CHAPTER 3

Lines in dimension 2

Contents

Unless otherwise stated, any coordinate system that we fix will be a right oriented orthonormal coordinate system (reference frame). In this document $Oxy = (O, \mathbf{i}, \mathbf{j})$ is a coordinate system of $\mathbb{E}^2$. Recall that $\mathbb{V}^2$ is the vector space of vectors which can be represented with points in $\mathbb{E}^2$. Recall also that we have a bijective map $\phi_O : \mathbb{E}^2 \to \mathbb{V}^2$ given by $\phi_O(P) = \overrightarrow{OP}$.

3.1 Describing curves in $\mathbb{E}^2$

There are essentially three ways in which we can describe curves (or, more generally, geometric objects). For curves in $\mathbb{E}^2$ these are the following

1. Via parametrizations, where, for a given interval I, you use a map $\varphi : I \to \mathbb{E}^2$ to specify for each $t \in I$ a point in $\mathbb{E}^2$. For example, the circle of radius 1 centered at the origin is given by $\phi : [0, 2\pi) \to \mathbb{E}^2$ with $\phi(t) = (\cos(t), \sin(t))$.

2. Via global equations, where you describe the curve as the set of all points whose coordinates satisfy a given equation. For example, the circle of radius 1 centered at the origin is the set $\{P(x, y) \in \mathbb{E}^2 : x^2 + y^2 = 1\}$.

3. Via geometric properties, where you describe the curve as the set of all points satisfying a geometric property. For example, the circle of radius 1 centered at the point O is the set $\{P \in \mathbb{E}^2 : P \text{ is at distance 1 from } O\}$.

Notice that:

- The first two descriptions depend on the coordinate system while the third one does not.

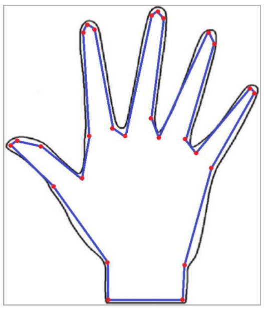
- The first two descriptions allow for concrete computational methods.

- In this course we focus on translating geometric properties into equations with respect to a coordinate system (a reference frame).

- One can think about the first approach as being local: when you vary the parameter t a little bit you move the point $\phi(t)$ a little bit, so this approach allows you to control the curve locally.

- One can think about the second approach as being global: indeed, you have one rule (one equation) for all points.

Computationally you will most often use an approximation of the curve that you are interested in. The simplest form of approximation is via line segments.



3.2 Lines in $\mathbb{E}^2$

3.2.1 Geometric description

Recall from Lecture 1 that we can describe lines as being sets of points S in $\mathbb{E}^2$ such that the set of vectors which can be represented by points in S form a 1-dimensional vector subspace of $\mathbb{V}^2$, i.e. for any $A \in S$

$$\phi_A(S) = \{\overrightarrow{AB} : B \in G\} \quad \text{is a 1-dimensional vector subspace of } \mathbb{V}^2.$$

3.2.2 Parametric equations - local description

If S is a line then for any two distinct points A, B in S the vector $\overrightarrow{AB}$ is called a *direction vector* of S. Since $\phi_A(S)$ is 1-dimensional, all direction vectors are linearly dependent and $\mathbf{v}$ is a direction vector for S if and only if it is linearly dependent on $\overrightarrow{AB}$. So, for any direction vector $\mathbf{v}$ of S there is a scalar $t \in \mathbb{R}$ such that

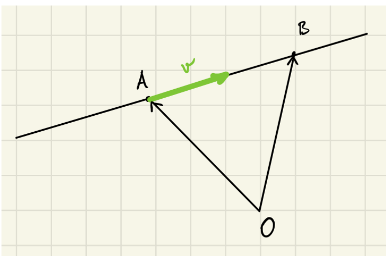
$$\overrightarrow{AB} = t\mathbf{v}.$$

Now, if you fix A and let B vary on the line then t varies in $\mathbb{R}$. Since ϕ_A is a bijection, the line S can be described as

$$S = \left\{ B \in \mathbb{E}^2 : \overrightarrow{AB} = t\mathbf{v} \text{ for some } t \in \mathbb{R} \right\}.$$

In this description the point A is arbitrary but fixed. We sometimes refer to it as the *base point*. Now, the coordinates of a point B are the components of the vector $\overrightarrow{OB}$ and, rearranging the above equation we have

$$\overrightarrow{OB} = \overrightarrow{OA} + t\mathbf{v}. \tag{3.1}$$



So, we can describe the line S as being the set of points B in $\mathbb{E}^2$ which satisfy Equation (3.1) for some $t \in \mathbb{R}$. This is called the *vector equation* of the line S.

- The vector equation depends on the choice of the base point A.
- The vector equation depends on the choice of the direction vector $\mathbf{v}$.
- Hence, a line does not have a unique vector equation.
- The vector equation does not depend on the coordinate system. In the above description O can be any point in $\mathbb{E}^2$.

If we write Equation (3.1) in coordinates (relative to the coordinate system Oxy) then we obtain

$$\begin{cases} x = x_A + tv_x \\ y = y_A + tv_y \end{cases} \quad \text{or, in matrix form} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_A \\ y_A \end{bmatrix} + t \begin{bmatrix} v_x \\ v_y \end{bmatrix} \tag{3.2}$$

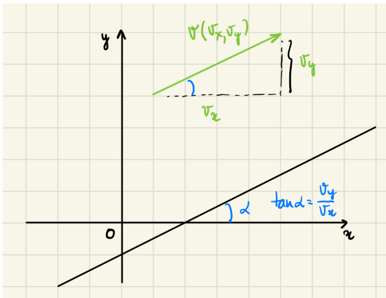
where $A = A(x_A, y_A)$, $\mathbf{v} = \mathbf{v}(v_x, v_y)$ and for different values t we obtain different points (x, y) on the line. The two equations in the system (3.2) are called *parametric equations* for the line S.

3.2.3 Cartesian equations - global description

It is possible to eliminate the parameter t in (3.2) by expressing it in both equations and setting the two expressions equal in order to obtain

$$\frac{x - x_A}{v_x} = \frac{y - y_A}{v_y}. \tag{3.3}$$

We refer to Equation (3.3) as *symmetric equation* of the line S. It could happen that v_x or v_y are zero. In that case, translate back to the parametric equations to understand what happens.



You can rearrange Equation (3.3) as

$$y = kx + m \quad \text{where} \quad k = \frac{v_y}{v_x} \quad \text{and} \quad m = -\frac{v_y}{v_x} x_A + y_A.$$

In this form we will call it *the equation where you can read-off the slope* k. Draw a picture to see that if $\alpha = \measuredangle(\mathbf{i}, \mathbf{v})$ then $k = \tan(\alpha)$. In fact, you can rearrange Equation (3.3) as

$$y - y_A = \tan(\alpha)(x - x_A).$$

If you like names, then you can call this *the equation of a line in* $\mathbb{E}^2$ *where you can read off the slope and a point on the line.*

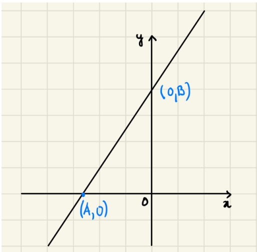
- We have just described a line with a linear equation (relative to the coordinate system Oxy).

Proposition 3.1. Every line in $\mathbb{E}^2$ can be described with a linear equation in two variables

$$ax + by + c = 0 \tag{3.4}$$

relative to a fixed coordinate system and any linear equation in two variables describes a line relative to a fixed coordinate system.

- Equation (3.4) is called a *Cartesian equation* of the line it describes.
- Notice that there are infinitely many Cartesian equations describing the same line, since you can multiply one equation by a non-zero constant.



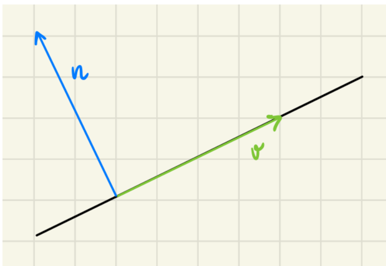
If a line is given with the Cartesian equation (3.4), you can rearrange it in the form

$$\frac{x}{A}+\frac{y}{B}=1 \quad \text{where} \quad A=-\frac{c}{a} \quad \text{and} \quad B=-\frac{c}{b}.$$

In this form we have the *equation of the line where we can read off the intersection points with the coordinate axes* since the line intersects Ox in $(A,0)$ and it intersects Oy in $(0,B)$.

3.2.4 Normal vectors

Normal vectors play an important role both in abstract geometry and in applications. Here we consider normal vectors of lines (in $\mathbb{E}^2$!).



The idea is very simple: we discussed in Lecture 2 that if we fix a vector $\mathbf{n}\in\mathbb{V}^2$ then $\mathbf{n}^\perp$ is a vector subspace of $\mathbb{V}^2$ and $\mathbb{V}^2=\langle\mathbf{n}\rangle\oplus\mathbf{n}^\perp$. The vector subspace $\langle\mathbf{n}\rangle$ is 1-dimensional (it is spanned by one vector) and $\dim(\mathbb{V}^2)=2$. Therefore $\mathbf{n}^\perp$ must have dimension 1. So, returning to the description of a line as a set of points of the form

$$S=\left\{B\in\mathbb{E}^2 : \overrightarrow{AB}=t\mathbf{v} \text{ for some } t\in\mathbb{R}\right\},$$

we may choose $\mathbf{n}$ such that a direction vector $\mathbf{v}$ for S is a generator of $\mathbf{n}^\perp$. In other words, we may choose $\mathbf{v}\in\mathbf{n}^\perp$ non-zero, i.e. we may choose $\mathbf{v}$ to be a non-zero vector orthogonal to $\mathbf{n}$. Then, we have

$$S=\left\{B\in\mathbb{E}^2 : \mathbf{n}\perp\overrightarrow{AB}\right\}=\left\{B\in\mathbb{E}^2 : \mathbf{n}\cdot\overrightarrow{AB}=0\right\},$$

where A is some point in S. In other words, the line S can be described as the set of point B such that

$$\mathbf{n} \cdot \overrightarrow{AB} = 0 \quad \text{or, equivalently} \quad \mathbf{n} \cdot (\overrightarrow{OB} - \overrightarrow{OA}) = 0$$

The vector $\mathbf{n}$ is perpendicular to the line S and any vector with this property is called a *normal vector* for the line S. If we look at this equation in coordinates, then $\overrightarrow{OA} = (x_A, y_A) = A \in S$ and with $\mathbf{n} = \mathbf{n}(n_x, n_y)$ we have

$$n_x(x - x_A) + n_y(y - y_A) = 0 \tag{3.5}$$

where (x, y) is an arbitrary point on the line S. This is a linear equation for S and since all other equations of S (in the coordinate system Oxy) are obtained form (3.5) by multiplying with a non-zero constant, we see that the coefficients of x and y in a Cartesian equation of S can be interpreted as the components of a normal vector.

3.2.5 Relative positions

- [Intersections] Assume we have two lines

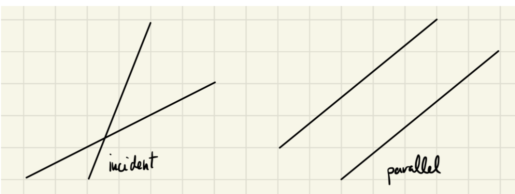
$$\ell_1 : a_1x + b_1y + c_1 = 0 \quad \text{and} \quad \ell_2 : a_2x + b_2y + c_2 = 0$$

In order to determine if they intersect or not one has to discuss the system:

$$\begin{cases} \ell_1 : a_1x + b_1y + c_1 & = & 0 \\ \ell_2 : a_2x + b_2y + c_2 & = & 0 \end{cases} \tag{3.6}$$

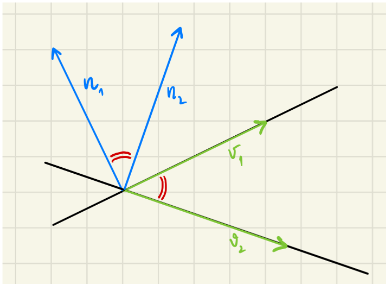
Recall Lecture 12 of your Algebra course last semester. In the plane the situation is very simple:

  - two lines either intersect, in a unique point, the coordinates of which will be a unique solution to (3.6); or
  - they don't intersect and (3.6) doesn't have solutions, in which case the lines are parallel; or
  - system (3.6) has infinitely many solutions in which cases $\ell_1 = \ell_2$.

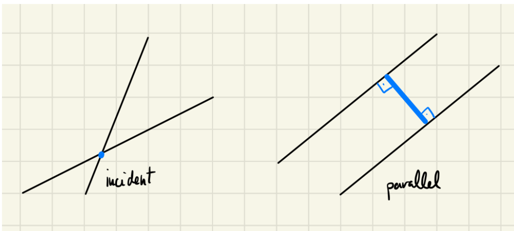


Notice that (a_1, b_1) is a normal vector for ℓ_1 and (a_2, b_2) is a normal vector for ℓ_2, so the last two cases above occur if these two vector are proportional, which is equivalent to $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$, i.e. the matrix of coefficients for the system (3.6) has rank 1.

- [Angles] Two lines define two angles: if $\mathbf{v}_1$ is a direction vector for ℓ_1 and if $\mathbf{v}_2$ is a direction vector for ℓ_2 then the two angles described by ℓ_1 and ℓ_2 are $\measuredangle(\mathbf{v}_1, \mathbf{v}_2)$ and $\measuredangle(-\mathbf{v}_1, \mathbf{v}_2)$. They are supplementary angles so if you can measure one of them you know the other one. Recall from Lecture 2 that
$$\cos \measuredangle(\mathbf{v}_1, \mathbf{v}_2) = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{\|\mathbf{v}_1\| \cdot \|\mathbf{v}_2\|}. \tag{3.7}$$
Notice also that the two angles can also be described with normal vectors: if $\mathbf{n}_1$ and $\mathbf{n}_2$ are normal vectors for ℓ_1 and ℓ_2 respectively, then the two angles between ℓ_1 and ℓ_2 are $\measuredangle(\mathbf{n}_1, \mathbf{n}_2)$ and $\measuredangle(-\mathbf{n}_1, \mathbf{n}_2)$. So, if these vectors are known we may calculate
$$\cos \measuredangle(\mathbf{n}_1, \mathbf{n}_2) = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\| \cdot \|\mathbf{n}_2\|}. \tag{3.8}$$
Clearly, if the two lines are parallel, then the two angles are 0° and 180°.



- [Distances]



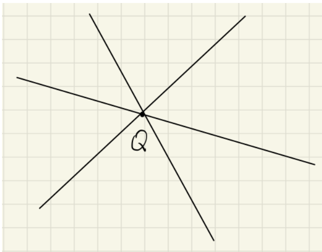
We first consider the distance from a point to a line

Proposition 3.2. Suppose you have a line $\ell : ax + by + c = 0$ and a point $P(x_P, y_P)$ in $\mathbb{E}^2$. The distance from P to ℓ is
$$d(P, \ell) = \frac{|ax_P + by_P + c|}{\sqrt{a^2 + b^2}}.$$

Now, considering the distances between two lines ℓ_1 and ℓ_2 we have the following cases:

- if ℓ_1 and ℓ_2 intersect then the distance between them is zero $d(\ell_1, \ell_2) = 0$.
- if ℓ_1 and ℓ_2 do not intersect, i.e. if they are parallel, then
$$d(\ell_1, \ell_2) = d(P_1, \ell_2) = d(\ell_1, P_2)$$
for any point $P_1 \in \ell_1$ and any point $P_2 \in \ell_2$.

3.3 Bundle of lines



Definition. Let $Q \in \mathbb{E}^2$. The set $\mathcal{L}_Q$ of all lines in $\mathbb{E}^2$ passing through Q is called a *bundle of lines* and Q is called the *center* of the bundle $\mathcal{L}_Q$.

Proposition 3.3. If $\ell_1 : a_1x + b_1y + c_1 = 0$ and $\ell_2 : a_2x + b_2y + c_2 = 0$ are two distinct lines in the bundle $\mathcal{L}_Q$, then $\mathcal{L}_Q$ consists of lines having equations of the form

$$\ell_{\lambda,\mu} : \lambda(a_1x + b_1y + c_1) + \mu(a_2x + b_2y + c_2) = 0.$$

where $\lambda, \mu \in \mathbb{R}$ not both zero.

- In particular, if $Q = Q(x_0, y_0)$, $\ell_1 : x = x_0$ and $\ell_2 : y = y_0$ then

$$\mathcal{L}_Q = \left\{ \ell_{\lambda,\mu} : \lambda(x - x_0) + \mu(y - y_0) = 0 : \lambda, \mu \in \mathbb{R} \text{ not both zero} \right\}.$$

- Bundles of lines are useful in praxis when a point Q is given as the intersection of two lines, but its coordinates are not known explicitly, and one wants to find the equation of a line passing through Q and satisfying some other conditions. For example, the condition that it contains some point P distinct from Q or that it is parallel to a given line.

- There is redundancy in the two parameters λ, μ, meaning that there are not two independent parameters here. If $\lambda \neq 0$ then one can divide the equation of $\ell_{\lambda,\mu}$ by λ to obtain

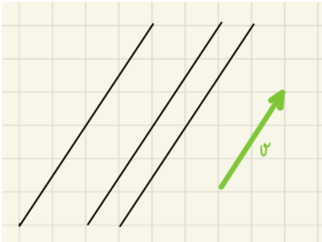
$$\ell_{1,t} : (a_1x + b_1y + c_1) + t(a_2x + b_2y + c_2) = 0.$$

  where $\frac{\mu}{\lambda} = t \in \mathbb{R}$. So $\ell_{1,\frac{\mu}{\lambda}}$ and $\ell_{\lambda,\mu}$ are in fact the same lines.

- A *reduced bundle* is the set of all lines $\mathcal{L}_Q$ passing through a common point Q from which we remove one line, i.e. it is $\mathcal{L}_Q \setminus \{\ell_2\}$ for some $\ell_2 \in \mathcal{L}_Q$. With the above notation and discussion, it is the set

$$\{\ell_{1,t} : (a_1x + b_1y + c_1) + t(a_2x + b_2y + c_2) = 0 : t \in \mathbb{R}\}$$

  The fact that we use one parameter instead of two, to describe almost all lines passing through Q, greatly simplifies calculations.

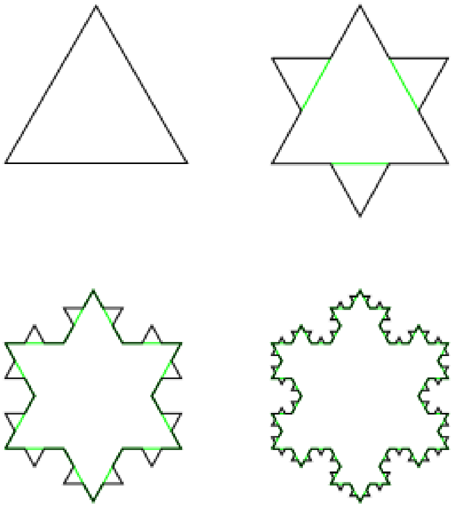


Definition. Let $\mathbf{v} \in \mathbb{V}^2$. The set $\mathcal{L}_{\mathbf{v}}$ of all lines in $\mathbb{E}^2$ with direction vector $\mathbf{v}$ is called an *improper bundle of lines*, and $\mathbf{v}$ is called a *direction vector* of the bundle $\mathcal{L}_{\mathbf{v}}$.

- The connection between bundles of lines and improper bundles of lines is best understood through projective geometry, where the improper bundle of lines is the set of all lines intersecting in the same point at infinity.

3.4 Strange curves in $\mathbb{E}^2$

There are 'curves' which can be described geometrically by a process, like the Koch curve (Koch snowflake):



It looks like a curve, but it has very little in common with the curves that you are familiar with:

- It has no tangent line, no derivatives.
- It has fractal dimension $\log_3(4) \sim 1.26186$ wherease as line segment has fractal dimension 1.

- It's better to call it a fractal than a curve.
- Notice that you describe it using line segments.