Solutions to hw2 homework on Convex Optimization
https://web.stanford.edu/class/ee364b/homework.html

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2.1 (8 points, 1 point per question)

Let f be a convex function with domain in R^n. We fix $x \in$ **int dom f** and $d \in R^n$. Recall the definition of the directional derivative of f at x along the direction d

$$f'(x, d) = \lim_{t \to 0} \frac{f(x + td) - f(x)}{t}$$

In this question we aim to show that $f'(x, d)$ exists and is finite, and that we have the following relationship between $\partial f(x)$ and $f'(x, d)$,

$$f'(x, d) = \sup_{g \in \partial f(x)} g^T d$$

(a) Show that the ratio $\frac{f(x+td)-f(x)}{t}$ is a nondecrasing function of $t > 0$. Deduce that $f'(x, d)$ exists and is either finite or equal to $-\infty$.
We know from the lectures that, since $x \in$ **int dom f**, the subdifferential set ∂f is non - empty, convex and compact.

Solution:

Proof of non - decreasing. Definition of subgradient is

$$f(z) \geq f(x) + g^T (z - x)$$

let $z = x + td$; then

$$f(x + td) \geq f(x) + g^T (x + td - x)$$

or

$$f(x + td) - f(x) \geq tg^T d$$

dividing both part of the inequality by t (as $t > 0$, we can do it) gives

$$\frac{f(x+td) - f(x)}{t} \geq g^T d$$

as the right - hand side of the equation is not depends of t, differentiating by t gives

$$\partial \frac{\frac{f(x+td)-f(x)}{t}}{\partial t} \geq 0$$

As the $\partial \frac{f'(x,d)}{\partial t} \geq 0$, it means the function $f'(x,d)$ is nondecreasing by variable t.

Proof of possible equality to $-\infty$.

The definition of convexity:

$$f(\theta x + (1-\theta)y)) \leq \theta f(x) + (1-\theta) f(y)$$

where $0 < \theta < 1$.

let $t = 1 - \theta$, $0 < t < 1$. then

$$f((1-t)x + ty)) \leq (1-t)f(x) + tf(y)$$

or

$$f(x + t(y-x))) \leq f(x) + t(f(y) - f(x))$$

as we can choose y any of the point in domain f, we can set $d = y - x$. Then

$$f(x+td) \leq f(x) + t(f(y) - f(x))$$

or

$$f(x+td) - f(x) \leq t(f(y) - f(x))$$

or

$$\frac{f(x+td) - f(x)}{t} \leq f(y) - f(x)$$

As $f(x)$ can be equal to ∞ on the domain of f, so $f'(x,d) = \frac{f(x+td)-f(x)}{t}$ can be less or equal than (for the infinity with sign minus it means strictly equal) $-\infty$ on the domain of f. This means that $f'(x,d)$ can be equal to $-\infty$ on domain of f.

(b) Let $g \in \partial f(x)$. Show that $f'(x,d) \geq g^T d$. Deduce that $f'(x,d)$ is finite and $f'(x,d) \geq \sup_{g \in \partial f(x)} g^T d$.

Solution:

We already shown that

$$f'(x,d) \geq g^T d$$

in part (a). We also shown in part (a) that

$$\frac{f(x+td) - f(x)}{t} \leq f(y) - f(x)$$

Second upper inequality means that $f'(x,d)$ is bounded from upper side (i.e it can't be equal to ∞), it means its value is finite.
As the first of upper inequalities is correct $\forall$ subgradients in domain of f, it means, that it is correct for the supremum of these subgradients in domain f. It means that

$$f'(x,d) \geq \sup_{g \in \partial f(x)} g^T d.$$