Solutions to hw5 homework on Convex Optimization https://web.stanford.edu/class/ee364a/homework.html

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5.17

Robust linear programming with polyhedral uncertainty. Consider the robust LP:

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & \sup\limits_{a \in P_i} a^T x \le b_i,\ i = 1, ..., m
\end{array}
$$

with variable $x \in R^n$, where $P_i = \{a : \ C_i a \preceq d_i\}$. The problem data are $a \in R^n$, $C_i \in R^{m_i \times n}$, $d_i \in R^{m_i}$, and $b \in R^m$. We asume the polyhedra P_i are nonempty. Show that this problem is equivalent to the LP:

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & d_i^T z_i \le b_i,\ i = 1, ..., m \\
& C_i z_i = x,\ i = 1, ..., m \\
& z_i \succeq 0,\ i = 1, ..., m
\end{array}
$$

with variables $x \in R^n$, $z_i \in R^{m_i}$, $i = 1, ..., m$. Hint: find the dual of the problem of maximizing $a_i^T x$ over $a_i \in P_i$ (with variable a_i).

Solution:

The problem of maximizing $a_i^T x$ over $a_i \in P_i$ (with variable a_i) is:

$$\begin{array}{ll} \text{maximize} & a_i^T x \\ \text{subject to} & a_i \in P_i, \text{where } P_i = \{a: \ C_i a \preceq d_i\} \end{array}$$

or

$$\begin{array}{ll} \text{minimize} & -a_i^T x \\ \text{subject to} & C_i a_i \preceq d_i \end{array}$$

The Lagrange dual of this problem is:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{m} \lambda_i d_i \\ \text{subject to} & C_i \lambda_i = x \\ & \lambda_i \succeq 0 \end{array}$$

The optimal value of this problem is less or equal to b_i, so we have the equivalent problem to our LP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & d_i^T \lambda_i \leq b_i, \ i = 1, ..., m \\ & C_i \lambda_i = x, \ i = 1, ..., m \\ & \lambda_i \succeq 0, \ i = 1, ..., m \end{array}$$

5.40

E - optimal experiment design. A variation on two optimal experiment design problems of exercise 5.10 is the E - optimal design problem:

$$\begin{array}{ll} \text{minimize} & \lambda_{max}(\sum_{i=1}^{p} x_i v_i v_i^T)^{-1} \\ \text{subject to} & x \succeq 0, \ \mathbf{1}^T x = 1 \end{array}$$

(See also §7.5.) Derive a dual for this problem first by reformulating it as:

$$\begin{array}{ll} \text{minimize} & 1/t \\ \text{subject to} & \sum_{i=1}^{p} x_i v_i v_i^T \succeq t\boldsymbol{I} \\ & x \succeq 0, \ \mathbf{1}^T x = 1 \end{array}$$

with variables $t \in R$, $x \in R^p$ and domain $R_{++} \times R^p$, and applying Lagrange duality. Simplify the dual problem as much as you can.

Solution: Let us introduce a variable $t \in R_{++}$. Then for a matrix A, inequality $\lambda_{max}(A^{-1}) \leq 1/t$ means $A^{-1} \preceq \frac{1}{t}I$, or $A \preceq tI$. Setting A to $\sum_{i=1}^{p} x_i v_i v_i^T$ we get a problem:

$$\begin{array}{ll} \text{minimize} & 1/t \\ \text{subject to} & \sum_{i=1}^{p} x_i v_i v_i^T \succeq t\boldsymbol{I} \\ & x \succeq 0, \ \mathbf{1}^T x = 1 \end{array}$$

The Lagrangian is:

$$L(x,t,Z,z) = 1/t - tr(Z(\sum_{i=1}^{p} x_i v_i v_i^T - t\boldsymbol{I})) - z^T x + \nu(\mathbf{1}^T x - 1) = 1/t + t\, tr(Z) + \sum_{i=1}^{p} x_i(-v_i^T Z v_i - z_i + \nu) - \nu$$

the infimum of x is bounded below only if $-v_i^T Z v_i - z_i + \nu = 0$. The

$$inf_\nu 1/t + t\, tr(Z) = \begin{cases} 2\sqrt{tr(Z)}, & Z \succeq 0 \\ -\infty, & \text{otherwise} \end{cases}$$

the dual function is

$$L(Z,z,\nu) = \begin{cases} 2\sqrt{tr(Z)} - \nu, & Z \succeq 0, \, -v_i^T Z v_i - z_i + \nu = 0, \\ -\infty, & \text{otherwise} \end{cases}$$

The dual problem:

$$\begin{array}{ll} \text{maximize} & 2\sqrt{tr(Z)} - \nu \\ \text{subject to} & v_i^T Z v_i + z_i \leq 0, \ i = 1, ..., p \\ & Z \succeq 0, \, \nu \geq 0 \end{array}$$

We can define $W = (1/\nu)Z$

$$\begin{array}{ll} \text{maximize} & 2\sqrt{\nu}\sqrt{tr(W)} - \nu \\ \text{subject to} & v_i^T W v_i \leq 1,\ i = 1, ..., p \\ & W \succeq 0,\ \nu \geq 0 \end{array}$$

Maximizing over ν we get $\nu = tr(W)$, so the problem is:

$$\begin{array}{ll} \text{maximize} & tr(W) \\ \text{subject to} & v_i^T W v_i \leq 1,\ i = 1, ..., p \\ & W \succeq 0 \end{array}$$

6.3

Formulate the following approximation problems as LPs, QPs, SOCPs, or SDPs. The problem data are $A \in R^{n\times m}$ and $b \in R^m$. The rows of A are denoted a_i^T.

(a) Deadzone-linear penalty approximation: minimize $\sum_{i=1}^m \phi(a_i^T x - b_i)$, where

$$\phi(u) = \begin{cases} 0, & |u| \leq a \\ u - [a|, & |u| > a \end{cases}$$

where $a > 0$.

(b) Log-barrier penalty approximation: minimize $\sum_{i=1}^m \phi(a_i^T x - b_i)$, where

$$\phi(u) = \begin{cases} -a^2 log(1 - (u/a)^2), & |u| < a \\ \infty, & |u| \geq a \end{cases}$$

with $a > 0$.

(c) Huber penalty approximation: minimize $\sum_{i=1}^m \phi(a_i^T x - b_i)$, where

$$\phi(u) = \begin{cases} u^2, & |u| \leq M \\ M(2|u| - M), & |u| > M \end{cases}$$

with $M > 0$.

(d) Log-Chebyshev approximation: minimize $max_{i=1,...,m}|log(a_i^T x) - log(b_i)|$. We assume $b \succ 0$. An equivalent convex from is

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & 1/t \leq a_i^T x/b_i \leq t,\ i = 1, ..., m \end{array}$$

with variables $x \in R^n$ and $t \in R$ and domain $R^n \times R_{++}$.

(e) Minimizing the sum of the largest k residuals:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{k} |r|_{[i]} \\ \text{subject to} & r = Ax - b \end{array}$$

where $|r|_{[1]} \geq |r|_{[2]} \geq, ..., \geq |r|_{[m]}$ are the numbers $|r_1|, |r_2|, ..., |r_m|$ sorted in decreasing order. (For $k = 1$ this reduces to l_∞ - norm approximation; for $k = m$ this reduces to l_1 norm approximation.) Hint: See exercise 5.19.