Solutions to hw1 homework on Convex Optimization https://web.stanford.edu/class/ee364b/homework.html

Andrei Keino

December 8, 2020

1.1 (3 points)

For each of the following convex functions, determine the subdifferential set at the specified point.

(a) $f(x_1, x_2, x_3) = max(|x_1|, |x_2|, |x_3|)$ at $(x_1, x_2, x_3) = (0, 0, 0)$.

(b) $f(x) = e^{|x|}$ (x is scalar)

(c) $f(x_1, x_2) = max(x_1 + x_2 - 1, x_1 - x_2 + 1)$ at $(x_1, x_2) = (1, 1)$.
Solution:
(a) There will be a gap in differential at the points $\{x_1 = \pm x_2, x_2 = \pm x_3, x_1 = \pm x_3\}$. Subdifferential set $g(0, 0, 0) = \{[-1, 1], [-1, 1], [-1, 1]\}$.

(b) There will be a gap in differential at the point $x = 0$. Subdifferential set $g(0) = [-e^0, e^0] = [-1, 1]$.

(c) There will be a gap in differential at the points $\{x_1 + x_2 - 1 = x_1 - x_2 + 1\}$. Subdifferential set $g(1, 1) = \{1, [-1, 1]\}$.

1.3 (2 points)

Convex functions that are not subdifferentiable. Verify that the following functions, defined on the interval $[0; 1)$, are convex, but not subdifferentiable at $x = 0$. (Hint: You can prove by contradiction.)

(a) $f(0) = 1$ and $f(x) = 0$ for $x > 0$.

(b) $f(x) = -x^p$ for some $p \in (0, 1)$

Solution.

(a) Proof by contradiction. Suppose what function $f(0) = 1$ and $f(x) = 0$ for $x > 0$. has a supporting hyperplane at point $x = 0$, and g is the subgradient of $f(x)$ in this point. Then at $x \geq 0$ the equation $f(x) \geq f(0) + gx$ must hold. For $x > 0$ this equation become $0 \geq 1 + gx$ or $gx \leq -1$ for $x \geq 0$. This is impossible, because at $x = 0$ we must have $0 \leq -1$ then.

(b) Proof by contradiction. Suppose what function $f(x) = -x^p$ for some $p \in (0, 1)$ has a supporting hyperplane at point $x = 0$, and g is the subgradient of $f(x)$ in this point. Then $\forall\ x \geq 0$ the equation $f(x) \geq f(0) + gx$ must hold. But this is impossible, as f(0) is ∞ (i.e. unlimited) and $f(x)$ has a limited value, i.e g should be unlimited in this case.

1.2 (7 points)

For each of the following convex functions, explain how to calculate a subgradient at a given x.

(a) $f(x) = max_{i=1,\ldots,m}(a_i^T x + b_i)$.

(b) $f(x) = max_{i=1,\ldots,m}(|a_i^T x + b_i|)$.

(c) $f(x) = max_{i=1,\ldots,m}(-log(a_i^T x+b_i)$. . You may assume x is in the domain of f.

(d) $f(x) = max_{0\leq t\leq 1}(p(t))$ where $p(t) = x_1 + x_2 t + \cdots + x_n t^{n-1}$.

(e) $f(x) = x_{[1]} + \cdots + x_{[k]}$ where $x_{[i]}$ denotes the $i-$ th largest element of x.

(f) $f(x) = min_{Ay\preceq b}(||x^2 - y^2||)$, , i.e., the square of the distance of x to the polyhedron defined by $Ay \preceq b$. You may assume that the inequalities $Ay \preceq b$. are strictly feasible. (Hint: You may use duality, and then use subgradient the rule for pointwise maximumum.

(g) $f(x) = max_{Ay\preceq b}(y^t x)$, x, i.e., the optimal value of an LP as a function of the cost vector. (You can assume that the polyhedron defined $Ay \preceq b$ is bounded.) (Hint: You may use the subgradient rule for pointwise maximum.

Solution.

(a) Find $k \in 1, \ldots, m$ such that $f(x) = a_k^T x + b_k$. Then subgradient at this point is a_k.

(b) Find $k \in 1, \ldots, m$ such that $f(x) = |a_k^T x + b_k|$. If $a_k^T x + b_k > 0$ then subgradient is a_k, if $a_k^T x + b_k < 0$ then subgradient is $-a_k$, if $a_k^T x + b_k = 0$ then subgradient is $[-|a_k|, |a_k|]$.

(c) Find $k \in 1, \ldots, m$ such that $f(x) = -log(a_k^T x + b_k)$. Then subgradient is $-1/(a_k^T x + b_k)$.

(d) Find t such that $f(x) = x_1 + x_2 t + \cdots + x_n t^{n-1}$. Then subgradient is $(1, t, \ldots, t^{n-1})$.

(e) Find $\{i_1, \ldots, i_k\}$ such that $f(x) = x_{i_1} + \cdots + x_{i_k}$. Then subgradient is $(a_1, \ldots, a_n)$ where $a_j = 1$ if $j \in \{i_1, \ldots, i_k\}$, $a_j = 0$ otherwise.

(f) $f(x)$ defined as optimal value of the problem

$$
\begin{array}{ll}
\text{minimize} & ||x - y||^2 \\
\text{subject to} & Ay \preceq b
\end{array}
$$

With variable y. We have dual problem because Slater's condition holds.

$$
\begin{array}{ll}
\text{minimize} & -1/4 z^T A A^T z + 1/2 z^T A x - b^T z \\
\text{subject to} & z \succeq 0
\end{array}
$$

From the global perturbation inequalities

$$f(x, b) \geq f(x^*, b) - z^{*T}(x - x^*)$$

By Slater's condition, we have strong duality and the dual optimum is attained.

Let $z*$ be the optimal dual solution for for the value of x at which we want a subgradient, i.e., $z \succeq 0$ and

$$f(x) = -1/4 z^{*T} A A^T z + 1/2 z^{*T} A x - b^T z^*$$

By weak duality we have for any $\hat{x}$,

$$f(\hat{x}) \geq -1/4 z^{*T} A A^T z + 1/2 z^{*T} A \hat{x} - b^T z^* = f(x) + A^T z^* (\hat{x} - x)$$

The KKT conditions for $y*$ to be the optimal point of the primal problem gives:

$$A^T z* = 2(x - y*)$$

Therefore $(x - y^*)$ is a subgradient at x.

(g) The set $\{y \mid Ay \preceq b\}$ is closed and bounded, i.e. compact.This means that the supremum in the definition of $f(x)$ is attained. Let $\hat{y} \in \{y \; Ay \succeq b\}$ be the value of y for which $f(x) = \hat{y}^t x$. Then $\hat{y}$ is a subgradient of f at x.