Covariance, correlation, and autocorrelation

1 Covariance and correlation

In the previous course, we defined the expectation and variance of a random variable X as $\mu_x = \mathrm{E}(X) = \int x \cdot f(x) dx$ and $\sigma_x^2 = \mathrm{Var}(X) = \mathrm{E}[(X - \mu)^2]$, where f is the probability density function (PDF) of X. If we have two random variables, we can also define the **covariance** between them as

$$\sigma_{xy} = \mathrm{Cov}(X, Y) = \mathrm{E}[(X - \mu_x)(Y - \mu_y)]$$

where $\mu_y = \mathrm{E}(Y)$. Notice that $\mathrm{Var}(X) = \mathrm{Cov}(X, X)$, so that the variance of X is just the covariance of X with itself.

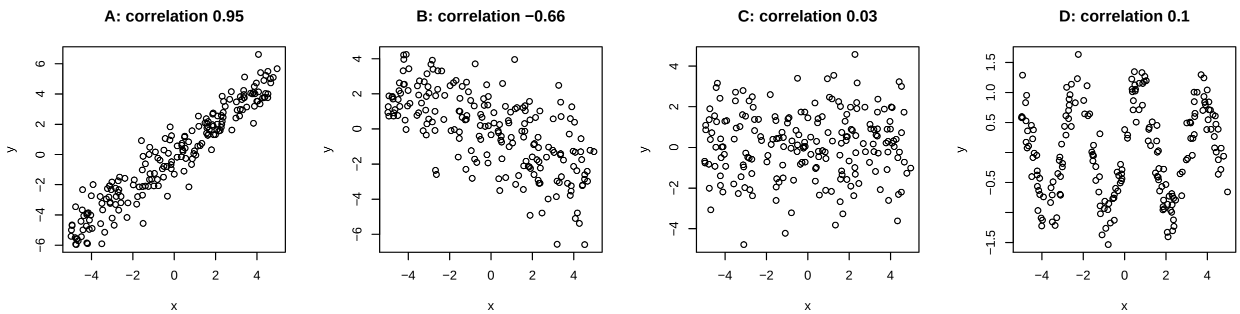
Correlation between X and Y is defined as

$$\rho_{xy} = \mathrm{Cor}(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X) \cdot \mathrm{Var}(Y)}} = \frac{\sigma_{xy}}{\sqrt{\sigma_x^2 \cdot \sigma_y^2}}.$$

Correlation is often more useful than covariance because it is standardized to be between -1 and 1, which makes it interpretable as a standard measure of *linear* relationship between any two variables. We emphasize linear because there can be dependence between two variables that is not linear, and those kinds of dependence would not necessarily result in strong correlation. Hence, think of correlation as measuring the strength of the linear relationship between variables.

Two independent random variables will have covariance (and correlation) equal to 0. However, the converse is not true. If X and Y have correlation at or near 0, they are not necessarily independent. We will see this in the example below.

Example: These four scatter plots demonstrate correlation in different scenarios. Plots A and B show strong linear relationships between the variables, so naturally their (estimated) correlations are high in magnitude. Plot C shows little relationship between the variables, and consequently the correlation is near 0. Despite the strong relationship between the variables in Plot D, the correlation is near 0. Correlation does not detect nonlinear relationships.



2 Autocorrelation

If we have a sequence of random variables $X_1, X_2, \ldots$ that are separated in time, as we did with the introduction to Markov chains, we can also think of the concept of **autocorrelation**, correlation of X_t with some past or future variable $X_{t-\ell}$. Formally, it is defined as

$$\mathrm{ACF}(X_t, X_{t-\ell}) = \frac{\mathrm{Cov}(X_t, X_{t-\ell})}{\sqrt{\mathrm{Var}(X_t) \cdot \mathrm{Var}(X_{t-\ell})}} \, .$$

If the sequence is stationary, so that the joint distribution of multiple Xs does not change with time shifts, then autocorrelation for two variables does not depend on the exact times t and $t - \ell$, but rather on the distance between them, ℓ. That is why the autocorrelation plots in the lesson on convergence of MCMC calculate autocorrelation in terms of lags.