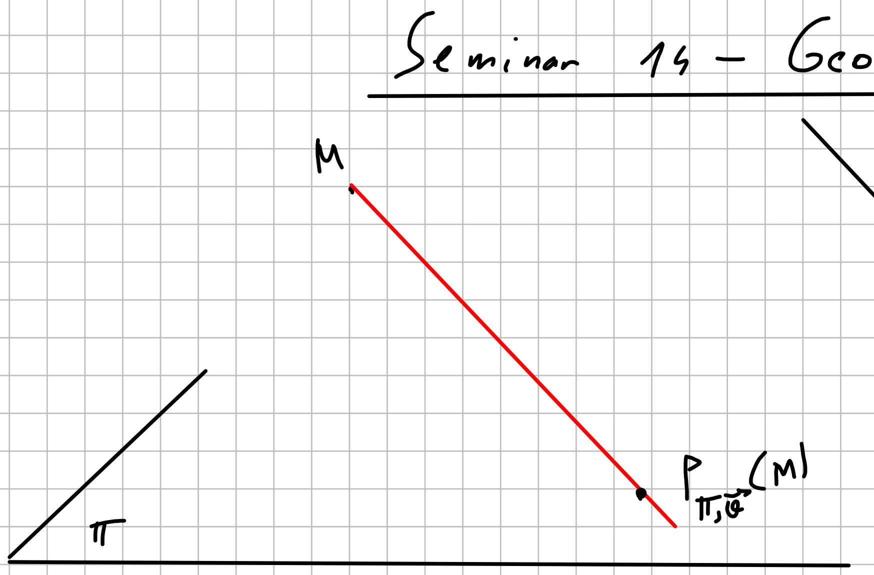
03.06.2023

Seminar 14 – Geometry



S7 **7.** Determine the orthogonal projection of the point $A(2,11,-5)$ on the plane $x+4y-3z+7=0$ by determining the matrix form of the projection. (Compare your result with the previous seminar.)

$$[\mathrm{Pr}_{H,\mathbf{v}}(P)]_{\mathcal{K}} = \underbrace{\frac{1}{\operatorname{lin}\varphi(\mathbf{v})}\left((\mathbf{v}^t\cdot\mathbf{a})\mathrm{Id}_n - \underbrace{\mathbf{v}\cdot\mathbf{a}^t}_{\mathbf{v}\otimes\mathbf{a}}\right)}_{\operatorname{lin}\mathrm{Pr}_{H,\vec{v}}}\cdot[P]_{\mathcal{K}} - \underbrace{\frac{a_{n+1}}{\operatorname{lin}\varphi(\mathbf{v})}[\mathbf{v}]_{\mathcal{K}}}$$

$$\varphi(x,y,z) = x+4y-3z+7$$

$$(\operatorname{lin}\varphi)(x,y,z) = x+4y-3z$$

$$v = \begin{pmatrix}1\\4\\-3\end{pmatrix} \qquad a = \begin{pmatrix}1\\4\\-3\end{pmatrix} \qquad a_4 = 7$$

$$[\mathrm{Pr}_\pi^\perp(A)] = \frac{1}{1+4\cdot4+(-3)\cdot(-3)}\cdot\left((1,4,-3)\cdot\begin{pmatrix}1\\4\\-3\end{pmatrix}\cdot I_3 - \begin{pmatrix}1\\4\\-3\end{pmatrix}\cdot\begin{pmatrix}1 & 4 & -3\end{pmatrix}\right)$$

$$\cdot\begin{pmatrix}2\\11\\-5\end{pmatrix} - \frac{7}{1+4\cdot4+(-3)\cdot(-3)}\cdot\begin{pmatrix}1\\4\\-3\end{pmatrix} =$$

$$= \frac{1}{26}\cdot\left(26I_3 - \begin{pmatrix}1 & 4 & -3\\4 & 16 & -12\\-3 & -12 & 9\end{pmatrix}\right)\cdot\begin{pmatrix}2\\11\\-5\end{pmatrix} - \frac{7}{26}\cdot\begin{pmatrix}1\\4\\-3\end{pmatrix}$$

S7 **9.** Determine the orthogonal projection of the line $\ell : 2x - y - 1 = 0 \cap x + y - z + 1 = 0$ on the plane $\pi : x + 2y - z = 0$ by determining the matrix form of the projection. (Compare your result with the previous seminar.)

Sol.:

$$[Pr_\pi^\perp(P)] = \frac{1}{(\text{lin}\,\varphi)(v)} \cdot \left(v^t \cdot a\, I_3 - v \cdot a^t \right) \cdot [P] - \frac{a_4}{\text{lin}\,\varphi} \cdot v$$

$\varphi(x,y,z) = x+2y-z \qquad \text{lin}\,\varphi = x+2y-z \qquad a_4 = 0$

$$v = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \qquad a = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \qquad P = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$[Pr_\pi^\perp(P)] = \frac{1}{1+2\cdot 2+(-1)(-1)} \cdot \left((1,2,-1) \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \cdot I_3 - \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \cdot (1\ 2\ -1) \right) \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{6}\left(6 I_3 - \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{pmatrix} \right) \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \frac{1}{6} \cdot \begin{pmatrix} 5 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 2 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\ell : \begin{cases} 2x - y - 1 = 0 \\ x + y - z + 1 = 0 \end{cases} \Leftrightarrow \begin{cases} x = t \\ y = 2x - 1 = 2t - 1 \\ t + 2t - 1 - z + 1 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = t \\ y = 2t - 1 \\ z = 3t \end{cases} \Rightarrow \text{parametric form}$$

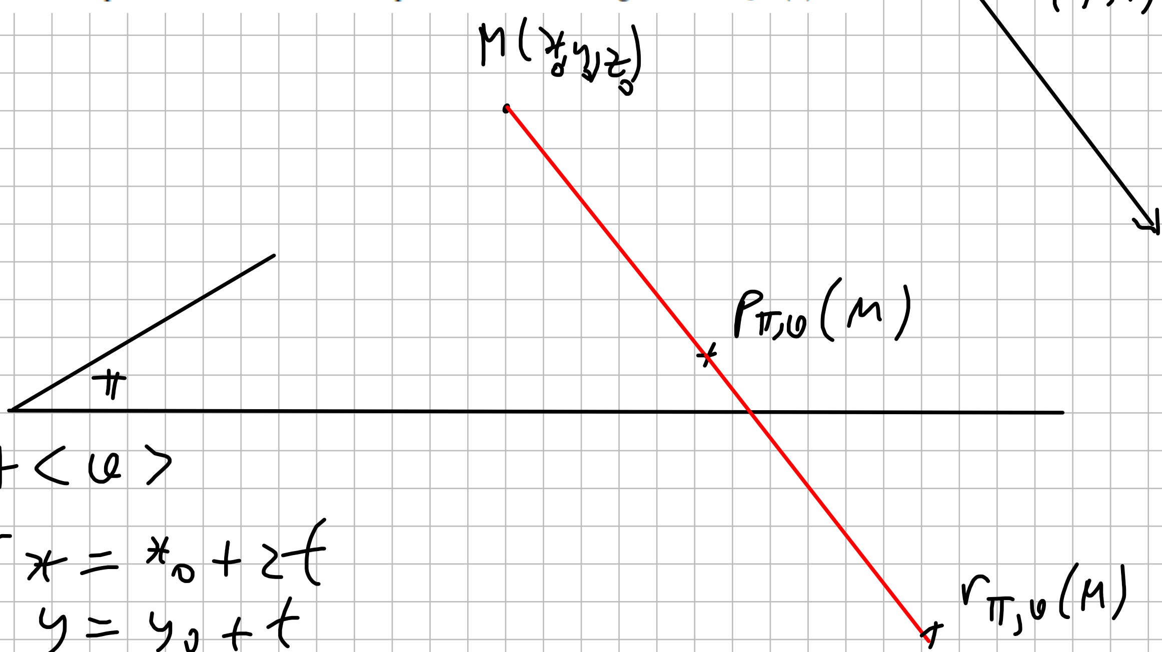
$$P^{\perp}_{\pi}(\ell): \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{6} \cdot \begin{pmatrix} 5 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 2 & 5 \end{pmatrix} \cdot \begin{pmatrix} t \\ 2t-1 \\ 3t \end{pmatrix}$$

$$\Rightarrow \begin{cases} x = \frac{1}{6}(5t-4t+2+3t) = \frac{1}{6}(4t+2) \\ y = \frac{1}{6}(-2t+4t-2+6t) = \frac{1}{6}(8t-2) \\ z = \frac{1}{6}(t+4t-2+15t) = \frac{1}{6}(20t-2) \end{cases}$$

57 **6.** Consider $\mathbf{v}(2,1,1) \in \mathbb{V}^3$ and $Q(2,2,2) \in \mathbb{E}^3$.

a) Give the matrix form for the parallel projection on the plane $\pi : z = 0$ along the line $Q + \langle \mathbf{v} \rangle$.

b) Give the matrix form for the parallel reflection in the plane $\pi : z = 0$ along the line $Q + \langle \mathbf{v} \rangle$.



Let $\ell_M = M + \langle v \rangle$

$$\ell_M: \begin{cases} x = x_0 + 2t \\ y = y_0 + t \\ z = z_0 + t \end{cases}$$

$$P_{\pi,v}(M): \begin{cases} x = x_0 + 2t \\ y = y_0 + t \\ z = z_0 + t \\ z = 0 \end{cases} \Rightarrow \begin{cases} t = -z_0 \\ x = x_0 - 2z_0 \\ y = y_0 - z_0 \\ z = 0 \end{cases}$$

So the matrix form of the projection is:

$$\underbrace{[P_{\pi,v}(M)]}_{=\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \cdot \underbrace{[M]}_{=\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}}$$

For the reflection:

$$r_{\Pi,v}(M) : \begin{cases} x = 2(x_0 - 2z_0) - x_0 \\ y = 2\cdot(y_0 - z_0) - y_0 \\ z = 2\cdot(0) - z_0 \end{cases}$$

because $[r_{\Pi,v}(M)] = 2\cdot[p_{\Pi,v}(M)] - (M)$

$$r_{\Pi,v}(M) : \begin{cases} x = x_0 - 4z_0 \\ y = y_0 - 2z_0 \\ z = -z_0 \end{cases}$$

$$\Rightarrow [r_{\Pi,v}(M)] = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{pmatrix} \cdot [M]$$

<u>Ex.</u>: If we have $\begin{cases} x = 5x_0 + 2y_0 - 5 \\ y = 4x_0 + y_0 - z_0 \\ z = 7x_0 + 8y_0 + 9y_0 + 1 \end{cases}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 & 2 & 0 \\ 4 & 1 & -1 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$$

Sn **10.** Using the classification of quadrics, decide what surfaces are described by the following equations.

a) $x^2+2y^2+z^2+xy+yz+zx=1$,

b) $xy+yz+zx=1$,

c) $x^2+xy+yz+zx=1$,

d) $xy+yz+zx=0$.

# of positive eigenvalues

# of negative eigenvalues

Case	$r=\operatorname{rank}Q$	$(p,r-p)$	k	equation	name
(a)	3	$(3,0)$	-1	$x^2+y^2+z^2-1=0$	**ellipsoid** (E)
(a)	3	$(2,1)$	-1	$x^2+y^2-z^2-1=0$	**hyperboloid of one sheet** (H1)
(a)	3	$(1,2)$	-1	$x^2-y^2-z^2-1=0$	**hyperboloid of two sheets** (H2)
(a)	3	$(0,3)$	-1	$-x^2-y^2-z^2-1=0$	imaginary ellipsoid (IE)
(a)	3	$(3,0)$	0	$x^2+y^2+z^2=0$	imaginary cone
(a)	3	$(2,1)$	0	$x^2+y^2-z^2=0$	(real, elliptic) **cone**
(a)	2	$(0,2)$	-1	$-x^2-y^2-1=0$	cylinder on imaginary ellipse
(a)	2	$(1,1)$	-1	$x^2-y^2-1=0$	**cylinder** on hyperbola
(a)	2	$(2,0)$	-1	$x^2+y^2-1=0$	**cylinder** on ellipse
(a)	2	$(0,2)$ or $(2,0)$	0	$-x^2-y^2=0$	cylinder on two complex lines
(a)	2	$(1,1)$	0	$x^2-y^2=0$	**cylinder** on two real lines
(a)	1	$(1,0)$	1	$x^2+1=0$	two complex planes
(a)	1	$(1,0)$	-1	$x^2-1=0$	**two real planes**
(a)	1	$(1,0)$	0	$x^2=0$	**a double plane**
(a)	1	$(0,1)$	-1	$x^2+1=0$	two complex planes
(a)	1	$(0,1)$	1	$x^2-1=0$	**two real planes**
(a)	1	$(0,1)$	0	$x^2=0$	**a double plane**
Case	$r=\operatorname{rank}Q$	$(p,r-p)$	k'	equation	name
(b)	2	$(2,0)$ or $(0,2)$	-1	$x^2+y^2-z=0$	**elliptic paraboloid** (EP)
(b)	2	$(1,1)$	-1	$x^2-y^2-z=0$	**hyperbolic paraboloid** (HP)
(b)	1	$(1,0)$	1	$x^2+y=0$	**cylinder** on parabola

a) $x^2+2y^2+z^2+xy+yz+zx+2x-4y+8z=1$

$$M(Q)=\begin{array}{c|c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} x\\ y\\ z\end{array} & \begin{pmatrix} 1 & \frac12 & \frac12 \\ \frac12 & 2 & \frac12 \\ \frac12 & \frac12 & 1 \end{pmatrix}\end{array} \qquad P_{M(Q)}=\begin{vmatrix} 1-X & \frac12 & \frac12 \\ \frac12 & 2-X & \frac12 \\ \frac12 & \frac12 & 1-X \end{vmatrix}=$$

$$=(2-X)\cdot(1-X)^2+\frac18+\frac18-\frac14(2-X)-\frac14(1-X)-\frac14(1-X)$$

$$=(2-X)(1-2X+X^2)+\cancel{\frac14}-\frac12+\frac{X}{4}-\cancel{\frac14}+\frac{X}{4}-\frac14+\frac{X}{4}$$

$$=2-4X+2X^2-X+2X^2-X^3+\frac{3X}{4}-\frac34$$

$$=-X^3+4X^2-4X+\frac54$$

$$\lambda_1=\frac52 \qquad \lambda_2=1 \qquad \lambda_3=\frac12$$

$$S(\lambda_1)=\left\{(x,y,z)\in\mathbb{R}^3 \;\middle|\; \begin{pmatrix} -\frac32 & \frac12 & \frac12 \\ \frac12 & -\frac12 & \frac12 \\ \frac12 & \frac12 & -\frac32 \end{pmatrix}\begin{pmatrix} x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}$$

$$\begin{cases} -3x+y+z=0 \\ x-y+z=0 \\ x+y-3z=0 \end{cases} \iff \begin{cases} y=x+z \\ -3x+x+z+z=0 \\ x+x+z-3z=0 \end{cases} \iff \begin{cases} y=x+z \\ -2x+2z=0 \\ 2x-2z=0 \end{cases}$$

$$\Rightarrow \begin{cases} x=z \\ y=2z \\ z=z \end{cases}$$

$\Rightarrow S(\lambda_1) = \langle (1,2,1) \rangle$

Choose $v_1 = \frac{1}{\sqrt{6}} (1,2,1)$

$S(\lambda_2) = \langle (1,-1,1) \rangle$

Choose $v_2 = \frac{1}{\sqrt{3}} (1,-1,1)$

$S(\lambda_3) = \langle (-1,0,1) \rangle$

Choose $v_3 = \frac{1}{\sqrt{2}} (-1,0,1)$

$$\Rightarrow M_{E,B} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & \sqrt{2} & -\sqrt{3} \\ 2 & -\sqrt{2} & 0 \\ 1 & \sqrt{2} & \sqrt{3} \end{pmatrix}$$

$$\det M_{E,B} = \frac{1}{(\sqrt{6})^3} \cdot \begin{vmatrix} 1 & \sqrt{2} & -\sqrt{3} \\ 2 & -\sqrt{2} & 0 \\ 1 & \sqrt{2} & \sqrt{3} \end{vmatrix} \overset{L_1 \leftarrow L_1 + L_3}{=}$$

$$= \frac{1}{(\sqrt{6})^3} \cdot \begin{vmatrix} 2 & 2\sqrt{2} & 0 \\ 2 & -\sqrt{2} & 0 \\ 1 & \sqrt{2} & \sqrt{3} \end{vmatrix} = \frac{1}{(\sqrt{6})^3} \cdot \sqrt{3} \cdot \begin{vmatrix} 2 & 2\sqrt{2} \\ 2 & -\sqrt{2} \end{vmatrix}$$

$$= \frac{1}{(\sqrt{6})^3} \cdot \sqrt{3} \cdot (-8\sqrt{2}) = \frac{1}{6\sqrt{6}} \cdot \sqrt{3} \cdot (-8\sqrt{2}) = \frac{4}{3}$$

$\det M_{E,B} = -1 \Rightarrow$ we choose $v_3 = \frac{1}{\sqrt{2}} (1,0,-1)$

$$M_{E,B} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & \sqrt{2} & -\sqrt{3} \\ 2 & -\sqrt{2} & 0 \\ 1 & \sqrt{2} & \sqrt{3} \end{pmatrix}$$

$$Q: (x \ y \ z) \cdot \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 2 & \frac{1}{2} \\ \frac{1}{2} & \frac{2}{2} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + (2 \ -4 \ 8) \begin{pmatrix} x \\ y \\ z \end{pmatrix} + 1 = 0$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = M_{E,B} \cdot \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

$$Q: \quad (x_1, y_1, z_1) \cdot \underbrace{M_{E,B}^{t} \cdot \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 2 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix} \cdot M_{E,B}}_{= \begin{pmatrix} 5/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}} \cdot \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + (2 \ -4 \ 8) \cdot M_{E,B} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + 1 = 0$$

$$M_{E,B} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & \sqrt{2} & -\sqrt{3} \\ 2 & -\sqrt{2} & 0 \\ 1 & \sqrt{2} & \sqrt{3} \end{pmatrix}$$

$$Q: \quad \frac{5}{2} x_1^2 + y_1^2 + \frac{1}{2} z_1^2 + \frac{1}{\sqrt{6}} \left((2 - 8 + 8) \quad (2\sqrt{2} - 4\sqrt{2} + 8\sqrt{2}) \quad (-2\sqrt{3} + 8\sqrt{3}) \right) \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + 1$$

$$\Rightarrow \frac{5}{2} x_1^2 + y_1^2 + \frac{1}{2} z_1^2 + \frac{2}{\sqrt{6}} x_1 + \frac{6\sqrt{2}}{\sqrt{6}} y_1 + \frac{6\sqrt{3}}{\sqrt{6}} z_1 + 1 = 0$$

$$\frac{5}{2} \left(x_1^2 + 2 \frac{2}{5\sqrt{6}} \cdot x_1 + \frac{4}{25 \cdot 6} \right) + \left(y_1^2 + 2 \cdot \frac{3\sqrt{2}}{\sqrt{6}} y_1 + \frac{18}{6} \right) +$$

$$+ \frac{1}{2} \left(z_1^2 + 2 \cdot \frac{6\sqrt{3}}{\sqrt{6}} \cdot z_1 + \frac{36 \cdot 3}{6} \right) + 1 - \frac{2}{5 \cdot 6} - 3 - \frac{36 \cdot 3}{12} = 9$$

$$\frac{5}{2} \left(x_1 + \frac{2}{5\sqrt{6}} \right)^2 + \left(y_1 + \frac{3\sqrt{2}}{\sqrt{6}} \right)^2 + \frac{1}{2} \cdot \left(z_1 + \frac{6\sqrt{3}}{\sqrt{6}} \right)^2 =$$

$$= 11 - \frac{1}{15} = \frac{164}{15}$$

$$x_2 = x_1 + \frac{2}{5\sqrt{6}} \qquad y_2 = y_1 + \frac{3\sqrt{2}}{\sqrt{6}} \qquad z_2 = z_1 + \frac{6\sqrt{3}}{\sqrt{6}}$$

$$\Rightarrow Q: \quad \frac{x_2^2}{\frac{164}{15} \cdot \frac{2}{5}} + \frac{y_2^2}{\frac{164}{15}} + \frac{z_2^2}{\frac{164}{15} \cdot 2} = 1$$

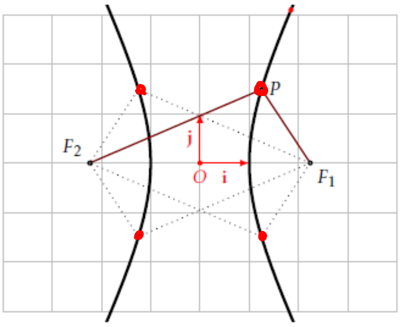
$\Rightarrow$ ellipsoid

570 **8.** Consider the hyperbola $\mathcal{H}: x^2 - \frac{y^2}{4} - 1 = 0$ with focal points F_1 and F_2. Find the points M situated on the hyperbola such that

a) The angle $\measuredangle F_1MF_2$ is right;

b) The angle $\measuredangle F_1MF_2$ is $60°$;

c) The angle $\measuredangle F_1MF_2$ is θ.



$a=1 \quad b=2$

$F_2(-c,0)\ ,\ F_1(c,0) \qquad c=\sqrt{a^2+b^2}=\sqrt{5}$

$M(x,y) \Rightarrow x^2-\frac{y^2}{4}=1$

$$\cos(\widehat{F_1MF_2}) = \frac{\overrightarrow{MF_1}\cdot\overrightarrow{MF_2}}{\|\overrightarrow{MF_1}\|\cdot\|\overrightarrow{MF_2}\|} = \frac{(c-x,-y)\cdot(-c-x,-y)}{\sqrt{(c-x)^2+y^2}\cdot\sqrt{(c+x)^2+y^2}}$$

$$\cos\theta = \frac{x^2-c^2+y^2}{\sqrt{(c-x)^2+y^2}\cdot\sqrt{(c+x)^2+y^2}} = \frac{x^2-5+4x^2-1}{\sqrt{(\sqrt5-x)^2+4x^2-1}\cdot\sqrt{(\sqrt5+x)^2+4x^2-1}}$$

De aici calcul ...

i) când $\theta=90^0$ avem $5x^2-6=0 \Rightarrow x^2=\frac{6}{5} \Rightarrow x=\pm\sqrt{\frac{6}{5}}$

$$\Rightarrow y = \begin{cases} 4\cdot\frac{6}{5}-1 = \frac{19}{5} \\ 4\cdot\left(-\frac{6}{5}\right)-1 = -\frac{29}{5}\end{cases} \Rightarrow \text{punctele sunt } \left(\sqrt{\frac{6}{5}}, \frac{19}{5}\right), \left(-\sqrt{\frac{6}{5}}, -\frac{29}{5}\right)$$

ii) când θ general

$$\Rightarrow \cos\theta = \frac{5x^2-6}{\sqrt{(\sqrt5-x)^2+4x^2-1}\cdot\sqrt{(\sqrt5+x)^2+4x^2-1}}$$

$$5x^2-6 = \sqrt{(5x^2+4-2\sqrt5x)(5x^2+4+2\sqrt5x)}\cdot\cos\theta$$

$$= \sqrt{(5x^2+4)^2-(2\sqrt5x)^2} = \sqrt{25x^4+20x^2+16}$$

$$\Rightarrow \quad 25x^4 - 60x^2 + 36 = (25x^4 + 20x^2 + 16) \cdot (\cos)^2\theta$$

Se calculează x

Dacă $\theta = 60^\circ$

$$\Rightarrow 50x^4 - 120x^2 + 72 = 25x^4 + 20x^2 + 16$$

$$\Rightarrow 25x^4 - 140x^2 + 56 = 0$$

$$(x^2)_{1,2} = \frac{140 \pm \sqrt{140^2 - 5600}}{50} =$$

$$= \frac{140 \pm 10\sqrt{196 - 56}}{50} = \frac{14 \pm \sqrt{140}}{5}$$

$$\Rightarrow x = \pm \sqrt{\frac{14 \pm \sqrt{140}}{5}}$$

S8

12. Verify that the matrices

$$A = \frac{1}{3}\begin{bmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \frac{1}{11}\begin{bmatrix} -9 & -2 & 6 \\ 6 & -6 & 7 \\ 2 & 9 & 6 \end{bmatrix}$$

belong to $SO(3)$. Moreover, determine the axis of rotation and the rotation angle.

Sol.: axis of rotation $= Fix(B)$

$$Fix(B) = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \frac{1}{11} \cdot \begin{pmatrix} -9 & -2 & 6 \\ 6 & -6 & 7 \\ 2 & 9 & 6 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\}$$

$$\begin{pmatrix} -20 & -2 & 6 \\ 6 & -17 & 7 \\ 2 & 9 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -20 & -2 & 6 & 0 \\ 6 & -17 & 7 & 0 \\ 2 & 9 & -5 & 0 \end{array}\right) \overset{L_1 \leftrightarrow L_3}{\sim} \left(\begin{array}{ccc|c} 2 & 9 & -5 & 0 \\ 6 & -17 & 7 & 0 \\ -20 & -2 & 6 & 0 \end{array}\right) \sim$$

$$\underset{L_3 \leftarrow L_3 + 10L_1}{\overset{L_2 \leftarrow L_2 - 3L_1}{\sim}} \left(\begin{array}{ccc|c} 2 & 9 & -5 & 0 \\ 0 & -44 & 22 & 0 \\ 0 & 88 & -44 & 0 \end{array}\right) \underset{L_2 \leftarrow \frac{1}{22} L_2}{\overset{L_3 \leftarrow L_3 + 2L_2}{\sim}}$$

$$\left(\begin{array}{ccc|c} 2 & 9 & -5 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \overset{L_1 \leftarrow L_1 + 5L_2}{\sim}$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \begin{cases} 2x - y = 0 \\ -2y + z = 0 \end{cases} \Rightarrow \begin{array}{l} y = 2x \\ -4x + z = 0 \Rightarrow z = 4x \end{array}$$

$$\Rightarrow Fix(B) = \langle (1, 2, 4) \rangle$$

$$2\cos\theta + 1 = \operatorname{Tr} R = \frac{1}{11}\left(-9-6+6\right) = -\frac{9}{11}$$

$$\Rightarrow \quad \cos\theta = -\frac{10}{11} \Rightarrow \theta = \pi - \arccos\frac{10}{11}$$

$$R = (O, b), \qquad b = (v_1, \dots, v_n)$$

$$R' = (O', b'), \qquad b' = (v_1', \dots, v_n')$$

$$[P]_{R'} = [\overrightarrow{O'P}]_{b'} = \overbrace{[id]_{b,b'}}^{M_{b',b} = M_{R',R}} \cdot [\overrightarrow{O'P}]_b = \left([v_1]_{b'}\ [v_2]_{b'} \cdots [v_n]_{b'}\right) \cdot$$

$$\cdot [\overrightarrow{OP} - \overrightarrow{OO'}]_b = M_{R',R} \cdot [\overrightarrow{OP}]_b - M_{R',R} \cdot [\overrightarrow{OO'}]_b$$

$$= M_{R',R} \cdot [P]_R - M_{R',R} \cdot [O']_R$$

$$= M_{R',R} \cdot [P]_R + \underbrace{M_{R',R}}_{= M_{b',b} = [id]_{b,b'}} \cdot [\overrightarrow{O'O}]_b$$

$$= M_{R',R} \cdot [P]_R + [\overrightarrow{O'O}]_{b'}$$

$$= M_{R',R} \cdot [P]_R + [O]_{R'}$$

S11 6. Consider the rotation R_{90° of $\mathbb{E}^2$ around the origin and the translation $T_{\mathbf{v}}$ of $\mathbb{E}^2$ with vector $\mathbf{v}(1,0)$.

a) Give the algebraic form of the isometries R_{90°, $T_{\mathbf{v}}$ and $T_{\mathbf{v}} \circ R_{90^\circ}$.

b) Determine the equations of the hyperbola $\mathcal{H} : \frac{x^2}{4} - \frac{y^2}{9} - 1 = 0$ and the parabola $\mathcal{P} : y^2 - 8x = 0$ after transforming them with R_{90° and with $T_{\mathbf{v}} \circ R_{90^\circ}$ respectively.

$$[R_\theta] = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$[R_\theta(P)] = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot [P]$$

$$[T_v(P)] = [P] + [v]$$

$$[(T_v \circ R_\theta)(P)] = [T_v(R_\theta(P))] = [R_\theta(P)] + [v] =$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot [P] + [v]$$

b) $P(x,y) \in \mathcal{H} \Rightarrow \frac{x^2}{4} - \frac{y^2}{9} = 1$

$$[R_\theta(P)] = \begin{pmatrix} \cos 90^\circ & -\sin 90 \\ \sin 90 & \cos 90 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

$$\Rightarrow \begin{cases} y = -x' \\ x = y' \end{cases}$$

So the equation of $R_{\frac{\pi}{2}}(\mathcal{H})$ is:

$$\frac{y'^2}{4} - \frac{x'^2}{9} = 1 \Rightarrow -\frac{x'^2}{9} + \frac{y'^2}{4} = 1$$

$$[T_v \circ R_{90}(P)] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1-y \\ x \end{pmatrix}$$

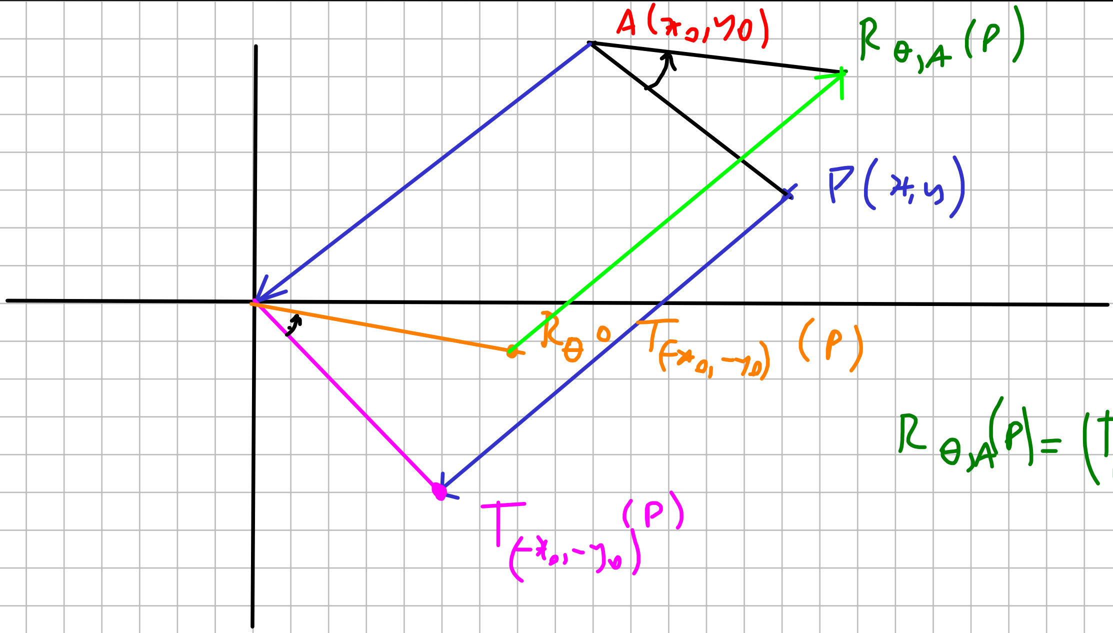
$$\Rightarrow \begin{cases} x' = 1-y \\ y' = x \end{cases} \Rightarrow \begin{cases} x = y' \\ y = 1-x' \end{cases}$$

$\Rightarrow (T_{\vec{u}} \circ R_{30^\circ})(\mathcal{H})$ is given by:

$$\frac{y'^2}{4} - \frac{(1-x')^2}{9} = 1$$

$$\Leftrightarrow 9y'^2 - 4x'^2 + 8x' - 4 - 36 = 0$$

$$-4x'^2 + 9y'^2 + 8x' - 40 = 0$$



$R_{\theta,A}(P) = ?$

$R_{\theta,A}(P) = (T_{(x_0,y_0)} \circ R_\theta \circ T_{(-x_0,-y_0)})(P)$