Solutions to hw3 homework on Convex Optimization https://web.stanford.edu/class/ee364a/homework.html

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3.15

A family of concave utility functions. For $0 < a \leq 1$ let

$$u_a(x) = \frac{x^a - 1}{a}$$

with $dom\, u_a = R_{++}$. We also define

$$u_0(x) = log(x)$$

(with $dom\, u_0 = R_{++}$).

(a) Show that for $x > 0$,

$$u_0(x) = \lim_{a \to 0} u_a(x)$$

Solution:
Using Taylor series for small values of a:

$$x^a = e^{a\, log(x)} \approx 1 + a\, log(x)$$

So, for small a

$$u_a(x) = \frac{x^a - 1}{a} \approx \frac{1 + a\, log(x) - 1}{a} = log(x)$$

(b) Show that u_a are concave, monotone increasing, and all satisfy $u_a(1) = 0$.

Solution:
Second derivative of u_a is

$$u_a''(x) = (a-1)x^{a-2} < 0\,\forall x \in dom\, u_a$$

i. e. u_a is concave as its second derivative is negative on all of its domain.

$$u_a'(x) = x^{a-1} > 0\,\forall x \in dom\, u_a$$

i. e. u_a is monotone increasing.

$u_a(1) = 0$: for large values of a it is evident from the definition of u_a, for small values of a it is evident from the definition of u_0.

3.25

Maximum probability distance between distributions. Let $p, q \in R$ represent two probability distributions on $\{1, ..., n\}$ (so $p, q \succeq 0$, $\mathbf{1}^T p = 1$, $\mathbf{1}^T q = 1$). We define the maximum probability distance $d_{mp}(p, q)$ between p and q as the maximum difference in probability assigned by p and q, over all events:

$$d_{mp}(p, q) = max\{|prob(p, C) - prob(q, C)|\,|\, C \subseteq \{1, ...n\}\}$$

Here $prob(p, C)$ is the probability of C under the distribution of p:
$prob(p, C) = \sum_{i \in C} p_i$.
Find a simple expression for $d_{mp}(p, q)$, involving $||p - q||_1 = \sum_{i=1}^{n} |p_i - q_i|$ and show that $d_{mp}(p, q)$, is a convex function on $R^n \times R^n$ (Its domain is $\{p, q \succeq 0,\, \mathbf{1}^T p = 1,\, \mathbf{1}^T q = 1\}$, but it has a natural extension to all of $R^n \times R^n$.
Solution:

I can't find such dependency. The Python code:

```
# -*- coding: utf-8 -*-
import numpy as np


N = 3

p = np.random.rand(N)
```

```
q = np.random.rand(N)

p = np.abs(p)
q = np.abs(q)

p = p / p.sum()
q = q / q.sum()

print('p = ', p)
print('q = ', q)

P_diff = np.zeros((N, N), dtype=np.double)

for i in range(N):
for k in range(N):
P_diff[i, k] = np.abs(p[i] - q[k])

print('P_diff = ', P_diff)
d_mp = np.max(P_diff)

print('d_mp = ', d_mp)

diff_norm = np.sum(np.abs(p - q))

print('diff_norm = %f'% diff_norm)

print('diff_norm / d_mp = ', diff_norm / d_mp)
```

for $N \leq 3$

`diff_norm / d_mp`

is 2, but for $N > 3$

`diff_norm / d_mp`

is a real number greater than 0.

3.55

Log-concavity of the cumulative distribution function of a log-concave probability density. In this problem we extend the result of exercise 3.54. Let $g(t) = exp(-h(t))$ be a differentiable log-concave probability density function, and let

$$f(x) = \int_{-\infty}^{x} g(t)\,dt = \int_{-\infty}^{x} e^{-h(t)}\,dt$$

(a) Express the derivatives of f in terms of the function h. Verify that $f(x)f''(x) < f'^2(x)$ if $h^{'}(x) > 0$

Solution:

The condition that $log(g(t))$ is concave means that $-h(x)$ is concave and $h(t)$ is convex, which means $h^{''} \geq 0$. So, we have:

$$f^{'}(x) = e^{-h(x)}$$

$$f^{''}(x) = -h^{'}(x)e^{-h(x)}$$

So,

$$f(x)f''(x) = -h^{'}(x)e^{-2h(x)}$$

and

$$f'^2(x) = e^{-2h(x)}$$

As $e^{-2h(x)} > 0$, and $h(x) \geq 0$ we have:

$$f(x)f''(x) = -h^{'}(x)e^{-2h(x)} < e^{-2h(x)} = f'^2(x)$$

(b) Assume that $h'(x) < 0$. Use the inequality

$$h(t) \geq h(x) + h^{'}(x)(t - x)$$

(which follows from convexity of h) to show that

$$\int_{-\infty}^{x} e^{-h(t)}\, dt \leq -\frac{e^{-h(x)}}{h^{'}(x)}$$

Solution:

As $h(t) \geq h(x) + h^{'}(x)(t - x)$ we have:

$$\int_{-\infty}^{x} e^{-h(t)}\, dt \leq \int_{-\infty}^{x} e^{-(h(x)+h^{'}(x)(t-x))}\, dt = e^{-h(x)+h'(x)x} \int_{-\infty}^{x} e^{-h'(x)t}\, dt = -\frac{e^{-h(x)}}{h'(x)}$$

A2.21

Symmetric convex functions of eigenvalues. A function $f : R^n \to R$ is said to be symmetric if it is invariant with respect to a permutation of its arguments, i.e., $f(x) = f(Px)$ for any permutation matrix P. An example of a symmetric function is $f(x) = log(\sum_{k=1}^{n} x_k)$.
In this problem we show that if $f : R^n \to R$ is convex and symmetric, then the function $g : S^n \to R$ defined as $g(X) = f(\lambda(X))$ is convex, where $\lambda(X) =$

$(\lambda_1(X), ..., \lambda_n(X))$ is the vector of eigenvalues of X. This implies, for example, that the function

$$g(X) = log(tr(e^X)) = log(\sum_{k=1}^{n} e^{\lambda_k(X)})$$

is convex on S^n.

(a) A square matrix S is doubly stochastic if its elements are nonnegative and all row sums and column sums are equal to one. It can be shown that every doubly stochastic matrix is a convex combination of permutation matrices. Show that if f is convex and symmetric and S is doubly stochastic, then

$$f(Sx) \leq f(x)$$

Solution:

As every doubly stochastic matrix is a convex combination of permutation matrices then

$$Sx = \sum_{k=1}^{n} \theta_k P_k x,$$

where $\sum_{k=1}^{n} \theta_k = 1$ and $\theta_k \geq 0$.
From convexity and symmetry of $f(X)$ we have:

$$f(Sx) = f(\sum_{k=1}^{n} \theta_k P_k x) \leq \sum_{k=1}^{n} \theta_k f(P_k x)$$

It means what $f(Sx)$ is convex by the definition of convexity.

(b) Let $Y = Q\, diag(\lambda)\, Q^T$ be an eigenvalue decomposition of $Y \in S^n$ with Q orthogonal. Show that $n \times n$ matrix S with elements $S_{ij} = Q_{ij}^2$ is doubly stochastic and that $diag(Y) = S\lambda$

Solution:
As Q is orthogonal it means $QQ^T = \boldsymbol{I}$, i. e. $\sum_{k=1}^{n} q_{ik}^2 = 1$. Also, from $Q^T Q = \boldsymbol{I}$ we have $\sum_{k=1}^{n} q_{ki}^2 = 1$. So, matrix with elements S_{ij} is doubly stochastic.
As $Y = Q\, diag(\lambda)\, Q^T$, then $Y_{ii} = \lambda_i \sum_{k=1}^{n} q_{ik} q_{ki} = \lambda_i S_{ii}$, so, $diag(Y) = S\lambda$.

(c) Use the results in parts (a) and (b) to show that if f is convex and symmetric and $X \in S^n$ then

$$f(\lambda(X)) = \sup_{V \in \mathcal{V}} f(diag(V^T X V))$$

Solution:

From (a) and (b) we have that for any symmetric X

$$f(diag(X)) \leq f(\lambda(X))$$

But if V is orthogonal, then $\lambda(X) = \lambda(VXV^T)$. It means that

$$f(diag(VXV^T)) \leq f(\lambda(X))$$

for all orthogonal V with equality if $V = Q$. It means that

$$f(\lambda(X)) = \sup_{V \in \mathcal{V}} f(diag(V^T XV))$$

A 12.1 a-c

FIR low-pass filter design. Consider the (symmetric, linear phase) finite impulse response (FIR) filter described by its frequency response

$$H(\omega) = a_0 + \sum_{k=1}^{N} a_k cos(k\omega)$$

where $\omega \in [0, \pi]$ is the frequency. The design variables in our problems are the real coefficients $a = \{a_0, ..., a_n\} \in R^{N+1}$, where N is called the order or length of the FIR filter. In this problem we will explore the design of a low-pass filter, with specifications:

- For $0 \leq \omega \leq \pi/3$, $0.89 \leq H(\omega) \leq 1.12$ i.e., the filter has about ± 1 Db ripple in the 'passband' $[0, \pi/3]$.
- For $\omega_c \leq \omega \leq \pi$,, $|H_\omega| \leq \alpha$. In other words, the filter achieves an attenuation given by α in the 'stopband' $[\omega_c, \pi]$. Here ω_c is called the filter 'cutoff frequency'.

(It is called a low-pass filter since low frequencies are allowed to pass, but frequencies above the cutoff frequency are attenuated.) These specifications are depicted graphically in the figure below.
For parts (a) - (c), explain how to formulate the given problem as a convex or quasiconvex optimization problem.

(a) Maximum stopband attenuation. We fix ω_c and N, and wish to maximize the stopband attenuation, i.e., minimize α.

Solution:
minimize α
subject to:

$f_1(\alpha) \leq 1.12$
$f_2(\alpha) \geq 0.89$
$f_3(\alpha) \leq \alpha$
$f_4(\alpha) \geq \alpha$
where:
$f_1(\alpha) = \sup_{0 \leq \omega \leq \pi/3} H(\omega)$
$f_2(\alpha) = \inf_{0 \leq \omega \leq \pi/3} H(\omega)$
$f_3(\alpha) = \sup_{\omega_c \leq \omega \leq \pi} H(\omega)$
$f_4(\alpha) = \inf_{\omega_c \leq \omega \leq \pi} H(\omega)$

The functions f_1, f_3 are convex and f_2, f_4 are concave. So, the problem is convex.

(b) Minimum transition band. We fix N and α and want to minimize ω_c i.e. we set the stopband attenuation and filter length, and wish to minimize the 'transition' band (between $pi/3$ and ω_c)

Solution:
This problem can be expressed as:
minimize $f_5(\alpha)$ subject to:
$f_1(\alpha) \leq 1.12$
$f_2(\alpha) \geq 0.89$
$f_5(a) \leq \alpha$
Where $f_1(\alpha), f_2(\alpha)$ are the same, and
$f_5(a) = \inf\{\Omega : -\alpha \leq H(\omega) \leq \alpha \text{ for } \Omega \leq \omega \leq \pi\}$

This problem is quasiconvex because f_1 is convex, f_2 is concave and f_5 is quasiconvex, its sublevel sets are:
$\{a|\, f_5(a) \leq \Omega\} = \{a| - \alpha \leq H(\omega) \leq \alpha \text{ for } \Omega \leq \omega \leq \pi\}$, i.e., the intersection of an infinite number of halfspaces.

(c) Shortest length filter. We fix ω_c and N and wish to find the smallest N that can meet the specfications, i.e., we seek the shortest length FIR filter that can meet the specifiations.

Solution:
This problem can be expressed as:
minimize $f_6(\alpha)$ subject to:
$f_1(\alpha) \leq 1.12$
$f_2(\alpha) \geq 0.89$
$f_3(\alpha) \leq \alpha$
$f_4(\alpha) \geq \alpha$
Where $f_1(\alpha), f_2(\alpha), f_3(\alpha), f_4(\alpha)$ are the same, and
$f_6(a) = min\{k|\, a_{k+1} =, ..., = a_N = 0\}$
the sublevel sets of $f_6(a)$ are affine sets,
$\{a|\, f_6(a) \leq k\} = \{a|\, a_{k+1} =, ..., = a_N = 0\}$

It means that $f_6(a)$ is affine function and the problem is quasiconvex.

A 16.1

A hypergraph with nodes $1, ..., m$ is a set of nonempty subsets of $\{1, 2, ..., m\}$ called edges. An ordinary graph is a special case in which the edges contain no more than two nodes. We consider a hypergraph with m nodes and assume coordinate vectors $x_j \in R^p$, $j = 1, ..., m$ are associated with the nodes. Some nodes are fixed and their coordinate vectors x_j are given. The other nodes are free, and their coordinate vectors will be the optimization variables in the problem. The objective is to place the free nodes in such a way that some measure of the physical size of the nets is small.

As an example application, we can think of the nodes as modules in an integrated circuit, placed at positions $x_j \in R^2$. Every edge is an interconnect network that carries a signal from one module to one or more other modules.

To define a measure of the size of a net, we store the vectors x_j as columns of a matrix $X \in R^{p \times m}$. For each edge S in the hypergraph, we use X_S to denote the $p \times |S|$ submatrix of X with the columns associated with the nodes of S. We define

$$f_S(X) = inf_y ||X_S - y\mathbf{1}^T|| \qquad (50)$$

as the size of the edge S, where $|| \cdot ||$ is a matrix norm, and $\mathbf{1}$ is a vector of ones of length S.

(a) Show that the optimization problem

$$minimize \sum_{edges\, S} f_S(X)$$

is convex in the free node coordinates x_j.

Solution:

Any norm is a convex function (matrix norm also is). Let's show that $inf_y||X_S - y\mathbf{1}^T||$ is a convex function of X.

Proof:

We should proof the [stability under partial minimization] first:
If $f(x, y) : R^m_x \times R^m_y$ is convex (as a function $z = f(x, y)$; this is called joint convexity) and the function $g(x) = \inf_y f(x, y)$ is proper, i.e. is $> -\infty$ everywhere and is finite at least at one point, then g is convex.

Proof:
`https://ljk.imag.fr/membres/Anatoli.Iouditski/cours/convex/chapitre_3.pdf`
p. 60
[*stability under partial minimization*]

We should prove that if $x, x' \in dom(g)$ and $x'' = \lambda x + (1-\lambda)x'$ with $\lambda \in [0,1]$, then $x'' \in dom(g)$ and $g(x'') \leq \lambda g(x) + (1-\lambda)g(x')$. Given positive ϵ we can find y, y' such that $(x,y) \in dom(f), (x',y') \in dom(f)$ and $g(x)+\epsilon > f(x,y)$, $g(x')+\epsilon > f(x',y')$ Taking weighted sum of these two inequalities, we get

$$\lambda g(x) + (1-\lambda)g(x') + \epsilon \leq \lambda f(x,y) + (1-\lambda)f(x',y') \geq$$

(since f is convex)

$$f(\lambda x + (1-\lambda x', \lambda y + (1-\lambda)y') = f(x'', \lambda y + (1-\lambda)y')$$

As $g(x) = \inf_y f(x,y)$, the concluding quantity in the chain $f(x'', \lambda y + (1-\lambda)y') \geq g(x'')$ and we get $g(x'') \leq \lambda g(x) + (1-\lambda)g(x') + \epsilon$. In particular $x'' \in dom(g)$. Moreover, since the resulting inequality is valid for all $\epsilon > 0$, we come to $g(x'') \leq \lambda g(x) + (1-\lambda)g(x')$
It means that $\inf_y f(x,y)$ is a convex function of x. A sum of convex functions is also a convex function. So, $inf_y ||X_S - y\mathbf{1}^T||$ is a convex function of X.

(b) The size $f_S(X)$ of a net S obviously depends on the norm used in the definition (50). We consider five norms.

- Frobenius norm:

$$||X_S - y\mathbf{1}^T||_F = (\sum_{j\in S}\sum_{i=1}^{p}(x_{ij}-y_i)^2)^{1/2}$$

- Maximum Euclidean column norm:

$$||X_S - y\mathbf{1}^T||_{2,1} = \max_{j\in S}(\sum_{i=1}^{p}(x_{ij}-y_i)^2)^{1/2}$$

- Maximum column sum norm:

$$||X_S - y\mathbf{1}^T||_{1,1} = \max_{j\in S}\sum_{i=1}^{p}|x_{ij}-y_i|$$

- Sum of absolute values norm:

$$||X_S - y\mathbf{1}^T||_{sav} = \sum_{j\in S}\sum_{i=1}^{p}|x_{ij}-y_i|$$

- Sum-row-max norm:

$$||X_S - y\mathbf{1}^T||_{srm} = \sum_{i=1}^{p}\max_{j\in S}|x_{ij}-y_i|$$

For which of these norms does f_S have the following interpretations?

(i) $f_S(X)$ is the radius of the smallest Euclidean ball that contains the nodes of S.

Solution: Item 2.

(ii) $f_S(X)$ is (proportional to) the perimeter of the smallest rectangle that contains the nodes of S:

$$f_S(X) = \frac{1}{4}\sum_{i=1}^{p}(\max_{j\in S} x_{ij} - \min_{j\in S} x_{ij})$$

Solution: Item 3.

(iii) $f_S(X)$ is the squareroot of the sum of the squares of the Euclidean distances to the mean of the coordinates of the nodes in S:

$$f_S(X) = (\sum_{j\in S} ||x_{ij} - \overline{x}||_2^2)^{1/2} \quad where \quad \overline{x}_i = \frac{1}{S}\sum_{k\in S} x_{ik}$$

Solution: Item 1.

(iv) $f_S(X)$ is the sum of the ‘1-distances to the (coordinate-wise) median of the coordinates of the nodes in S:

$$f_S(X) = \sum_{j\in S} ||x_{ij} - \hat{x}_i|| \quad where \quad \hat{x}_i = median(\{x_i k \,|\, k \in S\}),\, i = 1, ..., p$$

Solution: Item 4.