

Solutions to hw2 homework on Convex  
Optimization  
<https://web.stanford.edu/class/ee364b/homework.html>

Andrei Keino

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## 2.1 (8 points, 1 point per question)

Let  $f$  be a convex function with domain in  $R^n$ . We fix  $x \in \mathbf{int\,dom\,f}$  and  $d \in R^n$ . Recall the definition of the directional derivative of  $f$  at  $x$  along the direction  $d$

$$f'(x, d) = \lim_{t \rightarrow 0} \frac{f(x + td) - f(x)}{t}$$

In this question we aim to show that  $f'(x, d)$  exists and is finite, and that we have the following relationship between  $\partial f(x)$  and  $f'(x, d)$ ,

$$f'(x, d) = \sup_{g \in \partial f(x)} g^T d$$

(a) Show that the ratio  $\frac{f(x+td)-f(x)}{t}$  is a nondecreasing function of  $t > 0$ . Deduce that  $f'(x, d)$  exists and is either finite or equal to  $-\infty$ . We know from the lectures that, since  $x \in \mathbf{int\,dom\,f}$ , the subdifferential set  $\partial f$  is non - empty, convex and compact.

Solution:

**Proof of non - decreasing.** Definition of subgradient is

$$f(z) \geq f(x) + g^T(z - x)$$

let  $z = x + td$ ; then

$$f(x + td) \geq f(x) + g^T(x + td - x)$$

or

$$f(x + td) - f(x) \geq tg^T d$$

dividing both part of the inequality by  $t$  (as  $t > 0$ , we can do it) gives

$$\frac{f(x+td) - f(x)}{t} \geq g^T d$$

as the right - hand side of the equation is not depends of  $t$ , differentiating by  $t$  gives

$$\partial \frac{f(x+td) - f(x)}{t} \geq 0$$

**As the  $\frac{\partial f'(x,d)}{\partial t} \geq 0$ , it means the function  $f'(x,d)$  is nondecreasing by variable  $t$ .**

**Proof of possible equality to  $-\infty$ .**

The definition of convexity:

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$$

where  $0 < \theta < 1$ .

let  $t = 1 - \theta$ ,  $0 < t < 1$ . then

$$f((1 - t)x + ty) \leq (1 - t)f(x) + tf(y)$$

or

$$f(x + t(y - x)) \leq f(x) + t(f(y) - f(x))$$

as we can choose  $y$  any of the point in domain  $f$ , we can set  $d = y - x$ . Then

$$f(x + td) \leq f(x) + t(f(y) - f(x))$$

or

$$f(x + td) - f(x) \leq t(f(y) - f(x))$$

or

$$\frac{f(x + td) - f(x)}{t} \leq f(y) - f(x)$$

**As  $f(x)$  can be equal to  $\infty$  on the domain of  $f$ , so  $f'(x,d) = \frac{f(x+td) - f(x)}{t}$  can be less or equal than (for the infinity with sign minus it means strictly equal)  $-\infty$  on the domain of  $f$ .** This means that  $f'(x,d)$  can be equal to  $-\infty$  on domain of  $f$ .

(b) Let  $g \in \partial f(x)$ . Show that  $f'(x,d) \geq g^T d$ . Deduce that  $f'(x,d)$  is finite and  $f'(x,d) \geq \sup_{g \in \partial f(x)} g^T d$ .

Solution:

We already shown that

$$f'(x,d) \geq g^T d$$

in part (a). We also shown in part (a) that

$$\frac{f(x + td) - f(x)}{t} \leq f(y) - f(x)$$

Second upper inequality means that  $f'(x, d)$  is bounded from upper side (i.e it can't be equal to  $\infty$ ), it means its value is finite.

As the first of upper inequalities is correct  $\forall$  subgradients in domain of  $f$ , it means, that it is correct for the supremum of these subgradients in domain  $f$ . It means that

$$f'(x, d) \geq \sup_{g \in \partial f(x)} g^T d.$$

In the remaining part of this question, we will establish the converse inequality  $f'(x, d) \leq \sup_{g \in \partial f(x)} g^T d$ , by showing the existence of a subgradient  $g^* \in \partial f(x)$ , such that  $f'(x, d) \leq g^{*T} d$ . We introduce two following sets

$$C_1 = \{(z, t) \mid z \in \mathbf{dom} f, f(z) < t\}$$

$$C_2 = \{(y, v) \mid y = x + \alpha d, v = f(x) + \alpha f'(x, d), \alpha \geq 0\}$$

(c) Prove that  $C_2$  and  $C_2$  are nonempty, convex and disjoint.

Solution:

$C_1$  epigraph of the convex function, therefore it is nonempty and convex.

$C_2$  is the nonempty set, because it have at least one point, which corresponds to  $\alpha = 0, y = x, v = f(x)$ . It is also a convex set, because  $C_2^1 = \{y \mid y = x + \alpha d\}$  is a convex set as it is translated domain of  $f$  which is a convex set and  $C_2^2 = \{v \mid v = f(x) + \alpha f'(x, d), \alpha \geq 0\}$  is either a straight line or a beam or a segment.

Proof of disjointedness:

We should show that there is exists a nonzero vector  $(a, \beta) \in R^n \times R$  such as

$$a^T(x + \alpha d) + \beta(f(x) + \alpha f'(x, d)) \leq a^T z + \beta w$$

for all  $\alpha \geq 0, z \in \mathbf{dom} f$ , and  $f(z) < w$ .

Solution: As we shown earlier,

$$f'(x, d) \leq f(y) - f(x)$$

where  $x, y \in \mathbf{dom} f$ . As  $x, y$  can be any points in domain  $f$ , it follows that

$$f'(x, d) \leq \min_{z \in \mathbf{dom} f} (f(z)) - \max_{z \in \mathbf{dom} f} (f(z))$$

Lets just derive equation for  $\beta$ .

$$\beta(f(x) + \alpha f'(x, d) - w) \leq a^T(z - x - \alpha d)$$

or

$$\beta \leq \frac{a^T(z - x - \alpha d)}{f(x) + \alpha f'(x, d) - w}$$

I don't know how to solve items (e) - (g)

(h) Let  $A \in R^{m \times n}$ ,  $b \in R^m$ ,  $\lambda > 0$ , and fix a direction  $d \in R^n$ . Consider the function  $\frac{1}{2}\|Ax - b\|_2^2 + \lambda\|x\|_1$ . Compute  $f'(0, d)$ . Remark: you can either compute it directly by using the definition of the directional derivative, or, use the variational formula  $f'(0, d) = \sup_{g \in \partial f(0)} g^T d$ .

Solution:

$$\nabla\|x\|_1 = \text{sign}(x)$$

$$\nabla\|Ax - b\|_2^2 = \nabla((Ax - b)^T(Ax - b)) = 2(Ax - b)A^T$$

see <https://math.stackexchange.com/questions/606646/matrix-derivative-ax-btax-b> and <http://www.math.uwaterloo.ca/~hwolkowi/matrixcookbook.pdf>

So,

$$\nabla(\frac{1}{2}\|Ax - b\|_2^2 + \lambda\|x\|_1) = 2(Ax - b)A^T + \lambda\text{sign}(x)$$

Then

$$f'(0, d) = d^T((Ax - b)A^T + \lambda[-1, 1]_n)$$

where  $[-1, 1]_n$  is a vector in  $R^n$  with component values in range  $-1 \leq x_i \leq 1$ ,  $i \in 1, \dots, n$ .