Solutions to hw5 homework on Convex Optimization https://web.stanford.edu/class/ee364a/homework.html

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5.17

Robust linear programming with polyhedral uncertainty. Consider the robust LP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \sup\limits_{a \in P_i} a^T x \le b_i, \ i = 1, ..., m \end{array}$$

with variable $x \in R^n$, where $P_i = \{a : \ C_i a \preceq d_i\}$. The problem data are $a \in R^n$, $C_i \in R^{m_i \times n}$, $d_i \in R^{m_i}$, and $b \in R^m$. We asume the polyhedra P_i are nonempty. Show that this problem is equivalent to the LP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & d_i^T z_i \le b_i, \ i = 1, ..., m \\ & C_i z_i = x, \ i = 1, ..., m \\ & z_i \succeq 0, \ i = 1, ..., m \end{array}$$

with variables $x \in R^n$, $z_i \in R^{m_i}, \ i = 1, ..., m$. Hint: find the dual of the problem of maximizing $a_i^T x$ over $a_i \in P_i$ (with variable a_i).

Solution:

The problem of maximizing $a_i^T x$ over $a_i \in P_i$ (with variable a_i) is:

$$\begin{array}{ll} \text{maximize} & a_i^T x \\ \text{subject to} & a_i \in P_i, \text{where } P_i = \{a : \ C_i a \preceq d_i\} \end{array}$$

or

$$\begin{array}{ll} \text{minimize} & -a_i^T x \\ \text{subject to} & C_i a_i \preceq d_i \end{array}$$

The Lagrange dual of this problem is:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{m} \lambda_i d_i \\ \text{subject to} & C_i \lambda_i = x \\ & \lambda_i \succeq 0 \end{array}$$

The optimal value of this problem is less or equal to b_i, so we have the equivalent problem to our LP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & d_i^T \lambda_i \leq b_i, \ i = 1, ..., m \\ & C_i \lambda_i = x, \ i = 1, ..., m \\ & \lambda_i \succeq 0, \ i = 1, ..., m \end{array}$$

5.40

E - optimal experiment design. A variation on two optimal experiment design problems of exercise 5.10 is the E - optimal design problem:

$$\begin{array}{ll} \text{minimize} & \lambda_{max}(\sum_{i=1}^{p} x_i v_i v_i^T)^{-1} \\ \text{subject to} & x \succeq 0, \ \mathbf{1}^T x = 1 \end{array}$$

(See also §7.5.) Derive a dual for this problem first by reformulating it as:

$$\begin{array}{ll} \text{minimize} & 1/t \\ \text{subject to} & \sum_{i=1}^{p} x_i v_i v_i^T \succeq t\boldsymbol{I} \\ & x \succeq 0,\ \mathbf{1}^T x = 1 \end{array}$$

with variables $t \in R$, $x \in R^p$ and domain $R_{++} \times R^p$, and applying Lagrange duality. Simplify the dual problem as much as you can.

Solution: Let us introduce a variable $t \in R_{++}$. Then for a matrix A, inequality $\lambda_{max}(A^{-1}) \le 1/t$ means $A^{-1} \preceq \frac{1}{t}I$, or $A \preceq tI$. Setting A to $\sum_{i=1}^{p} x_i v_i v_i^T$ we get a problem:

$$\begin{array}{ll} \text{minimize} & 1/t \\ \text{subject to} & \sum_{i=1}^{p} x_i v_i v_i^T \succeq t\boldsymbol{I} \\ & x \succeq 0,\ \mathbf{1}^T x = 1 \end{array}$$

The Lagrangian is:

$$L(x,t,Z,z) = 1/t - tr(Z(\sum_{i=1}^{p} x_i v_i v_i^T - t\boldsymbol{I})) - z^T x + \nu(\mathbf{1}^T x - 1) = 1/t + t\, tr(Z) + \sum_{i=1}^{p} x_i(-v_i^T Z v_i - z_i + \nu) - \nu$$

the infimum of x is bounded below only if $-v_i^T Z v_i - z_i + \nu = 0$. The

$$inf_\nu 1/t + t\, tr(Z) = \begin{cases} 2\sqrt{tr(Z)}, & Z \succeq 0 \\ -\infty, & \text{otherwise} \end{cases}$$

the dual function is

$$L(Z,z,\nu) = \begin{cases} 2\sqrt{tr(Z)} - \nu, & Z \succeq 0,\ -v_i^T Z v_i - z_i + \nu = 0, \\ -\infty, & \text{otherwise} \end{cases}$$

The dual problem:

$$\begin{array}{ll} \text{maximize} & 2\sqrt{tr(Z)} - \nu \\ \text{subject to} & v_i^T Z v_i + z_i \le 0,\ i = 1, ..., p \\ & Z \succeq 0,\ \nu \ge 0 \end{array}$$

We can define $W = (1/\nu)Z$

$$\begin{array}{ll} \text{maximize} & 2\sqrt{\nu}\sqrt{tr(W)} - \nu \\ \text{subject to} & v_i^T W v_i \leq 1,\ i = 1, ..., p \\ & W \succeq 0,\ \nu \geq 0 \end{array}$$

Maximizing over ν we get $\nu = tr(W)$, so the problem is:

$$\begin{array}{ll} \text{maximize} & tr(W) \\ \text{subject to} & v_i^T W v_i \leq 1,\ i = 1, ..., p \\ & W \succeq 0 \end{array}$$

6.3

Formulate the following approximation problems as LPs, QPs, SOCPs, or SDPs. The problem data are $A \in R^{n \times m}$ and $b \in R^m$. The rows of A are denoted a_i^T.

(a) Deadzone-linear penalty approximation: minimize $\sum_{i=1}^m \phi(a_i^T x - b_i)$, where

$$\phi(u) = \begin{cases} 0, & |u| \leq a \\ |u| - a, & |u| > a \end{cases}$$

where $a > 0$.

(b) Log-barrier penalty approximation: minimize $\sum_{i=1}^m \phi(a_i^T x - b_i)$, where

$$\phi(u) = \begin{cases} -a^2 log(1 - (u/a)^2), & |u| < a \\ \infty, & |u| \geq a \end{cases}$$

with $a > 0$.

(c) Huber penalty approximation: minimize $\sum_{i=1}^m \phi(a_i^T x - b_i)$, where

$$\phi(u) = \begin{cases} u^2, & |u| \leq M \\ M(2|u| - M), & |u| > M \end{cases}$$

with $M > 0$.

(d) Log-Chebyshev approximation: minimize $max_{i=1,...,m} |log(a_i^T x) - log(b_i)|$. We assume $b \succ 0$. An equivalent convex from is

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & 1/t \leq a_i^T x / b_i \leq t,\ i = 1, ..., m \end{array}$$

with variables $x \in R^n$ and $t \in R$ and domain $R^n \times R_{++}$.

(e) Minimizing the sum of the largest k residuals:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{k} |r|_{[i]} \\ \text{subject to} & r = Ax - b \end{array}$$

where $|r|_{[1]} \geq |r|_{[2]} \geq, ..., \geq |r|_{[m]}$ are the numbers $|r_1|, |r_2|, ..., |r_m|$ sorted in decreasing order. (For $k = 1$ this reduces to l_∞ - norm approximation; for $k = m$ this reduces to l_1 norm approximation.) Hint: See exercise 5.19.

Solution:

(a) Introduce the variable $y \in R^m$ so that $|a_i^T x - b_i| \leq |a + y_i|$, $y_i \geq 0$ then we have equivalent problem:

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T y \\ \text{subject to} & -y - a\mathbf{1} \preceq Ax - b \preceq y + a\mathbf{1} \\ & y \succeq 0 \end{array}$$

(b)

Introduce the variable $y = Ax - b$, $t_i \leq (1 - y_i/a)(1 + y_i + a)$, and add inequality $-a \leq y_i \leq a$ this problem will transform to:

$$\begin{array}{ll} \text{maximize} & \prod_{i=1}^{m} t_i^2 \\ \text{subject to} & y = Ax - b \\ & t_i \leq (1 - y_i/a)(1 + y_i/a),\ i = 1, ..., m \\ & -1 \leq y_i/a \leq 1,\ i = 1, ..., m \end{array}$$

with variables $t, y \in R^m$, $x \in R^n$. Next assume that $m = 2^k$. It can be done without loss of generality, as in other case we always can add missing components of y_i as unity values ($a_i = 0$ and $b_i = -1$). Lets transform this problem for $m = 4$.

$$\begin{array}{ll} \text{maximize} & (t_1 t_2 t_3 t_4)^2 \\ \text{subject to} & y = Ax - b \\ & t_i \leq (1 - y_i/a)(1 + y_i/a),\ i = 1, ..., m \\ & -1 \leq y_i/a \leq 1,\ i = 1, ..., m \end{array}$$

this problem is equivalent to:

$$
\begin{array}{ll}
\text{maximize} & z_1 z_2 \\
\text{subject to} & \\
& z_1^2 \leq t_1 t_2 \\
& z_2^2 \leq t_3 t_4 \\
& y = Ax - b \\
& t_i \leq (1 - y_i/a)(1 + y_i/a),\ i = 1, ..., m \\
& -1 \leq y_i/a \leq 1,\ i = 1, ..., m
\end{array}
$$

and also as:

$$
\begin{array}{ll}
\text{maximize} & z \\
\text{subject to} & \\
& z^2 \leq z_1 z_2 \\
& z_1^2 \leq t_1 t_2 \\
& z_2^2 \leq t_3 t_4 \\
& y = Ax - b \\
& t_i \leq (1 - y_i/a)(1 + y_i/a),\ i = 1, ..., m \\
& -1 \leq y_i/a \leq 1,\ i = 1, ..., m
\end{array}
$$

as it is easy to show that $x^T x \leq yz$ where $x \in R^n$, $y, z \in R_+$ is equivalent to:

$$
\left\| \begin{bmatrix} x \\ y - z \end{bmatrix} \right\|_2 \leq y + z
$$

overriding the first three inequalities with their norm analogue we have:

$$
\begin{aligned}
&\text{minimize} && -z\\
&\text{subject to} && \\
& && \left\|\begin{bmatrix} z \\ z_1 - z_2\end{bmatrix}\right\|_2 \le z_1 + z_2\\
& && \left\|\begin{bmatrix} z_1 \\ t_1 - t_2\end{bmatrix}\right\|_2 \le t_1 + t_2\\
& && \left\|\begin{bmatrix} z_2 \\ t_3 - t_4\end{bmatrix}\right\|_2 \le t_3 + t_4\\
& && y = Ax - b\\
& && t_i \le (1 - y_i/a)(1 + y_i/a),\ i = 1, ..., m\\
& && -1 \le y_i/a \le 1,\ i = 1, ..., m
\end{aligned}
$$

which is Second Order Cone Program (SOCP).

(c)

Lets show that this problem is equivalent to QP:

$$
\begin{aligned}
&\text{minimize} && \sum_{i=1}^{m}(u_i^2 + 2Mv_i)\\
&\text{subject to} && -u - v \preceq Ax - b \preceq u + v\\
& && 0 \preceq u \preceq M\mathbf{1}\\
& && v \succeq 0
\end{aligned}
$$

Proof: Lets fix x in our QP. For the optimum point we must have $u_i + v_i = |a_i^T x - b_i|$. In other case, if $u_i + v_i > |a_i^T x b_i|$ and $0 \le u_i \le M$ and $v_i \ge 0$, then as u_i and v_i are not both zero, we can decrease u_i and/or v_i without violating the constraints and the objective will be decreased also. So, at the optimum we have:

$$v_i = |a_i^T x - b_i| - u_i$$

Eliminating v_i yields equivalent problem:

$$
\begin{aligned}
&\text{minimize} && \sum_{i=1}^{m}(u_i^2 - 2Mu_i + 2M|a_i^T x - b_i|)\\
&\text{subject to} && 0 \preceq u_i \preceq min(M,\ |a_i^T x - b_i|)
\end{aligned}
$$

It $M > |a_i^T x - b_i|$ the optimal choice for u_i is $|a_i^T x - b_i|$. In this case the objective function reduces to $|a_i^T x - b_i|^2$. Otherwise the optimal choice for u_i is M, and the objective function reduces to $2M|a_i^T x - b_i| - M^2$. So, we conclude that with

$\phi(a_i^T x - b_i)$ these problems are equivalent.
(c) The constraint $ta_i^T x \geq b_i$, $t \geq 0$, $a_i^T x \geq 0$ can be formulated as an LMI

$$\begin{bmatrix} t & \sqrt{b_i} \\ \sqrt{b_i} & a_i^T x \end{bmatrix} \succeq 0$$

or as follows:

$$\left\| \begin{bmatrix} 2\sqrt{b_i} \\ t - a_i^T x \end{bmatrix} \right\|_2 \leq t + a_i^T x$$

(e) As in exercise 5.19, we have a problem:

$$\begin{array}{ll} \text{minimize} & kt + \mathbf{1}z \\ \text{subject to} & -t\mathbf{1} - z \preceq Ax - b \preceq t\mathbf{1} + z \\ & z \succeq 0 \end{array}$$

where $x \in R^n$, $t \in R$, $z \in R^m$

6.8 a - b

Formulate the following robust approximation problems as LPs, QPs, SOCPs, or SDPs. For each subproblem, consider the l_1-, l_2-, and the $l_\infty-$ norms.

(a) Stochastic robust approximation with a finite set of parameter values, i.e., the sum of - norms problem

$$\text{minimize} \quad \sum_{i=1}^{k} p_i \left\| A_i x - b \right\|$$

where $p \succeq 0$ and $\mathbf{1}^T p = 1$. (See §6.4.1.)

(b) Worst-case robust approximation with coefficient bounds:

$$\text{minimize} \quad \sup_{A \in \mathcal{A}} \|Ax - b\|$$

where $\mathcal{A} = \{A \in R^{m \times n} | l_{ij} \leq a_{ij} \leq u_{ij}, i = 1, ..., m, j = 1, ..., n\}$. Here the uncertainty set is described by giving upper and lower bounds for the components of A. We assume $l_{ij} < u_{ij}$.

Solution:
(a) l_1 norm: Introduce a slack variable $y : |y_i| \succeq |A_i x - b|$. We have LP:

$$\begin{array}{ll} \text{minimize} & p^T y \\ \text{subject to} & -y_i \preceq A_i x - b \preceq y_i \end{array}$$

l_2 norm: Introduce a slack variable $y : y_i \succeq ||A_i x - b||_2$. We have SOCP:

$$\begin{array}{ll} \text{minimize} & p^T y \\ \text{subject to} & ||A_i x - b|| \preceq y_i \end{array}$$

l_∞ norm: We have LP:

$$\begin{array}{ll} \text{minimize} & p^T y \\ \text{subject to} & -y_i \mathbf{1} \preceq A_i x - b \preceq y_i \mathbf{1} \end{array}$$

(b)

$$\sup_{l_{ij} \leq a_{ij} \leq u_{ij}} |a_i^T x - b_i| = \sup_{l_{ij} \leq a_{ij} \leq u_{ij}} max(-a_i^T x + b_i, a_i^T x - b_i) = max(\sup_{l_{ij} \leq a_{ij} \leq u_{ij}} -a_i^T x + b_i, \sup_{l_{ij} \leq a_{ij} \leq u_{ij}} a_i^T x - b_i)$$

$$\sup_{l_{ij} \leq a_{ij} \leq u_{ij}} -a_i^T x + b_i = \overline{-a}_i^T x + b_i + v_i^T |x|$$

where $\overline{a}_{ij} = (u_{ij} + l_{ij})/2$, $v_{ij} = (u_{ij} - l_{ij})/2$ and

$$\sup_{l_{ij} \leq a_{ij} \leq u_{ij}} a_i^T x - b_i = \overline{a}_i^T x - b_i + v_i^T |x|$$

Therefore

$$\sup_{l_{ij} \leq a_{ij} \leq u_{ij}} |a_i^T x - b_i| = \bar{|}a_i^T x - b_i| + v_i^T |x|$$

(a) l_1 norm.

$$\text{minimize} \quad \sum_{i=1}^{m} (|\overline{a}_i x - b_i| + v_i^T |x|)$$

Introducing slack variables $y : |y_i| \geq |\overline{a}_i x - b_i|$ and $w : |w_i| \geq |x_i|$ we have LP:

$$
\begin{array}{ll}
\text{minimize} & \mathbf{1}^T(y+Vw) \\
\text{subject to} & -y \preceq \overline{A}x - b \preceq y \\
& -w \preceq x \preceq w
\end{array}
$$

l_2 norm.

$$
\text{minimize} \quad \sum_{i=1}^{m}(|\overline{a}_i x - b_i| + v_i^T|x|)^2
$$

introduce the same slack variables and variable y, w and the new variable t : $t \leq ||y+Vw||_2$ we have SOCP:

$$
\begin{array}{ll}
\text{minimize} & t \\
\text{subject to} & -y \preceq \overline{A}x - b \preceq y \\
& -w \preceq x \preceq w \\
& t \geq ||y+Vw||_2
\end{array}
$$

l_∞ norm:

$$
\text{minimize} \quad \max_{i=1,\ldots,m}(|\overline{a}_i x - b_i| + v_i^T|x|)^2
$$

this can be expressed as LP:

$$
\begin{array}{ll}
\text{minimize} & t \\
\text{subject to} & -y \preceq \overline{A}x - b \preceq y \\
& -w \preceq x \preceq w \\
& -t\mathbf{1} \preceq y+Vw \preceq t\mathbf{1}
\end{array}
$$

A5.4

Penalty function approximation. We consider the approximation problem

$$
\text{minimize} \quad \phi(Ax+b)
$$

where $a \in R^m \times n$, $b\ inR^m$, the variable is $x \in R^n$, and $\phi : R^m \rightarrow R$ is a convex penalty function that measures the quality of the approximation $Ax \approx b$. We will consider the following choices of penalty function:

(a) Euclidean norm.

$$\phi(y) = ||y||_2 = (\sum_{k=1}^{m} y_k^2)^{1/2}$$

(b) l_1 - norm:

$$\phi(y) = ||y||_1 = \sum_{k=1}^{m} |y_k|$$

(c) Sum of the largest $m/2$ absolute values:

$$\phi(y) = \sum_{k=1}^{m/2} |y_{[k]}|$$

where $y_{[1]}, y_{[2]}, ...$ denote the the absolute values of the components of y sorted in the decreasing order.

(d) A piecewise-linear penalty.

$$\phi(y) = \sum_{k=1}^{m} h(y_k), \quad h(u) = \begin{cases} 0, & |u| \leq 0.2 \\ |u| - 0.2, & 0.2 \leq |u| \leq 0.3 \\ 2|u| - 0.5, & |u| \geq 0.3 \end{cases}$$

(e) Huber penalty.

$$\phi(y) = \sum_{k=1}^{m} h(y_k), \quad h(u) = \begin{cases} u^2, & |u| \leq M \\ M(2|u| - M), & |u| \geq M \end{cases}$$

with $M = 0.2$

(f) Log-barrier penalty.

$$\phi(y) = \sum_{k=1}^{m} h(y_k), \quad h(u) = -log(1 - u^2), \quad \textbf{\textit{dom}}\ h = \{u \mid |u| < 1.\}$$

with $M = 0.2$

Here is the problem. Generate data A and b as follows:

```
m = 200;
n = 100;
A = randn(m,n);
b = randn(m,1);
b = b/(1.01*max(abs(b)));
```

(The normalization of b ensures that the domain of $\phi(Ax - b)$ is nonempty if we use the log-barrier penalty.) To compare the results, plot a histogram of the vector of residuals $y = Ax - b$ for each of the solutions x, using the Matlab command

```
hist(A*x-b,m/2);
```

Some additional hints and remarks for the individual problems:
(a) This problem can be solved using least-squares `(x=A\b)`.
(b) Use the CVX function `norm(y,1)`.
(c) Use the CVX function `norm_largest()`.
(d) Use CVX, with the overloaded `max()`, `abs()`, and `sum()` functions.
(e) Use the CVX function `huber()`.
(f) The current version of CVX handles the logarithm using an iterative procedure, which is slow and not entirely reliable. However, you can reformulate this problem as

$$\text{maximize} \qquad \prod_{k=1}^{m} ((1 - (Ax - b)_k)(1 + (Ax - b)_k))^{1/2m},$$

and use the CVX function `geo_mean()`.