Geometrie 11.9

Se consideră pătratul $ABCD$, de vârfuri $A(0,0)$, $B(2,0)$, $C(2,2)$, $D(0,2)$.
Demonstrați că patrulaterul $A'B'C'D'$, cu $A'\left(3-\frac{\sqrt{2}}{2}, -1-\frac{3\sqrt{2}}{2}\right)$, $B'\left(3+\frac{3\sqrt{2}}{2}, -1+\frac{\sqrt{2}}{2}\right)$, $C'\left(3+\frac{\sqrt{2}}{2}, -1+\frac{3\sqrt{2}}{2}\right)$, $D'\left(3-\frac{3\sqrt{2}}{2}, -1-\frac{\sqrt{2}}{2}\right)$ este un dreptunghi și indicați o secvență de transformări geometrice care transformă pătratul în dreptunghi.

$A'B'C'D'$ este dreptunghi $\Leftrightarrow \sphericalangle A' = \sphericalangle B' = \sphericalangle C' = \sphericalangle D' = 90^\circ$

$$A'B' = \sqrt{(x_{A'}-x_{B'})^2 + (y_{A'}-y_{B'})} =$$

$$= \sqrt{\left(3-\frac{\sqrt{2}}{2}-3-\frac{3\sqrt{2}}{2}\right)^2 + \left(-1-\frac{3\sqrt{2}}{2}+1-\frac{\sqrt{2}}{2}\right)^2}$$

$$= \sqrt{(-2\sqrt{2})^2 + (-2\sqrt{2})^2} = \sqrt{8+8} = 4$$

$$A'D' = \sqrt{\left(3-\frac{\sqrt{2}}{2}-3+\frac{3\sqrt{2}}{2}\right)^2 + \left(-1-\frac{3\sqrt{2}}{2}+1+\frac{\sqrt{2}}{2}\right)^2}$$

$$= \sqrt{(\sqrt{2})^2 + (-\sqrt{2})^2} = \sqrt{4} = 2$$

$$B'D' = \sqrt{\left(3+\frac{3\sqrt{2}}{2}-3+\frac{3\sqrt{2}}{2}\right)^2 + \left(-1+\frac{\sqrt{2}}{2}+1+\frac{\sqrt{2}}{2}\right)^2} = \sqrt{20} = 2\sqrt{5}$$

$$= \sqrt{(3\sqrt{2})^2 + (\sqrt{2})^2}$$

Pr. $\Delta A'B'D'$ dr. în A'

T.F.P $\Rightarrow B'D'^2 = A'B'^2 + A'D'^2$

$(2\sqrt{5})^2 = 16 + 4$

$20 = 20$ „A"

$\Rightarrow \angle A' = 90^\circ$ (1)

$C'D' = \sqrt{\left(\frac{2\sqrt{2}}{2}\right)^2 + \left(\frac{2\sqrt{2}}{2}\right)^2}$

$= \sqrt{8+8} = 4$

$C'B' = \sqrt{\left(-\frac{2\sqrt{2}}{2}\right)^2 + \left(\frac{2\sqrt{2}}{2}\right)^2}$

$= \sqrt{2+2} = 2$

$A'C' = \sqrt{\left(-\frac{2\sqrt{2}}{2}\right)^2 + \left(\frac{-6\sqrt{2}}{2}\right)^2}$

$= \sqrt{2+18} = 2\sqrt{5}$

Pr. $\Delta C'B'D'$ dr. în C'

T.F.P $\Rightarrow A'C'^2 = A'B'^2 + B'C'^2$

$20 = 16 + 4$

$20 = 20$ „A"

$\Rightarrow \angle C' = 90^\circ$ (2)

$(1), (2) \Rightarrow \sphericalangle B' = \sphericalangle D' = 90^\circ$

$\Rightarrow A'B'C'D'$ este dreptunghi

Aplicăm translația care să ducă centrul pătratului în centrul dreptunghiului.

Fie M centrul pătratului $\Rightarrow M(1, 1)$
Fie N centrul dreptunghiului $\Rightarrow N(3, -1)$

$$T(x, y) = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

$\vec{v}(x, y)$

$$\vec{MN} = (2, -2) \Rightarrow T(2, -2) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

Aplicăm scalarea neuniformă:

$$S(Q, s_x, s_y) = \begin{pmatrix} s_x & 0 & (1-s_x)\, q_1 \\ 0 & s_y & (1-s_y)\, q_2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\cancel{\left.\begin{matrix} A'B' = 4 \\ AB = 2 \end{matrix}\right\} \Rightarrow S_x = \dots}$$

$$\left.\begin{matrix} A'B' = 4 \\ AB = 2 \end{matrix}\right\} \Rightarrow s_x = \frac{A'B'}{AB} = 2$$

$$\left.\begin{matrix} B'C' = 2 \\ BC = 2 \end{matrix}\right\} \Rightarrow s_y = \frac{B'C'}{BC} = 1$$

$$S(3, -1, 2, 1) = \begin{pmatrix} 2 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rotatia: $\cancel{R(Q, \theta) = \left(\cos\theta \quad -\sin\theta \right.}$

$\mathbb{R}$

Translatia

$$\begin{cases} 3 = 1 + w_1 \\ -1 = 1 + w_2 \end{cases} \Rightarrow \begin{cases} w_1 = 2 \\ w_2 = -2 \end{cases}$$

$$T = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

I. $$\begin{pmatrix} x_{At} \\ y_{At} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_A \\ y_A \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \Rightarrow A_t(2, -2)$$

II. $$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} \Rightarrow B_t(4, -2)$$

III. $$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} \Rightarrow C_t(4, 0)$$

IV. $$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \Rightarrow D_t(2, 0).$$

Rotation : $N(3,-1)$

$$Rot(Q,\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & q_1(1-\cos\theta) + q_2 \sin\theta \\ \sin\theta & \cos\theta & -q_1 \sin\theta + q_2(1-\cos\theta) \\ 0 & 0 & 1 \end{bmatrix}$$

$$Rot(N,\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 3(1-\cos\theta) - \sin\theta \\ \sin\theta & \cos\theta & -3\sin\theta - (1-\cos\theta) \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta & 3 - 3\cos\theta - \sin\theta \\ \sin\theta & \cos\theta & -3\sin\theta - 1 + \cos\theta \\ 0 & 0 & 1 \end{pmatrix}$$

$$Rot(N,\theta) \cdot S(N,s) \cdot [A_t B_t C_t D_t] = [A' B' C' D'] \quad (1)$$

$$Rot(N,\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix}$$

$$Rot(N,\theta) \cdot S(N,s) = \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\cancel{= \begin{pmatrix} 2\cos\theta & -\sin\theta & -3\cos\theta + x \\ 2\sin\theta & \cos\theta & -3\sin\theta + y \end{pmatrix}}$$

$$= \begin{pmatrix} 2\cos\theta & -\sin\theta & -3\cos\theta + x \\ 2\sin\theta & \cos\theta & -3\sin\theta + y \\ 0 & 0 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} 2\cos\theta & -\sin\theta & -3\cos\theta+3-3\cos\theta-\sin\theta \\ 2\sin\theta & \cos\theta & -3\sin\theta-3\sin\theta-1+\cos\theta \\ 0 & 0 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} 2\cos\theta & -\sin\theta & -6\cos\theta-\sin\theta+3 \\ 2\sin\theta & \cos\theta & -6\sin\theta+\cos\theta-1 \\ 0 & 0 & 1 \end{pmatrix}$$

$Rot(N,\theta)\cdot S(N,s)\cdot(A_t\ B_t\ C_t\ D_t)$

$$=\begin{pmatrix} 2\cos\theta & -\sin\theta & a \\ 2\sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{pmatrix}\cdot$$

$$=\begin{pmatrix} 2\cos\theta & -\sin\theta & a \\ 2\sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} 2 & 4 & 4 & 2 \\ -2 & -2 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}=$$

$$=\begin{pmatrix} 2\cos\theta+2\sin\theta+a & 8\cos\theta+2\sin\theta+a & 8\cos\theta+a & 4\cos\theta+a \\ 4\sin\theta-2\cos\theta+b & 8\sin\theta-2\cos\theta+b & 8\sin\theta+b & 4\sin\theta+b \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\overset{not}{=} M$$

$$Din\ (1) \Rightarrow M=[A'B'C'D']=\begin{pmatrix} 3-\frac{\sqrt2}{2} & 3+\frac{3\sqrt2}{2} \\ -1-\frac{3\sqrt2}{2} & -1+\frac{\sqrt2}{2} \\ 1 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} 3-\frac{\sqrt2}{2} & 3+\frac{3\sqrt2}{2} & 3+\frac{\sqrt2}{2} & 3-\frac{3\sqrt2}{2} \\ -1-\frac{3\sqrt2}{2} & -1+\frac{\sqrt2}{2} & -1+\frac{3\sqrt2}{2} & -1-\frac{\sqrt2}{2} \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Din prima linie $\Rightarrow$

$$2\cos\theta + 2\sin\theta + a = 3 - \frac{\sqrt{2}}{2}$$

$$8\cos\theta + 2\sin\theta + a = 3 + \frac{3\sqrt{2}}{2}$$

$$\left.\begin{aligned} 8\cos\theta + a &= 3 + \frac{\sqrt{2}}{2} \\ 4\cos\theta + a &= 3 - \frac{3\sqrt{2}}{2} \end{aligned}\right\} \Rightarrow 4\cos\theta = \frac{\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}$$

$$\Rightarrow 4\cos\theta = \frac{4\sqrt{2}}{2} \Rightarrow \cos\theta = \frac{\sqrt{2}}{2}$$

$$\Rightarrow \theta = \frac{\pi}{4} = 45^\circ$$

$$R(N, 45^\circ) = R(3, -1, 45^\circ)$$

$$= \begin{pmatrix} \cos 45^\circ & -\sin 45^\circ & 3 \\ \sin 45^\circ & \cos 45^\circ & -1 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 3 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$T_R = T(2,-2) \cdot S(3,-1,2,1) \cdot R(3,-1,45°)$

$T\cancel{R} =$

$T(2,-2) \cdot S(3,-1,2,1) =$

$$= \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$T \cdot S \cdot R = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 3 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{2} & -\sqrt{2} & 5 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -3 \\ 0 & 0 & 1 \end{pmatrix}$$