Eigenvalues and eigenvectors.

1. Find the eigenvalues and eigenvectors of the following matrices in $\mathrm{Mat}_{2\times 2}(\mathbb{R})$:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

2. Consider the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Show that A doesn't have eigenvectors when considered in $\mathrm{Mat}_{n\times n}(\mathbb{R})$. Show that A is diagonalizable when considered in $\mathrm{Mat}_{n\times n}(\mathbb{C})$ and find the eigenvectors of A.

3. Give the eigenvalues of $\mathrm{lin}(\mathrm{Pr}_{H,\mathbf{v}})$, $\mathrm{lin}(\mathrm{Ref}_{H,\mathbf{v}})$. What can you say about the eigenvectors?

4. Let $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map

$$\phi(x,y,z) = (x+y-z, y+z, 2x).$$

Find the matrix $\mathrm{M}_{\boldsymbol{b},\boldsymbol{b}}(\phi)$ where

$$\boldsymbol{b} = \{(1,1,0), (-1,0,1), (1,1,1)\}.$$

5. Calculate the eigenvalues and their algebraic and geometric multiplicities for the following matrices in $\mathrm{Mat}_{3\times 3}(\mathbb{R})$, and deduce whether or not they are diagonalizable:

$$\begin{bmatrix} -6 & 2 & -5 \\ -4 & 4 & -2 \\ 10 & -3 & 8 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & -15 \\ 0 & 2 & 8 \end{bmatrix}.$$

6. Find the eigenvectors for each of the following symmetric matrices:

$$A = \begin{bmatrix} 73 & 36 \\ 36 & 52 \end{bmatrix}, \quad B = \begin{bmatrix} -94 & 180 \\ 180 & 263 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 128 & 240 \\ 240 & 450 \end{bmatrix}.$$

We will use these matrices to discuss examples of conic sections.

Rotations.

7. The vertices of a triangle are $A(1,1)$, $B(4,1)$ and $C(2,3)$. Determine the image of the triangle ABC under a rotation by $90°$ around C followed by an orthogonal reflection relative to the line AB.

8. Determine the sum-of-angles formulas for sine and cosine using rotation matrices.

9. Let T be the isometry obtained by applying a rotation of angle $-\frac{\pi}{3}$ around the origin after a transation with vector $(-2,5)$. Determine the inverse transformation, T^{-1}.

10. Determine the matrix form of a rotation with angle $45°$ having the same center of rotation as the rotation

$$f(\mathbf{x}) = \frac{1}{\sqrt{13}} \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

11. Determine the cosine of the angle of the rotation f given in the previous exercise and find the inverse rotation, f^{-1}.

12. Verify that the matrices

$$A = \frac{1}{3}\begin{bmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \frac{1}{11}\begin{bmatrix} -9 & -2 & 6 \\ 6 & -6 & 7 \\ 2 & 9 & 6 \end{bmatrix}$$

belong to $SO(3)$. Moreover, determine the axis of rotation and the rotation angle.

13. Show that an isometry is bijective.