Solutions to hw5 homework on Convex Optimization
https://web.stanford.edu/class/ee364a/homework.html

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5.17

Robust linear programming with polyhedral uncertainty. Consider the robust LP:

$$
\begin{aligned}
& \text{minimize} && c^T x \\
& \text{subject to} && \sup_{a \in P_i} a^T x \leq b_i, \ i = 1, ..., m
\end{aligned}
$$

with variable $x \in R^n$, where $P_i = \{a : \ C_i a \preceq d_i\}$. The problem data are $a \in R^n$, $C_i \in R^{m_i \times n}$, $d_i \in R^{m_i}$, and $b \in R^m$. We asume the polyhedra P_i are nonempty. Show that this problem is equivalent to the LP:

$$
\begin{aligned}
& \text{minimize} && c^T x \\
& \text{subject to} && d_i^T z_i \leq b_i, \ i = 1, ..., m \\
& && C_i z_i = x, \ i = 1, ..., m \\
& && z_i \succeq 0, \ i = 1, ..., m
\end{aligned}
$$

with variables $x \in R^n$, $z_i \in R^{m_i}$, $i = 1, ..., m$. Hint: find the dual of the problem of maximizing $a_i^T x$ over $a_i \in P_i$ (with variable a_i).

Solution:

The problem of maximizing $a_i^T x$ over $a_i \in P_i$ (with variable a_i) is:

$$\begin{array}{ll} \text{maximize} & a_i^T x \\ \text{subject to} & a_i \in P_i, \text{where } P_i = \{a : \; C_i a \preceq d_i\} \end{array}$$

or

$$\begin{array}{ll} \text{minimize} & -a_i^T x \\ \text{subject to} & C_i a_i \preceq d_i \end{array}$$

The Lagrange dual of this problem is:

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^{m} \lambda_i d_i \\ \text{subject to} & C_i \lambda_i = x \\ & \lambda_i \succeq 0 \end{array}$$

The optimal value of this problem is less or equal to b_i, so we have the equivalent problem to our LP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & d_i^T \lambda_i \leq b_i, \; i = 1, ..., m \\ & C_i \lambda_i = x, \; i = 1, ..., m \\ & \lambda_i \succeq 0, \; i = 1, ..., m \end{array}$$

5.40

E - optimal experiment design. A variation on two optimal experiment design problems of exercise 5.10 is the E - optimal design problem:

$$\begin{array}{ll} \text{minimize} & \lambda_{max}\left(\displaystyle\sum_{i=1}^{p} x_i v_i v_i^T\right)^{-1} \\ \text{subject to} & x \succeq 0, \; \mathbf{1}^T x = 1 \end{array}$$

(See also §7.5.) Derive a dual for this problem first by reformulating it as:

$$\text{minimize} \quad 1/t$$
$$\text{subject to} \quad \sum_{i=1}^{p} x_i v_i v_i^T \succeq t\mathbf{I}$$
$$x \succeq 0, \ \mathbf{1}^T x = 1$$

with variables $t \in R$, $x \in R^p$ and domain $R_{++} \times R^p$, and applying Lagrange duality. Simplify the dual problem as much as you can.

Solution: Let us introduce a variable $t \in R_{++}$. Then for a matrix A, inequality $\lambda_{max}(A^{-1}) \leq 1/t$ means $A^{-1} \preceq \frac{1}{t}I$, or $A \preceq tI$. Setting A to $\sum_{i=1}^{p} x_i v_i v_i^T$ we get a problem:

$$\text{minimize} \quad 1/t$$
$$\text{subject to} \quad \sum_{i=1}^{p} x_i v_i v_i^T \succeq t\mathbf{I}$$
$$x \succeq 0, \ \mathbf{1}^T x = 1$$

The Lagrangian is:

$$L(x, t, Z, z) = 1/t - tr(Z(\sum_{i=1}^{p} x_i v_i v_i^T - t\mathbf{I})) - z^T x + \nu(\mathbf{1}^T x - 1) = 1/t + t\,tr(Z) + \sum_{i=1}^{p} x_i(-v_i^T Z v_i - z_i + \nu) - \nu$$

the infimum of x is bounded below only if $-v_i^T Z v_i - z_i + \nu = 0$. The

$$inf_\nu 1/t + t\,tr(Z) = \begin{cases} 2\sqrt{tr(Z)}, & Z \succeq 0 \\ -\infty, & \text{otherwise} \end{cases}$$

the dual function is

$$L(Z, z, \nu) = \begin{cases} 2\sqrt{tr(Z)} - \nu, & Z \succeq 0, \ -v_i^T Z v_i - z_i + \nu = 0, \\ -\infty, & \text{otherwise} \end{cases}$$

The dual problem:

$$\text{maximize} \quad 2\sqrt{tr(Z)} - \nu$$
$$\text{subject to} \quad v_i^T Z v_i + z_i \leq 0, \ i = 1, ..., p$$
$$Z \succeq 0, \ \nu \geq 0$$

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We can define $W = (1/\nu)Z$

$$
\begin{array}{ll}
\text{maximize} & 2\sqrt{\nu}\sqrt{tr(W)} - \nu \\
\text{subject to} & v_i^T W v_i \leq 1, \ i = 1, ..., p \\
& W \succeq 0, \ \nu \geq 0
\end{array}
$$

Maximizing over ν we get $\nu = tr(W)$, so the problem is:

$$
\begin{array}{ll}
\text{maximize} & tr(W) \\
\text{subject to} & v_i^T W v_i \leq 1, \ i = 1, ..., p \\
& W \succeq 0
\end{array}
$$

6.3

Formulate the following approximation problems as LPs, QPs, SOCPs, or SDPs. The problem data are $A \in R^{n \times m}$ and $b \in R^m$. The rows of A are denoted a_i^T.

(a) Deadzone-linear penalty approximation: minimize $\sum_{i=1}^{m} \phi(a_i^T x - b_i)$, where

$$
\phi(u) = \begin{cases} 0, & |u| \leq a \\ |u| - a, & |u| > a \end{cases}
$$

where $a > 0$.

(b) Log-barrier penalty approximation: minimize $\sum_{i=1}^{m} \phi(a_i^T x - b_i)$, where

$$
\phi(u) = \begin{cases} -a^2 log(1 - (u/a)^2), & |u| < a \\ \infty, & |u| \geq a \end{cases}
$$

with $a > 0$.

(c) Huber penalty approximation: minimize $\sum_{i=1}^{m} \phi(a_i^T x - b_i)$, where

$$
\phi(u) = \begin{cases} u^2, & |u| \leq M \\ M(2|u| - M), & |u| > M \end{cases}
$$

with $M > 0$.

(d) Log-Chebyshev approximation: minimize $max_{i=1,...,m}|log(a_i^T x) - log(b_i)|$. We assume $b \succ 0$. An equivalent convex from is

$$
\begin{array}{ll}
\text{minimize} & t \\
\text{subject to} & 1/t \leq a_i^T x / b_i \leq t, \ i = 1, ..., m
\end{array}
$$

with variables $x \in R^n$ and $t \in R$ and domain $R^n \times R_{++}$.

(e) Minimizing the sum of the largest k residuals:

$$\text{minimize} \qquad \sum_{i=1}^{k} |r|_{[i]}$$

$$\text{subject to} \qquad r = Ax - b$$

where $|r|_{[1]} \geq |r|_{[2]} \geq, ..., \geq |r|_{[m]}$ are the numbers $|r_1|, |r_2|, ..., |r_m|$ sorted in decreasing order. (For $k = 1$ this reduces to l_∞ - norm approximation; for $k = m$ this reduces to l_1 norm approximation.) Hint: See exercise 5.19.

Solution:

(a) Introduce the variable $y \in R^m$ so that $|a_i^T x - b_i| \leq |a + y_i|$, $y_i \geq 0$ then we have equivalent problem:

$$\text{minimize} \qquad \mathbf{1}^T y$$
$$\text{subject to} \qquad -y - a\mathbf{1} \preceq Ax - b \preceq y + a\mathbf{1}$$
$$y \succeq 0$$

(b)

Introduce the variable $y = Ax - b$, $t_i \leq (1 - y_i/a)(1 + y_i + a)$, and add inequality $-a \leq y_i \leq a$ this problem will transform to:

$$\text{maximize} \qquad \prod_{i=1}^{m} t_i^2$$
$$\text{subject to} \qquad y = Ax - b$$
$$t_i \leq (1 - y_i/a)(1 + y_i/a), \ i = 1, ..., m$$
$$-1 \leq y_i/a \leq 1, \ i = 1, ..., m$$

with variables $t, y \in R^m$, $x \in R^n$. Next assume that $m = 2^k$. It can be done without loss of generality, as in other case we always can add missing components of y_i as unity values ($a_i = 0$ and $b_i = -1$). Lets transform this problem for $m = 4$.

$$\text{maximize} \qquad (t_1 t_2 t_3 t_4)^2$$
$$\text{subject to} \qquad y = Ax - b$$
$$t_i \leq (1 - y_i/a)(1 + y_i/a), \ i = 1, ..., m$$
$$-1 \leq y_i/a \leq 1, \ i = 1, ..., m$$

this problem is equivalent to:

maximize $\quad z_1 z_2$

subject to

$$z_1^2 \leq t_1 t_2$$
$$z_2^2 \leq t_3 t_4$$
$$y = Ax - b$$
$$t_i \leq (1 - y_i/a)(1 + y_i/a), \; i = 1, ..., m$$
$$-1 \leq y_i/a \leq 1, \; i = 1, ..., m$$

and also as:

maximize $\quad z$

subject to

$$z^2 \leq z_1 z_2$$
$$z_1^2 \leq t_1 t_2$$
$$z_2^2 \leq t_3 t_4$$
$$y = Ax - b$$
$$t_i \leq (1 - y_i/a)(1 + y_i/a), \; i = 1, ..., m$$
$$-1 \leq y_i/a \leq 1, \; i = 1, ..., m$$

as it is easy to show that $x^T x \leq yz$ where $x \in R^n$, $y, z \in R_+$ is equivalent to:

$$\left\| \begin{bmatrix} x \\ y - z \end{bmatrix} \right\|_2 \leq y + z$$

overriding the first three inequalities with their norm analogue we have:

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$$
\begin{aligned}
&\text{minimize} && -z \\
&\text{subject to} \\
&&& \left\| \begin{bmatrix} z \\ z_1 - z_2 \end{bmatrix} \right\|_2 \leq z_1 + z_2 \\
&&& \left\| \begin{bmatrix} z_1 \\ t_1 - t_2 \end{bmatrix} \right\|_2 \leq t_1 + t_2 \\
&&& \left\| \begin{bmatrix} z_2 \\ t_3 - t_4 \end{bmatrix} \right\|_2 \leq t_3 + t_4 \\
&&& y = Ax - b \\
&&& t_i \leq (1 - y_i/a)(1 + y_i/a),\ i = 1, ..., m \\
&&& -1 \leq y_i/a \leq 1,\ i = 1, ..., m
\end{aligned}
$$

which is Secon Order Cone Program (SOCP).