Solutions to hw2 homework on Convex Optimization https://web.stanford.edu/class/ee364a/homework.html

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May 30, 2020

3.1

Suppose $f : R \to R$ is convex and $a, b \in \boldsymbol{dom} f$ with $a < b$.

(a) Show what
$f(x) \leq \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b)$ for all $x \in [a, b]$

Definition of function convexity:

A function $f : R^n \to R$ is convex if $\boldsymbol{dom} f$ is a convex set and for all $x, y \in \boldsymbol{dom} f$ and $0 \leq \theta \leq 1$ we have
$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$

Solution:

Let $z = \theta x + (1-\theta)y,$ let $z, x, y \in [a, b]$,
let $\theta = \frac{b-z}{b-a}$, then $1 - \theta = \frac{z-a}{b-a}$,
therefore $\theta \in [0, 1]$.
Then by definition of convexity of f:

$$\boldsymbol{f(z)} = f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y) = \boldsymbol{\frac{b-z}{b-a} f(x) + \frac{z-a}{b-a} f(y)}$$

and then for $x = a$, $y = b$ we have:

$$f(z) \leq \frac{b-z}{b-a} f(a) + \frac{z-a}{b-a} f(b)$$

(1)
Proved.

(b) Show what

$$\frac{f(x)-f(a)}{x-a} \le \frac{f(b)-f(a)}{b-a} \le \frac{f(b)-f(x)}{b-x}$$

Solution:

First we show what

$$\frac{f(x)-f(a)}{x-a} \le \frac{f(b)-f(a)}{b-a}$$

Using the result from the inequality proved in (a), dividing the right and left parts of the inequality on $(x-a)$ and subtracting from the left and right parts of the inequality $\frac{f(a)}{x-a}$ we get

$$\frac{f(x)-f(a)}{x-a} \le \frac{f(b)-f(a)}{b-a}$$

Now second part: Using the result from the inequality proved in (a), dividing the right and left parts of the inequality on $(b-x)$ and subtracting from the left and right parts of the inequality $\frac{f(b)}{b-x}$ we get

$$\frac{f(b)-f(a)}{b-a} \le \frac{f(b)-f(x)}{b-x}$$

Proved.

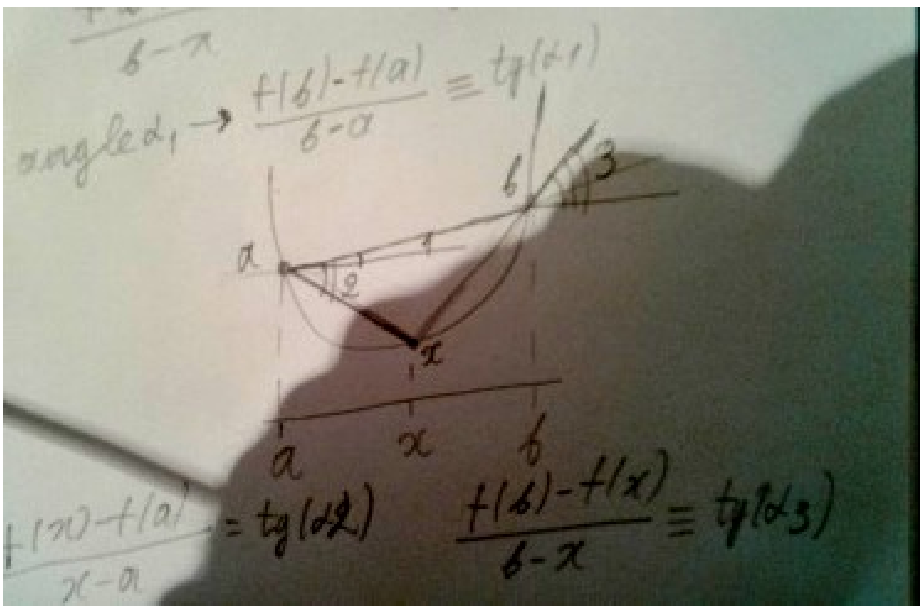

Figure 1: A sketch.