3.3.10 (a) $f \in Hom_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2)$

$\Rightarrow f: \mathbb{R}^3 \to \mathbb{R}^2$ - este aplicaţie liniară
f este aplicaţie liniară $\Leftrightarrow$
$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$, unde $\alpha, \beta \in \mathbb{R}$
şi $x = [x_1, x_2, x_3]$, $y = [y_1, y_2, y_3] \in \mathbb{R}^3$

$f[x_1, x_2, x_3] = [x_2, -x_1]$

$f(\alpha x + \beta y) = f(\alpha [x_1, x_2, x_3] + \beta [y_1, y_2, y_3])$
$= f[\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3] =$
$= [\alpha x_2 + \beta y_2, -(\alpha x_1 + \beta y_1)]$ (1)

$\alpha f(x) + \beta f(y) = \alpha f[x_1, x_2, x_3] + \beta f[y_1, y_2, y_3] =$
$= \alpha [x_2, -x_1] + \beta [y_2, -y_1] =$
$= [\alpha x_2, -\alpha x_1] + [\beta y_2, -\beta y_1] =$
$= [\alpha x_2 + \beta y_2, -\alpha x_1 - \beta y_1] =$
$= [\alpha x_2 + \beta y_2, -(\alpha x_1 + \beta y_1)]$ (2)

din (1) şi (2) $\Rightarrow f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$
$\Rightarrow f: \mathbb{R}^3 \to \mathbb{R}^2$ este aplicaţie liniară $\Rightarrow f \in Hom_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2)$

(b) $v = [[1,1,0], [0,1,1], [1,0,1]]^T$
este bază a lui $\mathbb{R}^3$; $\dim \mathbb{R}^3 = 3$ $\xrightarrow{\text{T alternativa}}$
$\Leftrightarrow \langle v \rangle = \mathbb{R}^3$?

Fie $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ $\quad 1 \leq \text{rang } A \leq 3$

$d_2 = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 \neq 0 \Rightarrow \text{rang } A = 2$

$$d_3 = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 1+1+0-0-0-0 = 2 \neq 0 \Rightarrow rang\, A = 3$$

$= \langle v \rangle = \mathbb{R}^3 \rightarrow$ v este bază a lui $\mathbb{R}^3$

$w = [[1,1],[1,-2]]^t$ este bază a lui $\mathbb{R}^2$,

$\dim \mathbb{R}^2 = 2 \xRightarrow{\text{T. alternativei}} \langle w \rangle = \mathbb{R}^2$?

Fie $A = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$ $\quad 1 \leq rang\, A \leq 2$

$$d_2 = \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = -2-1 = -3 \neq 0 \Rightarrow rang\, A = 2$$

$\Rightarrow \langle w \rangle = \mathbb{R}^2 \rightarrow$ w este bază a lui $\mathbb{R}^2$

sistem de vectori în e (baza canonică)

$[f]_{v,e} = [f(v)]_e \in M_{3\times 2}(\mathbb{R})$

$f[x_1, x_2, x_3] = (x_2, -x_1]$

$v = [[1,1,0],[0,1,1],[1,0,1]]^t = [v_1, v_2, v_3]^t$

$e = [[1,0],[0,1]]^t = [e_1, e_2]^t$

$f(v_1) = a_{11} \cdot e_1 + a_{12} e_2$

$f(v_2) = a_{21} \cdot e_1 + a_{22} e_2$ $\qquad [f]_{v,e} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$

$f(v_3) = a_{31} \cdot e_1 + a_{32} e_2$

$f[1,1,0] = [1,-1] = a_{1,1} \cdot [1,0] + a_{1,2} \cdot [0,1]$

$f[0,1,1] = [1,0] = a_{2,1} \cdot [1,0] + a_{2,2} [0,1]$

$f[1,0,1] = [0,-1] = a_{3,1} \cdot [1,0] + a_{32} [0,1]$

$[1,-1] = [a_{1,1}, a_{12}]$

$[1,0] = [a_{21}, a_{22}]$ $\quad \Rightarrow [f]_{v,e} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & -1 \end{pmatrix}$

$[0,-1] = [a_{31}, a_{32}]$

$f[k_1, k_2, k_3] = [k_2, -k_1]$

$[f]_{v,w} = [f(v)]_w \in M_{3\times 2}(\mathbb{R})$

$$[f]_{v,w} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

$v = [v_1, v_2, v_3]^t = [[1,1,0], [0,1,1], [1,0,1]]^t$

$w = [w_1, w_2]^t = [[1,1], [1,-2]]^t$

$f(v_1) = a_{11} \cdot w_1 + a_{12}\, w_2$

$f(v_2) = a_{21}\, w_1 + a_{22}\, w_2$

$f(v_3) = a_{31}\, w_1 + a_{32}\, w_2$

$[1,-1] = a_{11} \cdot [1,1] + a_{12}\, [1,-2]$

$[1,0] = a_{21} \cdot [1,1] + a_{22}\, [1,-2]$

$[0,-1] = a_{31} \cdot [1,1] + a_{32}\, [1,-2]$

$[1,-1] = [a_{11} + a_{12},\ a_{11} - 2a_{12}]$

$[1,0] = [a_{21} + a_{22},\ a_{21} - 2a_{22}]$

$[0,-1] = [a_{31} + a_{32},\ a_{31} - a_{32}]$

$$\begin{cases} a_{11} + a_{12} = 1 \\ a_{11} - 2a_{12} = -1 \quad | \cdot (-1) \end{cases}$$

$3a_{12} = 2 \qquad a_{12} = \frac{2}{3}$

$a_{11} = 1 - \frac{2}{3} = \frac{1}{3}$

$$\begin{cases} a_{21} + a_{22} = 1 \\ a_{21} - 2a_{22} = 0 \quad | \cdot (-1) \end{cases}$$

$3a_{22} = 1 \Rightarrow a_{22} = \frac{1}{3}$

$a_{21} = 1 - \frac{1}{3} = \frac{2}{3}$

$$\begin{cases} a_{31} + a_{32} = 0 \\ a_{31} - 2a_{32} = -1 \quad | \cdot (-1) \end{cases}$$

$$3a_{32} = 1 \quad \Rightarrow \quad a_{32} = \frac{1}{3}$$

$$a_{31} = 0 - \frac{1}{3}$$

$$[f]_{v,w} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

(c) $\mathrm{Ker}(f) = \{ x \in \mathbb{R}^3 \mid f(x) = 0 \}$

$\mathrm{Im}(f) = \{ f(x) \mid x \in \mathbb{R}^3 \}$

$\dim \mathbb{R}^3 = \dim \mathrm{Ker} f + \dim \mathrm{Im} f$

$\mathrm{Im} f = \mathbb{R}^2 \Rightarrow \dim \mathbb{R}^2 = 2$

$3 = \dim \ker f + 2$

$\dim \mathrm{Ker} f = 1$

$x = [x_1, x_2, x_3] \in \mathbb{R}^3 \qquad 0 = [0, 0] \in \mathbb{R}^2$

$f(x) = 0$

$f[x_1, x_2, x_3] = [0, 0]$

$[x\cancel{x}_1, -x_1] = [0, 0]$

$x_2 = 0$

$x_1 = 0$

$x_3 \in \mathbb{R}$

$\ker f = \{ [0, 0, x_3] \mid x_3 \in \mathbb{R} \}$

~~$\mathrm{Im} f = \mathbb{R}^2$~~

$\mathrm{Im} f = \mathbb{R}^2$

$e = [e_1, e_2]$ - baza canonică pentru $\mathbb{R}^2$

Bază pentru nucleu?.