Q4: Find a parametric representation of the plane containing three points $(1,0,0)$, $(2,1,0)$ and $(1,0,3)$. (20 points)

Method one:

1. Find the normal vector (8 points)

   Identify two vectors sitting on the plane:

   $$\vec{v}_1 = (2,1,0) - (1,0,0) = (1,1,0),$$

   $$\vec{v}_2 = (1,0,3) - (1,0,0) = (0,0,3).$$

   Normal vector:

   $$\mathbf{N} = \vec{v}_1 \times \vec{v}_2 = (3,-3,0).(or = \vec{v}_2 \times \vec{v}_1 = (-3,3,0))$$

2. Plane equation (6 points)

   Thus, the plane equation is given by,

   $$\begin{aligned}&\mathbf{N} \cdot [(x,y,z) - (1,0,0)] = 0\\ &3(x-1) - 3(y-0) + 0(z-0) = 0\\ &x - 1 - y = 0\end{aligned}$$

3. Parametric representation (6 points)

   From the equation above, we could parametrize the plane by

   $$\begin{cases} x = x \\ y = x - 1 \\ z = z \end{cases}$$

Note: two different parametrization variables should be used. Here we use x and z. You can have different notations.

Method two:

1. Plane equation (14 points)

   Consider the general equation for a plane:

   $$ax + by + cz + d = 0.$$

   Since the plane contains points $(1,0,0)$, $(2,1,0)$ and $(1,0,3)$, we obtain,

   $$\begin{aligned}&a + d = 0\\ &2a + b + d = 0\\ &a + 3c + d = 0\end{aligned}$$

The solution is,

$$\begin{cases} a = -d \\ b = d \\ c = 0 \end{cases}$$

Thus

$$\begin{aligned} -dx + dy + d = 0 \\ -x + y + 1 = 0 \end{aligned}$$

2. Parametric representation (6 points)

   From the equation above, we could parametrize the plane by

$$\begin{cases} x = x \\ y = x - 1 \\ z = z \end{cases}$$

Method three:

From a geometrical point of view, any vector starting from $(1, 0, 0)$ on the plane is a linear combination of $\vec{v}_1$ and $\vec{v}_2$ we have got above.(6 points)
i.e.

$$\begin{aligned} \vec{v} = (x, y, z) - (1, 0, 0) &= u\vec{v}_1 + v\vec{v}_2 \\ &= u(1, 1, 0) + v(0, 0, 3) \\ &= (u, u, 3v). \end{aligned}$$

Then we obtain,

$$\begin{cases} x = 1 + u \\ y = u \\ z = 3v. \qquad \text{(14 points)} \end{cases}$$