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In these notes we assume working knowledge of eigenvalues and eigenvectors. Have a look at your Algebra course from last semester.

6.1 Isometries

Definition. An *isometry* is a map $\phi : \mathbb{E}^n \rightarrow \mathbb{E}^n$ which preserves distances, i.e.

$$d(\phi(P), \phi(Q)) = d(P, Q)$$

for any points $P, Q \in \mathbb{E}^n$.

- One can show that isometries are affine transformations, i.e. they are elements in $\text{AGL}(\mathbb{E}^n)$.

Proposition 6.1. Let $\phi \in \text{AGL}(\mathbb{E}^n)$ be an affine transformation given by $\phi(\mathbf{x}) = A\mathbf{x} + b$ with respect to some orthonormal coordinate system. The following are equivalent:

1. ϕ is an isometry
2. $A^{-1} = A^t$.

Proposition 6.2. Let $A \in \text{Mat}_{n \times n}(\mathbb{R})$ be a matrix such that $A^t A = I_n$. Then $\det(A) \in \{\pm 1\}$.

Definition. A matrix $A \in \text{Mat}_{n \times n}(\mathbb{R})$ such that $A^t A = I_n$ is called *orthogonal*. The set of all such matrices are denoted by $O(n)$. The set of matrices in $O(n)$ with determinant 1 is denoted by $SO(n)$. Such matrices are called *special orthogonal*.

The set $O(n)$ is a subgroup of $\text{AGL}(\mathbb{R}^n)$ and $SO(n)$ is a normal subgroup of $O(n)$:

$$SO(n) \trianglelefteq O(n) \leq \text{AGL}(\mathbb{R}^n).$$

Let $\phi \in \text{AGL}(\mathbb{R}^n)$ be given by $\phi(\mathbf{x}) = A\mathbf{x} + b$ with respect to some orthonormal coordinate system. Then ϕ is called a *displacement*, or a *direct isometry*, if $A \in SO(n)$.

6.1.1 Rotations in dimension 2

Proposition 6.3. A matrix A is in $SO(2)$ if and only if A has the form

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

for some $\theta \in \mathbb{R}$.

Corollary 6.4. A direct isometry ϕ of \mathbb{E}^2 that fixes a point is either the identity or a rotation. Moreover, the angle θ of the rotation is such that

$$\cos(\theta) = \frac{\text{tr}(\text{lin}(\phi))}{2}.$$

6.1.2 Rotations in dimension 3

Theorem 6.5 (Euler). A direct isometry ϕ of \mathbb{E}^3 that fixes a point is either the identity or a rotation around an axis that passes through that point. Moreover, the angle θ of the rotation is such that

$$\cos(\theta) = \frac{\text{tr}(\text{lin}(\phi)) - 1}{2}.$$

6.1.3 Classification of isometries

Theorem 6.6 (Chasles). An isometry of the plane \mathbb{E}^2 is either a direct isometry, in which case it is

- the identity, or
- a translation, or
- a rotation;

or an indirect isometry, in which case it is

- a reflection, or

- a glidereflexion.

Theorem 6.7 (Euler). Any isometry of the 3-dimensional Euclidean space \mathbb{E}^3 is either a direct isometry, in which case it is

- the identity, or
- a translation, or
- a rotation around an axis, or
- a gliderotation (also called helical displacement);

or an indirect isometry, in which case it is

- a reflection, or
- a glidereflexion, or
- a rotation-reflection.

6.2 Spectral theorem

Proposition 6.8. Any set of mutually orthogonal vectors is linearly independent. In particular, any set of n mutually orthogonal vectors is a basis of \mathbb{V}^n .

Proposition 6.9. Let $e = \{e_1, \dots, e_n\}$ and $f = \{f_1, \dots, f_n\}$ be two bases of the vector space \mathbb{V}^n , and suppose that e is orthonormal. The basis f is orthonormal if and only if the base change matrix $M_{e,f}$ is orthogonal.

- Proposition 6.9, says that if we change the coordinate system from an orthonormal basis e to an orthonormal basis f then the base change matrix $M_{e,f}$ is in $O(n)$, i.e. the base change is an isometry.
- For a matrix $A \in \text{Mat}_{n \times n}(\mathbb{R})$ and a matrix $M \in O(n)$, since $M^{-1} = M^t$ we have

$$M^{-1}AM = M^tAM.$$

- The above observation is of fundamental importance for the isometric classification of quadrics. Suppose you have a quadratic equation

$$2x^2 - 6xy - 1y^2 = c \tag{6.1}$$

for some constant $c \in \mathbb{R}$. Then, you may write it in matrix form like this:

$$\begin{bmatrix} x & y \end{bmatrix} \underbrace{\begin{bmatrix} 2 & -3 \\ -3 & -1 \end{bmatrix}}_{=:A} \begin{bmatrix} x \\ y \end{bmatrix} = c$$

and if we change coordinates with a change of basis matrix M then

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = M \begin{bmatrix} x \\ y \end{bmatrix}$$

and thus, Equation (6.1) becomes

$$\begin{bmatrix} x' & y' \end{bmatrix} (M^t A M) \begin{bmatrix} x' \\ y' \end{bmatrix} = c$$

Now if M is orthogonal, i.e. $M \in O(n)$ (which by Proposition 6.9 happens when we go from one orthonormal coordinate system to another), then the above equation becomes

$$\begin{bmatrix} x' & y' \end{bmatrix} (M^{-1} A M) \begin{bmatrix} x' \\ y' \end{bmatrix} = c$$

The matrix A is the matrix of a linear map ϕ_A , and $M^{-1} A M$ is the matrix of the same linear map with respect to a different (orthonormal) basis. Thus, if we find an orthonormal basis with respect to which the matrix of ϕ_A is 'nice' (spoiler: diagonal), then with respect to that basis Equation (6.1) becomes

$$\lambda_1 x^2 + \lambda_2 y^2 = c$$

for some values $\lambda_1, \lambda_2 \in \mathbb{R}$ which are the eigenvalues of ϕ_A and of A .

- Notice that if we change the order of two vectors in the basis f then we change the sign of $\det(M_{e,f})$. Notice also that if we change the sign of one vector in f we change the sign of $\det(M_{e,f})$. So, changing orthonormal coordinate systems can be performed with matrices in $SO(n)$ if we give ourselves the freedom of interchanging two axes or of changing the direction of one axis.
- The above observation is of interest because coordinate changes performed with matrices in $SO(n)$ are displacements. Such transformations are close to our intuition since they correspond to the usual movements that we do/see in our surroundings.
- The next statements show how a symmetric operator (a linear map with a symmetric matrix) can be diagonalized with matrices in $SO(n)$.

Lemma 6.10. The characteristic polynomial of a symmetric matrix $A \in \text{Mat}_{n \times n}(\mathbb{R})$ has only real roots.

Theorem 6.11. (Spectral Theorem) Let $T : \mathbb{V}^n \rightarrow \mathbb{V}^n$ be a symmetric operator. There is an orthonormal basis of \mathbb{V}^n with respect to which the matrix of T is diagonal.

Theorem 6.12. For every real symmetric matrix $A \in \text{Mat}_{n \times n}(\mathbb{R})$ there is an orthogonal matrix $M \in O(n)$ such that $M^{-1} A M$ is diagonal.

Proposition 6.13. Let $T : \mathbb{V}^n \rightarrow \mathbb{V}^n$ be a symmetric operator on a Euclidean vector space. If λ and μ are two distinct eigenvalues of T then every eigenvector with eigenvalue λ is orthogonal to every eigenvector with eigenvalue μ .