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Gr. 212

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Ex 1) $x'(t) = -k\,x(t)$

$$\frac{dx}{dt} = -k\,x$$

$$\int \frac{dx}{x} = \int -k\,dt$$

$$\ln x = -kt + c_1$$

$$x = c_2\,e^{-kt}$$

$x(0) = x_0 \Rightarrow c_2 = x_0$

$x(t) = x_0 \cdot e^{-kt}$ – sol. gen.

Din ipoteză, rezultă:

$$x(5) = 10 \cdot e^{-k\cdot 5} = 2$$

$$e^{-5k} = 0,2 \quad | \ln$$

$$-5k = \ln 0,2$$

$$k = \frac{\ln 0,2}{-5} = \frac{\ln 5^{-1}}{-5} = \frac{\ln 5}{5}$$

$T_{1/2}$ = timp de înjumătățire $= \dfrac{\ln 2}{k} = 5\dfrac{\ln 2}{\ln 5} \simeq$

$\simeq 2,15$ ani

Ex. 2 |

1) $y'' - 2y' + 2y = x e^x$

$r^2 - 2r + 2 = 0$

~~$r_{1,2} = \frac{2 \pm ...}{2}$~~

$\Delta = 4 - 8 = -4$

$r_{1,2} = \frac{2 \pm 2i}{2}$ $\begin{cases} r_1 = 1 + i \\ r_2 = 1 - i \end{cases}$ $\Rightarrow$ $\alpha = 1$, $\beta = 1$

$y_0(x) = c_1 e^x \cos x + c_2 e^x \sin x$, $c_1, c_2 \in \mathbb{R}$

$f(x) = x e^x \Rightarrow Q_n(x) = x$

$t = 1 \neq$ răd. a ec. caract.

$y_p = e^x (ax + b)$

$y_p' = e^x (ax+b) + a e^x$

$y_p'' = e^x (ax+b) + a e^x + a e^x =$

$= e^x (ax+b) + 2a e^x$

$e^x (ax+b) + \cancel{2ae^x} - \cancel{2 e^x (ax+b)} - \cancel{2ae^x} +$

$+ \cancel{2 e^x (ax+b)} = x e^x$

$e^x (ax+b) = x e^x \Rightarrow$ $a = 1$, $b = 0$

p2

$y_p = x e^{x}$

$\Rightarrow y(x) = y_0 + y_p = c_1 e^{x} \cos x + c_2 e^{x} \sin x +$

$+ x e^{x}, \quad c_1, c_2 \in \mathbb{R}$

Ex 3) $\begin{cases} y'' - \dfrac{2e^{2x}}{e^{2x}+1} \cdot y' = \dfrac{2e^{2x}}{e^{2x}+1} \\ y(0) = 2 \\ y'(0) = 3 \end{cases}$

Fie $z = y' \Rightarrow z' = y''$

$$z' = \frac{2e^{2x}}{e^{2x}+1}(1+z)$$

$$\frac{dz}{dx} = \frac{2e^{2x}}{e^{2x}+1}(1+z) \quad \text{FVS}$$

i: $1 + z \neq 0$

$$\int \frac{dz}{z+1} = \int \frac{2e^{2x}}{e^{2x}+1}\, dx$$

$$\ln|z+1| = \ln(e^{2x}+1) + c$$

$$|z+1| = c_1 \cdot e^{\ln(e^{2x}+1)}$$

$$z + 1 = c_2 (e^{2x}+1)$$

$$z = c_2(e^{2x}+1) - 1$$

$y' = z \Rightarrow y = \int z \, dx = \int c_2 (e^{2x} + 1) - 1 \, dx =$

$= \int c_2 e^{2x} + \int c_2 - \int 1 =$

$= \frac{c_2}{2} e^{2x} + x c_2 - x + c_1 =$

$= \frac{c_2}{2} e^{2x} + x(c_2 - 1) + c_1$

$y'(0) = 3 \Rightarrow c_2(e^0 + 1) - 1 = 3$

$2c_2 = 4$

$c_2 = 2$

$y(0) = 2 \Rightarrow e^0 + 0 \cdot 1 + c_1 = 2$

$\Rightarrow 1 + c_1 = 2$

$c_1 = 1$

$\Rightarrow y(x) = e^{2x} + x + 1$

Ex. 4) $x'(t) = 2xy - x^3$

$y'(t) = -x^2 - y^5$

Fie $V(x, y) = x^2 + 2y^2$

$V(0,0) = 0$ şi $V(x,y) > 0, \forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$f_1(x,y) = 2xy - x^3$

$f_2(x,y) = -x^2 - y^5$

p4

$$\dot{V}(x,y) = \frac{\partial V}{\partial x} f_1 + \frac{\partial V}{\partial y} f_2 =$$

$$= 2x(2xy - x^3) + 4y(-x^2 - y^5) =$$

$$= \cancel{4x^2y} - 2x^4 \cancel{- 4yx^2} - 4y^6 =$$

$$= -2(x^4 + 2y^6) < 0, \forall (x,y) \in \mathbb{R} / \{(0,0)\}$$

$\Rightarrow X^* (0,0)$ - local asimptotic stabil

<u>Ex 5</u>) $x' = ax^2 - x^3 + 3a - 3x$

$$f(x) = x^2(a-x) + 3(a-x) =$$

$$= (x^2+3)(a-x)$$

$f(x) = 0 \Rightarrow x = a \in \mathbb{R}$ pct. de echil.

Folosim teorema stabilității în prima aprox.:

$$f'(x) = 2ax - 3x^2 - 3 =$$

$$= -3x^2 + 2ax - 3$$

$$f'(a) = -3a^2 + 2a^2 - 3 = -a^2 - 3$$

$-a^2 - 3 < 0, \forall a \in \mathbb{R} \Rightarrow x^* = a$

local asimptotic stabil

Ex7) $\begin{cases} x'(t) = 4y(t) \\ y'(t) = -2x(t) \end{cases}$

a) $y = \frac{x'}{4}$

$y' = \frac{x''}{4}$

$\frac{x''}{4} = -2x$

$\frac{x''}{4} + 2x = 0$

$x'' + 8x = 0$

$r^2 + 8 = 0$

$r = \pm i 2\sqrt{2} \Rightarrow \begin{matrix} \alpha = 0 \\ \beta = 2\sqrt{2} \end{matrix}$

$x = c_1 \cos \sqrt{8}\, t + c_2 \sin \sqrt{8}\, t$

$\cancel{y = \frac{x'}{4} =}$

$x' = -\sqrt{8}\, c_1 \sin \sqrt{8}\, t + \sqrt{8}\, c_2 \cos \sqrt{8}\, t$

$y = \frac{x'}{4} = \frac{\sqrt{8}}{4}(-c_1 \sin \sqrt{8}\, t + c_2 \cos \sqrt{8}\, t)$

Fie $x(0) = \eta_1$

$y(0) = \eta_2$

$x(0) = c_1 \Rightarrow c_1 = \eta_1$

$y(0) = \frac{\sqrt{8}}{4} c_2 = \eta_2 \Rightarrow c_2 = \frac{4\eta_2}{\sqrt{8}} =$

$= \frac{\cancel{4}^{2} \eta_2}{\cancel{2}\sqrt{2}} = \frac{2\sqrt{2}\eta_2}{2} = \sqrt{2}\eta_2$

$$\Rightarrow \begin{cases} x(t) = \eta_1 \cos\sqrt{8}t + \sqrt{2}\eta_2 \sin\sqrt{8}t \\ y(t) = \frac{\sqrt{8}}{4}(-\eta_1 \sin\sqrt{8}t + \sqrt{2}\eta_2 \cos\sqrt{8}t) \end{cases}$$

Fie fluxul $f : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$

$$f(t, \eta_1, \eta_2) = \Big(\eta_1 \cos\sqrt{8}t + \eta_2\sqrt{2}\sin\sqrt{8}t, \frac{\sqrt{8}}{4}(-\eta_1 \sin\sqrt{8}t + \eta_2\sqrt{2}\cos\sqrt{8}t)\Big)$$

b) $A = \begin{pmatrix} 0 & 4 \\ -2 & 0 \end{pmatrix}$

$|\lambda I_2 - A| = 0 \Leftrightarrow \begin{vmatrix} \lambda & -4 \\ 2 & \lambda \end{vmatrix} = 0$

$\lambda^2 + 8 = 0$

$\lambda_{1,2} = \pm 2\sqrt{2} i$

$Re(\lambda) = 0 \Rightarrow (0,0)$ local stabil

de tip centru

$$\frac{dx}{dt} = 4y$$

$$\frac{dy}{dt} = -2x$$

$$\frac{\frac{dx}{dt}}{\frac{dy}{dt}} = \frac{dx}{dy} = \frac{4y}{-2x} = \frac{2y}{-x}$$

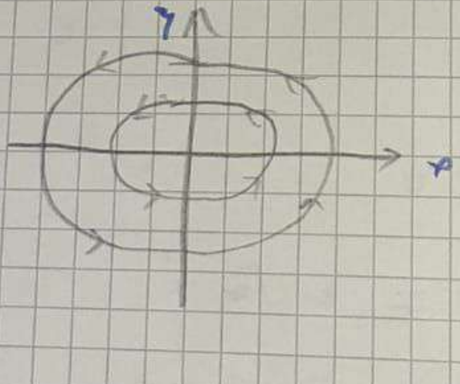
$$\frac{dx}{dy} = \frac{2y}{-x}$$

$$-x\,dx = 2y\,dy$$

$$\int -x\,dx = \int 2y\,dy$$

$$-\frac{x^2}{2} = y^2 + c,\quad c \in \mathbb{R}$$

$$\frac{x^2}{2} + y^2 = c_1$$



– portretul fazic

Ex. 6) $\begin{cases} y' = 4x^3 + 2y^2 \\ y(0) = 1 \end{cases}$

$$f(x, y) = y' = 4x^3 + 2y^2$$

$$y(x_0) = y^0 \Rightarrow x_0 = 0$$

$$y^0 = 1$$

$f : \mathbb{R}^2 \to \mathbb{R}$, f. cont.

$$y(x) = y^0 + \int_{x_0}^{x} f(s, y(s))\, ds =$$

$$= 1 + \int_0^x 4s^3 + 2y^2(s)\, ds$$

Șirul aprox. succesive:

$\cancel{y_0 \in [-a, a]}$

$\cancel{y_{n+1}(x) = 1 + \int_0^x 4s^3}$

$$y_{n+1}(x) = y^0 + \int_{x_0}^{x} f(s, y_n(s))\, ds$$

$$y_{n+1}(x) = 1 + \int_0^x 4s^3 + 2y_n^2(s)\, ds$$

Fie $y_0(x) \equiv 1$

$$y_1(x) = 1 + \int_0^x 4s^3 + 2y_0^2(s)\, ds =$$

$$= 1 + \int_0^x 4s^3 + 2\, ds = 1 + 2x + x^4$$

$$y_2(x) = 1 + \int_0^x 4s^3 + 2y_1^2(s)\, ds =$$

$$= 1 + \int_0^x 4s^3 + 2(1 + 2s + s^4)^2\, ds =$$

$$= 1 + \int_0^x 4s^3 + 2(1 + 4s^2 + s^8 + 4s + 2s^4 + 4s^5)\, ds =$$

$$= 1 + \int_0^x 4s^3 + 2 + 8s^2 + 2s^8 + 8s + 4s^4 + 8s^5\, ds =$$

$$= 1 + x^4 + 2x + \frac{8}{3}x^3 + \frac{2}{9}x^9 + 4x^2 +$$

$$+ \frac{4}{5}x^5 + \frac{4}{3}x^6 =$$

$$= \frac{2}{9}x^9 + \frac{4}{3}x^6 + \frac{4}{5}x^5 + x^4 + \frac{8}{3}x^3$$

$$+ 4x^2 + 2x + 1$$