Solutions to hw4 homework on Convex Optimization https://web.stanford.edu/class/ee364a/homework.html

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5.1

A simple example. Consider the optimization problem

$$\begin{array}{ll} \textit{minimize} & x^2 + 1 \\ \textit{subject to} & (x-2)(x-4) \leq 0 \end{array}$$

with variable $x \in R$

(a) Analysis of primal problem. Give the feasible set, the optimal value, and the optimal solution.

(b) Lagrangian and dual function. Plot the objective $x^2 + 1$ versus x. On the same plot, show the feasible set, optimal point and value, and plot the Lagrangian $L(x, \lambda)$ versus x for a few positive values of λ. Verify the lower bound property ($p^* \leq \inf_x L(x, \lambda) \ for \ \lambda \geq 0$). Derive and sketch the Lagrange dual function g.

(c) Lagrange dual problem. State the dual problem, and verify that it is a concave maximization problem. Find the dual optimal value and dual optimal solution λ^*. Does strong duality hold?

(d) Sensitivity analysis. Let $p^*(u)$ denote the optimal value of the problem

$$\begin{array}{ll} \textit{minimize} & x^2 + 1 \\ \textit{subject to} & (x-2)(x-4) \leq u \end{array}$$

as a function of the parameter u. Plot $p^*(u)$. Verify that $dp^*(0)/du = -\lambda^*$.

Solution:

(a)

The feasible set is $x \in [2, 4]$. The optimal solution is $x^* = 2$, the optimal value is $p^* = 5$.

(b)

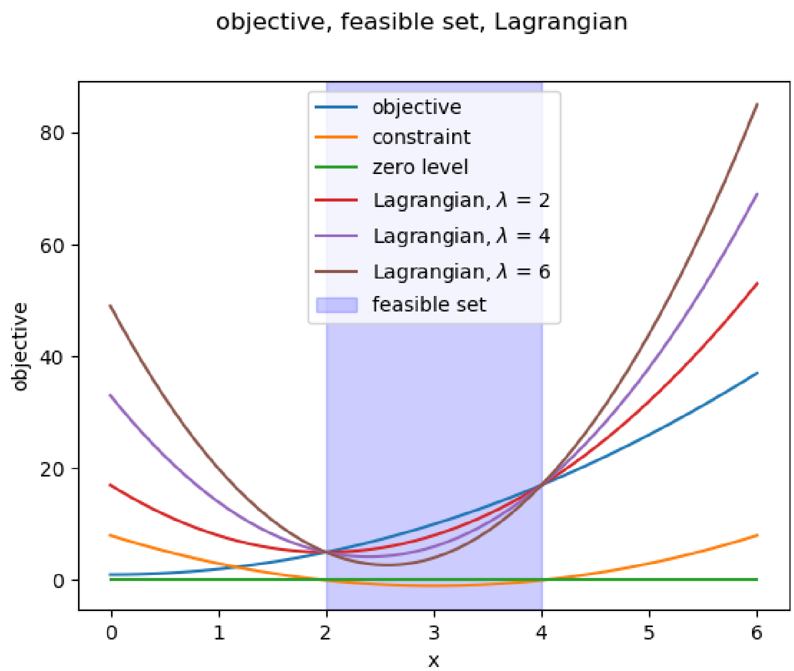

Figure 1: objective, feasible set, lagrangian for this problem.

It's easy to see from the Figure 1, that the Lagrangian values on the feasible set are less or equal than the objective values on the feasible set, i. e. $(p^* \geq \inf_x L(x, \lambda)\ for\ \lambda \geq 0)$.

(c)
The dual objective function for this problem can be found solving the constrained equation for the Lagrangian:

$$
\begin{aligned}
& g(\lambda) = inf_x(x^2 + 1 + \lambda(x-2)(x-4)) \\
& subject\ to \qquad \lambda \geq 0
\end{aligned}
$$

The Lagrangian reaches its minimum at the point $\tilde{x} = \frac{3\lambda}{\lambda+1}$. Then the dual objective itself is:

$$g(\lambda) = -\lambda + 10 - \frac{9}{(\lambda+1)}$$

The second derivative of the dual function is:

$$g^{''}(\lambda) = -18/(\lambda+1)^3$$

which is obviously less than zero for $\lambda \geq 0$, i.e. the dual function is concave for $\lambda \geq 0$.

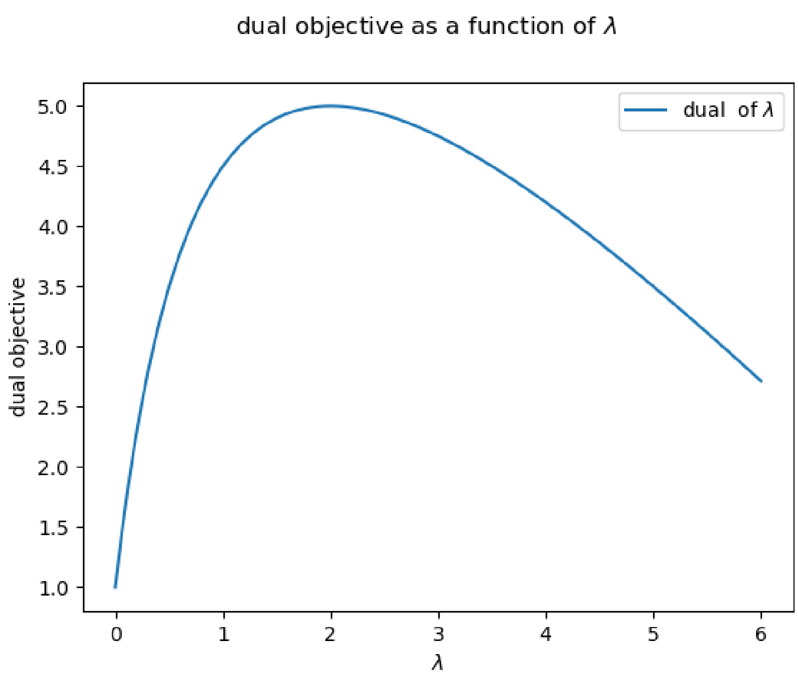

Figure 2: Dual objective for this problem.

The dual optimal value λ^* can be found solving the dual problem for the $g(\lambda)$:

$$\begin{array}{ll} \textit{maximize} & g(\lambda) \\ \textit{subject to} & \lambda \geq 0 \end{array}$$

or

$$\begin{array}{ll} dg(\lambda)/d\lambda = 0 & \\ \textit{subject to} & \lambda \geq 0 \end{array}$$

Solving the equation we found $\lambda^* = 2$; optimal value of g (i.e. $sup_\lambda\{g(\lambda)\}$) $g^* = 5$. We can see that $p^* = 5 = g^*$, i. e. strong duality holds.

(d)

Solving the constraints equation $(x-2)(x-4) = u$ we get:
$x_{1,2} = 3 \pm \sqrt{1+u},\ u \geq 1$. Then

$$p^*(u) = \begin{cases} \text{not exists,} & \text{if } u < -1 \\ u - 6\sqrt{1+u} + 11, & \text{if } -1 \leq u \leq 8 \\ 1, & \text{if } u > 8 \end{cases} \tag{1}$$

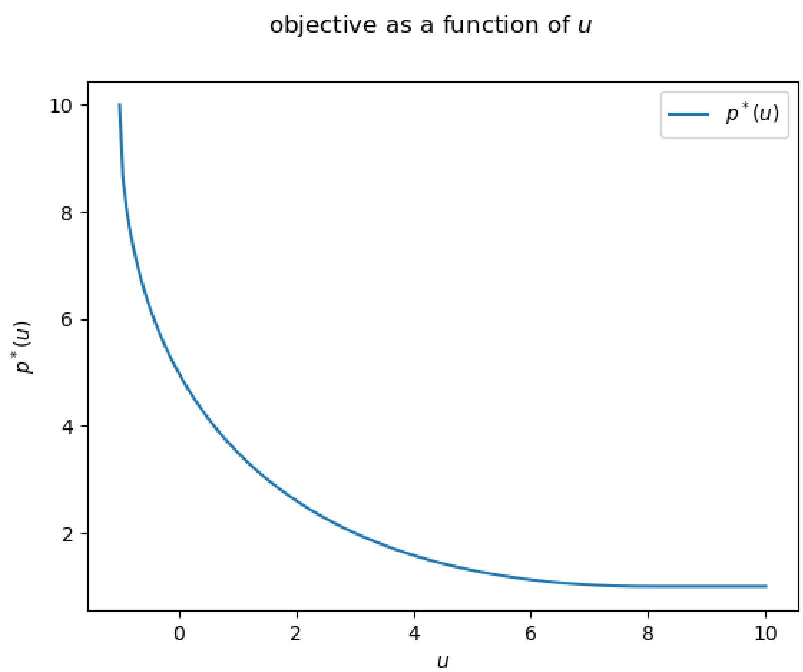

Figure 3: Graph of $p^*(u)$ for this problem.

And finally:

$$dp^*(u)/du = 1 - \frac{3}{\sqrt{1+u}}$$

Then $dp^*(0)/du = -2 = -\lambda^*$

5.3

Problems with one inequality constraint. Express the dual problem of

$$\begin{aligned}&\text{minimize } c^T x\\&\text{subject to } f(x) \le 0\end{aligned}$$

with $c \neq 0$ in terms of the conjugate f^*. Explain why the problem you give is convex. We do not assume f is convex.

Solution:
The dual problem of the task is:

$$\begin{aligned}&\text{maximize } \inf_x (c^T x + \lambda f(x))\\&\text{subject to } \lambda \ge 0\end{aligned}$$

The definition of the conjugate function is:

$$f^*(y) = \sup_x (y^T x - f(x)) = -\inf_x (f(x) - y^T x)$$

i.e. the dual problem can be reformulated as:

$$\begin{aligned}&\text{maximize } F(c, \lambda)\\&\text{subject to } \lambda \ge 0\end{aligned}$$

where $F(c, \lambda) = -\lambda f^*(-c/\lambda)$, where f^* is the conjugate function of f, and it is always convex. $F(c, \lambda)$ is the concave function, as it is the negative perspective of the convex function f^*.

5.12

Analytic centering. Derive a dual problem for

$$\text{minimize } -\sum_{i=1}^{m} log(b_i - a_i^T x)$$

with domain $\{x \mid a_i^T x \le b_i,\ i = 1, ..., m\}$. First introduce new variables y_i and equality constraints $y_i = b_i - a_i^T x$. (The solution of this problem is called the analytic center of the linear inequalities $a_i^T x \le b_i,\ i = 1, ..., m\}$. Analytic centers have geometric applications and play an important role in barrier methods.

Solution:

This problem is equivalent to:

$$
\begin{array}{ll}
\text{minimize} & -\sum_{i=1}^{m} log(y_i) \\
\text{subject to} & y + A^T x - b = 0
\end{array}
$$

where the matrix A composed from the rows a_i^T.
Then the Lagrangian is

$$L(x, y, \nu) = -\sum_{i=1}^{m} log(y_i) + \nu^T (y + A^T x - b)$$

The dual function is

$$g(v) = \inf_{x,y} (-\sum_{i=1}^{m} log(y_i) + \nu^T (y + A^T x - b))$$

The term $\nu^T A^T x$ is unbounded below as $x \to \infty$, so

$$g(\nu) = \begin{cases} \sum_{i=1}^{m} log(y_i) + m - \nu^T b, & \nu^T A^T = 0,\ \nu_i \geq 0 \\ -\infty, & \nu^T A^T \neq 0 \end{cases}$$

and the dual problem is:

$$
\begin{array}{ll}
\text{maximize} & \sum_{i=1}^{m} log(\nu_i) + m - \nu^T b \\
\text{subject to} & \nu^T A^T = 0
\end{array}
$$