3.1.42 Care dintre urm. aplicații sunt liniare.

(1) $f: R^3 \to R^3$, $f[x_1, x_2, x_3] = [x_1 - x_2, x_2 - x_3, x_3 - x_1]$

$f: V \to W$ aplicație vectorială $\Leftrightarrow$

$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$, $\forall x, y \in V$ și $\forall \alpha, \beta \in K$

$x = [x_1, x_2, x_3] \in R^3$

$y = [y_1, y_2, y_3] \in R^3$

$f(\alpha [x_1, x_2, x_3] + \beta [y_1, y_2, y_3]) =$

$\alpha f[x_1, x_2, x_3] + \beta f[y_1, y_2, y_3]$

$f(\alpha [x_1, x_2, x_3] + \beta [y_1, y_2, y_3]) =$

$= f([\alpha x_1, \alpha x_2, \alpha x_3] + [\beta y_1, \beta y_2, \beta y_3]) =$

$= f[\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3] =$

$= [\alpha x_1 + \beta y_1 - \alpha x_2 - \beta y_2; \alpha x_2 + \beta y_2 - \alpha x_3 - \beta y_3, \alpha x_3 + \beta y_3 - \alpha x_1 - \beta y_1]$

$= [\alpha(x_1 - x_2) + \beta(y_1 - y_2), \alpha(x_2 - x_3) + \beta(y_2 - y_3), \alpha(x_3 - x_1) + \beta(y_3 - y_1)]$ (1)

$\alpha f[x_1, x_2, x_3] + \beta f[y_1, y_2, y_3] =$

$= \alpha [x_1 - x_2, x_2 - x_3, x_3 - x_1] + \beta [y_1 - y_2, y_2 - y_3, y_3 - y_1] =$

$= [\alpha(x_1 - x_2), \alpha(x_2 - x_3), \alpha(x_3 - x_1)] + [\beta(y_1 - y_2), \beta(y_2 - y_3), \beta(y_3 - y_1)]$

$= [\alpha(x_1 - x_2) + \beta(y_1 - y_2), \alpha(x_2 - x_3) + \beta(y_2 - y_3), \alpha(x_3 - x_1) + \beta(y_3 - y_1)]$ (2)

Din (1) și (2) $\Rightarrow f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$, $\forall x, y \in R^3$, și $\forall \alpha, \beta \in K \Rightarrow$ Aplicația este liniară

(2) $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f[x_1, x_2, x_3] = [x_1 - 1, x_2 + 2, x_3 + 1]$

$f: \mathbb{R}^3 \to \mathbb{R}^3$ aplicație liniară $\iff$

$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \quad \forall x, y \in \mathbb{R}^3$

și $\forall \alpha, \beta \in K$

$x = [x_1, x_2, x_3]$

$y = [y_1, y_2, y_3]$

$f(\alpha[x_1, x_2, x_3] + \beta[y_1, y_2, y_3]) =$

$= \alpha f[x_1, x_2, x_3] + \beta f[y_1, y_2, y_3]$

$f(\alpha[x_1, x_2, x_3] + \beta[y_1, y_2, y_3]) =$

$= f([\alpha x_1, \alpha x_2, \alpha x_3] + [\beta y_1, \beta y_2, \beta y_3]) =$

$= f[\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3] =$

$= [\alpha x_1 + \beta y_1 - 1, \alpha x_2 + \beta y_2 + 2, \alpha x_3 + \beta y_3 + 1]$ (1)

$\alpha f[x_1, x_2, x_3] + \beta f[y_1, y_2, y_3] =$

$= \alpha[x_1 - 1, x_2 + 2, x_3 + 1] + \beta[y_1 - 1, y_2 + 2, y_3 + 1]$

$= [\alpha x_1 - \alpha, \alpha x_2 + 2\alpha, \alpha x_3 + \alpha] + [\beta y_1 - \beta, \beta y_2 + 2\beta, \beta y_3 + \beta]$

$= [\alpha x_1 - \alpha + \beta y_1 - \beta, \alpha x_2 + 2\alpha + \beta y_2 + 2\beta, \alpha x_3 + \alpha + \beta y_3 + \beta]$ (2)

Din (1) și (2) $\Rightarrow f(\alpha x + \beta y) \neq \alpha f(x) + \beta f(y)$

$\Rightarrow f: \mathbb{R}^3 \to \mathbb{R}^3$, $f[x_1, x_2, x_3] = [x_1 - 1, x_2 + 2, x_3 + 1]$

nu este aplicație liniară

(3) $f: \mathbb{R}^3 \to \mathbb{R}^2$

$f[x_1, x_2, x_3] = [2x_1 - 3x_2 + x_3, -x_1 + x_2 + 3x_3, x_1 + x_2 + x_3]$

Funcția $f: \mathbb{R}^3 \to \mathbb{R}^2$ nu este bine definită

deoarece $[2x_1 - 3x_2 + x_3, -x_1 + x_2 + 3x_3, x_1 + x_2 + x_3] \notin \mathbb{R}^2$

(4) $f: \mathbb{R}^2 \to \mathbb{R}^3$, $f[x_1, x_2] = [x_1 + x_2, x_1 - x_2, 2x_1 + x_2]$

$x = [x_1, x_2] \in \mathbb{R}^2$

$y = [y_1, y_2] \in \mathbb{R}^3$

$f: \mathbb{R}^2 \to \mathbb{R}^3$ aplicație liniară $\Leftrightarrow$

$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y), \forall x, y \in \mathbb{R}^2, \alpha, \beta \in K$

$f(\alpha [x_1, x_2] + \beta [y_1, y_2]) =$

$= f([\alpha x_1, \alpha x_2] + [\beta y_1, \beta y_2]) =$

$= f[\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2] =$

$= [\alpha(x_1 + x_2) + \beta(y_1 + y_2), \alpha(x_1 - x_2) + \beta(y_1 - y_2), \alpha(2x_1 + x_2) + \beta($
$2y_1 + y_2)]$ (1)

$\alpha f[x_1, x_2] + \beta f[y_1, y_2] = \alpha [x_1 + x_2, x_1 - x_2, 2x_1 + x_2] +$

$+ \beta [y_1 + y_2, y_1 - y_2, 2y_1 + y_2] =$

$= [\alpha(x_1 + x_2) + \beta(y_1 + y_2), \alpha(x_1 - x_2) + \beta(y_1 - y_2), \alpha(2x_1 + x_2) + \beta(2y_1 + y_2)]$ (2)

Din (1) și (2) $\Rightarrow f$ - aplicație liniară

(5) $f: \mathbb{R}^2 \to \mathbb{R}$, $f[x_1, x_2] = x_1^2 - x_2^2$

$x = [x_1, x_2] \in \mathbb{R}^2$

$y = [y_1, y_2] \in \mathbb{R}^2$

f - aplicație liniară $\Leftrightarrow f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$

$f(\alpha [x_1, x_2] + \beta [y_1, y_2]) =$

$= f[\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2] =$

$= (\alpha x_1 + \beta y_1)^2 - (\alpha x_2 + \beta y_2)^2 =$

$= \alpha^2 x_1^2 + \beta^2 y_1^2 + 2\alpha x_1 \beta y_1 - \alpha^2 x_2^2 - \beta^2 - y_2^2 -$
$- 2\alpha x_2 \beta y_2$ (1)

$\alpha f[x_1, x_2] + \beta f[y_1, y_2] = \alpha(x_1^2 - x_2^2) + \beta(y_1^2 - y_2^2)$

$= \alpha x_1^2 - \alpha x_2^2 + \beta y_1^2 - \beta y_2^2$ (2)

Din (1) și (2) $\Rightarrow f(\alpha x + \beta y) \neq \alpha f(x) + \beta f(y)$

$\Rightarrow f$ - nu este aplicație liniară

(6) $f: \mathbb{R}^2 \to \mathbb{R}^2$

$f[x_1, x_2] = [a_{1,1} x_1 + a_{2,1} x_2, a_{1,2} x_1 + a_{2,2} x_2]$
unde $a_{1,1}; a_{1,2}; a_{2,1}; a_{2,2} \in \mathbb{R}$ sunt fixate
$x = [x_1, x_2]$

$y = [y_1, y_2]$

f - aplicație liniară $\Leftrightarrow f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$
$\forall x, y \in \mathbb{R}^2$, $\alpha, \beta \in k$

$f(\alpha [x_1, x_2] + \beta [y_1, y_2]) = f[\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2]$

$= [a_{1,1}(\alpha x_1 + \beta y_1) + a_{1,2}(\alpha x_2 + \beta y_2), a_{2,1}(\alpha x_1 + \beta y_1) + a_{2,2}(\alpha x_2 + \beta y_2)]$ (1)

$\alpha f[x_1, x_2] + \beta f[y_1, y_2] =$

$= [\alpha(a_{1,1}x_1 + a_{1,2}x_2), \alpha(a_{2,1}x_1 + a_{2,2}x_2)] +$
$+ [\beta(a_{1,1}y_1 + a_{1,2}y_2), \beta(a_{2,1}y_1 + a_{2,2}y_2)] =$
$= [a_{1,1}(\alpha x_1 + \beta y_1) + a_{1,2}(\alpha x_2 + \beta y_2), a_{2,1}(\alpha x_1 + \beta y_1) + a_{2,2}(\alpha x_2 + \beta y_2)]$ (2)

Din (1) și (2) $\Rightarrow f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$
$\forall x, y \in \mathbb{R}^2, \alpha, \beta \in K \Rightarrow f$ - aplicație liniară.

3.1.29. $f : V \to W$ aplicație liniară

$\ker f = \{ x \in V \mid f(x) = 0 \}$

$\operatorname{Im} f = \{ f(x) \mid x \in V \}$

(1) $f[x_1, x_2, x_3] = [0, 0, 0]$

$[x_1 - x_2, x_2 - x_3, x_3 - x_1] = [0, 0, 0]$

$$S : \begin{cases} x_1 - x_2 = 0 \\ x_2 - x_3 = 0 \\ x_3 - x_1 = 0 \end{cases} \qquad A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

rang $A = 2$

x_1, x_2, nec. principale

$x_3 = \alpha$

$$\begin{cases} x_1 - x_2 = 0 \Rightarrow x_1 = x_2 \\ x_2 - \alpha = 0 \Rightarrow x_2 = \alpha \end{cases} \Rightarrow x_1 = \alpha$$

$S = \{ (\alpha, \alpha, \alpha) \mid \alpha \in \mathbb{R} \}$

$\operatorname{Im} f = \{ f[x_1, x_2, x_3] \mid [x_1, x_2, x_3] \in \mathbb{R}^3 \}$

$f[x_1, x_2, x_3] = [x_1 - x_2, x_2 - x_3, x_3 - x_1]$

$[m, n, \theta] \in \mathbb{R}^3 \in \operatorname{Im} f$

$m, n, \theta \in \mathbb{R}$

$[x_1 - x_2, x_2 - x_3, x_3 - x_1] = [m, n, \theta]$

$$S : \begin{cases} x_1 - x_2 = m \\ x_2 - x_3 = n \\ -x_1 + x_3 = \theta \end{cases} \qquad d_3 = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & +1 \end{vmatrix} =$$

(6) $\ker f = \{ x \in V \mid f(x) = 0 \}$

$x = [x_1, x_2] \in \mathbb{R}^2$

$f[x_1, x_2] = [0, 0] \in \mathbb{R}^2$

$$\begin{cases} a_{1,1} x_1 + a_{1,2} x_2 = 0 \\ a_{2,1} x_1 + a_{2,2} x_2 = 0 \end{cases}$$

$$x_1 = -\frac{a_{2,2} x_2}{a_{2,1}}$$

$$x_2 \left(\frac{a_{1,2} \cdot a_{2,1} - a_{2,2}}{a_{2,1}} \right) = 0 \quad \Rightarrow x_2 = 0 \Rightarrow x_1 = 0$$

$S = \{ [0, 0] \}$

$\operatorname{Im} f = \{ f(x) \mid x \in V \}$

$x = [x_1, x_2]$

$f[x_1, x_2] = [\alpha, \beta] \qquad \alpha, \beta \in \mathbb{R}$

$$\begin{cases} a_{1,1} x_1 + a_{1,2} x_2 = \alpha \\ a_{2,1} x_1 + a_{2,2} x_2 = \beta \end{cases}$$

$$x_1 = \frac{-a_{2,2} x_2 + \beta}{a_{2,1}}$$

$$\frac{a_{1,1} \beta}{a_{2,1}} + \frac{x_2 (a_{1,2} a_{2,1} - a_{2,2})}{a_{2,1}} = \alpha$$

$$x_2 = \left(\alpha - \frac{a_{1,1} \beta}{a_{2,1}} \right) \cdot \frac{a_{2,1}}{a_{1,2} \cdot a_{2,1} - a_{2,2}}$$

$$x_2 = \frac{a_{2,1} \alpha - a_{1,1} \beta}{a_{1,2} \cdot a_{2,1} - a_{2,2}}$$

$$x_1 = \frac{\dfrac{-a_{2,2} \cdot a_{2,1} \alpha + a_{1,1} \cdot a_{2,2} \beta}{a_{1,2} \cdot a_{2,1} - a_{2,2}} + \beta}{a_{2,1}}$$

(4) $\ker f = \{x \in V \mid f(x) = 0\}$

$$f[x_1, x_2] = [x_1 + x_2,\ x_1 - x_2,\ 2x_1 + x_2] \in \mathbb{R}^3$$

$$f[x_1, x_2] = [0, 0, 0]$$

$$S: \begin{cases} x_1 + x_2 = 0 \\ x_1 - x_2 = 0 \\ 2x_1 + x_2 = 0 \end{cases}$$

$x_1 = x_2$

$3x_1 = 0$

$S = \{[0, 0, 0]\}$

$\operatorname{Im} f = \{f(x) \mid x \in V\}$

$[m, n, \theta] \in \operatorname{Im} f$

$m, n, \theta \in \mathbb{R}$

$$S \begin{cases} x_1 + x_2 = m \\ x_1 - x_2 = n \\ 2x_1 + x_2 = \theta \end{cases}$$

$x_1 = m - x_2$ $\Rightarrow x_1 = \frac{2m}{2} - \frac{m-n}{2} = \frac{n-m}{2}$

$\cancel{-}2x_2 = n - m$

$x_2 = \frac{m-n}{2}$

$$S = \left\{\left[\frac{n-m}{2}, \frac{m-n}{2}\right]\right\}$$

$= 1 + 0 - 1 - 0 - 0 - 0) = 0$

$\text{rang } A = 2$

$$\bar{A} = \begin{pmatrix} 1 & -1 & 0 & m \\ 0 & 1 & -1 & n \\ -1 & 0 & 1 & 0 \end{pmatrix}$$

$$d'_3 = \begin{vmatrix} 1 & -1 & m \\ 0 & 1 & n \\ -1 & 0 & 0 \end{vmatrix} = 0 + 0 + n + m - 0 - 0 =$$
$$= m + n + 0$$

Dacă $m + n + 0 \neq 0 \Rightarrow \text{rang } A = 2 \neq 3 = \text{rang } \bar{A}$

$\Rightarrow$ sistemul e incompatibil

Dacă $m + n + 0 = 0 \Rightarrow (*)$

$$d''_3 = \begin{vmatrix} 1 & 0 & m \\ 0 & -1 & n \\ -1 & 1 & 0 \end{vmatrix} = -0 - m - n = -(m + n + 0)$$

$$d'''_3 = \begin{vmatrix} -1 & 0 & m \\ 1 & -1 & n \\ 0 & 1 & 0 \end{vmatrix} = +0 + m + n$$

$(*) \Rightarrow \text{rang } A = 2 = \text{rang } \bar{A}$

x_1, x_2 - nec. principale

$\alpha = x_3$ - nec. secundară

$$\begin{cases} x_1 - x_2 = m \\ x_2 - \alpha = n \end{cases}$$

$x_2 = n + \alpha$

$x_1 = m + n + \alpha$

$S = \{(m + n + \alpha, n + \alpha, \alpha) \mid \alpha \in \mathbb{R}\}$