Temă

3.3.10.

$f: \mathbb{R}^3 \to \mathbb{R}^2$, $f[x_1, x_2, x_3] = [x_2, -x_1]$

$v = [(1,1,0), (0,1,1), (1,0,1)]^*$

$w = [(1,1), (1,-2)]^*$

a) $f \in Hom_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2) \iff f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$

$\forall \alpha, \beta \in \mathbb{R}$ și $x, y \in \mathbb{R}^3$

$x = (x_1, x_2, x_3)$ și $y = (y_1, y_2, y_3)$

$$f(\alpha x + \beta y) = f(\alpha(x_1, x_2, x_3) + \beta(y_1, y_2, y_3))$$
$$= f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3)$$
$$= (\alpha x_2 + \beta y_2, -\alpha x_1 - \beta y_1)$$
$$= (\alpha x_2, -\alpha x_1) + (\beta y_2, -\beta y_1)$$
$$= \alpha(x_2, -x_1) + \beta(y_2, -y_1) = \alpha f(x) + \beta f(y) \Rightarrow$$

$\Rightarrow f \in Hom_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2)$

b) v bază în $\mathbb{R}^3$ și w bază în $\mathbb{R}^2$.

$v = \{v_1, v_2, v_3\}$, $w = \{w_1, w_2\}$

	v_1	v_2	v_3
e_1	$\boxed{1}$	0	1
e_2	1	1	0
e_3	0	1	1
v_1	1	0	1
e_2	0	$\boxed{1}$	-1
e_3	0	1	1
v_1	1	0	1
v_2	0	1	1
e_3	0	0	$\boxed{2}$
v_1	1	0	0
v_2	0	1	0
v_3	0	0	1

$\Rightarrow v$ este bază în $\mathbb{R}^3$

W - bază în $\mathbb{R}^2 \iff \det A \neq 0$

$$A = \begin{array}{c} \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \\ \quad w_1 \quad w_2 \end{array}$$

(card W = ~~card R~~ $\dim \mathbb{R}^2$)

$$\det A = \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = -3 \neq 0 \Rightarrow$$

$\Rightarrow \operatorname{rang} A = 2$ (maxim) $\Rightarrow W$ - bază în $\mathbb{R}^2$

$[f]_{v,e} = [f(v)]_e \in M_{3\times2}(R)$

$f(v) = f(<v_1, v_2, v_3>) = <f(v_1), f(v_2), f(v_3)>$

$f(<x>) = <f(x)>$, $\forall x$ - listă de vectori din R^n

$f(v) = <f(1,1,0), f(0,1,1), f(1,0,1)>$

$= <\underset{f_1}{(1,-1)}, \underset{f_2}{(1,0)}, \underset{f_3}{(0,-1)}>$

$[f]_{v,e} = [(f_1)_e, (f_2)_e, (f_3)_e]^t \in M_{3\times2}$.

$(f_1)_e = a_{11} e_1 + a_{12} e_2 \Leftrightarrow (1,-1) = a_{11}(1,0) + a_{12}(0,1) \Rightarrow$

$\Rightarrow \begin{cases} a_{11} = 1 \\ a_{12} = -1 \end{cases}$

$(f_2)_e = a_{21} e_1 + a_{22} e_2 \Rightarrow \begin{cases} a_{21} = 1 \\ a_{22} = 0 \end{cases}$

$(f_3)_e = a_{31} e_1 + a_{32} e_2 \Rightarrow \begin{cases} a_{31} = 0 \\ a_{32} = -1 \end{cases}$

$[f]_{v,e} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$

$[f]_{v,u} = [f(v)]_u \in M_{3\times2}(R) = [(f_1)_u, (f_2)_u, (f_3)_u]^t \in M_{3\times2}(R)$

$(f_1)_u = b_{11} u_1 + b_{21} u_2 \Leftrightarrow (1,-1) = b_{11}(1,1) + b_{12}(1,-2)$

$b_{11} + b_{12} = 1$

$\underline{b_{11} - 2b_{12} = -1} \quad (-)$

$3b_{12} = 2 \Rightarrow b_{12} = \frac{2}{3} \Rightarrow b_{11} = \frac{1}{3}$

$(f_2)_u = b_{21} u_1 + b_{22} u_2 \Rightarrow b_{21} = \frac{2}{3}, b_{22} = \frac{1}{3}$

$(f_3)_u = b_{31} u_1 + b_{32} u_2 \Rightarrow b_{31} = \frac{2}{3}, b_{32} = \frac{1}{3}$

$[f]_{v,u} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} \in M_{3\times2}(R)$

c) $\dim \operatorname{Ker} f = ?$ $\dim \operatorname{Im} f = ?$

bază $\operatorname{Ker} f = ?$ ~~dim Ker f~~ $\Rightarrow$ bază $\operatorname{Im} f = ?$

$\operatorname{Im} f = f(R^3)$

v - bază în R^3

$\Rightarrow \operatorname{Im} f = f(v) = \langle \underset{f_1}{(1,-1)}, \underset{f_2}{(1,0)}, \underset{f_3}{(0,-1)} \rangle$

Se observă că $f_1 = f_2 + f_3 \Rightarrow \operatorname{Im} f = \langle f_2, f_3 \rangle \Rightarrow$

$\Rightarrow B_{\operatorname{Im} f} = \{(1,0), (0,-1)\} \Rightarrow \dim \operatorname{Im} f = 2$

(bază Im f)

$\dim R^3 = \dim \operatorname{Im} f + \dim \operatorname{Ker} f \Rightarrow$

$\Rightarrow \dim \operatorname{Ker} f = \dim R^3 - \dim \operatorname{Im} f = 3 - 2 = 1 \Rightarrow$

$\Rightarrow \dim \operatorname{Ker} f = 1$

$f(v_1) = f(v_2) + f(v_3) \Leftrightarrow f(v_1) - f(v_2) - f(v_3) = 0$

$f(v_1 - v_2 - v_3) = 0 \Rightarrow \langle (1,-1,-1) \rangle \operatorname{Ker} f \Rightarrow$

$B_{\operatorname{Ker} f} = \{(1,-1,-1)\}$

3.3.11

$f: R^4 \to R^4$, $f[x_1, x_2, x_3, x_4] = [x_1 + 2x_2 + x_3 + x_4, 3x_1 + 7x_2 + 5x_3 + 2x_4, x_1 + 3x_2 + 3x_3, 4x_1 + 9x_2 + x_3 + 8x_4]$, e - bază canonică în R^4

a) $f \in End_R(R^4)$ sau $f \in Hom_R(R^4, R^4)$

$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y), \forall \alpha, \beta \in R$ și $x, y \in R^4$

$x = (x_1, x_2, x_3, x_4)$ și $y = (y_1, y_2, y_3, y_4)$

$f(\alpha x + \beta y) = f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3, \alpha x_4 + \beta y_4)$

$= (\alpha x_1 + \beta y_1 + 2\alpha x_2 + 2\beta y_2 + \alpha x_3 + \beta x_3 + \alpha x_4 + \beta x_4, 3\alpha x_1 + 3\beta y_1 + 7\alpha x_2 + 7\beta y_2 + 5\alpha x_3 + 5\beta x_3 + 2\alpha x_4 + 2\beta y_4, \alpha x_1 + \beta y_1 + 3\alpha x_2 + 3\beta y_2 + 3\alpha x_3 + 3\beta y_3, 4\alpha x_1 + 4\beta y_1 + 9\alpha x_2 + 9\beta x_2 + \alpha x_3 + \beta y_3 + 8\alpha x_4 + 8\beta x_4)$

$= (\alpha x_1 + 2\alpha x_2 + \alpha x_3 + \alpha x_4, 3\alpha x_1 + 7\alpha x_2 + 5\alpha x_3 + 2\alpha x_4, \alpha x_1 + 3\alpha x_2 + 3\alpha x_3, 4\alpha x_1 + 9\alpha x_2 + \alpha x_3 + 8\alpha x_4) + (\beta y_1 + 2\beta y_2 + \beta y_3 + \beta y_4, 3\beta y_1 + 7\beta y_2 + 5\beta y_3 + 2\beta y_4, \beta y_1 + 3\beta y_2 + 3\beta y_3, 4\beta y_1 + 9\beta y_2 + \beta y_3 + 8\beta y_4)$

$= \alpha(x_1+2x_2+x_3+x_4, 3x_1+7x_2+5x_3+2x_4, x_1+3x_2+3x_3, 4x_1+9x_2+x_3+8x_4)$

$+\beta(y_1+2y_2+y_3+y_4, 3y_1+7y_2+5y_3+2y_4, y_1+3y_2+3y_3, 4y_1+9y_2+y_3+8y_4)$

$= \alpha f(x) + \beta f(y) \Rightarrow f \in End_R(R^4)$

b) $[f]_e = [f(e)_e] \in M_{4\times4}(R)$

$f(e) = f(<e_1, e_2, e_3, e_4>) = <f(e_1), f(e_2), f(e_3), f(e_4)>$

~~$(= <(1,0,0,0), ()$~~

$= <\underset{f_1}{(1,3,1,4)}, \underset{f_2}{(2,7,3,9)}, \underset{f_3}{(1,5,3,1)}, \underset{f_3}{(1,2,0,8)}>$

$(f_1)_e = a_{11}e_1 + a_{12}e_2 + a_{13}e_3 + a_{14}e_4 \Rightarrow \begin{cases} a_{11}=1 & a_{13}=1 \\ a_{12}=3 & a_{14}=4 \end{cases}$

$(f_2)_e = 2e_1 + 7e_2 + 3e_3 + 9e_4.$

$(f_3)_e = 1e_1 + 5e_2 + 3e_3 + 1e_4$

$(f_4)_e = 1e_1 + 2e_2 + 0e_3 + 8e_4.$

$$[f]_e = \begin{pmatrix} 1 & 3 & 1 & 4 \\ 2 & 7 & 3 & 9 \\ 1 & 5 & 3 & 1 \\ 1 & 2 & 0 & 8 \end{pmatrix} \begin{array}{l} \to f_1 \\ \to f_2 \\ \to f_3 \\ \to f_4 \end{array}$$

c) $b = [b_1, b_2, b_3, b_4]$

$b_1 = e_1 = (1,0,0,0)$

$b_2 = e_1+e_2 = (1,1,0,0)$

$b_3 = e_1+e_2+e_3 \cancel{+e_4} = (1,1,1,0)$

$b_4 = e_1+e_2+e_3+e_4 = (1,1,1,1)$

$v = [1,2,-1,0]$

$[f]_b = [f(b)]_b \in M_4(R)$

$f(b) = <f(b_1), f(b_2), f(b_3), f(b_4)>$

$= < \underset{f_1}{(1,3,1,4)}, \underset{f_2}{(3,10,4,13)}, \underset{f_3}{(4,15,7,14)}, \underset{f_4}{(5,17,7,22)} >$

$(f_1)e = a_{11} b_1 + a_{12} b_2 + a_{13} b_3 + a_{14} b_4.$

$\cancel{b_1 = 1}$ $\quad$ $\cancel{a_{11} = 1}$

$\cancel{b_1 + b_2 =}$ $\quad$ $\cancel{a_{11} + a_{12} = 3}$

$\cancel{a_{11} + a_{12} +}$

$$\begin{cases} a_{11} + a_{12} + a_{13} + a_{14} = 7 \Rightarrow a_{11} = -5 \\ a_{12} + a_{13} + a_{14} = 3 \Rightarrow a_{12} = 6 \\ a_{13} + a_{14} = 1 \Rightarrow a_{13} = -3 \\ a_{14} = 4 \end{cases}$$

$(f_2)e = 13 b_4 - 7 b_3 + 17 b_2 - 14 b_1$

$(f_3)e = 14 b_4 - 7 b_3 + 22 b_2 - 18 b_1.$

$(f_4)e = 22 b_4 - 15 b_3 + 32 b_2 - 27 b_1.$

$$[f]_e = \begin{pmatrix} -5 & 6 & -3 & 4 \\ -14 & 17 & -7 & 13 \\ -18 & 22 & -7 & 14 \\ -27 & -15 & 32 & 22 \end{pmatrix}$$

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$(v)e = \alpha_1 b_1 + \alpha_2 b_2 + \alpha_3 b_3 + \alpha_4 b_4$

$$\Rightarrow \begin{cases} \alpha_1 = -1 & \alpha_3 = -1 \\ \alpha_2 = 3 & \alpha_4 = 0 \end{cases}$$

$$[v]_e = \begin{pmatrix} -1 & 3 & -1 & 0 \end{pmatrix}^t = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_e = \begin{pmatrix} -1 \\ 3 \\ -1 \\ 0 \end{pmatrix}_e$$

d) $\operatorname{Im} f = f(R^4) = f(e) = \langle f(e_1), f(e_2), f(e_3), f(e_4) \rangle$

$= \langle \underbrace{(1,3,1,4)}_{f_1}, \underbrace{(3,10,4,13)}_{f_2}, \underbrace{(4,15,7,14)}_{f_3}, \underbrace{(5,17,7,22)}_{f_4} \rangle$

	f_1	f_2	f_3	f_4
e_1	1	3	4	5
e_2	3	10	15	17
e_3	1	4	7	7
e_4	4	13	14	22

f_1	1	3	4	5
e_2	0	$\underline{1}$	3	2
e_3	0	1	3	2
e_4	0	1	-2	2
f_1	1	0	-5	-1
f_2	0	1	3	2.
e_3	0	0	0	0
e_4	0	0	$\underline{-5}$	0
f_1	1	0	0	-1
f_2	0	1	0	-2
e_3	0	0	0	0
f_4	0	0	1	0

CALCULE GREȘITE

$B_{\mathrm{Im} f} = \{f_1, f_2, f_4\} \Rightarrow \dim \mathrm{Im} f = 3$

$f_3 = -f_1 - 2f_2$

$\dim \mathbb{R}^4 = \dim \mathrm{Ker} f + \dim \mathrm{Im} f \Rightarrow \dim \mathrm{Ker} f = 1$

$f_3 = -f_1 - 2f_2 \Leftrightarrow f_3 + f_1 + 2f_2 = 0_{\mathbb{R}} \Leftrightarrow$

$f(e_3 + e_1 + 2e_2 + 0e_4) = 0_{\mathbb{R}} \Rightarrow$

$\Rightarrow B_{\mathrm{Ker} f} = \{(1, 2, 1, 0)\}$