CURS 13

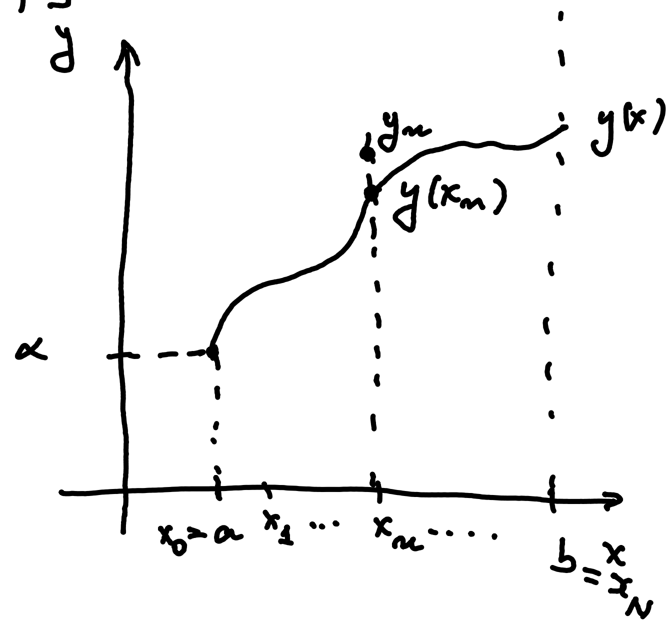
Metode de aproximare a soluțiilor ecuațiilor diferențiale

II Metode numerice

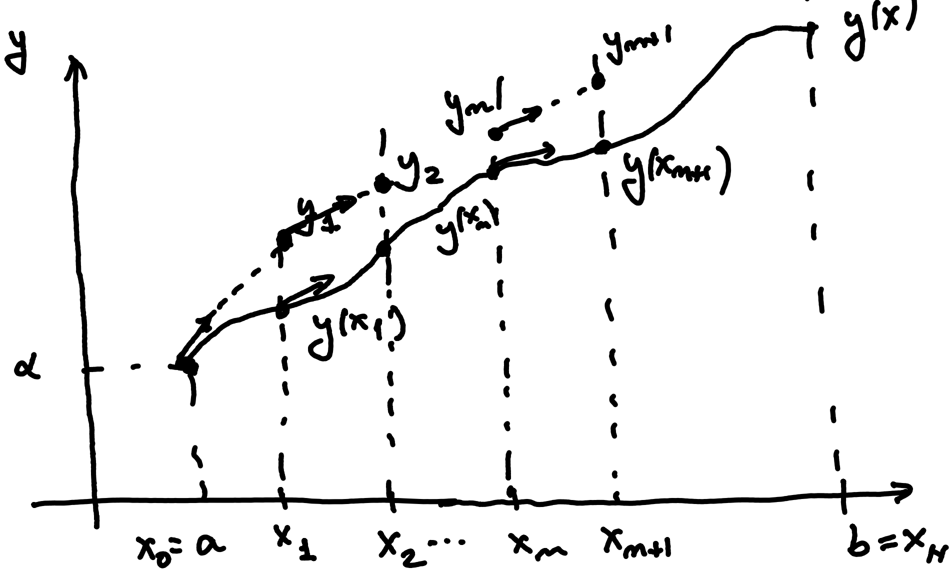
$$\begin{cases} y' = f(x,y), & x \in [a,b] \\ y(a) = \alpha \end{cases}$$

$$a = x_0 < x_1 < \ldots < x_n < \ldots < x_N = b$$

$$y(x_n) \simeq y_n, \quad n = 0, 1, \ldots, N$$



1) Metoda lui Euler



$x_0 = a \quad y(x_0) = \alpha = y_0$

$(x_0, y_0) \Rightarrow y'(x_0) = f(x_0, y(x_0)) = f(x_0, y_0)$

ec. dreptei ce pleacă din (x_0, y_0) și de panta $m = y'(x_0) = f(x_0, y_0)$

$$d_1 : y - y_0 = m(x - x_0) \Rightarrow y - y_0 = f(x_0, y_0)(x - x_0)$$

$(x_1, y_1) \in d_1$ – dreptei ce trece prin (x_0, y_0) și de pantă $m = y'(x_0)$

$$\Rightarrow y_1 - y_0 = f(x_0, y_0)(x_1 - x_0)$$

$\Rightarrow y_1 = y_0 + f(x_0, y_0)(x_1 - x_0) \qquad y_1 \simeq y(x_1)$

Se continuă procedeul având ca punct de pornire

$(x_1, y_1) \simeq (x_1, y(x_1))$

- se det. o aproximantă a tang. în $(x_1, y(x_1))$

$$y'(x_1) = f(x, \underset{y_1}{\underset{\simeq}{y(x_1)}}) \simeq f(x_1, y_1) = m$$

$$y - y_1 = m(x - x_1)$$

$$d_2 : y = y_1 + f(x_1, y_1)(x - x_1)$$

$(x_2, y_2) \in d_2$

$$\Rightarrow \boxed{y_2 = y_1 + f(x_1, y_1)(x_2 - x_1)}$$

$$y_2 \simeq y(x_2)$$

Procedeul se continuă până se ajunge în $x_N = b$.

(x_n, y_n)

$$y'(x_n) = f(x_n, \underbrace{y(x_n)}_{\simeq y_n}) \simeq f(x_n, y_n) = m$$

$$d_n: \quad y - y_n = m(x - x_n) \Rightarrow y - y_n = f(x_n, y_n)(x - x_n)$$

$(x_{n+1}, y_{n+1}) \in d_n$

$$y_{n+1} - y_n = f(x_n, y_n)(x_{n+1} - x_n)$$

$$\boxed{y_{n+1} = y_n + f(x_n, y_n)(x_{n+1} - x_n)}$$ formula lui Euler

$n = 0, \dots, N-1$

$$y_{n+1} \simeq y(x_{n+1})$$

În cazul nodurilor echidistante, adică:

$$x_{n+1} - x_n = h = \text{const} \qquad h - \text{pasul}$$

$$x_{n+1} = x_n + h \Rightarrow x_n = x_0 + nh$$

$$h = \frac{b-a}{N}$$

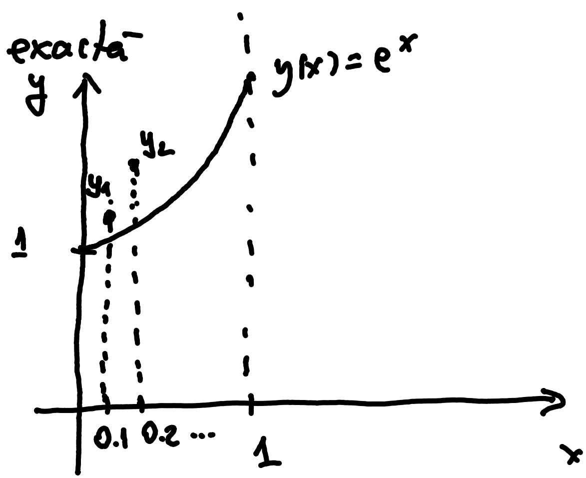
$$\Rightarrow \boxed{y_{n+1} = y_n + f(x_n, y_n) \cdot h} \quad \text{formula lui Euler cu pas echidistant}$$

$$n = 0, \dots, N-1$$

Exemplu

$$\begin{cases} y' = y \\ y(0) = 1 \end{cases} \qquad y(x) = e^x \text{ soluția exactă}$$

met. lui Euler cu pas $h = 0.1$ pe $[0,1]$



$h = 0.1 \Rightarrow \underline{N = 10}$

$x_0 = 0 \quad , y_0 = 1.$

$x_1 = 0.1$

$x_2 = 0.2$

$x_3 = 0.3$

$\vdots$

$x_{10} = \quad \underline{1}$

$f(x,y) = y$

$y_{n+1} = y_n + f(x_n, y_n) \cdot h \quad , n = 0, \dots, 9$

$y_{n+1} = y_n + y_n \cdot h$

$$\boxed{y_{n+1} = y_n (1+h)} \;, n = 0, \dots, 9$$

$y_1 = y_0 \cdot (1 + 0.1) = \underbrace{y_0}_{=1} \cdot 1.1 = 1.1$

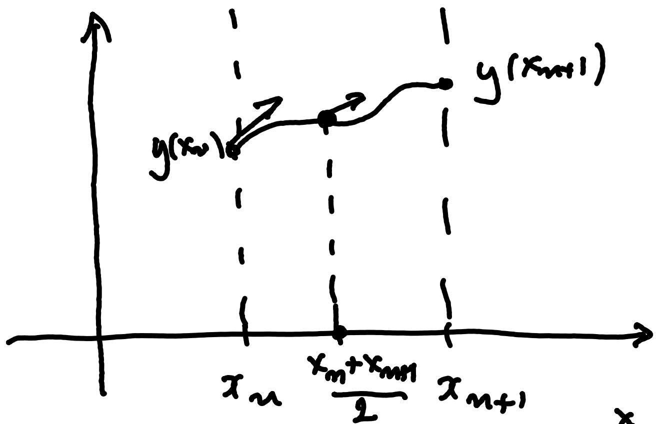
$y_2 = y_1 (1 + 0.1) = 1.1 \cdot 1.1 = (1.1)^2$

$y_3 = y_2 \cdot 1.1 = (1.1)^2 \cdot (1.1) = (1.1)^3$

$\vdots$

$y_{10} = (1.1)^{10}$

2) Metoda Euler modificată



$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}$$

$$y(x_{n+1}) - y(x_n) = \int_{x_n}^{x_{n+1}} y'(s)\,ds$$

$$y(x_{n+1}) - y(x_n) = \int_{x_n}^{x_{n+1}} f(s, y(s))\,ds.$$

$$y(x_{n+1}) = y(x_n) + (x_{n+1} - x_n) \cdot \underbrace{\left(\frac{1}{x_{n+1} - x_n} \cdot \int_{x_n}^{x_{n+1}} f(s, y(s))\,ds \right)}_{P_n}$$

$$P_n = \frac{1}{x_{n+1}-x_n} \int_{x_n}^{x_{n+1}} f(s, y(s))\, ds$$ — panta medie a soluției y pe $[x_n, x_{n+1}]$

- metoda Euler : $P_n \simeq y'(x_n) = f(x_n, y(x_n)) \simeq f(x_n, y_n)$
- metoda Euler modificată aproximează panta medie P_n printr-o aproximantă a pantei în mijl. interv. $[x_n, x_{n+1}]$.

$$P_n \simeq y'\left(\frac{x_n+x_{n+1}}{2}\right) = f\left(\frac{x_n+x_{n+1}}{2}, y\left(\frac{x_n+x_{n+1}}{2}\right)\right)$$

$$\Rightarrow \quad y_{n+1} = y_n + (x_{n+1}-x_n) \cdot f\left(\frac{x_n+x_{n+1}}{2}, \underbrace{y\left(\frac{x_n+x_{n+1}}{2}\right)}_{?}\right)$$

$y\left(\frac{x_n+x_{n+1}}{2}\right)$ — se aproximează prin metoda Euler clasică

$$y\left(\frac{x_n+x_{n+1}}{2}\right) \simeq y_n + f(x_n, y_n) \cdot \left(\frac{x_n+x_{n+1}}{2} - x_n\right) =$$

$$= y_n + f(x_n, y_n) \cdot \frac{x_{n+1}-x_n}{2}$$

$$\Rightarrow \boxed{y_{n+1} = y_n + (x_{n+1} - x_n) \cdot f\left(\frac{x_n + x_{n+1}}{2}, y_n + f(x_n, y_n) \cdot \frac{x_{n+1} - x_n}{2}\right)}$$

(x_n, y_n)

$$\begin{cases} K_1 = f(x_n, y_n) \\ K_2 = f\left(\frac{x_n + x_{n+1}}{2}, y_n + K_1 \cdot \frac{x_{n+1} - x_n}{2}\right) \\ y_{n+1} = y_n + K_2 \cdot (x_{n+1} - x_n) \end{cases}$$

Noduri echidistante: $x_{n+1} - x_n = h$

$$\begin{cases} x_n = x_0 + nh \\ K_1 = f(x_n, y_n) \\ K_2 = f\left(x_n + \frac{h}{2}, y_n + K_1 \cdot \frac{h}{2}\right) \\ y_{n+1} = y_n + K_2 \cdot h \end{cases}$$

3) Metode de tip Runge-Kutta

- sunt formate dintr-o succesiune de etape, fiecare etapă evaluând o val. aprox. a pantei soluției
- pasul final utilizează o medie ponderată a pantelor calculate în etapele anterioare.
- vom considera cazul nodurilor echidistante:

$$x_{n+1} - x_n = h \qquad h = \frac{b-a}{N}$$

(x_n, y_n):

$$K_1 = f(x_n, y_n)$$

$$K_2 = f(x_n + c_2 \cdot h,\ y_n + h \cdot a_{21} \cdot K_1)$$

$$K_3 = f(x_n + c_3 \cdot h,\ y_n + h\,(a_{31} K_1 + a_{32} \cdot K_2))$$

$$\vdots$$

$$K_s = f\left(x_n + c_s h,\ y_n + h \cdot \left(\sum_{j=1}^{s-1} a_{sj} \cdot K_j\right)\right)$$

$$\Rightarrow \boxed{y_{n+1} = y_n + h \cdot (b_1 K_1 + b_2 K_2 + \ldots + b_s \cdot K_s)}$$

$$
\begin{array}{c|ccccc}
0 & & & & & \\
c_2 & a_{21} & & & & \\
c_3 & a_{31} & a_{32} & & & \\
\vdots & \vdots & & & & \\
c_s & a_{s1} & a_{s2} & \dots & a_{ss-1} & \\
\hline
 & b_1 & b_2 & & b_{s-1} & b_s
\end{array}
$$

Metoda Runge-Kutta cu s stagii

Metoda Euler modificată este o metodă Runge-Kutta cu 2 stagii:

(RK2)
$$
\begin{array}{c|cc}
0 & & \\
\frac{1}{2} & \frac{1}{2} & \\
\hline
 & 0 & 1
\end{array}
$$

metodă Runge-Kutta cu 3 stagii:

(RK3)

$$\begin{array}{c|ccc} 0 & & & \\ \frac{1}{3} & \frac{1}{3} & & \\ \frac{2}{3} & 0 & \frac{2}{3} & \\ \hline & \frac{1}{4} & 0 & \frac{3}{4} \end{array}$$

(x_n, y_n)

$$K_1 = f(x_n, y_n)$$

$$K_2 = f\left(x_n + \frac{1}{3}h, y_n + \frac{1}{3}h \cdot K_1\right)$$

$$K_3 = f\left(x_n + \frac{2}{3}h, y_n + \frac{2}{3}h \cdot K_2\right)$$

$$y_{n+1} = y_n + h \cdot \left(\frac{K_1}{4} + \frac{3}{4} \cdot K_3\right)$$

(RK4)

$$
\begin{array}{c|cccc}
0 & & & & \\
\frac{1}{2} & \frac{1}{2} & & & \\
\frac{1}{2} & 0 & \frac{1}{2} & & \\
1 & 0 & 0 & 1 & \\
\hline
 & \frac{1}{6} & \frac{2}{6} & \frac{2}{6} & \frac{1}{6}
\end{array}
$$

(x_n, y_n)

$$K_1 = f(x_n, y_n)$$

$$K_2 = f\left(x_n + \frac{1}{2}h, \; y_n + \frac{h}{2} \cdot K_1\right)$$

$$K_3 = f\left(x_n + \frac{1}{2}h, \; y_n + \frac{h \cdot K_2}{2}\right)$$

$$K_4 = f\left(x_n + h, \; y_n + h \cdot K_3\right)$$

$$\boxed{y_{n+1} = y_n + h\left(\frac{K_1}{6} + \frac{2}{6} \cdot K_2 + \frac{2}{6}K_3 + \frac{1}{6}K_4\right)}$$