Solutions to hw2 homework on Convex Optimization https://web.stanford.edu/class/ee364b/homework.html

Andrei Keino

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2.1 (8 points, 1 point per question)

Let f be a convex function with domain in R^n. We fix $x \in$ **int dom f** and $d \in R^n$. Recall the definition of the directional derivative of f at x along the direction d

$$f'(x,d) = \lim_{t \to 0} \frac{f(x+td) - f(x)}{t}$$

In this question we aim to show that $f'(x,d)$ exists and is finite, and that we have the following relationship between $\partial f(x)$ and $f'(x,d)$,

$$f'(x,d) = \sup_{g \in \partial f(x)} g^T d$$

(a) Show that the ratio $\frac{f(x+td)-f(x)}{t}$ is a nondecrasing function of $t > 0$. Deduce that $f'(x,d)$ exists and is either finite or equal to $-\infty$.
We know from the lectures that, since $x \in$ **int dom f**, the subdifferential set ∂f is non - empty, convex and compact.

Solution:

Proof of non - decreasing. Definition of subgradient is

$$f(z) \geq f(x) + g^T(z - x)$$

let $z = x + td$; then

$$f(x+td) \geq f(x) + g^T(x + td - x)$$

or

$$f(x+td) - f(x) \geq tg^T d$$

dividing both part of the inequality by t (as $t > 0$, we can do it) gives

$$\frac{f(x+td)-f(x)}{t} \geq g^T d$$

The right-hand side of the upper equation can't be decreasing function of t, because the function f is convex, and it means that it's second differential is more or equal to zero.

Proof of possible equality to $-\infty$.
The definition of convexity:

$$f(\theta x + (1-\theta)y)) \leq \theta f(x) + (1-\theta)f(y)$$

where $0 < \theta < 1$.
let $t = 1-\theta$, $0 < t < 1$. then

$$f((1-t)x + ty)) \leq (1-t)f(x) + tf(y)$$

or

$$f(x + t(y-x))) \leq f(x) + t(f(y) - f(x))$$

as we can choose y any of the point in domain f, we can set $d = y - x$. Then

$$f(x+td) \leq f(x) + t(f(y) - f(x))$$

or

$$f(x+td) - f(x) \leq t(f(y) - f(x))$$

or

$$\frac{f(x+td)-f(x)}{t} \leq f(y) - f(x)$$

As $f(x)$ **can be equal to** ∞ **on the domain of** f**, so** $f'(x,d) = \frac{f(x+td)-f(x)}{t}$ **can be less or equal than (for the infinity with sign minus it means strictly equal)** $-\infty$ **on the domain of** f. This means that $f'(x,d)$ can be equal to $-\infty$ on domain of f.

(b) Let $g \in \partial f(x)$. Show that $f'(x,d) \geq g^T d$. Deduce that $f'(x,d)$ is finite and $f'(x,d) \geq \sup_{g\in\partial f(x)} g^T d$.

Solution:
We already shown that

$$f'(x,d) \geq g^T d$$

in part (a). We also shown in part (a) that

$$\frac{f(x+td)-f(x)}{t} \leq f(y) - f(x)$$

Second upper inequality means that $f'(x,d)$ is bounded from upper side (i.e it can't be equal to ∞), it means its value is finite.
As the first of upper inequalities is correct $\forall$ subgradients in domain of f, it means, that it is correct for the supremum of these subgradients in domain f. It means that

$$f'(x,d) \geq \sup_{g \in \partial f(x)} g^T d.$$