompunerea L U P
2) descompunerea Cholesky

} curs

3) rez. sist. lin. fol. L U P
4) rez. sis. lin. fol. Cholesky.