Solutions to hw1 homework on Convex Optimization
https://web.stanford.edu/class/ee364a/homework.html

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2.8

Which of the following sets S are polyhedra? If possible, express S in the form
$S = \{x | Ax \preceq b, Fx = g\}$
(a)
$S = \{y_1 a_1 + y_2 a_2 \,|\, -1 \leq y_1 \leq 1, -1 \leq y_2 \leq 1\}$ where $a_1, a_2 \in R^n$
Solution:
Yes, this is a polyhedron. Namely, this is a parallelogram with corners $-a_1 - a_2, a_1 - a_2, -a_1 + a_2, a_1 + a_2$
(b)
$S = \{x \in R^n \,|\, x \succeq 0, \, \mathbf{1}^T x = 1, \, \sum_{i=1}^{n} x_i a_i = b_1, \, \sum_{i=1}^{n} x_i a_i^2 = b_2\}$, where $a_1, ..., a_n \in R$ and $b_1, b_2 \in R$
Solution:
Yes, this is a polyhedron. It's defined by inequality and three equality constraints.
(c)
$S = \{x \in R^n \,|\, x \succeq 0, \, x^T y < 1, \, for\, all\, y\, with\, ||y||_2 = 1\}$
Solution:
No, it's not a polyhedron. It's something with a spherical shape.
(d)
$S = \{x \in R^n \,|\, x \succeq 0, \, x^T y < 1, \, for\, all\, y\, with\, \sum_{i=1}^{n} |y| = 1\}$
Solution:
First prove what $[x_k| < 1 \forall\, k \in \{1...n\}$:
1. Suppose what $x_i = 1/y_i$ for $i : y_i = max\{y_k\}$ and $x_k = 0 \,\forall\, k \neq i$. Then $x^T y = 1$, that is shouldn't be.
But if $[x_k| < 1$ then
$x^T y = \sum_{i=1}^{n} x_i y_i \leq \sum_{i=1}^{n} |x_i||y_i| \leq \sum_{i=1}^{n} |y_i| = 1$,
so the inequality $x^T y < 1$ holds for all $[x_k| < 1$.
So, yes, this is a polyhedron, equilateral rhombus for 2d.

2.13

Consider the set of rank-k outer products, defined as:
$\{XX^T : X \in R^{n\times k}, \boldsymbol{rank}\, X = k\}$.
Describe its conic hull in simple terms.
Solution:
As XX^T is a positive semi-definite matrix of rank k, we have a conic combination of a positive semi-definite matrices of rank k and dimension $k \times k$. The conic combination of such matrices cannot have rang less than k, because this is a linear combination of full - rank matrices consisting of linearly independent vectors. So, conic combination of positive semi-definite matrices of rank k is also a positive semi-definite matrix of rank of k.

2.22

Finish the proof of the separating hyperplane theorem in §2.5.1: Show that a separating hyperplane exists for two disjoint convex sets C and D. You can use the result proved in §2.5.1, i.e., that a separating hyperplane exists when there exist points in the two sets whose distance is equal to the distance between the two sets.
Hint. If C and D are disjoint convex sets, then the set $\{x - y, \ | \, x \in C, y \in D\}$ is convex and does not contain the origin.
Solution:
Prove first what the set $S = \{x - y, \ | \, x \in C, y \in D\}$ is convex:
As the sets C and D is disjoint, $0 \notin S$.There are two cases:
First case:
$0 \notin \boldsymbol{cl}\, S$ Then the result proved in the §2.5.1 is applied to sets 0 and S, i. e. exists a matrix $a \neq 0$ such what $a^T(x - y) > 0 \, \forall x \in C, \forall y \in D$, i. e. $a^T x > a^T y \ \forall x \in C, \forall y \in D$
Second case:
Assume $0 \in \boldsymbol{cl}\, S$. As $0 \notin S$, the zero point should be on the boundary of S.
If S has empty inferior, it contained in a hyperplane $\{z \,|\, a^T z = 0\}$, in other words, $a^T x = a^T y \ \forall x \in C, \forall y \in D$, and the separating hyperplane is trivial.
Now if S has nonempty inferior, consider the set $S_{-\epsilon} = \{z \,|\, B(z, \epsilon) \subseteq S\}$ where $B(z, \epsilon)$ is the Ecludian ball with radius $\epsilon > 0$ and the center in z. $S_{-\epsilon}$ is a subset of S, it is closed and convex, and does not contain 0. So, by result in §2.5.1 it is separated from 0 by at least one separating hyperplane with normal vector $a(\epsilon)$: $a(\epsilon)^T z > 0 \, \forall \, z \in S(-\epsilon)$. We can assume what $||a_\epsilon||_2 = 1$. Now let $\epsilon_k, k = 1, 2, ...$ be a sequence of positive values with $\lim_{x\to\infty} \epsilon_k = 0$. As $||a_{\epsilon_k}||_2 = 1$, the sequence $a(\epsilon_k)$ contains a convergent subsequence with limit $\overline{a}$. So we have $a(\epsilon_k)^T z > 0$ for all $z \in S_{-\epsilon_k}$ for all k, so $\overline{a}_{\epsilon_k} z > 0$ for all $z \in \boldsymbol{int}\, S$. As $z = x - y$, it means what $\overline{a}x > \overline{a}y$ for all $x \in C$ for all $y \in D$.

A1.5

Dual and intersection of cones. Let C and D be closed convex cones. In this problem we will show what
$(C \cap D)^* = C^* + D^*$.
Here + denotes set addition: $C^* + D^*$ is the set $\{u + v \mid u \in C^*, v \in D^*\}$
In other words, the dual of the intersection of two closed convex cones is the sum of the dual cones.

(a) Show what $(C \cap D)^*$ and $C^* + D^*$ are convex cones. (In fact, these are closed, but we won't ask you to show this.)
Solution:
Let $x \in (C \cap D)$. It means what $x \in C$ and $x \in D$. It implies $\theta x \in C$ and $\theta x \in D$ for any $\theta \geq 0$. Therefore $\theta x \in D \cap C$ for any $\theta \geq 0$. This implies what the $D \cap C$ is a cone. As intersection of a convex sets is convex, it implies what the $D \cap C$ is convex also. So, we proved what intersection of two convex cones is a convex cone also.
As C^* and D^* are closed convex cones, then $C^* + D^*$ is a conic hull of $C^* \cup D^*$ and therefore is a convex cone.

(b) Show what $(C \cap D)^* \supseteq C^* + D^*$
Solution:
Let $x \in C^* + D^*$. We can write $x = u + v$ where $u \in C^*$ and $v \in D^*$. Then, by definition of dual cone, $u^T y \geq 0$ for all $y \in C$ and $v^T y \geq 0$ for all $y \in D$, it means $x^T y = u^T y + v^T y \geq 0$ for all $y \in C \cap D$. It shows what x is in the dual cone of $C \cap D$, i.e $x \in C^* \cap D^*$, and so $(C \cap D)^* \supseteq C^* + D^*$.

(c) Show what $(C \cap D)^* \subseteq C^* + D^*$
Solution:
We showed in (a) what $C \cap D$ and $C^* + D^*$ are closed convex cones. Therefore $(C \cap D)^{**} = (C \cap D)$ and $C^* + D^* = (C^* + D^*)^{**}$.
It means
$(C \cap D)^* \subseteq C^* + D^* \Longleftrightarrow C \cap D \supseteq (C^* + D^*)^*$
Suppose $x \in (C^* + D^*)^*$. $x^T y \geq 0$ for all $y = u + v$, $u \in C^*$, $v \in D^*$. It can be written as $x^T u + v^T v \geq 0$ for all $u \in C^*$, $v \in D^*$. As $0 \in C^*$ and $0 \in D^*$, taking $v = 0$ we get $x^T u \geq 0$, taking $u = 0$ we get $x^T v \geq 0$ This implies $x \in C^{**} = C$ and $x \in D^{**} = D$, i.e $x \in C \cap D$. We have shown what $(C \cap D)^* \subseteq C^* + D^*$ and $(C \cap D)^* \supseteq C^* + D^*$. It means $(C \cap D)^* = C^* + D^*$.

(d) Show that the dual of the polyhedral cone $V = \{x | Ax \geq 0\}$ can be expressed as $V^* = \{A^T v | v \geq 0\}$.
Solution:
Using the previous result we can write:
$V^* = \{x | a_1^T x \geq 0\} + \{x | a_2^T x \geq 0\} + ... + \{x | a_m^T x \geq 0\}$.
The dual of $\{x | a_i^T x \geq 0\}$ is $\{\theta a_i | \theta \geq 0\}$, so we get

$V^* = \{\theta a_1 | \theta \geq 0\} + ... + \{\theta a_m | \theta \geq 0\} = \{\theta_1 a_1 \geq 0 + ... + \theta_m a_m \succeq 0 | \theta_{1...m} \geq 0\}$, which can be written as $V^* = \{A^T v | v \geq 0\}$.

A1.9

Correlation matrices. Determine if the following subsets of S^n are convex: ...
Solution:
See example 2.15 in §2.4.1. The positive semidefinite cone S^n is a proper cone in S^n, therefore it is convex, besides all.

Alternatively, let $A, B \in S^n$. Then for $0 \leq \theta \leq 1$,
$x^T(\theta A + (1-\theta)B)x = \theta x^T Ax + (1-\theta)x^T Bx$.
As $A, B \in S^n$, $x^T Ax \geq 0 \ \forall x$, and $x^T Bx \geq 0 \ \forall x$.
Therefore $x^T(\theta A + (1-\theta)B)x = \theta x^T Ax + (1-\theta)x^T Bx \geq 0 \ \forall x$ as a sum of two nonegative numbers. It means what set of positive semidefinite matrices S^n is a convex one.

As the set of a correlation matrices (and a covariance matrices) is a subset of S^n, it implies what set of a correlation (or covariance) matrices is convex.