Worksheet 2

Practical Lab Numerical Computing

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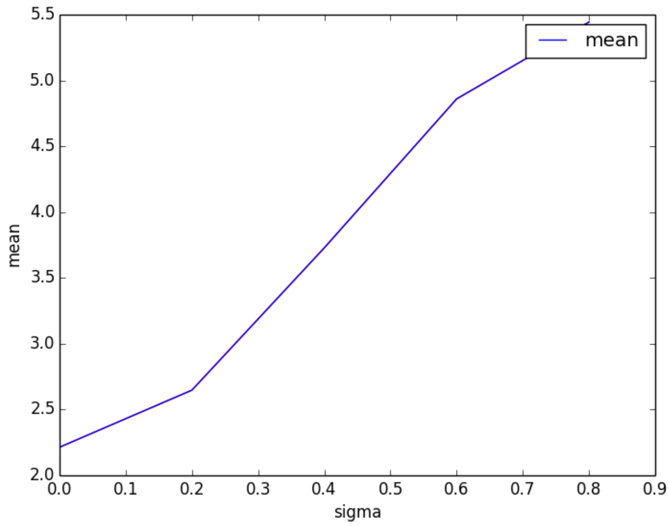
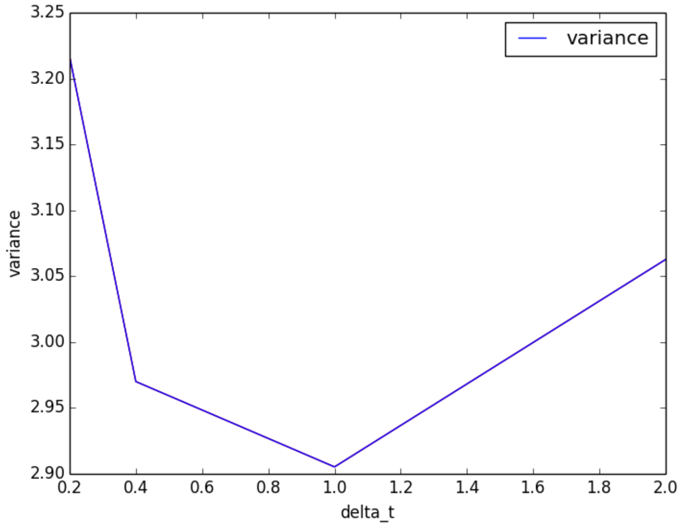
Task 1



Task 2



Task 3

To prove that

$$\mathbb{E}[V_{call}(S_T, 0)] = S(0)\exp(\mu T)\Phi(\sigma\sqrt{T} - \chi) - K\Phi(-\chi)$$

we use that by change-of-variable with $t := -t$ we get

$$\begin{aligned}
& \mathbb{E}[V_{call}(S_T, 0)] \\
= \; & \frac{1}{\sqrt{2\pi}} \int_{\chi}^{\infty} \left(S(0)\exp\left((\mu - \frac{1}{2}\sigma^2) \cdot T + \sigma\sqrt{T}s \right) - K \right) \exp\left(-\frac{s^2}{2} \right) ds \\
= \; & \frac{1}{2\pi} \int_{\chi}^{\infty} \left(S(0)\exp\left((\mu - \frac{1}{2}\sigma^2) \cdot T + \sigma\sqrt{T}s \right) \right) \exp\left(-\frac{s^2}{2} \right) ds - K\frac{1}{2\pi} \int_{\chi}^{\infty} \exp\left(-\frac{t^2}{2} \right) dt \\
= \; & \Psi - K\frac{1}{2\pi} \int_{-\infty}^{-\chi} \exp\left(-\frac{t^2}{2} \right) dt \\
= \; & \Psi - K\Phi(-\chi)
\end{aligned}$$

with

$$\Psi := \frac{1}{2\pi} \int_{\chi}^{\infty} \left(S(0)\exp\left((\mu - \frac{1}{2}\sigma^2) \cdot T + \sigma\sqrt{T}s \right) \right) \exp\left(-\frac{s^2}{2} \right) ds.$$

Now we prove that

$$\Psi = S(0)\exp(\mu T)\Phi(\sigma\sqrt{T} - \chi).$$

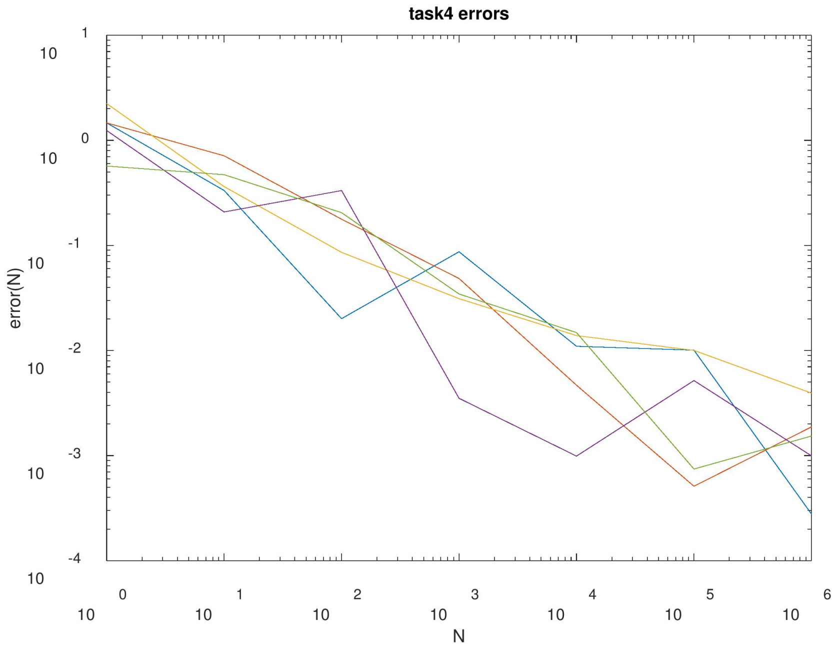
Again, we use a change-of-variable $z := t + \sigma\sqrt{T}$ and get

$$\begin{aligned}
& \frac{1}{\sqrt{2\pi}} \exp\left(\frac{\sigma^2}{2}T\right) \int\limits_{-\infty}^{\sigma\sqrt{T}-\chi} \exp\left(-\frac{t^2}{2}\right) dt \\
= & \frac{1}{\sqrt{2\pi}} \exp\left(\frac{\sigma^2}{2}T\right) \int\limits_{\chi-\sigma\sqrt{T}}^{\infty} \exp\left(-\frac{t^2}{2}\right) dt \\
= & \frac{1}{\sqrt{2\pi}} \exp\left(\frac{\sigma^2}{2}T\right) \int\limits_{\chi}^{\infty} \exp\left(-\frac{(\sigma\sqrt{T}-z)^2}{2}\right) dz \\
= & \frac{1}{\sqrt{2\pi}} \int\limits_{\chi}^{\infty} \exp\left(-\frac{z^2}{2} + z\sigma\sqrt{T}\right) dz.
\end{aligned}$$

We have

$$\begin{aligned}
\Psi &= \frac{1}{\sqrt{2\pi}} S(0) \exp\left((\mu - \frac{\sigma^2}{2}) \cdot T\right) \int_{\chi}^{\infty} \exp\left(-\frac{s^2}{2} + \sigma\sqrt{T}s\right) ds \\
&= S(0)\exp(\mu T) \exp\left(-\frac{\sigma^2}{2}T\right) \frac{1}{\sqrt{2\pi}} \int_{\chi}^{\infty} \exp\left(-\frac{s^2}{2} + \sigma\sqrt{T}s\right) ds \\
&= S(0)\exp(\mu T) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\sigma^2}{2}T\right) \exp\left(\frac{\sigma^2}{2}T\right) \int\limits_{-\infty}^{\sigma\sqrt{T}-\chi} \exp\left(-\frac{s^2}{2}\right) ds \\
&= S(0)\exp(\mu T)\Phi(\sigma\sqrt{T}-\chi).
\end{aligned}$$

Task 4



Task 5

We have $\Phi^{-1}(0) = -\infty$ and $\Phi^{-1}(1) = \infty$ where $\Phi^{-1} : (0,1) \to (-\infty, \infty)$ is the inverse cumulative distribution function. As an integral of a positive continious function, Φ is a bijection, continious and differentiable. That means Φ is a diffeomorphism. Φ^{-1} is also differentiable, so we can use the transformation theorem with the change-of-variable $t = \Phi(s)$. We have $|\det(D\Phi(s))| = \frac{1}{\sqrt{2\pi}} \exp(-\frac{s^2}{2})$ and

$$\begin{aligned}
& \int_0^1 f(\Phi^{-1}(t))dt \\
= \quad & \int_{\Phi^{-1}(0)}^{\Phi^{-1}(1)} f(\Phi^{-1}(\Phi(s))) \cdot |\det(D\Phi(s))| ds \\
= \quad & \int_{-\infty}^{\infty} f(s) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{s^2}{2}\right) ds \\
= \quad & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s) \exp\left(-\frac{s^2}{2}\right) ds
\end{aligned}$$

what proves formula (7).

Task 6

In case of the trapezoidal-rule, the set of nodes of level l is a subset of the nodes of level $l+1$. Furthermore, the 2^l additional values lay exactly half way in between the nodes of level l. (Except for the first and the last node, which lie half way in between 0 and the first node of level l and in between the last node of level l and the end of the interval.)

Task 7

In case of the Gauß-Legendre Quadrature, the nodes of level l are not a subset of the nodes of level $l+1$. At the edges of the interval, the nodes are denser. According to Satz 1.18 from Einführung in die Grundlagen der Numerik, node $x_i^{(N_l)}$ from level l lays between the nodes $x_i^{(N_{l+1})}, x_{i+1}^{(N_{l+1})}$ from level $l+1$.

The nodes are the roots of a three-term-recurrence relation of the form $p_{n+1}(t) = (t-\alpha_n)p_n(t) - \beta_n^2 p_{n-1}(t),\ n \geq 0$. The roots are the eigenvalues of a tridiagonal $(N_l \times N_l)$-dimensional matrix of the form

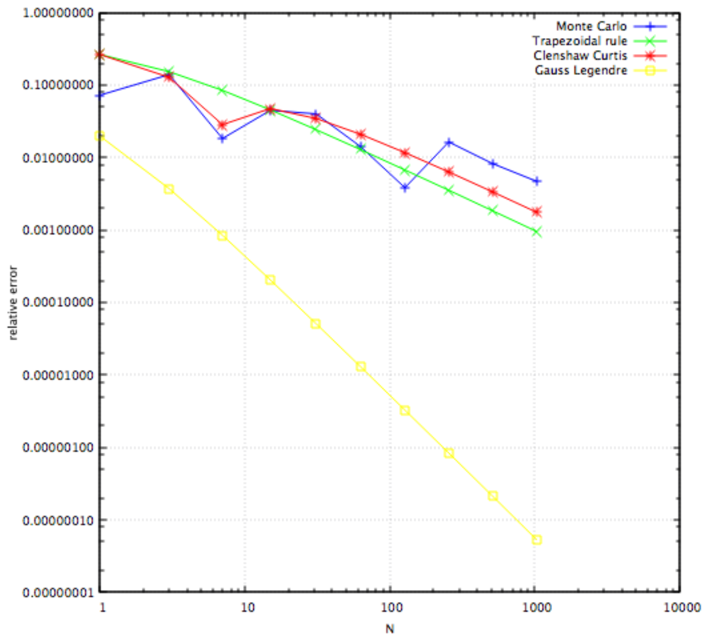
$$\begin{bmatrix} \alpha_0 & \beta_1 & 0 & ... & ... & ... \\ \beta_1 & \alpha_1 & \beta_2 & ... & ... & ... \\ 0 & \beta_2 & a_2 & \beta_3 & ... & ... \\ 0 & ... & ... & ... & ... & 0 \\ ... & ... & 0 & \beta_{N_l-2} & \alpha_{N_l-2} & \beta_{N_l-1} \\ ... & ... & ... & 0 & \beta_{N_l-1} & \alpha_{N_l-1} \end{bmatrix}$$

The weights are the first entry of the eigenvectors to the eigenvalues calculated for the nodes. Alternitavely, they can be recieved by taking $\Lambda_{n+1}(x_i)$ of the Christoffel-function, with $x_i, i = 1, ..., n$ nodes.

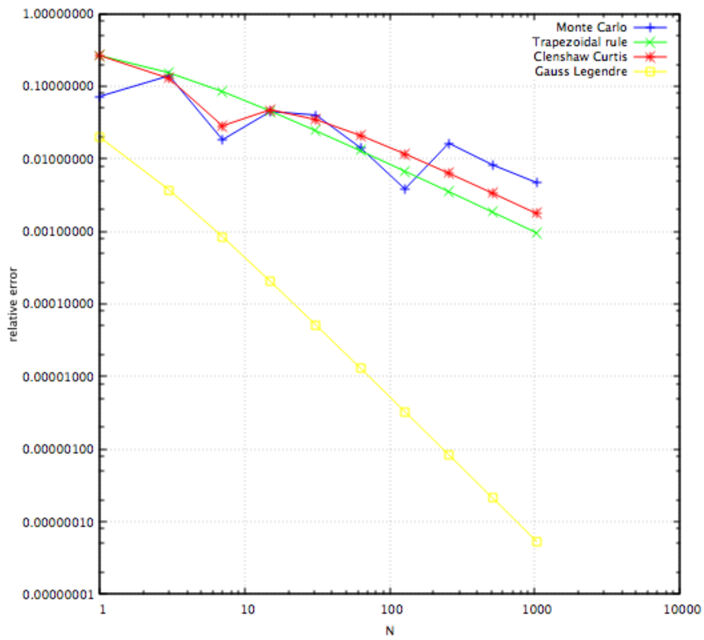
Task 8

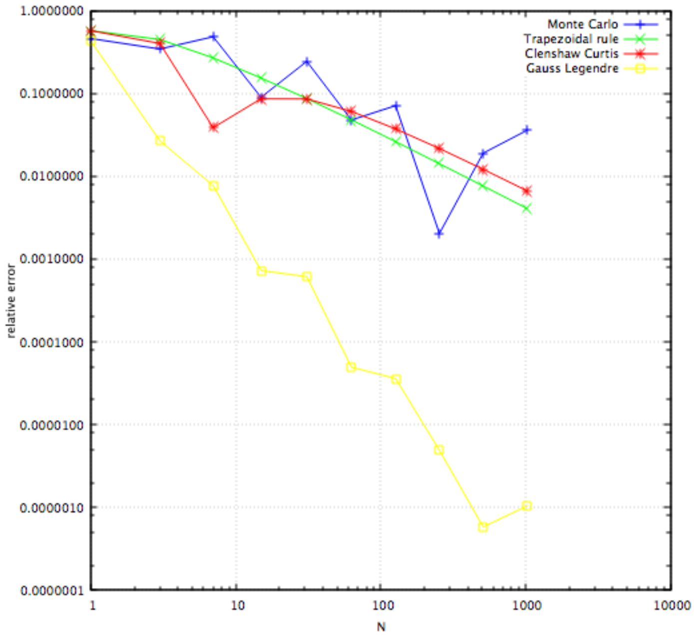
The Clenshaw-Curtis Quadrature rule uses nested nodes, the nodes of each level are a subset of the nodes of higher levels. Similarly to Gauß-Legendre quadrature, the number of nodes is higher at the edges of the integral.

Task 9



Task 10





1.0000000
0.1000000
0.0100000
0.0010000
0.0001000
0.0000100
0.0000010
0.0000001
relative error
1
10
100
1000
10000
N
Monte Carlo
Trapezoidal rule
Clenshaw Curtis
Gauss Legendre