

Solutions to hw2 homework on Convex
Optimization
<https://web.stanford.edu/class/ee364b/homework.html>

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2.1 (8 points, 1 point per question)

Let f be a convex function with domain in R^n . We fix $x \in \mathbf{int\,dom\,f}$ and $d \in R^n$. Recall the definition of the directional derivative of f at x along the direction d

$$f'(x, d) = \lim_{t \rightarrow 0} \frac{f(x + td) - f(x)}{t}$$

In this question we aim to show that $f'(x, d)$ exists and is finite, and that we have the following relationship between $\partial f(x)$ and $f'(x, d)$,

$$f'(x, d) = \sup_{g \in \partial f(x)} g^T d$$

(a) Show that the ratio $\frac{f(x+td)-f(x)}{t}$ is a nondecreasing function of $t > 0$. Deduce that $f'(x, d)$ exists and is either finite or equal to $-\infty$. We know from the lectures that, since $x \in \mathbf{int\,dom\,f}$, the subdifferential set ∂f is non - empty, convex and compact.

Solution:

Proof of non - decreasing. Definition of subgradient is

$$f(z) \geq f(x) + g^T(z - x)$$

let $z = x + td$; then

$$f(x + td) \geq f(x) + g^T(x + td - x)$$

or

$$f(x + td) - f(x) \geq tg^T d$$

dividing both part of the inequality by t (as $t > 0$, we can do it) gives

$$\frac{f(x+td) - f(x)}{t} \geq g^T d$$

as the right - hand side of the equation is not depends of t , differentiating by t gives

$$\partial \frac{f(x+td) - f(x)}{t} \geq 0$$

As the $\frac{\partial f'(x,d)}{\partial t} \geq 0$, it means the function $f'(x,d)$ is nondecreasing by variable t .

Proof of possible equality to $-\infty$.

The definition of convexity:

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$$

where $0 < \theta < 1$.

let $t = 1 - \theta$, $0 < t < 1$. then

$$f((1 - t)x + ty) \leq (1 - t)f(x) + tf(y)$$

or

$$f(x + t(y - x)) \leq f(x) + t(f(y) - f(x))$$

as we can choose y any of the point in domain f , we can set $d = y - x$. Then

$$f(x + td) \leq f(x) + t(f(y) - f(x))$$

or

$$f(x + td) - f(x) \leq t(f(y) - f(x))$$

or

$$\frac{f(x + td) - f(x)}{t} \leq f(y) - f(x)$$

As $f(x)$ can be equal to ∞ on the domain of f , so $f'(x,d) = \frac{f(x+td) - f(x)}{t}$ can be less or equal than (for the infinity with sign minus it means strictly equal) $-\infty$ on the domain of f . This means that $f'(x,d)$ can be equal to $-\infty$ on domain of f .

(b) Let $g \in \partial f(x)$. Show that $f'(x,d) \geq g^T d$. Deduce that $f'(x,d)$ is finite and $f'(x,d) \geq \sup_{g \in \partial f(x)} g^T d$.

Solution:

We already shown that

$$f'(x,d) \geq g^T d$$

in part (a). We also shown in part (a) that

$$\frac{f(x + td) - f(x)}{t} \leq f(y) - f(x)$$

Second upper inequality means that $f'(x, d)$ is bounded from upper side (i.e it can't be equal to ∞), it means its value is finite.

As the first of upper inequalities is correct \forall subgradients in domain of f , it means, that it is correct for the supremum of these subgradients in domain f . It means that

$$f'(x, d) \geq \sup_{g \in \partial f(x)} g^T d.$$

In the remaining part of this question, we will establish the converse inequality $f'(x, d) \leq \sup_{g \in \partial f(x)} g^T d$, by showing the existence of a subgradient $g^* \in \partial f(x)$, such that $f'(x, d) \leq g^{*T} d$. We introduce two following sets

$$C_1 = \{(z, t) \mid z \in \mathbf{dom} f, f(z) < t\}$$

$$C_2 = \{(y, v) \mid y = x + \alpha d, v = f(x) + \alpha f'(x, d), \alpha \geq 0\}$$

(c) Prove that C_2 and C_2 are nonempty, convex and disjoint.

Solution:

C_1 epigraph of the convex function, therefore it is nonempty and convex.

C_2 is the nonempty set, because it have at least one point, which corresponds to $\alpha = 0, y = x, v = f(x)$. It is also a convex set, because $C_2^1 = \{y \mid y = x + \alpha d\}$ is a convex set as it is translated domain of f which is a convex set and $C_2^2 = \{v \mid v = f(x) + \alpha f'(x, d), \alpha \geq 0\}$ is either a straight line or a beam or a segment.

Proof of disjointedness:

We should show that there is exists a nonzero vector $(a, \beta) \in R^n \times R$ such as

$$a^T(x + \alpha d) + \beta(f(x) + \alpha f'(x, d)) \leq a^T z + \beta w$$

for all $\alpha \geq 0, z \in \mathbf{dom} f$, and $f(z) < w$.

Solution: As we shown earlier,

$$f'(x, d) \leq f(y) - f(x)$$

where $x, y \in \mathbf{dom} f$. As x, y can be any points in domain f , it follows that

$$f'(x, d) \leq \min_{z \in \mathbf{dom} f} (f(z)) - \max_{z \in \mathbf{dom} f} (f(z))$$

Lets just derive equation for β .

$$\beta(f(x) + \alpha f'(x, d) - w) \leq a^T(z - x - \alpha d)$$

or

$$\beta \leq \frac{a^T(z - x - \alpha d)}{f(x) + \alpha f'(x, d) - w}$$

(e) Prove that β must be strictly positive. Define $\tilde{a} = \frac{a}{\beta}$. Show that

$$\tilde{a}^T(x + \alpha d) + f(x) + \alpha f'(x, d) \leq \tilde{a}^T z + w$$

for all $\alpha \geq 0$, $z \in \mathbf{dom} f$ and $f(z) < w$.

Proof:

Let's choose $\tilde{a} = g$. Then we have:

$$g^T(x + \alpha d) + f(x) + \alpha f'(x, d) \leq g^T z + w$$

or

$$g^T(x - z) + f(x) - w + \alpha(g^T d + f'(x, d)) \leq 0$$

as $f(z) < w$ we can replace w with $f(z)$ without a violation of inequality. We get then:

$$g^T(x - z) + f(x) - f(z) + \alpha(g^T d + f'(x, d)) \leq 0$$

the first three terms in equation are

$$g^T(x - z) + f(x) - f(z)$$

and they are less or equal to zero, because of the subgradient definition. the last term of the equation is

$$\alpha(g^T d + f'(x, d))$$

or, as $\alpha \geq 0$, the rest is to prove that

$$g^T d + f'(x, d) \leq 0$$

This term also is less or equal to zero, because of the earlier derived inequalities:

$$f'(x, d) \leq \min_{z \in \mathbf{dom} f} (f(z)) - \max_{z \in \mathbf{dom} f} (f(z))$$

and

$$f'(x, d) \geq g^T d$$

so $g^T d \leq f'(x, d) \leq 0$, and assertion (e) is proved.