CHAPTER 9

Canonical equations of real quadrics

Contents

Here we look at the main examples of *quadrics* in $\mathbb{E}^3$, sometimes called *quadratic surfaces*. These are surfaces which satisfy a quadratic equation of the form

$$\mathcal{S} : q_{11}x^2 + q_{22}y^2 + q_{33}z^2 + 2q_{12}xy + 2q_{13}xz + 2q_{23}yz + a_1x + a_2y + a_3z + c = 0. \tag{9.1}$$

Notice that an equation as above may not define a surface. It could happen that there are no solutions or that the coordinates of only one point satisfy a given quadratic equation.

9.1 Ellipsoid

9.1.1 Canonical equation - global description

An *ellipsoid* is a surface which (in some coordinate system) satisfies an equation of the form

$$\mathcal{E}_{a,b,c} : \underbrace{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}}_{\varphi(x,y,z)} = 1 \quad \text{so} \quad \mathcal{E}_{a,b,c} = \varphi^{-1}(1)$$

for some positive constants $a, b, c \in \mathbb{R}$.

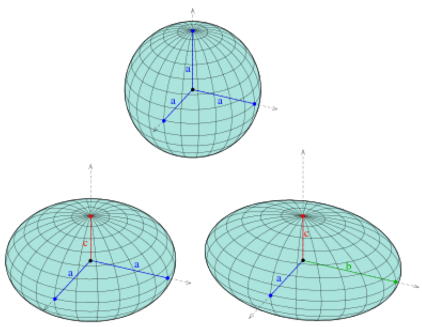


Figure 9.1: Ellipsoid[1]

If we fix one variable, we obtain intersections with planes parallel to the coordinate axes. For this surface the intersections are either ellipses, points or the empty set. Check this for $z = h$ and deduce the axes of the ellipses that you obtain.

[1] Image source: Wikipedia

9.1.2 Tangent planes

Using the gradient or the algebraic method we obtain the tangent plane at a point $p = (x_p, y_p, z_p) \in \mathcal{E}_{a,b,c}$ to be

$$T_p\mathcal{E}_{a,b,c} : \frac{x_p x}{a^2} + \frac{y_p y}{b^2} + \frac{z_p z}{c^2} = 1.$$

We will check this with the algebraic method. Let us consider a line passing through p in parametric form

$$l : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix} + t \underbrace{\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}}_{=v} \quad \Leftrightarrow \quad l = p + \langle v \rangle.$$

Recall that the tangent plane $T_p\mathcal{E}_{a,b,c}$ is the union of all lines intersecting the quadric $\mathcal{E}_{a,b,c}$ at p in a special way, i.e. in a double point. This is why we investigate the intersection of our quadric with l. How do we obtain the intersection $\mathcal{E}_{a,b,c} \cap l$? We look at those points of l which satisfy the equation of the ellipsoid, i.e. we look for solutions t for the equation

$$\varphi(\begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix} + t \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}) = 1 \quad \Leftrightarrow \quad \frac{(x_p + tv_x)^2}{a^2} + \frac{(y_p + tv_y)^2}{b^2} + \frac{(z_p + tv_z)^2}{c^2} - 1 = 0.$$

If we rearrange the left-hand side as a polynomial in t we get

$$\Big(\frac{v_x^2}{a^2} + \frac{v_y^2}{b^2} + \frac{v_z^2}{c^2}\Big)t^2 + 2\Big(\frac{x_p v_x}{a^2} + \frac{y_p v_y}{b^2} + \frac{z_p v_z}{c^2}\Big)t + \underbrace{\frac{x_p^2}{a^2} + \frac{y_p^2}{b^2} + \frac{z_p^2}{c^2}}_{=1} - 1 = 0$$

$$\Leftrightarrow \quad \underbrace{\Big(\frac{v_x^2}{a^2} + \frac{v_y^2}{b^2} + \frac{v_z^2}{c^2}\Big)}_{\neq 0}t^2 + 2\Big(\frac{x_p v_x}{a^2} + \frac{y_p v_y}{b^2} + \frac{z_p v_z}{c^2}\Big)t = 0.$$

This equation in t admits the solution $t = 0$. That is clear, the point on l corresponding to $t = 0$ is the point p which, by assumption, lies on $\mathcal{E}_{a,b,c}$. Furthermore, the second solution will correspond to a second point of intersection. However, the line l is tangent to our quadric if p is a *double point of intersection*. This happens if and only if the above equation has $t = 0$ as double solution, i.e. if and only if

$$\frac{x_p v_x}{a^2} + \frac{y_p v_y}{b^2} + \frac{z_p v_z}{c^2} = 0.$$

How do we interpret this? In the Euclidean setting this can be interpreted as saying that $(\frac{x_p}{a^2}, \frac{y_p}{b^2}, \frac{z_p}{c^2})$ is perpendicular to the direction vector v of the line. All lines which are tangent to the surface and contain p, need to satisfy this condition, so

$$T_p\mathcal{E}_{a,b,c} : \begin{bmatrix} \frac{x_p}{a^2} \\ \frac{y_p}{b^2} \\ \frac{z_p}{c^2} \end{bmatrix} \cdot (\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix}) = 0 \quad \Leftrightarrow \quad \frac{x_p x}{a^2} + \frac{y_p y}{b^2} + \frac{z_p z}{c^2} - 1 = 0.$$

Deduce this equation also with the gradient.

9.1.3 Parametrizations - local description

A parametrization of this surface is

$$\begin{cases} x(\theta_1,\theta_2) = a\cos(\theta_1)\cos(\theta_2) \\ y(\theta_1,\theta_2) = b\sin(\theta_1)\cos(\theta_2) \\ z(\theta_1,\theta_2) = c\sin(\theta_2) \end{cases} \quad \theta_1 \in [0,2\pi[\quad \theta_2 \in [-\frac{\pi}{2},\frac{\pi}{2}[$$

Why? Check this and deduce a parametrization of the tangent plane $T_p\mathcal{E}_{a,b,c}$ at the point

$$p = \begin{bmatrix} x(\theta_{1,p},\theta_{2,p}) \\ y(\theta_{1,p},\theta_{2,p}) \\ z(\theta_{1,p},\theta_{2,p}) \end{bmatrix} \in \mathcal{E}_{a,b,c}.$$

9.2 Elliptic Cone

9.2.1 Canonical equation - global description

An *elliptic cone* is a surface which (in some coordinate system) satisfies an equation of the form

$$\mathcal{C}_{a,b,c} : \underbrace{\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}}_{\varphi(x,y,z)} = 0 \quad \text{so} \quad \mathcal{E}_{a,b,c} = \varphi^{-1}(0)$$

for some positive constants $a,b,c \in \mathbb{R}$.

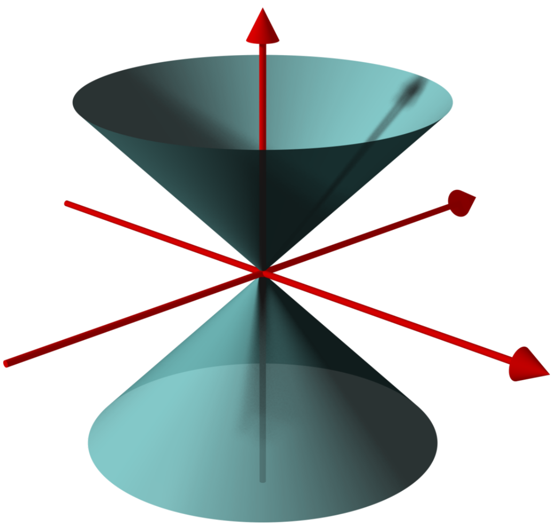


Figure 9.2: Elliptic cone[2]

If we fix one variable, we obtain intersections with planes parallel to the coordinate axes. For this surface the intersections are either ellipses, hyperbolas or a point. Check this for $z = h$ and deduce the axes of the ellipses that you obtain.

[2]Image source: Wikipedia

9.2.2 Conic sections

Above we noticed that the intersection of an elliptic cone with planes parallel to the coordinate axes are quadratic surfaces (possibly degenerate). In fact we have the following result.

Proposition 9.1. The intersection of an elliptic cone with an arbitrary plane is a (possibly degenerate) quadratic curve.

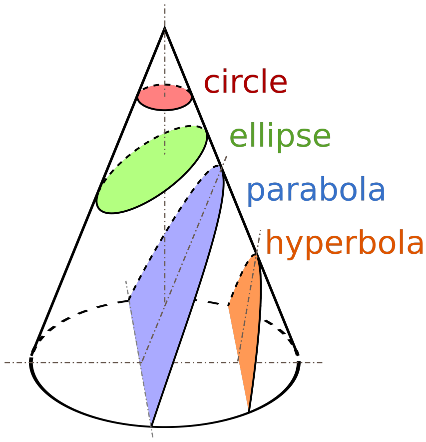


Figure 9.3: Conic sections[3]

9.2.3 Tangent planes

As in the case of the ellipsoid, using the gradient or the algebraic method we obtain the tangent plane at a point $p = (x_p, y_p, z_p) \in \mathcal{C}_{a,b,c}$ to be

$$T_p\mathcal{C}_{a,b,c} : \frac{x_p x}{a^2} + \frac{y_p y}{b^2} - \frac{z_p z}{c^2} = 0.$$

9.2.4 $\mathcal{C}_{a,b,c}$ as ruled surface

A *ruled surface* is a surface $\mathcal{S}$ such that for any point p on the surface S there is a line l which containes p and which is contained in S:

$$\forall p \in \mathcal{S}, \exists \text{ a line } l \text{ such that } p \in l \text{ and } l \subseteq S.$$

If we denote by $\mathcal{L}$ the family of all these lines, it is easy to see that the surface is the union of them:

$$\mathcal{S} = \bigcup_{l \in \mathcal{L}} l.$$

[3]Image source: Wikipedia

The lines in $\mathcal{L}$ are called *rectilinear generators of the surface* $\mathcal{S}$. We refer to them as *generators* since we don't consider here non-rectilinear generators.

So, how is the cone a ruled surface? Fix a point $(x_0, y_0, z_0) \in \mathcal{C}_{a,b,c}$ and notice that for any $t \in \mathbb{R}$ we have

$$\frac{(tx_0)^2}{a^2} + \frac{(ty_0)^2}{b^2} - \frac{(tz_0)^2}{c^2} = t^2\left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} - \frac{z_0^2}{c^2}\right) = 0.$$

Thus, the line $\{(tx_0, ty_0, tz_0) : t \in \mathbb{R}\}$, which passes through the given point and the origin is contained in $\mathcal{C}_{a,b,c}$. The set of all lines obtained in this way form the generators $\mathcal{L}$ of the cone.

9.2.5 Parametrizations - local description

Describing a parametrization of an elliptic cone can be generalized to any planar curve. Suppose you have a parametrization of a curve in the plane Oxy. In our case, for the ellipse

$$\begin{cases} x(\theta) = a\cos(\theta) \\ y(\theta) = b\sin(\theta) \end{cases} \quad \theta \in [0, 2\pi[.$$

You want to rescale this curve with the height such that when the height $z = 0$ you have a point, and for all other values of z you have a rescaled versions of your curve:

$$\begin{cases} x(\theta, h) = ha\cos(\theta) \\ y(\theta, h) = hb\sin(\theta) \\ z(\theta, h) = hc \end{cases} \quad \theta \in [0, 2\pi[\quad h \in \mathbb{R}.$$

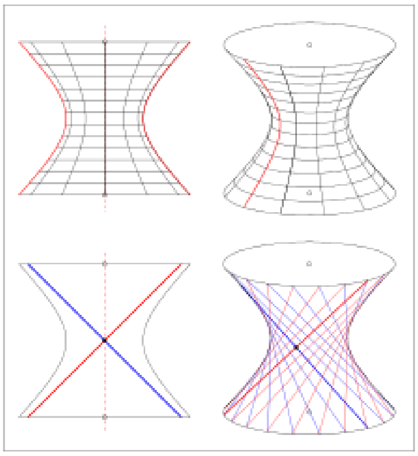
9.3 Hyperboloid of one sheet

9.3.1 Canonical equation - global description

A *hyperboloid of one sheet* is a surface which (in some coordinate system) satisfies an equation of the form

$$\mathcal{H}^1_{a,b,c} : \underbrace{\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2}}_{\varphi(x,y,z)} = 1 \quad \text{so} \quad \mathcal{H}^1_{a,b,c} = \varphi^{-1}(1) \tag{9.2}$$

for some positive constants $a, b, c \in \mathbb{R}$.



(a) Hyperboloid of one sheet[4]

(b) Kobe Port Tower

If we fix one variable, we obtain intersections with planes parallel to the coordinate axes. For this surface the intersections are either ellipses, hyperbolas or two lines. Check this for $y = h$ and deduce the axes of the hyperbolas that you obtain.

9.3.2 Tangent planes

Using the gradient or the algebraic method we obtain the tangent plane at a point $p = (x_p, y_p, z_p) \in \mathcal{H}^1_{a,b,c}$ to be

$$T_p\mathcal{H}^1_{a,b,c} : \frac{x_p x}{a^2} + \frac{y_p y}{b^2} - \frac{z_p z}{c^2} = 1.$$

We will check this with the algebraic method. Let us consider a line passing through p in parametric form

$$l : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix} + t \underbrace{\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}}_{=v} \quad \Leftrightarrow \quad l = p + \langle v \rangle.$$

Recall that the tangent plane $T_p\mathcal{H}^1_{a,b,c}$ is the union of all lines intersecting the quadric $\mathcal{H}^1_{a,b,c}$ at p in a special way, i.e. in a double point. This is why we investigate the intersection of our quadric with l. How do we obtain the intersection $\mathcal{H}^1_{a,b,c} \cap l$? We look at those points of l which satisfy the equation of our hyperboloid, i.e. we look for solutions t for the equation

$$\varphi(\begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix} + t \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}) = 1 \quad \Leftrightarrow \quad \frac{(x_p + tv_x)^2}{a^2} + \frac{(y_p + tv_y)^2}{b^2} - \frac{(z_p + tv_z)^2}{c^2} - 1 = 0. \tag{9.3}$$

[4]Image source: Wikipedia

If we rearrange the left-hand side as a polynomial in t we get

$$\Big(\frac{v_x^2}{a^2}+\frac{v_y^2}{b^2}-\frac{v_y^2}{c^2}\Big)t^2+2\Big(\frac{x_pv_x}{a^2}+\frac{y_pv_y}{b^2}-\frac{z_pv_z}{c^2}\Big)t+\underbrace{\frac{x_p^2}{a^2}+\frac{y_p^2}{b^2}-\frac{z_p^2}{c^2}}_{=1}-1=0$$

$$\Leftrightarrow\quad \underbrace{\Big(\frac{v_x^2}{a^2}+\frac{v_y^2}{b^2}-\frac{v_y^2}{c^2}\Big)}_{=0?}t^2+2\Big(\frac{x_pv_x}{a^2}+\frac{y_pv_y}{b^2}-\frac{z_pv_z}{c^2}\Big)t=0. \tag{9.4}$$

This equation in t admits the solution $t=0$. That is clear, the point on l corresponding to $t=0$ is the point p which, by assumption, lies on $\mathcal{H}^1_{a,b,c}$. Furthermore, a second solution will correspond to a second point of intersection. However, the line l is tangent to our quadric if p is a *double point of intersection*. This happens if and only if the above equation has $t=0$ as double solution, i.e. if and only if

$$\frac{v_x^2}{a^2}+\frac{v_y^2}{b^2}-\frac{v_y^2}{c^2}\neq 0 \quad\text{and}\quad \frac{x_pv_x}{a^2}+\frac{y_pv_y}{b^2}-\frac{z_pv_z}{c^2}=0.$$

How do we interpret the second condition? In the Euclidean setting this can also be interpreted as saying that $(\frac{x_p}{a^2},\frac{y_p}{b^2},-\frac{z_p}{c^2})$ is perpendicular to the direction vector v of the line. In both cases, all lines which are tangent to the surface and contain p, need to satisfy this equations, so the tangent plane is

$$T_p\mathcal{H}^1_{a,b,c}:\begin{bmatrix}\frac{x_p}{a^2}\\ \frac{y_p}{b^2}\\ -\frac{z_p}{c^2}\end{bmatrix}\cdot\left(\begin{bmatrix}x\\y\\z\end{bmatrix}-\begin{bmatrix}x_p\\y_p\\z_p\end{bmatrix}\right)=0 \quad\Leftrightarrow\quad \frac{x_px}{a^2}+\frac{y_py}{b^2}-\frac{z_pz}{c^2}-1=0.$$

Deduce this equation also with the gradient.

How do we interpret the first condition? If

$$\frac{v_x^2}{a^2}+\frac{v_y^2}{b^2}-\frac{v_y^2}{c^2}=0 \tag{9.5}$$

then equation (9.4) is linear, and has one simple solution $t=0$. This means that l intersects $\mathcal{H}^1_{a,b,c}$ only once (it punctures the surface in one point). Such lines are not tangent to the surface. How can we visualize this? Let us start with what we know: the vector $v=(v_x,v_y,v_z)$ satisfies the equation (9.5), so we can think of it as the position vector of some point on the cone $\mathcal{C}_{a,b,c}$. How does this cone relate to our hyperboloid? Our surface $\mathcal{H}^1_{a,b,c}$ is the union of hyperbolas (revolving on ellipses around the z-axis) and if we take the union of all the asymptotes to these hyperbolas we get $\mathcal{C}_{a,b,c}$ (see Figure 9.5).

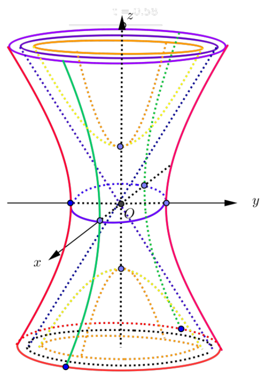


Figure 9.5: Hyperboloid and asymptotic cone[5]

This should help to see that, when the vector $v = (v_x, v_y, v_z)$ satisfies equation (9.5), the line l is parallel to a line contained in the cone $\mathcal{C}_{a,b,c}$. It will therefore intersect $\mathcal{H}^1_{a,b,c}$ in at most one point.

Notice also that if $l \subseteq \mathcal{C}_{a,b,c}$, it will not intersect $\mathcal{H}^1_{a,b,c}$ at all, but this cannot happen in our setting because we chose the point p such that it lies both on l and on our quadric. In fact, the related question 'does a given line l intersect $\mathcal{H}^1_{a,b,c}$?' can be answered by investigating equation (9.3) without the assumption that p lies on the surface $\mathcal{H}^1_{a,b,c}$. How would you do this?

9.3.3 $\mathcal{H}^1_{a,b,c}$ as ruled surface

Here is a fact: the hyperboloid with one sheet is a *doubly ruled surface* (this is visible in Figure 9.4a). Hmm.. I know what a ruled surface is, because cones and cylinders are ruled surfaces, but, 'doubly ruled'? *Doubly ruled* just means that it is a ruled surface in two ways. So, there are two distinct families of lines, $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively, such that

$$\mathcal{H}^1_{a,b,c} = \bigcup_{l \in \mathcal{L}_1} l \quad \text{and} \quad \mathcal{H}^1_{a,b,c} = \bigcup_{l \in \mathcal{L}_2} l.$$

One way to see where the two families of lines come from is to rearrange Equation (9.2):

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \quad \Leftrightarrow \quad \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{y^2}{b^2} \quad \Leftrightarrow \quad \left(\frac{x}{a} - \frac{z}{c}\right)\left(\frac{x}{a} + \frac{z}{c}\right) = \left(1 - \frac{y}{b}\right)\left(1 + \frac{y}{b}\right) \tag{9.6}$$

Now, assume that the factors in the last equation are not 0, then we can divide to obtain

$$\Leftrightarrow \quad \frac{\frac{x}{a} - \frac{z}{c}}{1 - \frac{y}{b}} = \frac{1 + \frac{y}{b}}{\frac{x}{a} + \frac{z}{c}} = \frac{\mu}{\lambda}$$

for some parameters λ and μ. We introduced these parameters in order to separate the above equation:

$$\Leftrightarrow \quad l_{\lambda,\mu} : \begin{cases} \lambda\left(\frac{x}{a} - \frac{z}{c}\right) = \mu\left(1 - \frac{y}{b}\right) \\ \mu\left(\frac{x}{a} + \frac{z}{c}\right) = \lambda\left(1 + \frac{y}{b}\right) \end{cases}.$$

[5]Prof. C. Pintea - lecture notes

What we end up with is a system of two equations, which are linear in x, y, z and which depend on the parameters λ and μ. For each fixed pair of parameters, λ and μ, we get a line which we denote with $l_{\lambda,\mu}$. Reading the above deduction backwards it is easy to see that all points on such a line satisfy the equation of $\mathcal{H}^1_{a,b,c}$. So, we have a family of lines contained in your hyperboloid.

We assumed that the factors in (9.6) are not zero. In fact, you only divide by two of them, so .. if one of those two is zero, you can flip the above fraction and divide by the other two. That will lead to the same family of lines $\mathcal{L}_1 = \{l_{\lambda,\mu} : \lambda, \mu \in \mathbb{R}, \lambda^2 + \mu^2 \neq 0\}$.

OK, what about $\mathcal{L}_2$? The second family of generators (these lines are called generators), is obtained if you group the terms differently:

$$\frac{\frac{x}{a} - \frac{z}{c}}{1 + \frac{y}{b}} = \frac{1 - \frac{y}{b}}{\frac{x}{a} + \frac{z}{c}} = \frac{\mu}{\lambda}.$$

Then, you obtain:

$$\tilde{l}_{\lambda,\mu} : \begin{cases} \lambda\left(\frac{x}{a} - \frac{z}{c}\right) = \mu\left(1 + \frac{y}{b}\right) \\ \mu\left(\frac{x}{a} + \frac{z}{c}\right) = \lambda\left(1 - \frac{y}{b}\right) \end{cases}.$$

As above, one can check that points on these lines satisfy the equation of our hyperboloid.

One important thing to notice is that, although we write down two parameters, λ and μ, we don't necessarily get distinct lines for distinct parameters: $l_{\lambda,\mu} = l_{t\lambda,t\mu}$ for any nonzero scalar t. So, in fact, $\mathcal{L}_1$ depends on one parameter. More concretely

$$\mathcal{L}_1 = \left\{ l_\alpha : \begin{cases} \left(\frac{x}{a} - \frac{z}{c}\right) = \alpha\left(1 - \frac{y}{b}\right) \\ \alpha\left(\frac{x}{a} + \frac{z}{c}\right) = \left(1 + \frac{y}{b}\right) \end{cases} \right\} \bigcup \left\{ l_\infty : \begin{cases} 0 = \left(1 - \frac{y}{b}\right) \\ \left(\frac{x}{a} + \frac{z}{c}\right) = 0 \end{cases} \right\}$$

and similarly for $\mathcal{L}_2$.

9.3.4 Parametrizations - local description

Two parametrizations of this surface are

$$\sigma_1(\theta_1, \theta_2) = \begin{bmatrix} a\sqrt{1+\theta_2^2}\cos(\theta_1) \\ b\sqrt{1+\theta_2^2}\sin(\theta_1) \\ c\theta_2 \end{bmatrix} \quad \text{and} \quad \sigma_2(\theta_1, \theta_2) = \begin{bmatrix} a\cosh(\theta_2)\cos(\theta_1) \\ b\cosh(\theta_2)\sin(\theta_1) \\ c\sinh(\theta_2) \end{bmatrix}$$

for $\theta_1 \in [0, 2\pi[$ and $\theta_2 \in \mathbb{R}$. Why? The parameter θ_1 is used to rotate on ellipses the curve obtained for $\theta_1 = 0$. What is this curve that we 'rotate'? Check this and deduce a parametrization of the tangent plane $T_p\mathcal{H}^1_{a,b,c}$ at the point

$$p = \begin{bmatrix} x(\theta_{1,p}, \theta_{2,p}) \\ y(\theta_{1,p}, \theta_{2,p}) \\ z(\theta_{1,p}, \theta_{2,p}) \end{bmatrix} \in \mathcal{H}^1_{a,b,c}$$

where $\theta_{1,p}$ and $\theta_{2,p}$ are the parameters of the point p, i.e. $p = p(x(\theta_{1,p}, \theta_{2,p}), y(\theta_{1,p}, \theta_{2,p}), z(\theta_{1,p}, \theta_{2,p}))$.

9.4 Hyperboloid of two sheets

9.4.1 Canonical equation - global description

A *hyperboloid of two sheets* is a surface which (in some coordinate system) satisfies an equation of the form

$$\mathcal{H}^2_{a,b,c} : \underbrace{\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2}}_{\varphi(x,y,z)} = -1 \quad \text{so} \quad \mathcal{H}^2_{a,b,c} = \varphi^{-1}(-1)$$

for some positive constants $a, b, c \in \mathbb{R}$.

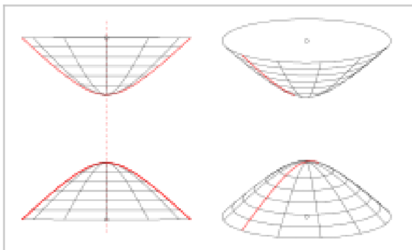


Figure 9.6: Hyperboloid of two sheets[6]

If we fix one variable, we obtain intersections with planes parallel to the coordinate axes. For this surface the intersections are either ellipses, hyperbolas or the empty set. Check this for $y = h$ and deduce the axes of the hyperbolas that you obtain.

9.4.2 Tangent planes

Using the gradient or the algebraic method we obtain the tangent plane at a point $p = (x_p, y_p, z_p) \in \mathcal{H}^2_{a,b,c}$ to be

$$T_p\mathcal{H}^2_{a,b,c} : \frac{x_p x}{a^2} + \frac{y_p y}{b^2} - \frac{z_p z}{c^2} = -1.$$

We will check this with the algebraic method. Let us consider a line passing through p in parametric form

$$l : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix} + t \underbrace{\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}}_{=v} \quad \Leftrightarrow \quad l = p + \langle v \rangle.$$

Recall that the tangent plane $T_p\mathcal{H}^2_{a,b,c}$ is the union of all lines intersecting the quadric $\mathcal{H}^2_{a,b,c}$ at p in a special way, i.e. in a double point. This is why we investigate the intersection of our quadric with l. How do we obtain the intersection $\mathcal{H}^2_{a,b,c} \cap l$? We look at those points of l which satisfy the equation

[6] Image source: Wikipedia

of our hyperboloid, i.e. we look for solutions t for the equation

$$\varphi(\begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix} + t \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}) = -1 \quad \Leftrightarrow \quad \frac{(x_p + tv_x)^2}{a^2} + \frac{(y_p + tv_y)^2}{b^2} - \frac{(z_p + tv_z)^2}{c^2} + 1 = 0.$$

If we rearrange the left-hand side as a polynomial in t we get

$$\Big(\frac{v_x^2}{a^2} + \frac{v_y^2}{b^2} - \frac{v_y^2}{c^2}\Big)t^2 + 2\Big(\frac{x_p v_x}{a^2} + \frac{y_p v_y}{b^2} - \frac{z_p v_z}{c^2}\Big)t + \underbrace{\frac{x_p^2}{a^2} + \frac{y_p^2}{b^2} - \frac{z_p^2}{c^2}}_{=-1} + 1 = 0$$

$$\Leftrightarrow \quad \underbrace{\Big(\frac{v_x^2}{a^2} + \frac{v_y^2}{b^2} - \frac{v_y^2}{c^2}\Big)}_{=0?}t^2 + 2\Big(\frac{x_p v_x}{a^2} + \frac{y_p v_y}{b^2} - \frac{z_p v_z}{c^2}\Big)t = 0. \tag{9.7}$$

This equation in t admits the solution $t = 0$. That is clear, the point on l corresponding to $t = 0$ is the point p which, by assumption, lies on $\mathcal{H}^2_{a,b,c}$. Furthermore, a second solution will correspond to a second point of intersection. However, the line l is tangent to our quadric if p is a *double point of intersection*. This happens if and only if the above equation has $t = 0$ as double solution, i.e. if and only if

$$\frac{v_x^2}{a^2} + \frac{v_y^2}{b^2} - \frac{v_y^2}{c^2} \neq 0 \quad \text{and} \quad \frac{x_p v_x}{a^2} + \frac{y_p v_y}{b^2} - \frac{z_p v_z}{c^2} = 0.$$

How do we interpret the second condition? Similar to the case of the hyperboloid of one sheet:

$$T_p\mathcal{H}^2_{a,b,c} : \begin{bmatrix} \frac{v_x}{a^2} \\ \frac{v_y}{b^2} \\ -\frac{v_x}{c^2} \end{bmatrix} \cdot (\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix}) = 0 \quad \Leftrightarrow \quad \frac{x_p x}{a^2} + \frac{y_p y}{b^2} - \frac{z_p z}{z^2} + 1 = 0.$$

Deduce this equation also with the gradient.

How do we interpret the first condition? If

$$\frac{v_x^2}{a^2} + \frac{v_y^2}{b^2} - \frac{v_y^2}{c^2} = 0 \tag{9.8}$$

then equation (9.7) is linear, and has one simple solution $t = 0$. This means that l intersects $\mathcal{H}^2_{a,b,c}$ only once (it punctures the surface in one point). Such lines are not tangent to the surface. How can we visualize this? Let us start with what we know: the vector $v = (v_x, v_y, v_z)$ satisfies the equation (9.8), so we can think of it as the position vector of some point on the cone $\mathcal{C}_{a,b,c}$. How does this cone relate to our hyperboloid? Our surface $\mathcal{H}^2_{a,b,c}$ is the union of hyperbolas and if we take the union of all the asymptotes to these hyperbolas we get $\mathcal{C}_{a,b,c}$ (see Figure 9.5). So, when l is parallel to a line contained in $\mathcal{C}_{a,b,c}$, it will intersect $\mathcal{H}^2_{a,b,c}$ in at most one point. Notice also that if $l \subseteq \mathcal{C}_{a,b,c}$, it will not intersect $\mathcal{H}^2_{a,b,c}$ at all, but this cannot happen because we chose the point p such that it lies both on l and on our quadric.

9.4.3 Parametrizations - local description

Two parametrizations of this surface are

$$\sigma_1(\theta_1,\theta_2)=\begin{bmatrix} a\sqrt{\theta_2^2-1}\cos(\theta_1)\\ b\sqrt{\theta_2^2-1}\sin(\theta_1)\\ c\theta_2\end{bmatrix}\quad\text{and}\quad \sigma_2(\theta_1,\theta_2)=\begin{bmatrix} a\sinh(\theta_2)\cos(\theta_1)\\ b\sinh(\theta_2)\sin(\theta_1)\\ \varepsilon c\cosh(\theta_2)\end{bmatrix}$$

for $\theta_1\in[0,2\pi[$, $\theta_2\in\mathbb{R}$ and $\varepsilon\in\{\pm1\}$. Why? The parameter θ_1 is used to 'rotate' on ellipses the curve obtained for $\theta_1=0$. What is this curve? Check this and deduce a parametrization of the tangent plane $T_p\mathcal{H}^2_{a,b,c}$ at the point

$$p=\begin{bmatrix} x(\theta_{1,p},\theta_{2,p})\\ y(\theta_{1,p},\theta_{2,p})\\ z(\theta_{1,p},\theta_{2,p})\end{bmatrix}\in\mathcal{H}^2_{a,b,c}$$

where $\theta_{1,p}$ and $\theta_{2,p}$ are the parameters of the point p, i.e. $p=p(x(\theta_{1,p},\theta_{2,p}),y(\theta_{1,p},\theta_{2,p}),z(\theta_{1,p},\theta_{2,p}))$. Notice also that with σ_2 we have a parametrization for each sheet of this hyperboloid, with $\varepsilon=1$ we get one sheet and with $\varepsilon=-1$ we get the other sheet. One should also be careful with where the parameters live: for σ_1 you want to choose θ_2 in $]-\infty,-1]\cup[1,\infty[$ so that the square root is defined.

9.5 Elliptic paraboloid

9.5.1 Canonical equation - global description

An *elliptic paraboloid* is a surface which (in some coordinate system) satisfies an equation of the form

$$\mathcal{P}^e_{a,b}:\underbrace{\frac{x^2}{a}+\frac{y^2}{b}-2z}_{\varphi(x,y,z)}=0\quad\text{so}\quad \mathcal{P}^e_{a,b}=\varphi^{-1}(0)$$

for some positive constants $a,b\in\mathbb{R}$.

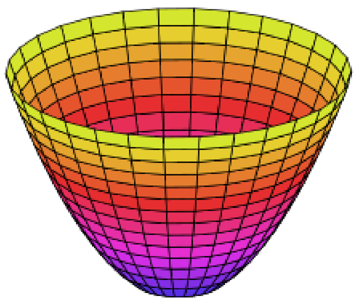


Figure 9.7: Elliptic paraboloid[7]

[7]Image source: Wikipedia

If we fix one variable, we obtain intersections with planes parallel to the coordinate axes. For this surface the intersections are either ellipses, parabolas or the empty set. Check this for $y = h$ and see what parabolas you obtain.

9.5.2 Tangent planes

Using the gradient or the algebraic method we obtain the tangent plane at a point $p = (x_p, y_p, z_p) \in \mathcal{P}^e_{a,b}$ to be

$$T_p\mathcal{P}^e_{a,b} : \frac{x_p x}{a} + \frac{y_p y}{b} - z_p - z = 0.$$

We will check this with the algebraic method. Let us consider a line passing through p in parametric form

$$l : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix} + t \underbrace{\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}}_{=v} \quad \Leftrightarrow \quad l = p + \langle v \rangle.$$

Recall that the tangent plane $T_p\mathcal{P}^e_{a,b}$ is the union of all lines intersecting the quadric $\mathcal{P}^e_{a,b}$ at p in a special way, i.e. in a double point. This is why we investigate the intersection of our quadric with l. How do we obtain the intersection $\mathcal{P}^e_{a,b} \cap l$? We look at those points of l which satisfy the equation of our paraboloid, i.e. we look for solutions t for the equation

$$\varphi(\begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix} + t \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}) = 0 \quad \Leftrightarrow \quad \frac{(x_p + t v_x)^2}{a} + \frac{(y_p + t v_y)^2}{b} - 2(z_p + t v_z) = 0.$$

If we rearrange the left-hand side as a polynomial in t we get

$$\Big(\frac{v_x^2}{a} + \frac{v_y^2}{b}\Big)t^2 + 2\Big(\frac{x_p v_x}{a} + \frac{y_p v_y}{b} - v_z\Big)t + \underbrace{\frac{x_p^2}{a} + \frac{y_p^2}{b} - 2z_p}_{=0} = 0$$

$$\Leftrightarrow \quad \underbrace{\Big(\frac{v_x^2}{a} + \frac{v_y^2}{b}\Big)}_{\neq 0}t^2 + 2\Big(\frac{x_p v_x}{a} + \frac{y_p v_y}{b} - v_z\Big)t = 0. \tag{9.9}$$

This equation in t admits the solution $t = 0$. That is clear, the point on l corresponding to $t = 0$ is the point p which, by assumption, lies on $\mathcal{P}^e_{a,b}$. Furthermore, a second solution will correspond to a second point of intersection. However, the line l is tangent to our quadric if p is a *double point of intersection*. This happens if and only if the above equation has $t = 0$ as double solution, i.e. if and only if

$$\frac{x_p v_x}{a} + \frac{y_p v_y}{b} - v_z = 0.$$

How do we interpret this condition? Similar to the quadrics treated in the previous sections, so

$$T_p\mathcal{P}^e_{p,q} : \begin{bmatrix} \frac{x_p}{a} \\ \frac{y_p}{b} \\ -1 \end{bmatrix} \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix} \right) = 0 \quad \Leftrightarrow \quad \frac{x_p x}{a} + \frac{y_p y}{b} - z_p - z = 0.$$

Deduce this equation also with the gradient.

9.5.3 Parametrizations - local description

A parametrization of this surface is

$$\sigma(\theta_1, \theta_2) = \begin{bmatrix} \sqrt{a\theta_2}\cos(\theta_1) \\ \sqrt{b\theta_2}\sin(\theta_1) \\ \theta_2/2 \end{bmatrix} \quad \theta_1 \in [0, 2\pi[\quad \theta_2 \in [0, \infty[$$

Why? Check this and deduce a parametrization of the tangent plane $T_p\mathcal{P}^e_{a,b}$ at the point

$$p = \begin{bmatrix} x(\theta_{1,p}, \theta_{2,p}) \\ y(\theta_{1,p}, \theta_{2,p}) \\ z(\theta_{1,p}, \theta_{2,p}) \end{bmatrix} \in \mathcal{P}^e_{a,b}.$$

where $\theta_{1,p}$ and $\theta_{2,p}$ are the parameters of the point p, i.e. $p = p(x(\theta_{1,p}, \theta_{2,p}), y(\theta_{1,p}, \theta_{2,p}), z(\theta_{1,p}, \theta_{2,p}))$.

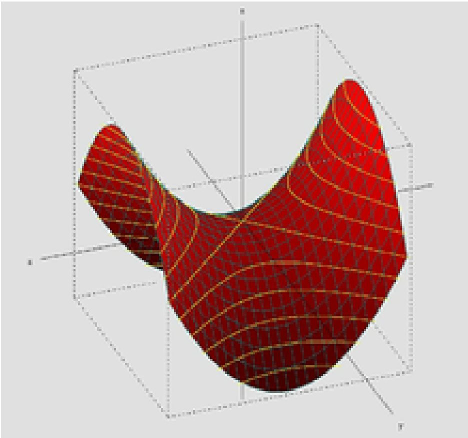
9.6 Hyperbolic paraboloid

9.6.1 Canonical equation - global description

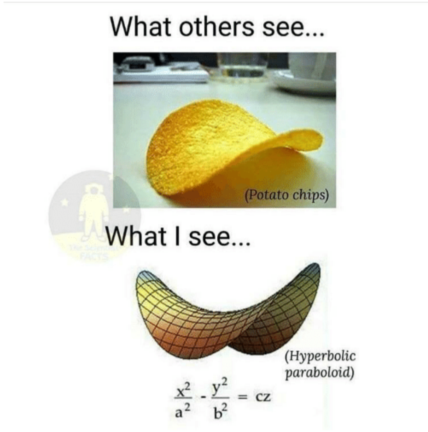
A *hyperbolic paraboloid* is a surface which (in some coordinate system) satisfies an equation of the form

$$\mathcal{P}^h_{a,b} : \underbrace{\frac{x^2}{a} - \frac{y^2}{b} - 2z}_{\varphi(x,y,z)} = 0 \quad \text{so} \quad \mathcal{P}^h_{a,b} = \varphi^{-1}(0) \tag{9.10}$$

for some positive constants $a, b \in \mathbb{R}$.



(a) Hyperbolic paraboloid[8]



(b) Potato chips[9]

If we fix one variable, we obtain intersections with planes parallel to the coordinate axes. For this surface the intersections are either parabolas, hyperbolas or two lines. Check this for $y = h$ and deduce the parabolas that you obtain.

9.6.2 Tangent planes

Using the gradient or the algebraic method we obtain the tangent plane at a point $p = (x_p, y_p, z_p) \in \mathcal{P}^h_{a,b}$ to be

$$T_p\mathcal{P}^h_{a,b} : \frac{x_p x}{a} - \frac{y_p y}{b} - z_p - z = 0.$$

We will check this with the algebraic method. Let us consider a line passing through p in parametric form

$$l : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix} + t \underbrace{\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}}_{=v} \quad \Leftrightarrow \quad l = p + \langle v \rangle.$$

Recall that the tangent plane $T_p\mathcal{P}^h_{a,b}$ is the union of all lines intersecting the quadric $\mathcal{P}^h_{a,b}$ at p in a special way, i.e. in a double point. This is why we investigate the intersection of our quadric with l. How do we obtain the intersection $\mathcal{P}^h_{a,b} \cap l$? We look at those points of l which satisfy the equation of our paraboloid, i.e. we look for solutions t for the equation

$$\varphi(\begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix} + t \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}) = 0 \quad \Leftrightarrow \quad \frac{(x_p + t v_x)^2}{a} - \frac{(y_p + t v_y)^2}{b} - 2(z_p + t v_z) = 0.$$

[6]Image source: Wikipedia
[9]Image source: https://me.me/

If we rearrange the left-hand side as a polynomial in t we get

$$\Big(\frac{v_x^2}{a}-\frac{v_y^2}{b}\Big)t^2+2\Big(\frac{x_pv_x}{a}-\frac{y_pv_y}{b}-v_z\Big)t+\underbrace{\frac{x_p^2}{a}-\frac{y_p^2}{b}-2z_p}_{=0}=0$$

$$\Leftrightarrow\quad \underbrace{\Big(\frac{v_x^2}{a}-\frac{v_y^2}{b}\Big)}_{=0?}t^2+2\Big(\frac{x_pv_x}{a}-\frac{y_pv_y}{b}-v_z\Big)t=0. \tag{9.11}$$

This equation in t admits the solution $t=0$. That is clear, the point on l corresponding to $t=0$ is the point p which, by assumption, lies on $\mathcal{P}^h_{a,b}$. Furthermore, a second solution will correspond to a second point of intersection. However, the line l is tangent to our quadric if p is a *double point of intersection*. This happens if and only if the above equation has $t=0$ as double solution, i.e. if and only if

$$\frac{v_x^2}{a}-\frac{v_y^2}{b}\neq 0\quad\text{and}\quad \frac{x_pv_x}{a}-\frac{y_pv_y}{b}-v_z=0.$$

How do we interpret the second condition? Similar to the quadrics treated in the previous sections, so

$$T_p\mathcal{P}^h_{p,q}:\begin{bmatrix}\frac{x_p}{a}\\-\frac{y_p}{b}\\-1\end{bmatrix}\cdot\Big(\begin{bmatrix}x\\y\\z\end{bmatrix}-\begin{bmatrix}x_p\\y_p\\z_p\end{bmatrix}\Big)=0\quad\Leftrightarrow\quad\frac{x_px}{a}-\frac{y_py}{b}-z_p-z=0.$$

Deduce this equation also with the gradient.

How do we interpret the first condition? If

$$\frac{v_x^2}{a}-\frac{v_y^2}{b}=0\quad\Leftrightarrow\quad\Big(\frac{v_x}{\sqrt{a}}-\frac{v_y}{\sqrt{b}}\Big)\Big(\frac{v_x}{\sqrt{a}}+\frac{v_y}{\sqrt{b}}\Big)=0 \tag{9.12}$$

then equation (9.11) is linear, and has one simple solution $t=0$. This means that l intersects $\mathcal{P}^h_{a,b}$ only once (it punctures the surface in one point). Such lines are not tangent to the surface. How can we visualize this? Let us start with what we know: the vector $v=(v_x,v_y,v_z)$ satisfies the equation (9.12), so we can think of it as the position vector of some point on the cylinder $\mathrm{Cyl}(\mathcal{A},\mathbf{k})$ where $\mathcal{A}$ is the union of two lines given by the equation (9.12). These two lines are the intersection of our quadric $\mathcal{P}^h_{a,b}$ with the coordinate plane $z=0$ (they are visible in Figure 9.8a). Three things can happen here:

1. l is one of the lines in $\mathcal{A}$, then all points of l lie in $\mathcal{P}^h_{a,b}$, so we have infinitely many solutions t for equation (9.11), or
2. l is one of the lines in $\mathcal{A}$ translated in the positive direction of the z-axis, in which case l will not intersect the surface $\mathcal{P}^h_{a,b}$ (this cannot happen for our choice of l because we assume that $p\in l\cap\mathcal{P}^h_{a,b}$), or
3. l is parallel to one of the lines in $\mathcal{A}$ (excluding the previous two cases), in which case l will puncture the surface $\mathcal{P}^h_{a,b}$ in a simple point (it will not be tangent to the surface).

9.6.3 $\mathcal{P}^h_{a,b}$ as ruled surface

Here is another fact: the hyperbolic paraboloid is a *doubly ruled surface* (like the hyperboloid of one sheet). In other words, there are two families of lines, $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively, such that

$$\mathcal{P}^h_{a,b} = \bigcup_{l\in\mathcal{L}_1} l \quad \text{and} \quad \mathcal{P}^h_{a,b} = \bigcup_{l\in\mathcal{L}_2} l.$$

Every point on this surface lies on one line in $\mathcal{L}_1$ and on one line in $\mathcal{L}_2$. The generators containing the *saddle point* (with our equation, this point is the origin of the coordinate system) are visible in Figure 9.8a.

Again, one way to see where the two families of lines come from is to rearrange (9.10)

$$\frac{x^2}{a} - \frac{y^2}{b} - 2z = 0 \quad \Leftrightarrow \quad \frac{x^2}{a} - \frac{y^2}{b} = 2z \quad \Leftrightarrow \quad \left(\frac{x}{\sqrt{a}} - \frac{y}{\sqrt{b}}\right)\left(\frac{x}{\sqrt{a}} + \frac{y}{\sqrt{b}}\right) = 2z \tag{9.13}$$

Similar to the case of the hyperboloid of one sheet, we can introduce two parameters λ and μ, in order to separate the above equation:

$$\Leftrightarrow \quad l_{\lambda,\mu} : \begin{cases} \lambda\left(\frac{x}{\sqrt{a}} - \frac{y}{\sqrt{b}}\right) = 2\mu z \\ \mu\left(\frac{x}{\sqrt{a}} + \frac{y}{\sqrt{b}}\right) = \lambda \end{cases}.$$

What we end up with is a system of two equations, which are linear in x, y, z and which depend on the parameters λ and μ. For each fixed pair of parameters, λ and μ, we get a line which we denote with $l_{\lambda,\mu}$. It is easy to check that all points on such a line satisfy the equation of $\mathcal{P}^h_{a,b}$. This is the first family of lines $\mathcal{L}_1 = \{l_{\lambda,\mu} : \lambda, \mu \text{ not both zero}\}$.

The second family of generators, $\mathcal{L}_2$, is obtained if you group the terms differently:

$$\tilde{l}_{\lambda,\mu} : \begin{cases} \lambda\left(\frac{x}{\sqrt{a}} + \frac{y}{\sqrt{b}}\right) = 2\mu z \\ \mu\left(\frac{x}{\sqrt{a}} - \frac{y}{\sqrt{b}}\right) = \lambda \end{cases}.$$

As above, one can check that points on these lines satisfy the equation of our paraboloid.

Again, one important thing to notice is that, although we write down two parameters, λ and μ, we don't necessarily get distinct lines for distinct parameters: $l_{\lambda,\mu} = l_{t\lambda,t\mu}$ for any nonzero scalar t. So, in fact, $\mathcal{L}_1$ depends on one parameter. More concretely

$$\mathcal{L}_1 := \left\{ l_\alpha : \begin{cases} \left(\frac{x}{\sqrt{a}} - \frac{y}{\sqrt{b}}\right) = 2\alpha z \\ \alpha\left(\frac{x}{\sqrt{a}} + \frac{y}{\sqrt{b}}\right) = 1 \end{cases} \right\} \bigcup \left\{ l_\infty : \begin{cases} 0 = 2z \\ \left(\frac{x}{\sqrt{a}} + \frac{y}{\sqrt{b}}\right) = 0 \end{cases} \right\}$$

and similarly for $\mathcal{L}_2$. You might have noticed that l_∞ is one of the lines visible in Figure 9.8a, since it lies in the plane $z = 0$. The other one, $\tilde{l}_\infty$, belongs to the family $\mathcal{L}_2$.

9.6.4 Parametrizations - local description

A parametrization of this surface is

$$\sigma_2(\theta_1,\theta_2)=\begin{bmatrix}\sqrt{a}\theta_1\\ \sqrt{b}\theta_2\\ \frac{1}{2}(\theta_1^2-\theta_2^2)\end{bmatrix}\quad \theta_1,\theta_2\in\mathbb{R}$$

Check this and deduce a parametrization of the tangent plane $T_p\mathcal{P}^h_{a,b}$ at the point

$$p=\begin{bmatrix}x(\theta_{1,p},\theta_{2,p})\\ y(\theta_{1,p},\theta_{2,p})\\ z(\theta_{1,p},\theta_{2,p})\end{bmatrix}\in\mathcal{P}^h_{a,b}.$$

where $\theta_{1,p}$ and $\theta_{2,p}$ are the parameters of the point p, i.e. $p=p(x(\theta_{1,p},\theta_{2,p}),y(\theta_{1,p},\theta_{2,p}),z(\theta_{1,p},\theta_{2,p}))$.