# AG

## CURS 12

## Complement la cursul precedent

### Teorema lui Petersen

$G=(V,E)$ graf cubic fără punți $\Rightarrow$ $G$ conține un cuplaj perfect (1-factor)

$G$ conex $\quad e=xy\in E$

$e$ punte $\Leftrightarrow$ $G-e$ neconex ex:

într-un arbore, $\forall$ muchie este punte

ex:

$G$ conex, $\forall$ muchie e punte $\Rightarrow$ $G$ arbore.

Dem:

Amintim T 1-factorilor (TUTTE)

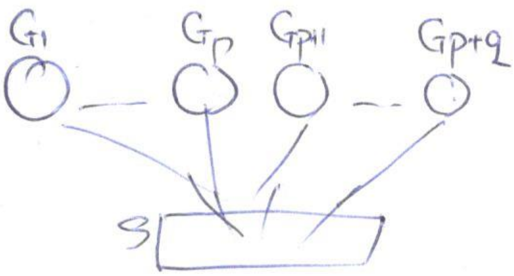
$G=(V,E)$ admite 1-factor (cupl. perfect) $\Leftrightarrow$

$\Leftrightarrow \forall S \subsetneq V: \nabla(G-S)\le |S|$

Fie $S\subsetneq V$

$$G-S=G_1+G_2+\dots+G_p+G_{p+1}+\dots+G_{p+q}$$

$$|V_i| = \begin{cases} 1(2) & 1\le i\le p \\ 0(2) & p+1\le i\le p+q \end{cases}$$



$o(G-S) \overset{3}{=} p \leq |S|$

$\sum$ gr. vf din $V_i$

Avem $\forall i \in \{1, \dots p+q\}: 2|E_i| + |[V_i, S]| = 3 \cdot |V_i|$ (1)

$\forall i \in \{1 \dots p\}: |[V_i, S]| \equiv 1 (2)$ $\quad$ (2) $\Leftarrow$ (1), $|V_i| \equiv 1 (2)$

$\forall i \in \{1, \dots p\}: |[V_i, S]| \geq 3$ $\quad$ (3) $\Leftarrow$ (2) G nu are punți

Am $3|S| = 2|E(G[S])| + \sum\limits_{1 \leq i \leq p} |[V_i, S]| + \sum\limits_{p+1 \leq i \leq p+q} |[V_i, S]| \geq$

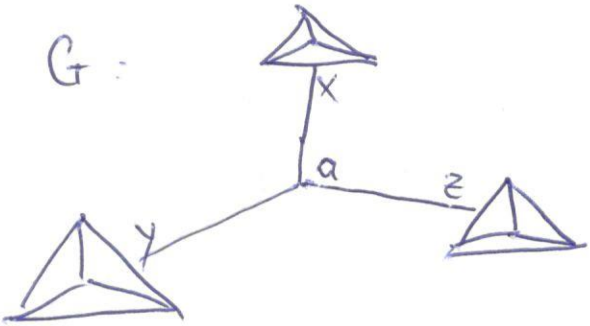
$\geq \sum\limits_{1 \leq i \leq p} |\overset{\geq 3}{[V_i, S]}| \overset{(3)}{\geq} 3p$

$|S| \geq p = o(G-S)$ $\square$

exerc: $\forall$ G cubic cu cel mult o punte conține 1-factor

$\forall$ G cubic cu cel mult 2 punți conține 1-factor.

G :




G nu conț. 1-factor

## Polinoame cromatice

1. $G = (V, E)$ graf simplu

$K \in \mathbb{N}_{\geq 1}$

$c : V \to \{1, 2, 3, \dots K\}$ K - colorare.

c K - colorare proprie $\Leftrightarrow c(x) \neq c(y)$ $\forall x, y \in E$

$P_k(G) = |\{c \mid c: V \to \{1, \dots k\},\ c\ K\text{-colorare proprie}\}|$

2. $P_1$ a) $P_k(\bar{K}_n) = k^n$

b) $P_k(K_n) = k(k-1)(k-2)\dots(k-n+1)$

c) $P_k(P_n) = k(k-1)^{n-1}$

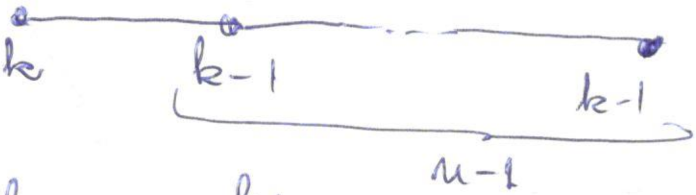
d) $P_k(T_n) = k(k-1)^{n-1}.$

e) $P_k(C_n) = (k-1)^n + (-1)^n(k-1)$

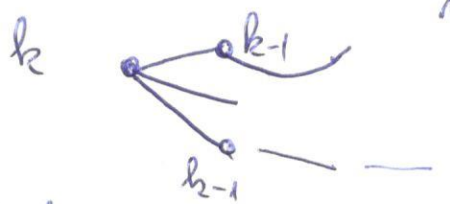
Dem a) Evident // nr de funcții

b)


c)




d)




variante de colorări

e) Inducție după $n \geq 3$

$J_3$ adev. $P_k(C_3) = P_k(K_3) = K(K-1)(K-2) = (K-1)^3 + (-1)^3(K-1)$

$J(n-1) \Rightarrow J(n) \quad n \geq 4$

Fie $C_n = [x_1, x_2, x_3 \dots x_{n-1}, x_n, x_1]$

$C_{n-1} = [x_1, x_2, x_3 \dots x_{n-1}, x_1]$

$P_n = [x_1, x_2, x_3 \dots x_{n-1}, x_n].$

$P_k(C_n) = P_k(P_n) - P_k(C_{n-1}) = K(K-1)^{n-1} - (K-1)^{n-1} - (-1)^{n-1}(K-1) =$

$$=(K-1)^n+(-1)^n(K-1) \ \square$$

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$P_2$ a) $G=(V,E)$ $\quad e=xy\in E \quad K\geq 1$

$$P_k(G)=P_k(G-e)-P_k(G*e)$$

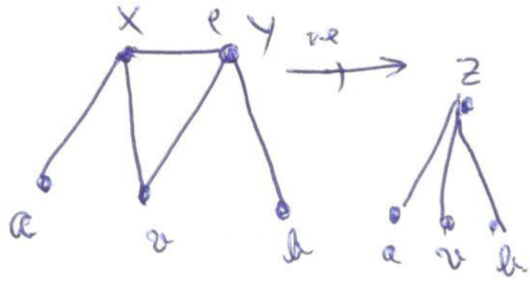
b) $G=(V,E)$, $e=xy\in V^{(2)}-E$ $k\geq 1$.

$$P_K(G)=P_K(G+e)+P_K(G*e)$$

<u>Dem</u>:

a) vezi e) $P_1$.

$$\overset{x}{\cdot} \quad e \quad \overset{y}{\cdot} \longrightarrow \overset{z}{\cdot}$$




b) $\left.\begin{array}{l} G+e\to G \\ G\to G-e \end{array}\right\}$ din a)

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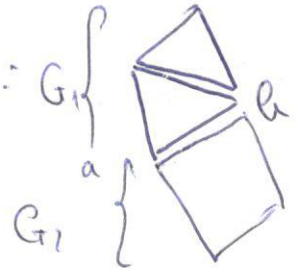
$P_3$: $\left.\begin{array}{l} G_1, G_2 \text{ grafuri}, K\geq 1. \\ V_1\cap V_2 \\ G_1[V_1\cap V_2]=G_2[V_1\cap V_2]\sim K_{|V_1\cap V_2|} \end{array}\right| \Rightarrow$

$$\Rightarrow \boxed{P_k(G_1\cup G_2)\cdot P_k(G_1\cap G_2)=P_k(G_1)\cdot P_k(G_2)}$$

Dem :

ex :




$G_1 [V_1 \cap V_2]$ : 

$\sim K_2$

$G_2 [V_1 \cap V_2]$.

$H = G_1 [V_1 \cap V_2] = G[V_1 \cap V_2]$ $\quad H \sim K_{|V_1 \cap V_2|} = K_m$

$m = |V_1 \cap V_2|$

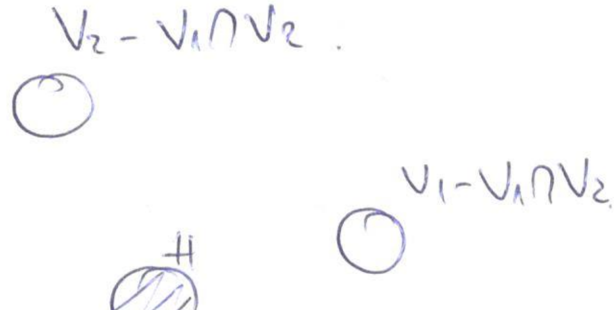
$P_k(H) = k(k-1)(k-2) \dots (k-m+1) \quad (2)$

H : $\Delta$ $\quad$ Fie $P_k(G_1) = k(k-1) \dots (k-m+1) \cdot t_1 \quad (3)$

$P_k(G_2) = k(k-1) \dots (k-m+1) t_2 \quad (4)$

$P_k(G_1 \cup G_2) = k(k-1) \dots (k-m+1) t_1 t_2 \quad (5)$

$(2), (3), (4), (5) \Rightarrow (1)$




$T_4$ : $G = (V, E)$ graf simplu

$n = |V|$ $\quad k \geq 1$.

$P_k(G)$ polinom de gr. n în k. cu coef. întregi de semne alternante cu coef lui $k^n$ egal cu 1 și termenul liber egal cu zero.

$$P_k(G) = k^n - a_{n-1} k^{n-1} + a_{n-2} k^{n-2} - \dots + (-1)^{n-2} \cdot a_2 k^2 + (-1)^{n-1} \cdot a_1 k^1$$

Dem

Ind după $m = |E|$

$I(0)$ adev: $P_k(\overline{K_n}) = k^n$ (v. $P_1$ a))

$I(m-1) \rightarrow I(m)$, $m \geq 1$

Fie $e \in E$

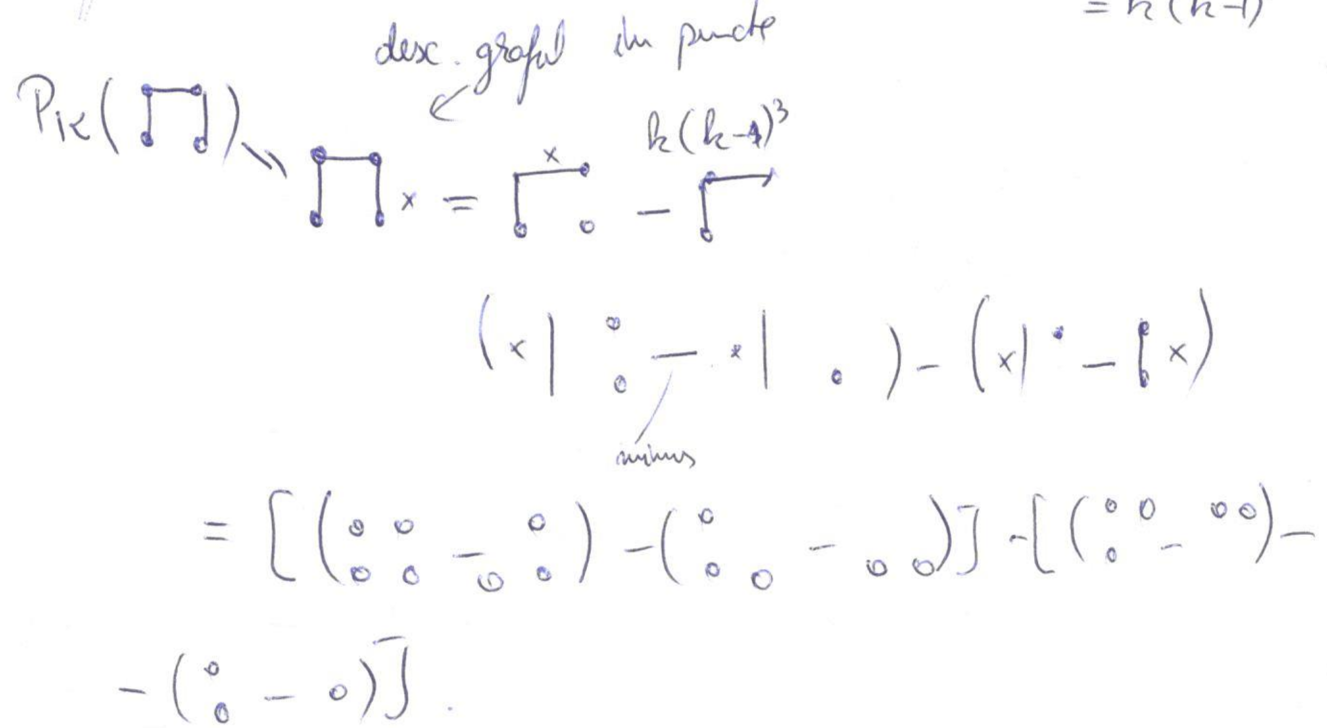
$$P_k(G) = P_k(G-e) - P_k(G*e) \overset{I(m-1)}{=} k^n - b_{n-1} \cdot k^{n-1} + b_{n-2} k^{n-2} - \dots + (-1)^{n-2} b_2 k^2 + (-1)^{n-1} b_1 \cdot k^1 - k^{n-1} + c_{n-2} \cdot k^{n-2} - \dots + (-1)^{n-2} c_2 k^2 + (-1)^{n-1} c_1 \cdot k^1$$

$$= k^n - (b_{n-1}+1) k^{n-1} + (b_{n-2} + c_{n-2}) k^{n-2} - \dots + (-1)^{n-2} (b_2 + c_2) k^2 + (-1)^{n-1} (b_1 + c_1) k^1$$

LT.

Ex:

$$P_k(\overline{K}_4) - 3P_k(\overline{K}_3) + 3P_k(\overline{K}_2) - P_k(K_1) = k^4 - 3k^3 + 3k^2 - k = k(k-1)^3$$



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