Lab 1 – Formula lui Taylor

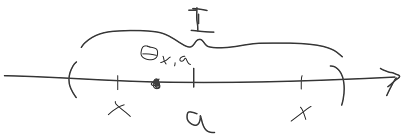
Wednesday, March 1, 2023 10:22 AM

I = interval deschis

C^k(I) = {f : I -> R: există f^(i), i = 0,k; f^(k) cont.}

T1 (formula lui Taylor cu rest Lagrange)

f ∈ C^n+1(I), a ∈ I => f(x) = Tnf(x) + Rnf(x), unde Tnf(x) = polinom Taylor de ordin n $T_n f(x)$, $R_n f(x) =$ restul



$$T_n f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$R_n f(x) = \frac{f^{(n+1)}(\theta_{x,a})}{(n+1)!}(x-a)^{n+1}, \quad \theta_{x,a} = \text{pct. intermediar este intre } x \text{ si } a$$

```
>> help @sym/taylor
```

$$T_n f(x) = \text{taylor}(f, x, a, \text{'order'}, n+1)$$

Serii de puteri MacLaurin (a = 0)

$$e^x = \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots, \quad x \in \mathbb{R}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^n}{(2n+1)!} x^{2n+1} + \dots, \quad x \in \mathbb{R}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + \frac{(-1)^n}{(2n)!} x^{2n} + \dots, \quad x \in \mathbb{R}$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + \frac{(-1)^{n-1}}{n} x^n + \dots, \quad |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \dots + (-1)^n x^n + \dots, \quad |x| < 1$$

$$\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \dots, \quad |x| < 1$$

$$(1+x)^k = 1 + \sum_{n=0}^{\infty} \binom{k}{n} x^n, \quad |x| < 1$$

$k \in \mathbb{R}$

ex: $(1+x)^{-\frac{1}{2}} = \frac{1}{\sqrt{1+x}}$

T2 (criteriul lui Leibniz pentru serii alternante)

$$S = \sum_{k=0}^{\infty} (-1)^k \cdot a_k, \quad (a_k)_{k\in\mathbb{N}} \text{ desc cu } \lim = 0$$
$$(a_k \searrow 0, k \to 0)$$

$$S \text{ conv.}, \quad S_n = \sum_{k=0}^{n} (-1)^k a_k \Rightarrow |S - S_n| \leq a_{n+1}$$

T3 (teor. Lui Abel pt. serii)

$$f(x) = \sum_{k=0}^{\infty} a_k x^k, \quad |x| < r$$

$-r \quad 0 \quad r$

$$\sum_{k=0}^{\infty} a_k \cdot r^k \text{ conv.} \Rightarrow \lim_{x \nearrow r} f(x) = \sum_{k=0}^{\infty} a_k r^k$$

(*) Pb 5 $\ln 2 \simeq ?$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + \frac{(-1)^{n-1}}{n} x^n + \ldots, \quad |x| < 1$$

Seria în $x = 1$ $\quad 1 - \frac{1}{2} + \frac{1}{3} - \ldots + \frac{(-1)^{n-1}}{n} + \ldots$ este alternantă

$T_2 \Rightarrow$ Seria e convergentă

$$T_3 \Rightarrow \underbrace{\lim_{x \nearrow 1} \ln(1+x)}_{\ln(2)} = 1 - \frac{1}{2} + \frac{1}{3} - \ldots + \frac{(-1)^{n-1}}{n} + \ldots$$

$$T_2 \Rightarrow \left| \ln 2 - \left(1 - \frac{1}{2} + \frac{1}{3} - \ldots + \frac{(-1)^{n-1}}{n}\right) \right| \leq \frac{1}{n+1}$$

? $\quad \frac{1}{n+1} < \frac{1}{10^5} \Rightarrow n \geq 10^5$

ex. $\overbrace{1{,}000001}^{p_1} \simeq 1$ $\quad$ cu precizia de 5 zecimale

$\underbrace{0{,}999999}_{p_2} \simeq 1$

$$|p_2 - 1| < \frac{1}{10^5}$$

$$|p_1 - 1| < \frac{1}{10^5}$$

$$f(x) = \ln\left(\frac{1+x}{1-x}\right)$$

```
>> solve((1+x)/(1-x)==2)
ans = (sym) 1/3
```

Temă Pb. 8 completând codurile: taylor_coef.m și pade_sym.m

Lab 2 – Virgulă flotantă

miercuri, 8 martie 2023 10:25

ex. $7_{10} = 2^2 + 2^1 + 2^0 = 111_2 = 2^2 \times 1,11$

$5,75_{10} = 2^2 + 2^0 + 2^{-1} + 2^{-2} = 101,11_2 = 2^2 \times 1,0111$

$9,125 = 2^3 + 2^0 + 2^{-3} = 1001,001_2 = 2^3 \times 1,001001$

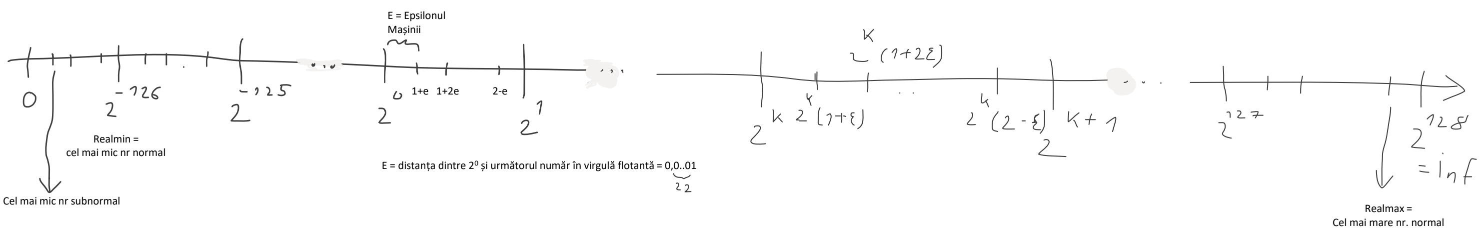
normalizare

$0,625 = 2^{-1} + 2^{-3} = 0,101 = 2^{-1} \times 1,01$

Reprezentarea numerelor în virgulă flotantă pe 32 de biți (format single)

poz binare

1 Semn 0: + 1: -	2 3 ... 8 9 Exponent $e_{10} = (e_1 \dots e_8)_2$ $E = e_{10} - 127$ -> 2^E x	10 11 .. 31 32 Mantisă/ semnificant $b_0,b_1b_2...b_{22}b_{23}$	Denumire de numere
0: +	$(0\ 0\ ..\ 0\ 0)_2$ -> 2^{-126} x	$0,b_1b_2...b_{22}b_{23}$ € $[0, 2^{-126})$	Subnormale/ denormalizate (excepție)
0: +	$(0\ 0\ ..\ 0\ 1)_2$ -> 2^{-126} x $(0\ 1\ ..\ 1\ 1)_2$ -> 2^0 x $(1\ 1\ ..\ 1\ 0)_2$ -> 2^{127} x	$1,b_1b_2...b_{22}b_{23}$ € $[2^{-126}, 2^{-125})$ $1,b_1b_2...b_{22}b_{23}$ € $[2^0, 2^1)$ $1,b_1b_2...b_{22}b_{23}$ € $[2^{127}, 2^{128})$	
	$(1\ 1\ ..\ 1\ 1)_2$ -> Inf, dacă $(1\ 1\ ..\ 1\ 1)_2$ -> NaN,	$b_1 = b_2 = ... = b_{22} = b_{23} = 0$ altfel	+/- Infinit -> excepție Not a Number



Format double pe 64 de bii
11 biți expo
52 biți mantisă

Temă: de completat codurile pentru
mysin, mycos -> seria Taylor cu input x subunitar € (0,1);
sinred, cosred -> reduc x la cadranul I + folosesc sin x € (pi/4, pi/2) = cos (pi/2- x);

Lab 3 – Condiționarea unei probleme

miercuri, 15 martie 2023 10:38

Nr. de condiționare

$$x,\ x^* \longrightarrow \boxed{f} \longrightarrow y,\ y^*$$

$$|\underbrace{x - x^*}_{\Delta x}| = \text{er abs.}$$

$$|\underbrace{y - y^*}_{\Delta y}| = -\,''\,-$$

$$\left|\frac{\delta y}{\delta x}\right| \leq \text{cond } f(x)$$

$$|\delta x| = \frac{|\Delta x|}{x} = \text{er rel.}$$

$-\,''\,-\ y$

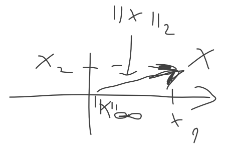
$f : \mathbb{R}^m \to \mathbb{R}^n$

$x = [x_1 \quad x_m]^T$

Ex de norme („p-norme")

- norma Euclidiană norm(x)

$$\|x\|_2 = \sqrt{x_1^2 + .. + x_m^2}$$



- norma Cebîșev: norm(x,Inf)

$$\|x\|_\infty = \max\{|x_1|, \ , |x_m|\}$$

- norma taxi/Manhattan norm(x,1)

$$\|x\|_1 = |x_1| + . + |x_m|$$

$A \in M_{n,m}(\mathbb{R})$

$$\|A\|_p = \max_{\substack{x \in \mathbb{R}^m \\ x \neq 0_m}} \frac{\|Ax\|_p}{\|x\|_p} \rightarrow \text{normă matric}$$

Def pt nr de cont pt f = (f1, ..., fn)

1) $$\Gamma(x) = \left(\frac{x_j \frac{\partial f_i}{\partial x_j}(x)}{f_i(x)}\right)_{\substack{i=\overline{1,n} \\ j=\overline{1,m}}} \qquad \text{cond}_p f(x) = \|\Gamma(x)\|_p$$

2) $$\frac{\partial f}{\partial x}(x) = \left(\frac{\partial f_i}{\partial x_j}(x)\right)_{\substack{i=\overline{1,n} \\ j=\overline{1,m}}} \qquad \text{cond}_p f(x) = \frac{\|x\|_p \cdot \left\|\frac{\partial f}{\partial x}\right\|_p}{\|f(x)\|_p}$$

Ex. $\left.\begin{array}{l} \cdot f\ \mathbb{R} \to \mathbb{R},\ f \in C^1(\mathbb{R}) \\ x \in \mathbb{R}\ \ f \ldots \end{array}\right\} \Rightarrow \text{cond } f(x) = \left|\frac{x f'(x)}{f(x)}\right|$

Ex. $f: \mathbb{R} \to \mathbb{R}$, $f \in C^1(\mathbb{R})$, $x \in \mathbb{R}$, $f(x) \neq 0$ $\Rightarrow$ $\operatorname{cond} f(x) = \left| \dfrac{x f'(x)}{f(x)} \right|$

$\downarrow$

Def 1)

Sist. Lin. $Ay = b$, $A \in M_{n,n}(\mathbb{R})$, det. $A \neq 0$ $\Rightarrow$ $\exists!\, y$ sol. unică

$$y = A^{-1} b$$

$$y = f(b) = A^{-1} b$$

$$b \longrightarrow \boxed{f} \longrightarrow y$$

$$\operatorname{cond}_p f(b) = \frac{\|b\|_p \, \|A^{-1}\|_p}{\|A^{-1} \cdot b\|_p}$$

$\downarrow$

Def. 2

$$\operatorname{cond}_p A = \max_{\substack{b \in \mathbb{R}^m \\ b \neq 0_m}} \operatorname{cond}_p f(b)$$

cond(A,p)

$$\operatorname{cond}_p A = \|A\|_p \cdot \|A^{-1}\|_p$$

Temă: $p(x) = x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_0$

Ec alg p(x) = 0 are o răd nenulă și simplă: $\xi(a) \neq 0$, $p(\xi(a)) = 0$, $p'(\xi(a)) \neq 0$

Vezi notițe

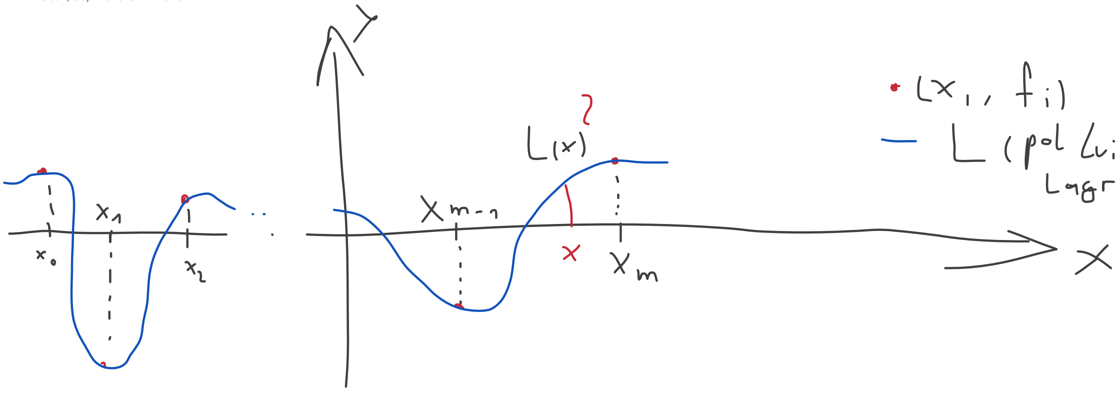
Lab 4 – Sisteme liniare

miercuri, 22 martie 2023 10:21

Rezolvarea sistemelor liniare

Lab 6 - Interpolare Lagrange = "Trece prin puncte"

miercuri, 5 aprilie 2023 10:10



• (x_i, f_i)

— L (pol lui Lagrange) = polinom de interpolare de grad minim

Input:
- nodes (x0 x1 .. xm) distincte!
- values (f0 f1 .. fm)
- X

Output:
- L(X)

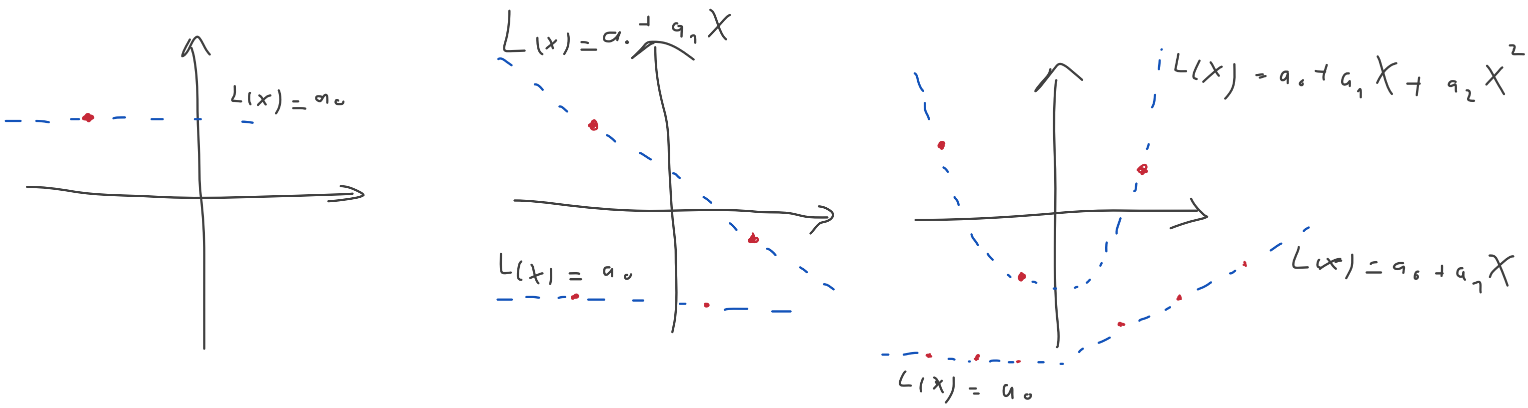
Punctele aparțin L dacă L(xi) = fi, i=0 .. m ==> interpolare (*)

$$L(X) = a_0 + a_1 X + \quad + a_m X^m$$

grad L $\leq$ m = nr noduri - 1

$$(*) \Leftrightarrow \underbrace{\begin{bmatrix} 1 & x_0 & & x_0^m \\ 1 & x_1 & \dots & x_1^m \\ & \cdot & & \\ 1 & x_m & & x_m^m \end{bmatrix}}_{\text{Vandermonde}} \begin{bmatrix} a_0 \\ a_1 \\ \\ a_m \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_m \end{bmatrix}$$

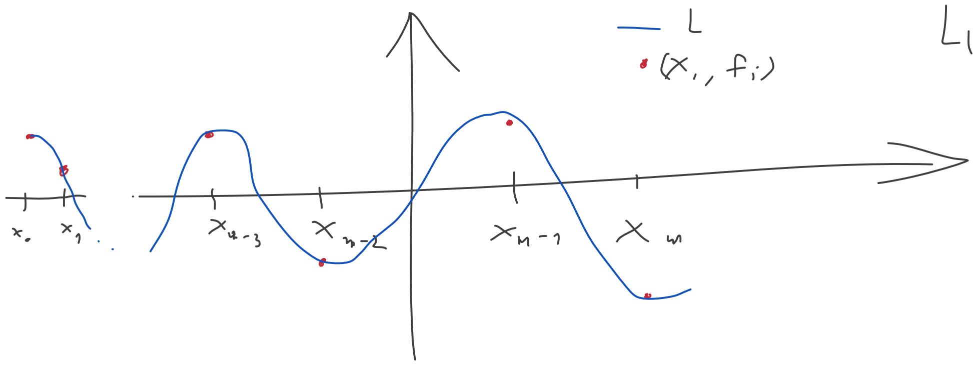
Alternativă : Formula baricentrică



Temă: implementăm simbolic (noduri simbolice, valori simbolice, X simbolic, ca și cum calculăm de mână pol. Lagrange) formula clasică pentru pol. Lui Lagrange folosind (1) și (2) din fișierul Lagrange.pdf, pg. 1

Lab 7 – Interpolare Lagrange (cu formula lui Newton)

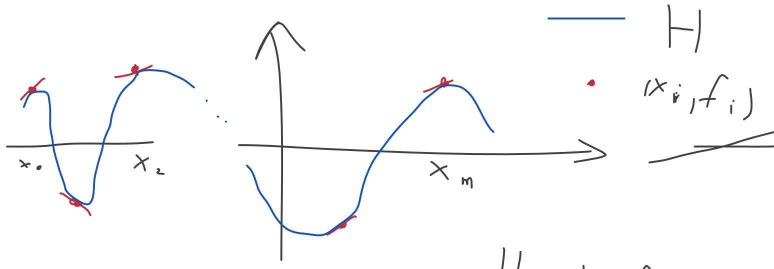
miercuri, 12 aprilie 2023 10:12

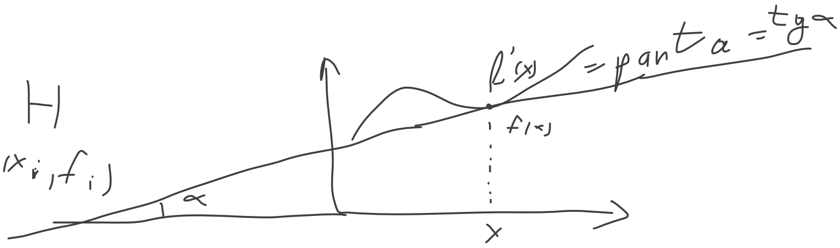


$L(x_i) = f_i, \quad i=\overline{0,n}$

gr. $L \leq$ nr. cond. $- 1$

Interpolare Hermite cu noduri duble





$H(x_i) = f_i, \quad i=\overline{0,n}$

$H'(x_i) = df_i$

gr. $H \leq \underbrace{\text{nr. cond.} - 1}_{2n+1}$

Input: x, f, df

Output: H(X), H'(X)

Temă: vectorizați funcția Hermite (test data viitoare, graficul distanță viteză)

Lab 8 – Interpolare cubică spline

miercuri, 26 aprilie 2023 10:25

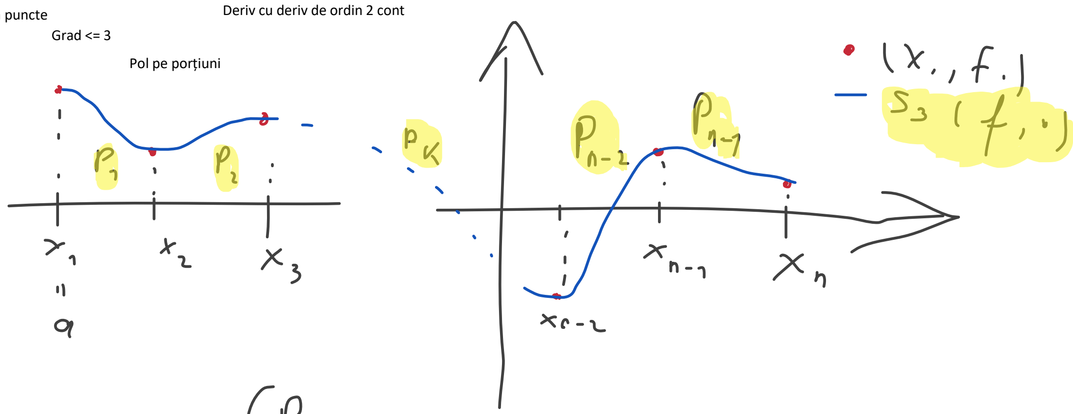
Interpolare cubică spline de clasă C^2

Trece prin puncte

Grad <= 3

Pol pe porțiuni

Deriv cu deriv de ordin 2 cont



$$S_3(f,x) = \begin{cases} P_1(x), & x \in [x_1, x_2] \\ P_2(x), & x \in [x_2, x_3] \\ \vdots \\ P_{n-1}(x), & x \in [x_{n-1}, x_n] \end{cases}$$

$S_3(f,\cdot)$, $S_3'(f,\cdot)$, $S_3''(f,\cdot)$ - cont

$$P_i(x_i) = f_i, \quad P_i(x_{i+1}) = f_{i+1}, \quad i = \overline{1, n-1}$$

$$P_i'(x_i) = P_{i-1}'(x_i), \quad i = \overline{2, n-1}$$

$$P_i''(x_i) = P_{i-1}''(x_i), \quad i = \overline{2, n-1}$$

$$P_i(x) = c_{i,0} + c_{i,1}(x - x_i) + c_{i,2}(x - x_i)^2 + c_{i,3}(x - x_i)^3, \quad i = \overline{1, n-1}$$

Nr. coef nec.: $4(n-1) = 4n-4$

Nr ec./cond.: $2(n-1) + n-2 + n-2 = 4n-6$

Adăugăm 2 cond./ ec. :

- Complet: $P_1'(x_1) = f'(a)$, $P_{n-1}'(x_n) = f'(b)$
- Natural: $P_1''(x_1) = 0$, $P_{n-1}''(x_n) = 0$
- Deriv. Secunde: $P_1''(x_1) = f''(a)$, $P_{n-1}''(x_n) = f''(b)$
- De Boor: $P_1'''(x_2) = P_2'''(x_2)$, $P_{n-2}'''(x_{n-1}) = P_{n-1}'''(x_{n-1})$

Lab 9 – Metoda celor mai mici pătrate

miercuri, 3 mai 2023 10:09

$\langle f, g \rangle =$ Prod. Scalar dintre f și g (vectori/ funcții)

$\| f - g \| =$ Distanța dintre f și g = norma pt. f - g

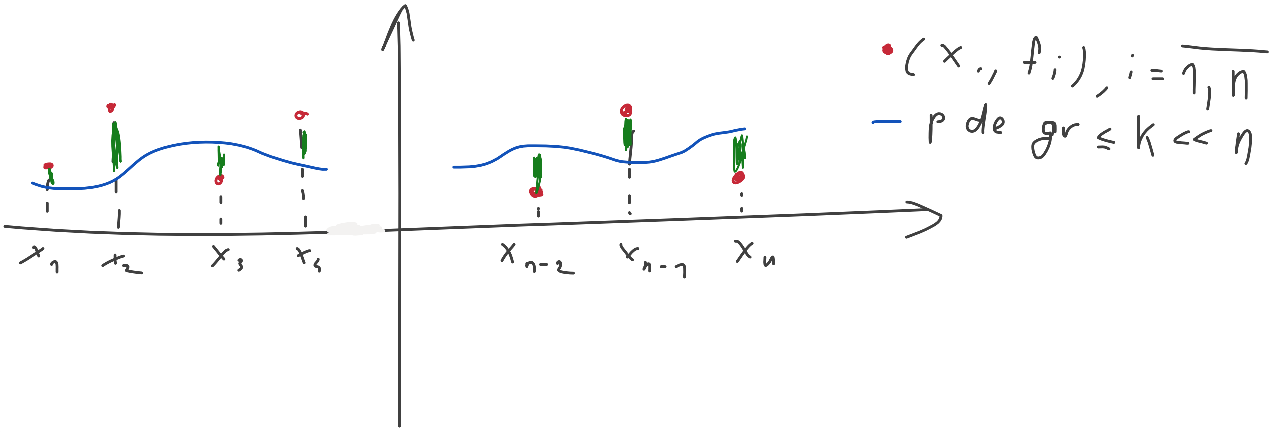
$= \sqrt{\langle f-g, f-g \rangle}$

Ex. $\langle v, w \rangle = v_1 w_1 + \dots + v_n \cdot w_n =$ Prod. Scalar Euclidian

$\| v \| = \sqrt{v_1^2 + \quad + v_n^2} =$ Norma Euclidiană

Ex.. $\langle f, g \rangle = \int_a^b f(x)\, g(x)\, dx$

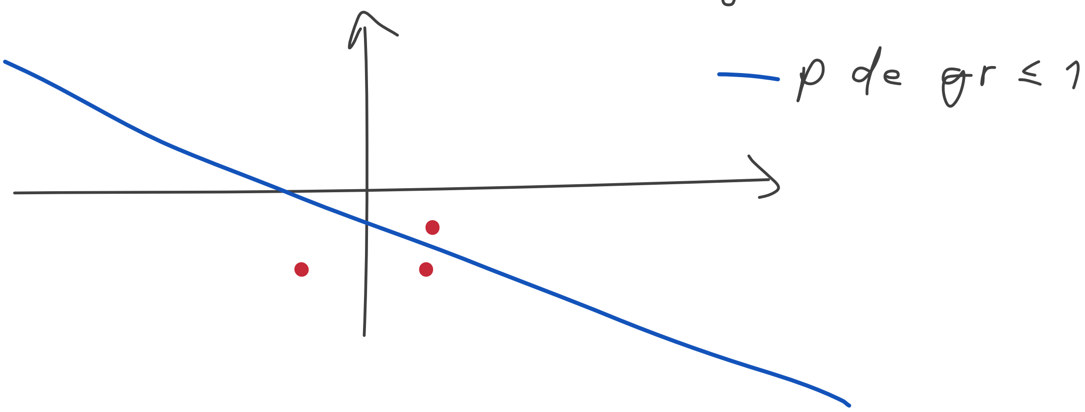
M.c m m p discretă (norma Eucl.)



? Căutăm p căciulă (p de î) pol. De gr. <= k a. î. $\| \hat{p} - f \| = \min\limits_{p \in \mathbb{P}_k} \| p - f \|$

$\uparrow$

spațiul pol de gr. ≤ k



$p(x) = c_k x^k + \dots + c_1 x + c_0$

$$\begin{cases} p(x_1) = c_k x_1^k + \dots + c_1 x_1 + c_0 \approx f_1 \\ \vdots \\ p(x_n) = c_k \cdot x_n^k + \dots + c_1 x_n + c_0 \approx f_n \end{cases} \quad \Bigg| \quad \begin{matrix} k+1 \text{ nec.} \\ \wedge\wedge \\ n \text{ ec.} \end{matrix}$$

$$\underbrace{\begin{bmatrix} x_1^k & \dots & x_1 & 1 \\ \vdots & & & \\ x_n^k & \dots & x_n & 1 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} c_k \\ \vdots \\ c_1 \\ c_0 \end{bmatrix}}_{c} \simeq \underbrace{\begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}}_{f}$$

Sist. Supradeterminat: A * c ~= f , $A \in \mathcal{M}_{n,m}(\mathbb{R})$, $n \geq m \ (= k+1)$

Căutăm $\hat{c} \in \mathbb{R}^m$ a.î. $\|A \cdot \hat{c} - f\| = \min_{c \in \mathbb{R}^m} \|A \cdot c - f\|$

<u>Desc. QR</u> $\forall A \in \mathcal{M}_{n,m}(\mathbb{R}), n \geq m,$

$\exists Q \in \mathcal{M}_{n,n}(\mathbb{R}),\ Q^{-1} = Q^T,\ \exists R \in \mathcal{M}_{n,m}(\mathbb{R})$ tr. sup.

$$R = \begin{array}{c} \overbrace{\hphantom{xxxx}}^{m} \\ \begin{bmatrix} 0 \ \nabla \\ 0 \end{bmatrix} \end{array} \begin{matrix} \}\, m \\ \}\, n-m \end{matrix} \quad \text{a.î.} \quad A = Q \cdot R$$

- $Q^{-1} = Q^T \Rightarrow \|Q \cdot v\| = \|v\|, \forall v \in \mathbb{R}^n$
- $\|A c - f\| = \|Q \cdot R c - f\| = \|Q (R c - Q^T f)\| =$

$$= \|\underbrace{R c}_{\substack{y \\ \parallel}} - \underbrace{Q^T f}_{\substack{b \\ =}}\| = \left\| \begin{bmatrix} \hat{y} - \hat{b} \\ -\tilde{b} \end{bmatrix} \right\|$$

R

$$\begin{bmatrix} 0 & \\ & 0 \end{bmatrix} \begin{bmatrix} c \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \big\} \hat{y} \\ \big\} 0 \end{matrix} \qquad \begin{bmatrix} b_1 \\ \vdots \\ b_m \\ b_{m+1} \\ \vdots \\ b_n \end{bmatrix} \begin{matrix} \big\} \hat{b} \\ \big\} \tilde{b} \end{matrix}$$

Sol. Sist. Supradet. A * c ~= f
se obț. Pt. Y căciulă = b căciulă

$$\Updownarrow$$

$$R(1:m,:) \cdot \hat{c} = b(1:m)$$

$$\boxed{\hat{c} = R(1:m,:) \backslash b(1:m)}$$

$$\|A\hat{c} - f\| = \underbrace{\min_{c \in \mathbb{R}^m} \|A \cdot c - f\|}_{\|\tilde{b}\| = \|b(m+1:n)\|}$$

Lab 10 – Cuadraturi

miercuri, 10 mai 2023 10:03

Temă: de trecut pseudocod cuadr. Adaptivă în Octave, apoi să testăm

Lab 11 – Cuadraturi Gauss

miercuri, 17 mai 2023 10:12

Vezi caiet

Lab 12 - Ecuații neliniare

miercuri, 24 mai 2023 10:58

F(x) = 0 => x = ? Răd. Pt. f

(f zero -> ec. Nelin. , f solve -> sist. De ec. Nelin.)

Met. Lui Newton

D = tg. La gr. F în Xn(xn, f(xn))

Obs.: f aparține C^2(I), f'' nu se anulează pe intervalul I (f = convexă/ concavă) atunci:

Orice x0 aparține I cu f(x0) * f''(x0) > 0 este pct. Bun de pornire pentru metoda lui Newton

D: y – f(xn) = f'(xn) (x-xn)

D intersect. Ox = {Xn+1 (xn+1, 0) }

0 – f(xn) = f' (xn) (x- xn)

Xn+1 = xn – f(xn) / f'(xn)

Met. Aprox. Succesive (fi(x) = x)

Xn+1 = fi(xn), x0 aparține I, fi : I -> I
Dacă xn+1 = fi(xn) --converge--> alpha = fi(alpha), n->Inf, fi aparține C^p(I),
Fi'(alpha) = ... = fi^(p-1)(alpha) = 0, fi^(p)(alpha) /= 0,
Atunci:
-ord. De conv. Este p.
-er. Asimpt. Este fi^p(alpha)/p!

Ex.: Met. Lui Newton (f(x) = 0)

Xn+1 = xn – f(xn)/f'(xn) -> alpha, n-> Inf (f(alpha) = 0, f'(alpha) /= 0)
....