Worksheet 2

Practical Lab Numerical Computing

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May 22, 2017

Task 1

Task 2

Task 3

To prove that

$$\mathbb{E}[V_{call}(S_T,0)] = S(0)\exp(\mu T)\Phi(\sigma\sqrt{T}-\chi) - K\Phi(-\chi)$$

we use that by change-of-variable with $t := -t$ we get

$$\begin{aligned}
& \mathbb{E}[V_{call}(S_T,0)] \\
= \; & \frac{1}{\sqrt{2\pi}}\int_\chi^\infty \left(S(0)\exp\left((\mu-\frac{1}{2}\sigma^2)\cdot T + \sigma\sqrt{T}s\right) - K\right)\exp\left(-\frac{s^2}{2}\right)ds \\
= \; & \frac{1}{2\pi}\int_\chi^\infty \left(S(0)\exp\left((\mu-\frac{1}{2}\sigma^2)\cdot T + \sigma\sqrt{T}s\right)\right)\exp\left(-\frac{s^2}{2}\right)ds - K\frac{1}{2\pi}\int_\chi^\infty \exp\left(-\frac{t^2}{2}\right)dt \\
= \; & \Psi - K\frac{1}{2\pi}\int_{-\infty}^{-\chi}\exp\left(-\frac{t^2}{2}\right)dt \\
= \; & \Psi - K\Phi(-\chi)
\end{aligned}$$

with

$$\Psi := \frac{1}{2\pi}\int_\chi^\infty \left(S(0)\exp\left((\mu-\frac{1}{2}\sigma^2)\cdot T + \sigma\sqrt{T}s\right)\right)\exp\left(-\frac{s^2}{2}\right)ds.$$

Now we prove that

$$\Psi = S(0)\exp(\mu T)\Phi(\sigma\sqrt{T}-\chi).$$

Again, we use a change-of-variable $z := t + \sigma\sqrt{T}$ and get

$$\begin{aligned}
& \frac{1}{\sqrt{2\pi}} \exp\left(\frac{\sigma^2}{2}T\right) \int\limits_{-\infty}^{\sigma\sqrt{T}-\chi} \exp\left(-\frac{t^2}{2}\right) dt \\
= & \frac{1}{\sqrt{2\pi}} \exp\left(\frac{\sigma^2}{2}T\right) \int\limits_{\chi-\sigma\sqrt{T}}^{\infty} \exp\left(-\frac{t^2}{2}\right) dt \\
= & \frac{1}{\sqrt{2\pi}} \exp\left(\frac{\sigma^2}{2}T\right) \int\limits_{\chi}^{\infty} \exp\left(-\frac{(\sigma\sqrt{T}-z)^2}{2}\right) dz \\
= & \frac{1}{\sqrt{2\pi}} \int\limits_{\chi}^{\infty} \exp\left(-\frac{z^2}{2} + z\sigma\sqrt{T}\right) dz.
\end{aligned}$$

We have

$$\begin{aligned}
\Psi &= \frac{1}{\sqrt{2\pi}} S(0) \exp\left((\mu - \frac{\sigma^2}{2}) \cdot T\right) \int_{\chi}^{\infty} \exp\left(-\frac{s^2}{2} + \sigma\sqrt{T}s\right) ds \\
&= S(0)\exp(\mu T) \exp\left(-\frac{\sigma^2}{2}T\right) \frac{1}{\sqrt{2\pi}} \int_{\chi}^{\infty} \exp\left(-\frac{s^2}{2} + \sigma\sqrt{T}s\right) ds \\
&= S(0)\exp(\mu T) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\sigma^2}{2}T\right) \exp\left(\frac{\sigma^2}{2}T\right) \int\limits_{-\infty}^{\sigma\sqrt{T}-\chi} \exp\left(-\frac{s^2}{2}\right) ds \\
&= S(0)\exp(\mu T)\Phi(\sigma\sqrt{T}-\chi).
\end{aligned}$$

Task 4

Task 5

We have $\Phi^{-1}(0) = -\infty$ and $\Phi^{-1}(1) = \infty$ where $\Phi^{-1} : (0,1) \to (-\infty, \infty)$ is the inverse cumulative distribution function. As an integral of a positive continious function, Φ is a bijection, continious and differentiable. That means Φ is a diffeomorphism. Φ^{-1} is also differentiable, so we can use the transformation theorem with the change-of-variable $t = \Phi(s)$. We have $|\det(D\Phi(s))| = \frac{1}{\sqrt{2\pi}} \exp(-\frac{s^2}{2})$ and

$$\begin{aligned}
& \int_0^1 f(\Phi^{-1}(t))dt \\
= & \int_{\Phi^{-1}(0)}^{\Phi^{-1}(1)} f(\Phi^{-1}(\Phi(s))) \cdot |\det(D\Phi(s))| ds \\
= & \int_{-\infty}^{\infty} f(s) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{s^2}{2}\right) ds \\
= & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s) \exp\left(-\frac{s^2}{2}\right) ds
\end{aligned}$$

what proves formula (7).

Task 6

Task 7

Task 8

Task 9

Task 10