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Lucrare scrisă la analiză matematică
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1,5 ① Studiați existența derivatelor după direcție ale funcției: $f(x,y) = \sqrt[3]{x+y^2}$ în punctul $(0,0)$. Este funcția derivabilă parțial în acest punct?! Justificați

1,5 ② Calculați integrala improprie:

$$\int_3^\infty \frac{1}{x^2-x-2}\,dx$$

2 ③ Fie funcția $f:(0,+\infty)^2 \to \mathbb{R}$,
$f(x,y) = x\sqrt{y} + \frac{y}{\sqrt{x}}$. Determinați ct. $\alpha \in \mathbb{R}$ a.î.

$$\alpha \cdot \frac{x^2}{y^2} \cdot \frac{\partial^2 f}{\partial x^2}(x,y) + 2\frac{\partial^2 f}{\partial y^2}(x,y) + \frac{x}{y} \cdot \frac{\partial^2 f}{\partial x \partial y}(x,y) = 0, \quad \forall (x,y) \in (0,+\infty)^2$$

2 ④ Calculați:

a) $\lim\limits_{n\to\infty} \dfrac{1+\frac{1}{2}+\ldots+\frac{1}{n}}{\ln n}$

h) Studiați convergența seriei cu t. p.:

$$\sum_{n=1}^{\infty} \left(\frac{1+\frac{1}{2}+\ldots+\frac{1}{n}}{n} \right)^{a}, \quad a>0, \text{ în funcție de}$$

valorile parametrului $a \in \mathbb{R}$

Rezolvare

(2) $$\int_3^{\infty} \frac{1}{x^2-x-2}\,dx = \int_3^{\infty} \frac{1}{x^2-x+\frac{1}{4}-\frac{1}{4}-2}\,dx =$$

$$= \int_3^{\infty} \frac{1}{\left(x-\frac{1}{2}\right)^2-\frac{9}{4}}\,dx = \int_3^{\infty} \frac{1}{u^2(x)-\left(\frac{3}{2}\right)^2}\,dx,$$

$$u(x) = x-\frac{1}{2}$$

$$\Rightarrow \int = \frac{1}{2}\cdot\frac{2}{3}\ln\left|\frac{u(x)-\frac{3}{2}}{u(x)+\frac{3}{2}}\right|\Bigg|_3^{\infty} =$$

$$= \frac{1}{3}\ln\left|\frac{x-2}{x+1}\right|\Bigg|_3^{\infty} = \frac{1}{3}\left(\ln\left|\underbrace{\frac{x-2}{x+1}}_{\to 1}\right|_{x\to\infty} - \ln\frac{1}{4}\right) =$$

$$= \frac{1}{3}\left(\ln \underbrace{1}_{\to 0} - \ln\frac{1}{4}\right) = -\frac{1}{3}\ln\frac{1}{4} = \frac{1}{3}\cdot\ln 4$$

③ $$\frac{\partial f}{\partial x}(x,y) = \sqrt{y} + y\left(x^{-\frac{1}{2}}\right)' =$$

$$= \sqrt{y} - \frac{y}{2}\, x^{-\frac{3}{2}}$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = 0 + \frac{3}{2}\cdot\frac{y}{2}\cdot x^{-\frac{5}{2}} = \frac{3y}{4}\cdot x^{-\frac{5}{2}} =$$

$$= \frac{3y}{4x^2\sqrt{x}} \quad \Big/ \cdot \left(\alpha \cdot \frac{x^2}{y^2}\right)$$

$$\alpha \cdot \frac{x^2}{y^2} \cdot \frac{\partial^2 f}{\partial x^2}(x,y) = \frac{3\alpha y \cancel{x^2}}{4\cancel{x^2} y^2 \sqrt{x}} = \frac{3\alpha}{4y\sqrt{x}} \quad ①$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{1}{2} y^{-\frac{1}{2}} - \frac{1}{2} x^{-\frac{3}{2}} = \frac{1}{2\sqrt{y}} - \frac{1}{2x\sqrt{x}} =$$

$$= \frac{x\sqrt{x} - \sqrt{y}}{2x\sqrt{x}\sqrt{y}} \quad \Big/ \cdot \frac{x}{y}$$

$$\frac{x}{y} \cdot \frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{x(x\sqrt{x} - \sqrt{y})}{2xy\sqrt{x}\sqrt{y}} = \frac{x\sqrt{x} - \sqrt{y}}{2y\sqrt{y}\sqrt{x}}$$

②

$$\frac{\partial f}{\partial y}(x,y) = \frac{x}{2}\cdot y^{-\frac{1}{2}} + \frac{1}{\sqrt{x}}$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = -\frac{1}{2}\cdot\frac{x}{2}\cdot y^{-\frac{3}{2}} = \frac{-x}{4y\sqrt{y}} \quad \Big/ \cdot 2$$

$$2\,\frac{\partial^2 f}{\partial y^2}(x,y) = -\frac{x}{2y\sqrt{y}} \quad ③$$

Adunând termenii:

$$1 + 2 + 3 = \frac{3\alpha}{4y\sqrt{x}} + \frac{x\sqrt{x} - \sqrt{y}}{2y\sqrt{y}\sqrt{x}} - \frac{x}{2\sqrt{y}\,y} = 0$$

$$\forall (x, y) \in (0, +\infty)^2$$

$$\frac{3\alpha\sqrt{y} + \cancel{2x\sqrt{x}} - 2\sqrt{y} - \cancel{2x\sqrt{x}}}{4y\sqrt{x}\sqrt{y}} = 0 \Rightarrow$$

$$\Rightarrow 3\alpha\sqrt{y} - 2\sqrt{y} = 0$$

$$\left.\begin{array}{l} \sqrt{y}\,(3\alpha - 2) = 0 \\ \sqrt{y} \neq 0 \end{array}\right| \Rightarrow \begin{array}{l} 3\alpha - 2 = 0 \\ \alpha = \frac{2}{3} \in \mathbb{R} \end{array}$$

(4) a) Observăm:

$a_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$

$b_n = \ln n$

$b_{n+1} - b_n = \ln \frac{n+1}{n}$

$n+1 > n$

$\frac{n+1}{n} > 1, \; n \in \mathbb{N}$

$\ln\left(\frac{n+1}{n}\right) > \ln 1 = 0 \Rightarrow b_{n+1} - b_n > 0, \; \forall n \in \mathbb{N}$

$\Rightarrow b_n$ - strict monoton

$\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \ln n = +\infty \Rightarrow b_n$ - divergent

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \lim_{n\to\infty} \frac{1 + \frac{1}{2} + \dots + \frac{1}{n+1} - 1 - \frac{1}{2} - \dots - \frac{1}{n}}{\ln(n+1) - \ln n} =$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\ln\left(\frac{n+1}{n}\right)} = \lim_{n\to\infty} \frac{1}{\ln\left(\frac{n+1}{n}\right)^{n+1}} =$$

$$= \lim_{n\to\infty} \frac{1}{\ln\left[\left(1 + \frac{1}{n}\right)^{n+1}\right]} = \lim_{n\to\infty} \frac{1}{\ln\left[\underbrace{\left(1+\frac{1}{n}\right)^{n}}_{e}{}^{\cdot \frac{n+1}{n}}\right]} =$$

$$= \lim_{n\to\infty} \frac{1}{\ln\left(e^{\frac{n+1}{n}}\right)} = \lim_{n\to\infty} \frac{1}{\ln e} = \frac{1}{1} = 1$$

$, 1 \in \overline{\mathbb{R}}$

$\Rightarrow \exists \lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 1$

h) $x_n = \left(\dfrac{1+\frac{1}{2}+\ldots+\frac{1}{n}}{n}\right)^a$

Fie $y_n = \dfrac{1}{n^a}$, .

$\lim\limits_{n\to\infty} \dfrac{x_n}{y_n} = \dfrac{\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)^a}{\cancel{n^a}} \cdot \cancel{n^a} =$

$= \left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)^a \to \infty > 0$ grao

dacă $a=1$

$\Rightarrow x_n$ divergent, $c=1$