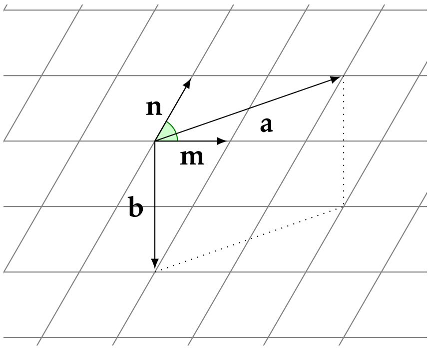
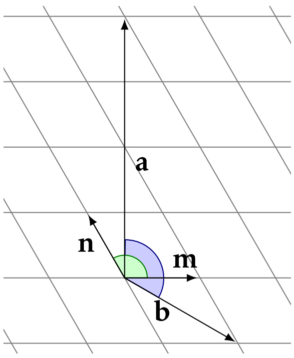
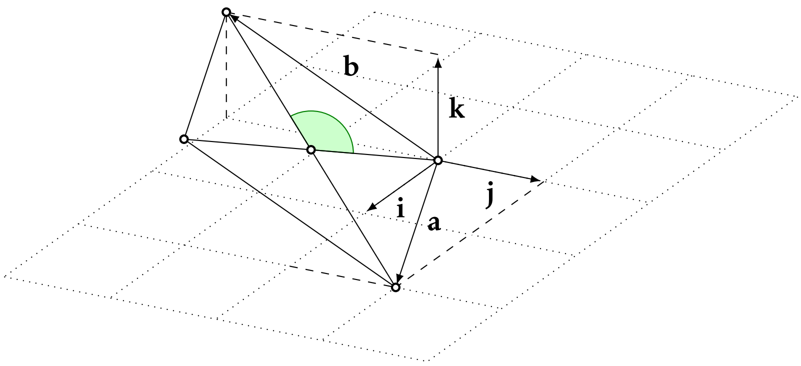
1. Let $\mathbf{m}$ and $\mathbf{n}$ be two unit vectors such that $\measuredangle(\mathbf{m},\mathbf{n}) = 60°$. Determine the length of the diagonals in the parallelogram spanned by the vectors $\mathbf{a} = 2\mathbf{m}+\mathbf{n}$ and $\mathbf{b} = \mathbf{m}-2\mathbf{n}$.



2. Let $\mathbf{m}$ and $\mathbf{n}$ be two unit vectors such that $\measuredangle(\mathbf{m},\mathbf{n}) = 120°$. Determine the angle between the vectors $\mathbf{a} = 2\mathbf{m}+4\mathbf{n}$ and $\mathbf{b} = \mathbf{m}-\mathbf{n}$.



3. You are given two vectors $\mathbf{a}(2,1,0)$ and $\mathbf{b}(0,-2,1)$ with respect to an orthonormal basis. Determine the angles between the diagonals of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.



4. Let $(\mathbf{i},\mathbf{j},\mathbf{k})$ be an orthonormal basis. Consider the vectors $\mathbf{q} = 3\mathbf{i}+\mathbf{j}$ and $\mathbf{p} = \mathbf{i}+2\mathbf{j}+\lambda\mathbf{k}$ with $\lambda \in \mathbb{R}$. Determine λ such that the cosine of the angle $\measuredangle(\mathbf{p},\mathbf{q})$ is $5/12$.

5. Using the scalar product, prove the Cauchy-Bunyakovsky-Schwarz inequality, i.e. show that for any $a_1,\dots,a_n,b_1,\dots,b_n \in \mathbb{R}$ we have

$$(a_1b_1+\cdots+a_nb_n)^2 \le (a_1^2+\cdots+a_n^2)(b_1^2+\cdots+b_n^2).$$

6. Let ABC be a triangle. Show that

$$\overrightarrow{AB}^2+\overrightarrow{AC}^2-\overrightarrow{BC}^2 = 2\overrightarrow{AB}\cdot\overrightarrow{AC}$$

and deduce the law of cosines in a triangle.

7. Let $ABCD$ be a tetrahedron. Show that

$$\cos(\measuredangle(\overrightarrow{AB},\overrightarrow{CD})) = \frac{AD^2+BC^2-AC^2-BD^2}{2\cdot AB\cdot CD}.$$

This is a $3D$-version of the law of cosine.

8. Let $ABCD$ be a rectangle. Show that for any point O

$$\overrightarrow{OA}\cdot\overrightarrow{OC} = \overrightarrow{OB}\cdot\overrightarrow{OD} \quad \text{and} \quad \overrightarrow{OA}^2+\overrightarrow{OC}^2 = \overrightarrow{OB}^2+\overrightarrow{OD}^2.$$

9. Consider the vector $\mathbf{v}$ which is perpendicular on $\mathbf{a}(4,-2,-3)$ and on $\mathbf{b}(0,1,3)$. If $\mathbf{v}$ describes an acute angle with Ox and $\|\mathbf{v}\| = 26$ determine the components of $\mathbf{v}$.

10. Show that the Gram-Schmidt orthogonalization process yields an orthonormal basis.

11. In an orthonormal basis, consider the vectors $\mathbf{v}_1(0,1,0)$, $\mathbf{v}_2(2,1,0)$ and $\mathbf{v}_3(-1,0,1)$. Use the Gram-Schmidt process to find an orthonormal basis containing $\mathbf{v}_1$.

12. In $\mathbb{E}^2$, show that the orthogonal reflection of a vector $\mathbf{b}$ parallel to $\mathbf{a}$ is

$$\mathrm{Ref}^{\|}_{\mathbf{a}}(\mathbf{b}) = \mathbf{b}-2\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a} = \mathbf{b}-2\,\mathrm{Pr}^{\perp}_{\mathbf{a}}(\mathbf{b}).$$

Show that the orthogonal reflection of a vector $\mathbf{b}$ in the vector $\mathbf{a}$ is

$$\mathrm{Ref}^{\perp}_{\mathbf{a}}(\mathbf{b}) = -\mathbf{b}+2\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a} = -\mathbf{b}+2\,\mathrm{Pr}^{\perp}_{\mathbf{a}}(\mathbf{b}) = -\mathrm{Ref}^{\|}_{\mathbf{a}}(\mathbf{b}).$$

13. Let $\mathbf{v} \in \mathbb{V}^n$ be a vector. Show that

a) The set $\mathbf{v}^{\perp}$ is a vector subspace of $\mathbb{V}^n$.

b) There is a basis $\mathbf{v},\mathbf{v}_2,\mathbf{v}_3,\dots,\mathbf{v}_n$ of $\mathbb{V}^n$ with $\mathbf{v}_2,\dots,\mathbf{v}_{n-1}$ a basis of $\mathbf{v}^{\perp}$.

14. Fix $\mathbf{v} \in \mathbb{V}^3$ and let $\phi : \mathbb{V}^3 \to \mathbb{R}$ be the map $\phi(\mathbf{w}) = \mathbf{v}\cdot\mathbf{w}$. Is the map linear? Explain why. Give the matrix of ϕ relative to an orthonormal basis. What changes if we define ϕ by $\phi(\mathbf{w}) = \mathbf{w}\cdot\mathbf{v}$?