24.05.2016.

AG
CURS 13

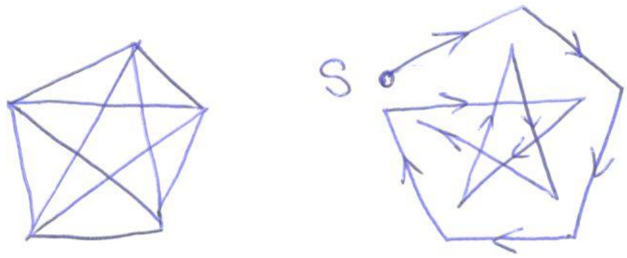
Linii hamiltoniene și euleriene

1. Linii euleriene

$G=(V,E)$ graf. neorient

Euler (1736 - Königsberg)

ciclu eulerian = ciclu simplu care conține toate muchiile



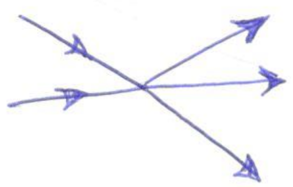
gradul trb. să fie par.

$G=(V,E)$ graf orientat E multiset peste V^2

$$x \xrightarrow{e} y$$

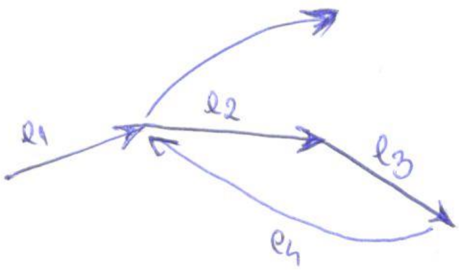
$e^- = x$ origine

$e^+ = y$ terminus



$$\begin{cases} E^+(x) = \{ e \mid \overset{e \in E}{e^-} = x \} \\ E^-(x) = \{ e \mid e \in E,\ e^+ = x \} \end{cases}$$

$$\underline{d(x) = d^-(x) + d^+(x)} \qquad \begin{cases} d^+(x) = |E^+(x)| \\ d^-(x) = |E^-(x)| \end{cases}$$



$e_1, e_2, e_3 \; \text{----} \; e_p$

1.2. T(Euler) Fie $G=(V,E)$ gf. orientat, conex

G este eulerian $= \forall\, x \in V: \boxed{d^-(x) = d^+(x)}$

Dem:

"$\Longrightarrow$" Fie C un circ. eulerian

$x \in V$

Avem $E(C) = E(G)$

$C = [e_1 e_2 \text{ — } e_{|E|}]$

$e_1^- = e_{|E|}^+$

$\exists\; i$ a.î: $x = e_i^+ \Rightarrow e_{i+1}^- = e_i^+ = x$

$f: E_G^-(x) \longrightarrow E_G^+(x)$

$e_i \longrightarrow e_{i+1}$

f bijecție și $|E^-(x)| = |E^+(x)|$

$d^-(x) = d^+(x)$

C eulerian $\Rightarrow E_C^-(x) = E_G^-(x)$

$E_C^+(x) = E_G^+(x)$

"$\Leftarrow$"

G conex

$d_G^-(x) = d_G^+(x), \forall x \in V$

~~Fie~~

a) $|V|=1 \xrightarrow{E\neq\emptyset}$

$C=[e_1 e_2 \dots e_p]$

b) $|V|\geq 2$

b_1 $\forall x\in V$ $\exists C$ circ. simplu care cont. x.

Pas 1: Fie $e_1 \in E^+(x)$

Pas i: $e_1, e_2 \dots e_{i-1}$ (alese deja), alegem e_i cu prop $e_i^- = e_{i-1}^+$

Dacă $e_{i-1} = x$: $C=[e_1 \dots e_{i-1}]$ Stop.

Dacă $e_{i-1} \neq x$ atunci:

$\xrightarrow{e_{i-1}}\xrightarrow{e_i}$

$|E| < \infty$

$C=[e_1 \dots e_p]$

Fie $C=[e_1, \dots e_p]$ circ. simplu cu nr max. de arce.

$G' = (V, \underbrace{E(G)-E(C)}_{E'})$

Cazul $E(G)=E(C) \Longrightarrow C$ eulerian STOP

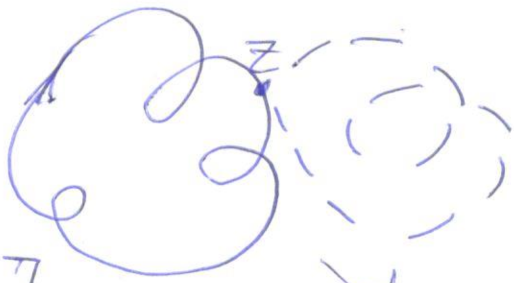
Cazul $E(G) \supsetneq E(C)$

$G' = G_1 + G_2 + \ldots + G_k$ desc în compl. conexe.

Am $\forall\ 1 \le i \le k,\ \forall x \in V_i: d^-_{G_i}(x) = d^+_{G_i}(x)$

G conex $\Rightarrow \exists\, i \in \{1, \ldots k\}$ cu $V_i \cap V(C) \neq \emptyset$
$\Rightarrow z$

Fie C_i circ. simplu din G_i care conține z



pt că e conex.

$C' = [z \xrightarrow{C} z \xrightarrow{C_i} z]$

C' ~~este~~ circ. simplu în G și $|E(C')| = |E(C)| + |E(C_i)|$ $> |E(C)|$ ↯

$\Rightarrow E(G) = E(C) \Rightarrow$ deci C circ eulerian.

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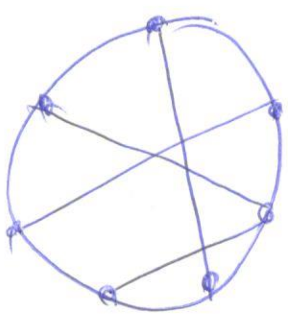
2. $G = (V, E)$ graf simplu

C ciclu hamiltonian $\Leftrightarrow$ C ciclu hamilton care conține toate vf. lui G



<u>L_1</u> $G = (V, E)$ g simplu hamiltonian

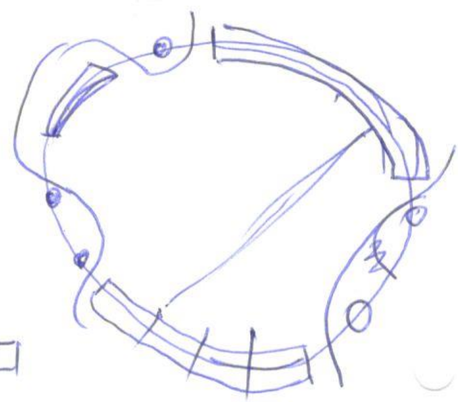
$\forall\, S \subseteq V$:

$c(G-S) \le |S|$

<u>Dem</u>: Fie C ciclu hamilton

$c(G-S) \le c(G-S) \le |S|$ □

$G - S \geq C - S$



2.2.

T2) Fie $G = (A_1 \dot{\cup} \ldots \dot{\cup} A_p, E)$ $p \geq 2$.
un graf p-partit complet

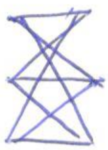
G hamiltonian $\iff |A_p| \leq |A_1| + \ldots + |A_{p-1}|$
unde $|A_p| \geq |A_1|, \ldots |A_{p-1}|$

$a_i = |A_i| \quad 1 \leq i \leq p$.

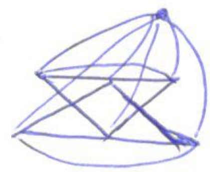
$G \sim K_{a_1 \ldots a_p}$ graful p-partit complet.

ex:

K_{33}



$K_{2,2,3}$



A_1 A_2 A_3

$K_{2,4}$

Obs $K_n \sim K_{\underbrace{1,1 \ldots 1}_{n}}$

Dem

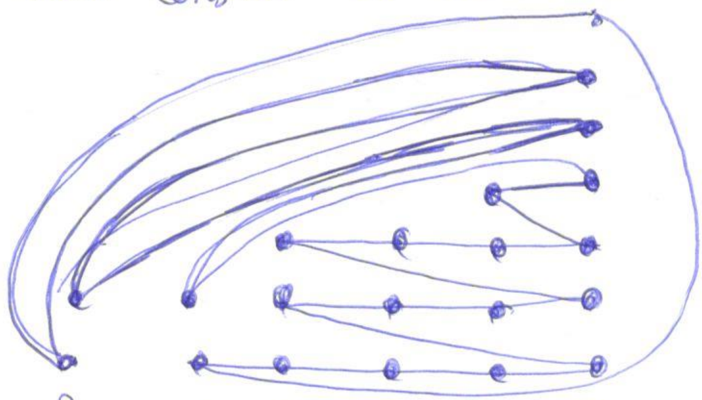
$\rightarrow$ ~~...~~ Conform L_1 Fie $S = A_1 \cup \ldots \cup A_{p-1}$

$$\underbrace{c(\underbrace{G - (A_1 \cup \ldots \cup A_{p-1})}_{A_p})}_{|A_p|} \leq \underbrace{|A_1 \cup \ldots A_{p-1}|}_{|A_1| + \ldots + |A_{p-1}|}$$

$$A_p \leq |A_1| + \ldots + |A_{p-1}|$$

"$\Longleftarrow$" $|A_p| \le |A_1| + \dots + |A_{p-1}|$, $p \ge 2$.

Vom construi un ciclu hamilt în G



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2.3. Condiția suficientă a lui Dirac de hamiltonicitate a unui graf

Teoremă (Dirac)

Fie $G = (V, E)$ gf. hamilton $|V| = n \ge 3$

$\underbrace{\left[\forall x \in V : d_G(x) \ge \frac{n}{2} \right]}_{(1)} \Rightarrow G$ hamilton

Dem

Pp abs (R.A) G nu e hamiltonian.

K_n hamiltonian $\Rightarrow \exists E \subseteq E' \subsetneq V^{(2)}$

cu $G' = (V, E')$: nu este hamiltonian

$G' + e$ hamiltonian

$\forall e \in V^{(2)} - E'$

Acest graf G' este nehamiltonian muchii-maximal.

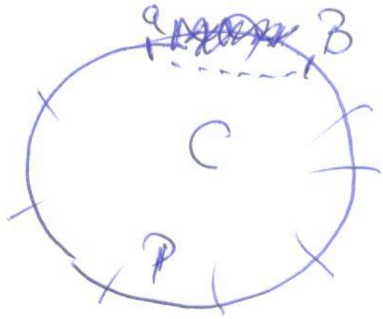
Obs:

(1) se conservă la adăug. de muchii.

$\Rightarrow$ G' are prop(1)

Fie $a, b \in V$ cu $a, b \notin E'$

$G' + ab$ hamiltonian



Fie $P = [a \xrightarrow{C} b]$ lanțul hamiltonian din G' indus de C.

Notăm

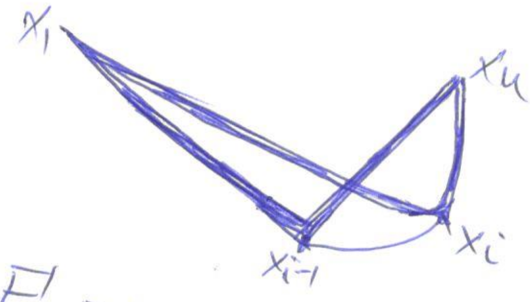
$P = [x_1, x_2, \ldots, x_n]$ $\begin{cases} x_1 = a \\ x_n = b \end{cases}$

Avem:

$d_{G'}(x_1) = |N_{G'}(x_1)| \geq n/2$

$d_{G'}(x_n) = |N_{G'}(x_n)| \geq n/2$.

Obs



$x_1 x_i \in E' \Rightarrow x_n x_{i-1} \notin E'$ (astfel s-ar forma un ciclu hamilt în G')

Ok

RS

$$\Rightarrow N_{G'}(x_n) \subseteq \{x_1, \dots, x_{n-1}\} - \{x_{i-1} \mid x_1 x_i \in E'\}$$

$$\cancel{- \{x_{i-1} \mid x_1 x_i \in E'\}}$$

$$|N_{G'}(x_n)| \leq n - 1 - |N_{G'}(x_1)|$$

$$\underbrace{\underset{\frac{n}{2}}{\underset{\vee\!\!|}{d_{G'}(x_n)}} + \underset{\frac{n}{2}}{\underset{\vee\!\!|}{d_{G'}(x_1)}}}_{n} \leq n - 1 < n$$

$$\Rightarrow n < n \quad \text{ↄ} \quad \square$$