Temă:

3.1.42.

(1) $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x_1, x_2, x_3) = (x_1 - x_2, x_2 - x_3, x_3 - x_1)$

$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$

$x, y \in \mathbb{R}^3$, $x = (x_1, \cancel{x_2}, x_3)$, $y = (y_1, y_2, y_3)$

$$f(\alpha x + \beta y) = f(\alpha(x_1, x_2, x_3) + \beta(y_1, y_2, y_3))$$
$$= f((\alpha x_1, \alpha x_2, \alpha x_3) + (\beta y_1, \beta y_2, \beta y_3))$$
$$= f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3)$$
$$= [\alpha(x_1 - x_2) + \beta(y_1 - y_2), \alpha(x_2 - x_3) + \beta(y_2 - y_3), \alpha(x_3 - x_1) + \beta(y_3 - y_1)]$$
$$= [[\alpha(x_1 - x_2), \alpha(x_2 - x_3), \alpha(x_3 - x_1)] + [\beta(y_1 - y_2), \beta(y_2 - y_3), \beta(y_3 - y_1)]$$
$$= \alpha(x_1 - x_2, x_2 - x_3, x_3 - x_1) + \beta(y_1 - y_2, y_2 - y_3, y_3 - y_1)$$
$$= \alpha f(x) + \beta f(y) \Rightarrow f: \mathbb{R}^3 \to \mathbb{R}^3 \text{ - aplic. lin.}$$

$\operatorname{Ker} f = \{x \in \mathbb{R}^3 \mid f(x) = 0\}$

$f(x) = 0$, $x = (x_1, x_2, x_3)$ $\Leftrightarrow (x_1 - x_2, x_2 - x_3, x_3 - x_1) = (0, 0, 0) \Rightarrow$

$\Rightarrow x_1 = x_2 = x_3$

$\operatorname{Ker} f = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = x_2 = x_3\}$

$\operatorname{Im} f = \{f(x) \mid x \in \mathbb{R}^3\}$, $v = (v_1, v_2, v_3)$

$f(x) = v \Leftrightarrow (x_1 - x_2, x_2 - x_3, x_3 - x_1) = (v_1, v_2, v_3)$

$$\begin{array}{l} x_1 - x_2 = v_1 \\ x_2 - x_3 = v_2 \\ \underline{x_3 - x_1 = v_3} \quad (+) \\ v_1 + v_2 + v_3 = 0 \end{array}$$

$\operatorname{Im} f = \{v = f(x) = (v_1, v_2, v_3) \mid x \in \mathbb{R}^3, \text{ şi } v_1 + v_2 + v_3 = 0\}$

2) $f:\mathbb{R}^3\to\mathbb{R}^3,\ f(x_1,x_2,x_3)=(x_1-1,x_2+2,x_3+1)$

$x,y\in V$

$f(\alpha x+\beta y)=f(\alpha x_1+\beta y_1,\alpha x_2+\beta y_2,\alpha x_3+\beta y_3)$

$=(\alpha x_1+\beta y_1-1,\alpha x_2+\beta y_2+2,\alpha x_3+\beta y_3+1)\quad(*)$

$\alpha f(x)+\beta f(y)=\cancel{(\alpha x_1+\beta y_1-2\alpha,\alpha x_2+\beta y_2-2\alpha,\alpha}$

$=(\alpha x_1+\beta y_1-\alpha-\beta,\ \alpha x_2+\beta y_2-2-\beta,\ \alpha x_3+\beta y_3-\alpha-\beta)\quad(**)$

$(*)\neq(**)\Rightarrow f:\mathbb{R}^3\to\mathbb{R}^3$ - nu e aplic. lin.

3) $f:\mathbb{R}^3\to\mathbb{R}^3,\ f(x_1,x_2,x_3)=(2x_1-3x_2+x_3,-x_1+x_2+3x_3,x_1+x_2+x_3)$

$f(\alpha x+\beta y)=f(\alpha x_1+\beta y_1,\alpha x_2+\beta y_2,\alpha x_3+\beta y_3)$

$=(2\alpha x_1+2\beta y_1-3\alpha x_2-3\beta y_2+\alpha x_3+\beta y_3,\ -\alpha x_1-\beta y_1+\alpha x_2+\beta y_2+3\alpha x_3+3\beta y_3,$

$\alpha x_1+\beta y_1+\alpha x_2+\beta y_2+\alpha x_3+\beta y_3)$

$=(2\alpha x_1-3\alpha x_2+\alpha x_3,\ -\alpha x_1+\alpha x_2+3\alpha x_3,\ \alpha x_1+\alpha x_2+\alpha x_3)+$

$(2\beta y_1-3\beta y_2+\beta y_3,\ -\beta y_1+\beta y_2+3\beta y_3,\ \beta y_1+\beta y_2+\beta y_3)$

$=\alpha(2x_1-3x_2+x_3,-x_1+x_2+3x_3,x_1+x_2+x_3)+\beta(2y_1-3y_2+y_3,-y_1+y_2+3y_3,y_1+y_2+y_3)$

$=\alpha f(x)+\beta f(y)\Rightarrow f:\mathbb{R}^3\to\mathbb{R}^3$ - apl. lin.

$f(x)=0\Rightarrow\begin{cases}2x_1-3x_2+x_3=0\\-x_1+x_2+3x_3=0\\x_1+x_2+x_3=0\end{cases}\qquad A=\begin{pmatrix}2&-3&1\\-1&1&3\\1&1&1\end{pmatrix}$

$\det A=-1,\quad d_x=d_y=d_z=0\Rightarrow x_1=x_2=x_3=0$

$\operatorname{Ker} f=\{0\}$

$f(x)=v\Rightarrow\begin{cases}2x_1-3x_2+x_3=v_1\\-x_1+x_2+3x_3=v_2\\x_1+x_2+x_3=v_3\end{cases}$, A - același

$\det A=-1\Rightarrow$ S.C.D. - sol. unică

$d_x=\begin{pmatrix}v_1&-3&1\\v_2&1&3\\v_3&1&1\end{pmatrix}\qquad d_y=\begin{pmatrix}2&v_1&1\\-1&v_2&3\\1&v_3&1\end{pmatrix}\qquad d_z=\begin{pmatrix}2&-3&v_1\\-1&1&v_2\\1&1&v_3\end{pmatrix}$

$x_1=\dfrac{d_x}{\det A}=2v_1-4v_2+10v_3\quad x_2=\dfrac{d_y}{\det A}=-4v_1+v_2+7v_3\quad x_3=\dfrac{d_z}{\det A}=2v_1+5v_2+v_3$

$\operatorname{Im} f = \{v = f(x) = (v_1, v_2, v_3) \mid x \in R^3$, și $x = (2v_1 - 4v_2 + 10v_3, -4v_1 - v_2 + 7v_3, 2v_1 + 5v_2 + v_3)$

4) $f: R^2 \to R^3$, $f(x_1, x_2) = (x_1 + x_2, x_1 - x_2, 2x_1 + x_2)$

$$f(\alpha x + \beta y) = f(\alpha x_1 + \beta y_1 + \alpha x_2 + \beta y_2, \alpha x_1 + \beta y_1 + \alpha x_2 - \beta y_2, 2\alpha x_1 + 2\beta x_1 + \alpha x_2 + \beta y_2)$$
$$= (\alpha x_1 + \alpha x_2, \alpha x_1 - \alpha x_2, 2\alpha x_1 + x_2) + (\beta y_1 + \beta y_2, \beta y_1 - \beta y_2, 2\beta y_1 + \beta y_2)$$
$$= \alpha(x_1 + x_2, x_1 - x_2, 2x_1 + x_2) + \beta(y_1 + y_2, y_1 - y_2, 2y_1 + y_2)$$
$$= \alpha f(x) + \beta f(y) \Rightarrow f: R^2 \to R^3 \text{ - e apl. lin.}$$

$f(x) = 0 \Rightarrow \begin{cases} x_1 + x_2 = 0 \\ x_1 - x_2 = 0 \\ 2x_1 + x_2 = 0 \end{cases} \Rightarrow x_1 = x_2 = 0$

$\operatorname{Ker} f = \{0\}$

$f(x) = v \in R^3 \Rightarrow \begin{cases} x_1 + x_2 = v_1 \\ x_1 - x_2 = v_2 \\ 2x_1 + x_2 = v_3 \end{cases} \Rightarrow x_1 = \frac{v_1 + v_2}{2}$

$x_2 = \frac{2v_1 - v_2}{2}$

$x_2 = v_3$

$\frac{2v_1 - v_2}{2} = v_3 \Leftrightarrow 2v_3 - 2v_1 + v_2 = 0$

$\operatorname{Im} f = \{v = f(x) = (v_1, v_2, v_3) \mid x \in R^2$ și $-2v_1 + v_2 + 2v_3 = 0\}$

5. $f: R^2 \to R$, $f(x_1, x_2) = x_1^2 - x_2^2$.

$$f(\alpha x + \beta y) = f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2)$$
$$= (\alpha x_1 + \beta y_1)^2 - (\alpha x_2 + \beta y_2)^2.$$
$$= \alpha^2 x_1^2 + 2\alpha\beta x_1 y_1 + \beta^2 y_1^2 - \alpha^2 x_2^2 - 2\alpha\beta x_2 y_2 - \beta^2 y_2^2$$
$$= \alpha^2 (x_1^2 - x_2)^2 + \beta^2 (y_1^2 - y_2)^2 + 2\alpha\beta(x_1 y_1 - x_2 y_2)$$
$$\neq \alpha f(x) + \beta f(y) \Rightarrow f: R^2 \to R \text{ nu e apl. lin.}$$

6) $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = (\alpha_{1,1}x_1 + \alpha_{2,1}x_2, \alpha_{1,2}x_1 + \alpha_{2,2}x_2)$, $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$

$$f(\alpha x + \beta y) = f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2)$$

$$= (\alpha_{11}\alpha x_1 + \alpha_{11}\beta y_1 + \alpha_{21}\alpha x_2 + \alpha_{21}\beta y_2, \alpha_{12}\alpha x_1 + \alpha_{12}\beta y_1 + \alpha_{22}\alpha x_2 + \alpha_{22}\beta y_2)$$

$$= (\alpha_{11}\alpha x_1 + \alpha_{21}\alpha x_2, \alpha_{12}\alpha x_1 + \alpha_{22}\alpha x_2) + (\alpha_{11}\beta y_1 + \alpha_{21}\beta y_2, \alpha_{12}\beta y_1 + \alpha_{22}\beta y_2)$$

$$= \alpha(\alpha_{11}x_1 + \alpha_{21}x_2, \alpha_{12}x_1 + \alpha_{22}x_2) + \beta(\alpha_{11}y_1 + \alpha_{21}y_2, \alpha_{12}y_1 + \alpha_{22}y_2)$$

$$= \alpha f(x) + \beta f(y) \Rightarrow f: \mathbb{R}^2 \to \mathbb{R}^2 \text{ - o apl. lin.}$$

$$f(x) = 0 \Rightarrow \begin{array}{l} \alpha_{11}x_1 + \alpha_{21}x_2 = 0 \\ \underline{\alpha_{12}x_1 + \alpha_{22}x_2 = 0} \ (+) \end{array}$$

$$x_1(\alpha_{11} + \alpha_{12}) + x_2(\alpha_{21} + \alpha_{22}) = 0$$

$$x_1 = x_2 = 0$$

$$\alpha_{11} = -\alpha_{12} \text{ și } \alpha_{21} = -\alpha_{22}$$

Dacă $\alpha_{11} \neq -\alpha_{12}$ și $\alpha_{21} \neq -\alpha_{22}$, $\operatorname{Ker} f = \{0\}$.

Dacă $\alpha_{11} = -\alpha_{12}$ și $\alpha_{21} = -\alpha_{22}$, $\operatorname{Ker} f = \mathbb{R}^2$

$$f(x) = v \Rightarrow x_1(\alpha_{11} + \alpha_{12}) + x_2(\alpha_{21} + \alpha_{22}) = v_1 + v_2$$

sau
$$(x_1 \ x_2) \times \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = (v_1 \ v_2)$$

$$\operatorname{Im} f = \{v = f(x) = (v_1, v_2) \mid x \in \mathbb{R}^2 \text{ și } x_1(\alpha_{11} + \alpha_{12}) + x_2(\alpha_{21} + \alpha_{22}) = v_1 + v_2\}$$