Global Attractors of Non-autonomous Lattice Dynamical Systems

David Cheban, Andrei Sultan

Institutul de Matematică și Informatică "Vladimir Andrunachievici"

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Denote by $\mathbb{R} := (-\infty, \infty)$, $\mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}$ and ℓ_2 the Hilbert space of all two-sided sequences $\xi = (\xi_i)_{i\in\mathbb{Z}}$ ($\xi_i \in \mathbb{R}$) with

$$\sum_{i\in\mathbb{Z}} |\xi_i|^2 < +\infty \tag{1}$$

and equipped with the scalar product

$$\langle \xi, \eta \rangle := \sum_{i\in\mathbb{Z}} \xi_i \eta_i . \tag{2}$$

Let $(\mathfrak{B}, |\cdot|)$ be a Banach space with the norm $|\cdot|$, $C(\mathbb{R}, \mathfrak{B})$ be the space of all continuous functions $f : \mathbb{R} \to \mathfrak{B}$ equipped with the distance

$$d(f_1, f_2) := \sup_{L>0} \min\{\max_{|t|\le L} |f_1(t) - f_2(t)|, L^{-1}\}. \tag{3}$$

The metric space $(C(\mathbb{R}, \mathfrak{B}), d)$ is complete and the distance d, defined by (3), generates on the space $C(\mathbb{R}, \mathfrak{B})$ the compact-open topology

Let $h \in \mathbb{R}$, $f \in C(\mathbb{R}, \mathfrak{B})$, $f^h(t) := f(t+h)$ for any $t \in \mathbb{R}$ and $\sigma : \mathbb{R} \times C(\mathbb{R}, \mathfrak{B}) \to C(\mathbb{R}, \mathfrak{B})$ be a mapping defined by $\sigma(h, f) := f^h$ for any $(h, f) \in \mathbb{R} \times C(\mathbb{R}, \mathfrak{B})$. Then [2, Ch.I] the triplet $(C(\mathbb{R}, \mathfrak{B}), \mathbb{R}, \sigma)$ is a shift dynamical system (or Bebutov's dynamical system) on he space $C(\mathbb{R}, \mathfrak{B})$. By $H(f)$ the closure in the space $C(\mathbb{R}, \mathfrak{B})$ of $\{f^h |\ h \in \mathbb{R}\}$ is denoted.

We study the compact global attractors of the systems

$$u_i' = \nu(u_{i-1} - 2u_i + u_{i+1}) - \lambda u_i + F(u_i) + f_i(t) \ (i \in \mathbb{Z}), \qquad (4)$$

where $\lambda > 0$, $F \in C(\mathbb{R}, \mathbb{R})$ and $f \in C(\mathbb{R}, \ell_2)$ $(f(t) := (f_i(t))_{i \in \mathbb{Z}}$ for any $t \in \mathbb{R})$.
The system (4) can be considered as a discrete (see, for example, [1], [6] and the bibliography therein) analogue of a reaction-diffusion equation in $\mathbb{R}$:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial^2 x} - \lambda u + F(u) + f(t,x), \qquad (5)$$

where grid points are spaced h distance apart and $\nu = D/h^2$.

Condition (**C1**). The function $f \in C(\mathbb{R}, \mathfrak{B})$ is translation-compact, i.e., the set $\{f^h | \ h \in \mathbb{R}\}$ is pre-compact in the space $C(\mathbb{R}, \mathfrak{B})$.

Lemma

[7, 8] The following statements are equivalent:

1. *the function $f \in C(\mathbb{R}, \mathfrak{B})$ is translation-compact;*
2. *the set $Q := \overline{f(\mathbb{R})}$ is compact in $\mathfrak{B}$ and the function $f \in C(\mathbb{R}, \mathfrak{B})$ is uniformly continuous.*

Condition (**C2**). The function $F \in C(\mathbb{R}, \mathbb{R})$ is Lipschitz continuous on bounded sets and $F(0) = 0$.
Condition (**C3**). $sF(s) \leq -\alpha s^2$ for any $s \in \mathbb{R}$.

Definition

A function $F \in C(Y \times \mathfrak{B}, \mathfrak{B})$ is said to be Lipschitzian) on every bounded subsets from $\mathfrak{B}$ uniformly with respect to $y \in Y$ if for any bounded set $B \subset \mathfrak{B}$ there exists a constant L_B such that

$$|F(y, v_1) - F(y, v_2)| \leq L_B |v_1 - v_2| \tag{6}$$

for any $v_1, v_2 \in B \subset \mathfrak{B}$.

For any $u = (u_i)_{i\in\mathbb{Z}}$, the discrete Laplace operator Λ is defined [6, Ch.III] from ℓ_2 to ℓ_2 component wise by $\Lambda(u)_i = u_{i-1} - 2u_i + u_{i+1}$ $(i \in \mathbb{Z})$. Define the bounded linear operators D^+ and D^- from ℓ_2 to ℓ_2 by $(D^+u)_i = u_{i+1} - u_i$, $(D^-u)_i = u_{i-1} - u_i$ $(i \in \mathbb{Z})$.
Note that $\Lambda = D^+D^- = D^-D^+$ and $\langle D^-u, v\rangle = \langle u, D^+v\rangle$ for any $u, v \in \ell_2$ and, consequently, $\langle \Lambda u, u\rangle = -|D^+u|^2 \leq 0$. Since Λ is a bounded linear operator acting on the space ℓ_2, it generates a uniformly continuous semi-group on ℓ_2.
Under the Conditions (C1) and (C2) the system of differential equations (4) can be written in the form of an ordinary differential equation

$$u' = \nu\Lambda u + \Phi(u) + f(t) \tag{7}$$

Theorem

*Under the Conditions (**C1**)-(**C3**) the following statements hold:*

1. *for any* $(v, g) \in \ell_2 \times H(f)$ *there exists a unique solution* $\varphi(t, v, g)$ *of the equation*
$$u' = \nu\Lambda u + \Phi(u) + g(t) \quad (g \in H(f)) \tag{8}$$
passing through the point v *at the initial moment* $t = 0$ *and defined on the semi-axis* $\mathbb{R}_+ := [0, +\infty)$*;*
2. $\varphi(0, v, g) = v$ *for any* $(v, g) \in \ell_2 \times H(f)$*;*
3. $\varphi(t + \tau, v, g) = \varphi(t, \varphi(\tau, v, g), g^\tau)$ *for any* $t, \tau \in \mathbb{R}_+$, $v \in \ell_2$ *and* $g \in H(f)$*;*
4. *the mapping* $\varphi : \mathbb{R}_+ \times \ell_2 \times H(f) \to \ell_2$ $((t, v, g) \to \varphi(t, v, g))$ *for any* $(t, v, g) \in \mathbb{R}_+ \times \ell_2 \times H(f)$ *is continuous.*

Definition

A cocycle φ is said to be asymptotically compact if for any bounded subset $B \subset \mathfrak{B}$ there exists a compact subset $K = K(B) \subset \mathfrak{B}$ such that the compact subset K attracts the bounded set B, that is,

$$\lim_{t\to+\infty} \sup_{y\in Y} \beta(\varphi(t, B, y), K) = 0. \tag{9}$$

Theorem

*Under the Conditions (**C1**)-(**C3**) the cocycle* $\langle \ell_2, \varphi, (H(f), \mathbb{R}, \sigma)\rangle$ *generated by the equation (7) is asymptotically compact.*

Theorem

*Under the Conditions (**C1**)-(**C3**) there exists a closed ball* $B[0,r] := \{\xi \in \ell_2 |\ |\xi| \le r\}$ *such that for any bounded subset* $B \subset \ell_2$ *there exist a positive number* $L = L(B)$ *such that* $\varphi(t,B,Y) \subseteq B[0,r]$ *for any* $t \ge L(B)$*, where* $\varphi(t,M,Y) := \{\varphi(t,u,y) |\ u \in M,\ y \in Y\}$.

Let $\langle \mathfrak{B}, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ (or shortly φ) be a cocycle over dynamical system $(Y, \mathbb{R}, \sigma)$ with the compact phase space Y.
Let A and B be two bounded subsets from $\mathfrak{B}$. Denote by $\rho(a, b) := |a - b|$ $(a, b \in \mathfrak{B})$, $\rho(a, B) := \inf_{b \in B} \rho(a, b)$ and

$$\beta(A, B) := \sup_{a \in A} \rho(a, B). \tag{10}$$

Definition

A cocycle φ is said to be asymptotically compact if for any bounded subset $B \subset \mathfrak{B}$ there exists a compact subset $K = K(B) \subset \mathfrak{B}$ such that the compact subset K attracts the bounded set B, that is,

$$\lim_{t \to +\infty} \sup_{y \in Y} \beta(\varphi(t, B, y), K) = 0. \tag{11}$$

Theorem

*Under the Conditions (**C1**)-(**C3**) the cocycle $\langle \ell_2, \varphi, (H(f), \mathbb{R}, \sigma)\rangle$ generated by the equation (7) is asymptotically compact.*

Definition

A family $\{I_y|\ y \in Y\}$ of compact subsets I_y of $\mathfrak{B}$ is said to be a compact global attractor for the cocycle $\langle \mathfrak{B}, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ if the following conditions are fulfilled:

1. the set
$$\mathcal{I} := \bigcup \{I_y|\ y \in Y\} \tag{12}$$
is precompact;
2. the family of subsets $\{I_y|\ y \in Y\}$ is invariant, i.e., $\varphi(t, I_y, y) = I_{\sigma(t,y)}$ for any $(t, y) \in \mathbb{R}_+ \times Y$;
3. $$\lim_{t \to +\infty} \sup_{y \in Y} \beta(\varphi(t, M, y), \mathcal{I}) = 0 \tag{13}$$

for any compact subset M from $\mathfrak{B}$.

Definition

A cocycle φ is said to be dissipative if there exists a bounded subset $K \subset \mathfrak{B}$ such that for any bounded subset $B \subset \mathfrak{B}$ there exists a positive number $L = L(B)$ such that $\varphi(t, B, Y) \subseteq K$ for any $t \geq L(B)$, where $\varphi(t, B, Y) := \{\varphi(t, u, y) | \ (u, y) \in B \times Y\}$.

Theorem

[4, Ch.II] Assume that the metric space Y is compact and the cocycle $\langle \mathfrak{B}, \varphi, (Y, \mathbb{R}, \sigma) \rangle$ is dissipative and asymptotically compact. Then the cocycle φ has a compact global attractor.

Theorem

*Under the Conditions (**C1**)-(**C3**) the equation (7) (the cocycle φ generated by the equation (7)) has a compact global attractor $\{I_g | \ g \in H(f)\}$.*

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