GLOBAL ATTRACTORS OF NON-AUTONOMOUS LATTICE DYNAMICAL SYSTEMS

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Abstract. The aim of this paper is studying the compact global attractors for non-autonomous lattice dynamical systems of the form $u_i' = \nu(u_{i-1} - 2u_i + u_{i+1}) - \lambda u_i + f(u_i) + f_i(t)$ $(i \in \mathbb{Z},\ \lambda > 0)$. We prove their dissipativness, asymptotic compactness and then the existence of compact global attractors.

1. Introduction

Denote by $\mathbb{R} := (-\infty, \infty)$, $\mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}$ and ℓ_2 the Hilbert space of all two-sided sequences $\xi = (\xi_i)_{i\in\mathbb{Z}}$ $(\xi_i \in \mathbb{R})$ with

$$\sum_{i\in\mathbb{Z}} |\xi_i|^2 < +\infty \tag{1}$$

and equipped with the scalar product

$$\langle \xi, \eta \rangle := \sum_{i\in\mathbb{Z}} \xi_i \eta_i. \tag{2}$$

Let $(\mathfrak{B}, |\cdot|)$ be a Banach space with the norm $|\cdot|$, $C(\mathbb{R}, \mathfrak{B})$ be the space of all continuous functions $f : \mathbb{R} \to \mathfrak{B}$ equipped with the distance

$$d(f_1, f_2) := \sup_{L>0} \min\{\max_{|t|\le L} |f_1(t) - f_2(t)|, L^{-1}\}. \tag{3}$$

The metric space $(C(\mathbb{R}, \mathfrak{B}), d)$ is complete and the distance d, defined by (3), generates on the space $C(\mathbb{R}, \mathfrak{B})$ the compact-open topology.

Let $h \in \mathbb{R}$, $f \in C(\mathbb{R}, \mathfrak{B})$, $f^h(t) := f(t + h)$ for any $t \in \mathbb{R}$ and $\sigma : \mathbb{R} \times C(\mathbb{R}, \mathfrak{B}) \to C(\mathbb{R}, \mathfrak{B})$ be a mapping defined by $\sigma(h, f) := f^h$ for any $(h, f) \in \mathbb{R} \times C(\mathbb{R}, \mathfrak{B})$. Then [2, Ch.I] the triplet $(C(\mathbb{R}, \mathfrak{B}), \mathbb{R}, \sigma)$ is a shift dynamical system (or Bebutov's dynamical system) on he space $C(\mathbb{R}, \mathfrak{B})$. By $H(f)$ the closure in the space $C(\mathbb{R}, \mathfrak{B})$ of $\{f^h |\ h \in \mathbb{R}\}$ is denoted. compare results pullback attractors In this note we study the compact global attractors of the systems

$$u_i' = \nu(u_{i-1} - 2u_i + u_{i+1}) - \lambda u_i + F(u_i) + f_i(t)\ (i \in \mathbb{Z}), \tag{4}$$

where $\lambda > 0$, $F \in C(\mathbb{R}, \mathbb{R})$ and $f \in C(\mathbb{R}, \ell_2)$ $(f(t) := (f_i(t))_{i\in\mathbb{Z}}$ for any $t \in \mathbb{R})$.

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The system (4) can be considered as a discrete (see, for example, [1], [6] and the bibliography therein) analogue of a reaction-diffusion equation in $\mathbb{R}$:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial^2 x} - \lambda u + F(u) + f(t,x), \tag{5}$$

where grid points are spaced h distance apart and $\nu = D/h^2$.

This study continues the first author's works devoted to the study of compact global attractors of non-autonomous dynamical systems [2] and compact attractors of lattice dynamical systems [1] (autonomous systems) and compact pullback attractors [6] (for non-autonomous systems).

The paper is organized as follows. In the second section we show that under some conditions the equation (4) generates a cocycle which plays a very important role in the study of the asymptotic properties of the equation (4). In the third section we prove that under some conditions the existence of an absorbing set for the equation (4). The fourth section is dedicated to the study the asymptotically compactness of the cocycle generated by the equation (4). In the fifth section we study the problem of existence of a compact global attractor for the equation (4).

2. Cocycles

Consider a non-autonomous system

$$u_i' = \nu(u_{i-1} - 2u_i + u_{i+1}) - \lambda u_i + F(u_i) + f_i(t) \ (i \in \mathbb{Z}). \tag{6}$$

Below we use the following conditions.

Condition (**C1**). The function $f \in C(\mathbb{R}, \mathfrak{B})$ is translation-compact, i.e., the set $\{f^h |\ h \in \mathbb{R}\}$ is pre-compact in the space $C(\mathbb{R}, \mathfrak{B})$.

Lemma 2.1. [7, 8] *The following statements are equivalent:*

(i) *the function $f \in C(\mathbb{R}, \mathfrak{B})$ is translation-compact;*
(ii) *the set $Q := \overline{f(\mathbb{R})}$ is compact in $\mathfrak{B}$ and the function $f \in C(\mathbb{R}, \mathfrak{B})$ is uniformly continuous.*

Condition (**C2**). The function $F \in C(\mathbb{R}, \mathbb{R})$ is continuously differentiable, $F'(x)$ is globally bounded in $\mathfrak{L}$ and $F(0) = 0$. Denote by $\widetilde{F} : \ell_2 \to \ell_2$ the Nemytskii operator generated by F, i.e., $\widetilde{F}(\xi)_i := F(\xi_i)$ for any $i \in \mathfrak{N}$.

Condition (**C2.1**). The function $F \in C(\mathbb{R}, \mathbb{R})$ is Lipschitz continuous on bounded sets.

Condition (**C2.2**). $sF(s) \leq -\alpha s^2$.

Definition 2.2. *A function $F \in C(Y \times \mathfrak{B}, \mathfrak{B})$ is said to be globally Lipschitzian (respectively locally Lipschitzian) with respect to variable $u \in \mathfrak{B}$ uniformly with respect to $y \in Y$ if there exists a positive constant L (for any bounded set $\mathfrak{B} \subset \mathfrak{L}$ there exists a constant $L_{\mathfrak{B}}$) such that*

$$|F(y, u_1) - F(y, u_2)| \leq L|u_1 - u_2| \tag{7}$$

Respectively,

$$|F(y, v_1) - F(y, v_2)| \leq L_{\mathfrak{B}}|v_1 - v_2| \tag{8}$$

for any $u_1, u_2 \in \mathfrak{B}$ *and* $y \in Y$ *(respectively* $v_1, v_2 \in \mathfrak{B} \subset \mathfrak{L}$*).*

Definition 2.3. *The smallest constant* L *(respectively* $L_{\mathfrak{B}}$*) with the property (7) is called Lipshchitz constant of function* F *(notation* $Lip(F)$*, respectively* $Lip_{\mathfrak{B}}(F)$*).*

Let $\mathfrak{G} \subset \mathfrak{L}$, denote by $CL(Y \times \mathfrak{G}, \mathfrak{B})$ the Banach space of any Lipschitzian functions $F \in C(Y \times \mathfrak{G}, \mathfrak{B})$ equipped with the norm

$$||F||_{CL} := \max_{y \in Y} |F(y, 0)| + Lip_{\mathfrak{G}}(F).$$

Lemma 2.4. [2] *Under the Condition* ***(C2)*** *it is well defined the mapping* $\widetilde{F} : \ell_2 \to \ell_2$ *and*

$$|\widetilde{F}(\xi) - \widetilde{F}(\eta)| \leq Lip_{\mathfrak{B}}(F)|\xi - \eta| \tag{9}$$

for any $\xi, \eta \in \ell_2$*.*

For any $u = (u_i)_{i \in \mathbb{Z}}$, the discrete Laplace operator Λ is defined [6, Ch.III] from ℓ_2 to ℓ_2 component wise by $\Lambda(u)_i = u_{i-1} - 2u_i + u_{i+1}$ ($i \in \mathbb{Z}$). Define the bounded linear operators D^+ and D^- from ℓ_2 to ℓ_2 by $(D^+u)_i = u_{i+1} - u_i$, $(D^-u)_i = u_{i-1} - u_i$ ($i \in \mathbb{Z}$).

Note that $\Lambda = D^+D^- = D^-D^+$ and $\langle D^-u, v\rangle = \langle u, D^+v\rangle$ for any $u, v \in \ell_2$ and, consequently, $\langle \Lambda u, u\rangle = -|D^+u|^2 \leq 0$. Since Λ is a bounded linear operator acting on the space ℓ_2, it generates a uniformly continuous semi-group on ℓ_2.

Under the Conditions (C1) and (C2) the system of differential equations (6) can be written in the form of an ordinary differential equation

$$u' = \nu\Lambda u + \Phi(u) + f(t) \tag{10}$$

in the Banach space $\mathfrak{B} = \ell_2$, where $\Phi(u) := -\lambda u + \widetilde{F}(u)$ and $\Lambda(u)_i := u_{i-1} - 2u_i + u_{i+1}$ for any $u = (u_i)_{i \in \mathbb{Z}} \in \ell_2$. Along with equation (10) we consider also it H-class, i.e., the family equations

$$u' = \nu\Lambda u + \Phi(u) + g(t), \tag{11}$$

where $g \in H(f)$.

The family of equations (11) can be rewritten as follows

$$u' = F(\sigma(t, g), u) \quad (g \in H(f)), \tag{12}$$

where $F : H(f) \times \ell_2 \to \ell_2$ is defined by $F(g, u) := \nu\Lambda u + \Phi(u) + g(0)$. It easy to see that $F(\sigma(t, g), u) = \nu\Lambda u + \Phi(u) + g(t)$ for any $(t, u, g) \in \mathbb{R} \times \mathfrak{B} \times H(f)$.

Let $(Y, \mathbb{R}, \sigma)$ be a dynamical system on the metric space Y.

Lemma 2.5. [2] *Assume that the function* $\mathcal{F} \in C(\mathbb{R} \times \mathfrak{B}, \mathfrak{B})$ *satisfies the (global) Lipschitz condition with the Lipschitz constant* $L_{\mathcal{F}}$*, then every function* $\mathcal{G} \in H(\mathcal{F})$ *satisfies the (global) Lipschitz condition with the Lipschitz constant* $L_{\mathcal{G}} \leq L_{\mathcal{F}}$*.*

Let Y be a complete metric space, $(Y, \mathbb{R}, \sigma)$ be a dynamical system on Y and Λ be some complete metric space of linear closed operators acting into Banach space $\mathfrak{B}$. Consider the following linear differential equation

$$x' = A(\sigma(t, y))x, \quad (y \in Y) \tag{13}$$

where $A \in C(Y, \Lambda)$. We assume that the following conditions are fulfilled for equation (13):

a. for any $u \in \mathfrak{B}$ and $y \in Y$ equation (13) has exactly one solution that is defined on $\mathbb{R}_+$ and satisfies the condition $\varphi(0, u, y) = u$;
b. the mapping $\varphi : (t, u, y) \to \varphi(t, u, y)$ is continuous in the topology of $\mathbb{R}_+ \times \mathfrak{B} \times Y$.

Denote by $U(t, y) := \varphi(t, \cdot, y)$ for any $(t, y) \in \mathbb{R}_+ \times Y$.

Consider an evolutionary differential equation

$$u' = A(\sigma(t, y))u + F(\sigma(t, y), u) \quad (y \in Y) \tag{14}$$

in the Banach space $\mathfrak{B}$, where F is a nonlinear continuous mapping (”small” perturbation) acting from $Y \times \mathfrak{B}$ into $\mathfrak{B}$.

Definition 2.6. *A function $u : [0, a) \mapsto \mathfrak{B}$ is said to be a weak (mild) solution of equation (14) passing through the point $x \in \mathfrak{B}$ at the initial moment $t = 0$ (notation $\varphi(t, x, y)$) if $u \in C([0, T], \mathfrak{B})$ and satisfies the integral equation*

$$u(t) = U(t, y)x + \int_0^t U(t - s, \sigma(s, y))F(\sigma(s, y), u(s))ds \tag{15}$$

for any $t \in [0, T]$ and $0 < T < a$.

Theorem 2.7. [3, Ch.VI] *Suppose that the function $F \in C(Y \times \mathfrak{B}, \mathfrak{B})$ is globally Lipschitzian.*

Then for any $(x, y) \in \mathfrak{B} \times Y$ there exists a unique solution $\varphi(t, x, y)$ of the equation (14) defined on the semi-axis $[0, +\infty)$ with the conditions: $\varphi(0, x, y) = x$ and the mapping $\varphi : [0, +\infty) \times \mathfrak{B} \times Y \to \mathfrak{B}$ $((t, x, y) \mapsto \varphi(t, x, y))$ is continuous.

Theorem 2.8. *Under the Conditions (**C1**) and (**C2**) the following statements hold:*

(i) *for any $(v, g) \in \ell_2 \times H(f)$ there exists a unique solution $\varphi(t, v, g)$ of the equation (11) passing through the point v at the initial moment $t = 0$ and defined on the semi-axis $\mathbb{R}_+ := [0, +\infty)$;*
(ii) *$\varphi(0, v, g) = v$ for any $(v, g) \in \ell_2 \times H(f)$;*
(iii) *$\varphi(t + \tau, v, g) = \varphi(t, \varphi(\tau, v, g), g^\tau)$ for any $t, \tau \in \mathbb{R}_+$, $v \in \ell_2$ and $g \in H(f)$;*
(iv) *the mapping $\varphi : \mathbb{R}_+ \times \ell_2 \times H(f) \to \ell_2$ $((t, v, g) \to \varphi(t, v, g))$ for any $(t, v, g) \in \mathbb{R}_+ \times \ell_2 \times H(f))$ is continuous.*

Proof. Assume that the Conditions (**C1**) and (**C2**) are fulfilled. Consider the equation (12), where $F(g, u) := \nu\Lambda u + \Phi(u) + g(0)$ for any $(u, g) \in \ell_2 \times H(f)$. It easy to check that under the conditions of Theorem the mapping F possesses the following properties:

(i) F is continuous;

(ii) the mapping F is Lipschitzian in $u \in \ell_2$ uniformly with respect to $g \in H(f)$ with the Lipschitz constant $L_F \le L_{\mathcal{F}}$;

(iii) there exists a positive constant A such that

$$|F(g,0)| \le A \tag{16}$$

for any $g \in H(f)$.

Now to finish the proof of Theorem it suffices to apply Theorem 2.7. $\square$

Lemma 2.9. *Assume that (C1) and (C2.2) holds and $g \in H(f)$. Then, for every $T > 0$, any solution u of the problem (4) and ($u(0) = u_0 \in \ell^2$) satisfies*

$$\|u(t)\| \le C\,, \quad \text{for all } 0 \le t \le T\,,$$

where C is a constant depending only on the data $(\lambda, \|f\|, \|u_0\|)$ and T.

Proof. *Taking the inner product of (4) with u in ℓ^2, by (6) and (8) we find that*

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \nu\|Bu\|^2 + \lambda\|u\|^2 = -(f(u),u) + (g(t),u)\,. \tag{l1.1}$$

Since

$$|(g(t),u)| \le \|g(t)\|\|u\| \le \frac{1}{2}\lambda\|u\|^2 + \frac{1}{2\lambda}\|g(t)\|^2\,,$$

using (C2.2) we get that

$$\frac{d}{dt}\|u\|^2 + 2\nu\|Bu\|^2 + \lambda\|u\|^2 \le \frac{1}{\lambda}\|g(t)\|^2 \le \frac{C}{\lambda} \tag{l1.2}$$

Then Lemma 2.1 follows from (l1.2) and Gronwall's lemma.

Theorem 2.10. [3, Ch.VI] [9, Ch.II] *Under the Conditions (C1) and (C2.1) the following statements hold:*

(i) *for any $(v,g) \in \ell_2 \times H(f)$ there exists a unique solution $\varphi(t,v,g)$ of the equation (11) passing through the point v at the initial moment $t = 0$ and defined on the semi-axis $\mathbb{R}_+ := [0,+\infty)$;*

(ii) *$\varphi(0,v,g) = v$ for any $(v,g) \in \ell_2 \times H(f)$;*

(iii) *$\varphi(t+\tau,v,g) = \varphi(t,\varphi(\tau,v,g),g^\tau)$ for any $t,\tau \in \mathbb{R}_+$, $v \in \ell_2$ and $g \in H(f)$;*

(iv) *the mapping $\varphi : \mathbb{R}_+ \times \ell_2 \times H(f) \to \ell_2$ $((t,v,g) \to \varphi(t,v,g))$ for any $(t,v,g) \in \mathbb{R}_+ \times \ell_2 \times H(f)$ is continuous.*

Proof. Assume that the Conditions (**C1**) and (**C2.1**) are fulfilled. Consider the equation (12), where $F(g,u) := \nu\Lambda u + \Phi(u) + g(0)$ for any $(u,g) \in \ell_2 \times H(f)$. It easy to check that under the conditions of Theorem the mapping F possesses the following properties:

(i) F is continuous;

(ii) the mapping F is locally Lipschitzian in $u \in \ell_2$ uniformly with respect to $g \in H(f)$ with the Lipschitz constant $L_F \le L_{\mathcal{F}}$;

(iii) there exists a positive constant A such that

$$|F(g,0)| \le A \tag{17}$$

for any $g \in H(f)$.

Now to finish the proof of Theorem it suffices to apply Lemma 2.9 and [9, Ch.II] 2.7. □

Let Y be a complete metric space and $(Y, \mathbb{R}, \sigma)$ be a dynamical system on Y.

Definition 2.11. *Recall* [2, Ch.I] *that $\langle \mathbb{B}, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ is said to be a cocycle over $(Y, \mathbb{R}, \sigma)$ with the fiber $\mathfrak{B}$ if φ is a continuous mapping acting from $\mathbb{R}_+ \times \mathfrak{B} \times Y \to \mathfrak{B}$ satisfying the following conditions:*

(i) $\varphi(0, u, y) = v$ *for any* $(v, y) \in \mathfrak{B} \times Y$*;*
(ii) $\varphi(t+\tau, u, y) = \varphi(t, \varphi(\tau, u, t), \sigma(\tau, y))$ *for any* $(t, \tau \in \mathbb{R}_+$ *and* $(u,) \in \mathfrak{B} \times Y$.

Corollary 2.12. *Under the conditions of Theorem 2.8 the equation (10) (respectively, the family of equations (11)) generates a cocycle $\langle \ell_2, \varphi, (H(f), \mathbb{R}, \sigma)\rangle$ over the shift dynamical system $(H(f), \mathbb{R}, \sigma)$ with the fiber ℓ_2.*

Proof. This statement directly follows from Theorem 2.8 and Definition 2.11. □

3. Existence of an absorbing set

Condition (**C3**). There exist positive numbers α and γ such that $F(s)s \leq -\alpha s^2 + \gamma$ for any $s \in \mathbb{R}$.

Theorem 3.1. *Under the Conditions (**C1**)-(**C3**) there exists a closed ball $B[0, r] := \{\xi \in \ell_2 |\ |\xi| \leq r\}$ such that for any bounded subset $B \subset \ell_2$ there exist a positive number $L = L(B)$ such that $\varphi(t, B, Y) \subseteq B[0, r]$ for any $t \geq L(B)$, where $\varphi(t, M, Y) := \{\varphi(t, u, y)|\ u \in M,\ y \in Y\}$.*

Lemma 3.2 (Gronwall)**.** *If $y' \leq -\alpha y + \beta, \alpha > 0, \beta > 0 \Rightarrow y(t) = y(0)e^{-\alpha t} + \frac{\beta}{\alpha}(1 - e^{-\alpha t})$*

Proof of 3.1: Taking the inner product of equation (10) with u gives:

$$\begin{aligned}
\frac{d}{dt}\|u\|^2 &= 2\nu \langle \Lambda u, u\rangle + 2\langle \Phi(u), u\rangle + 2\langle g(t), u\rangle \\
&\leq 2\sum_{i\in\mathbb{Z}} u_i f(u_i) + 2\sum_{i\in\mathbb{Z}} g(t)_i u_i \\
&\leq -\alpha\|u\|^2 + \frac{\|g\|^2}{\alpha} \\
&\leq -\alpha\|u\|^2 + \frac{C(g)}{\alpha} \leq -\alpha\|u\|^2 + \frac{C}{\alpha} \qquad (18)
\end{aligned}$$

where the penultimate step follows from Young's inequality, and $C(g) \leq C$ since $g \in H(f)$. Hence, Gronwall's lemma implies that

$$\|u(t)\|^2 \leq \|u_0\|^2 e^{-\alpha t} + \frac{\|C\|^2}{\alpha^2}\left(1 - e^{-\alpha t}\right), \quad t \geq 0.$$

Define the closed ball Q in ℓ^2 by

$$Q := \left\{ u \in \ell^2 : \|u\|^2 \leq R^2 := 1 + \frac{\|C\|^2}{\alpha^2} \right\}.$$

4. Asymptotically compactness of the cocycle generated by the equation (4)

Let $\langle \mathfrak{B}, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ (or shortly φ) be a cocycle over dynamical system $(Y, \mathbb{R}, \sigma)$ with the compact phase space Y.

Let A and B be two bounded subsets from $\mathfrak{B}$. Denote by $\rho(a,b) := |a-b|$ $(a, b \in \mathfrak{B})$, $\rho(a, B) := \inf_{b\in B} \rho(a,b)$ and

$$\beta(A,B) := \sup_{a\in A} \rho(a,B). \tag{19}$$

Definition 4.1. *A cocycle φ is said to be asymptotically compact if for any bounded subset $B \subset \mathfrak{B}$ there exists a compact subset $K = K(B) \subset \mathfrak{B}$ such that the compact subset K attracts the bounded set B, that is,*

$$\lim_{t\to+\infty} \sup_{y\in Y} \beta(\varphi(t, B, y), K) = 0. \tag{20}$$

Lemma 4.2. [2, Ch.I] *Assume that Y is a compact metric space and φ is a cocycle over dynamical system $(Y, \mathbb{R}, \sigma)$ with the fiber $\mathfrak{B}$.*

Then the following statements are equivalent:

- (i) *the cocycle φ is asymptotically compact;*
- (ii) *from any sequence $\{\varphi(t_n, u_n, y_n)\}_{n\in\mathbb{N}}$ (with bounded $\{u_n\} \subset \mathfrak{B}$ and $t_n \to +\infty$ as $n \to \infty$) can be extracted a convergent subsequence $\{\varphi(t_{k_n}, u_{k_n}, y_{k_n})\}$.*

Theorem 4.3. *Under the Conditions **(C1)**-**(C3)** the cocycle $\langle \ell_2, \varphi, (H(f), \mathbb{R}, \sigma)\rangle$ generated by the equation (10) is asymptotically compact.*

We will prove this statement using the ideas and methods elaborated in the work ([6], Chapter 3.2). Consider a smooth function $\xi : \mathbb{R}^+ \to [0,1]$ satisfying

$$\xi(s) = \begin{cases} 0, & 0 \leq s \leq 1, \\ \in [0,1], & 1 \leq s \leq 2, \\ 1, & s \geq 2 \end{cases}$$

and note that there exists a constant C_0 such that $|\xi'(s)| \leq C_0$ for all $s \geq 0$. Then for a fixed $k \in \mathbb{N}$ (its value will be specified later), define

$$\xi_k(s) = \xi\left(\frac{s}{k}\right) \quad \text{for all} \quad s \in \mathbb{R}_+.$$

Given $\mathbf{u} \in \ell^2$, define $\mathbf{v} \in \ell^2$ componentwise as

$$v_i := \xi_k(|i|)u_i \quad \text{for} \quad i \in \mathbb{Z}.$$

Taking the inner product of equation (11) with $\mathbf{v}$ gives

$$\frac{d}{dt}\langle \mathbf{u},\mathbf{v}\rangle + \nu\langle \mathrm{D}^+\mathbf{u},\mathrm{D}^+\mathbf{v}\rangle = \langle \Phi(\mathbf{u}),\mathbf{v}\rangle + \langle \mathbf{g}(\mathbf{t}),\mathbf{v}\rangle,$$

that is

$$\frac{d}{dt}\sum_{i\in\mathbb{Z}}\xi_k(|i|)|u_i|^2 + 2\nu\langle \mathrm{D}^+\mathbf{u},\mathrm{D}^+\mathbf{v}\rangle = 2\sum_{i\in\mathbb{Z}}\xi_k(|i|)u_i f(u_i) + 2\sum_{i\in\mathbb{Z}}\xi_k(|i|)g(t)_i u_i \tag{21}$$

Each term in equation (21) will now be estimated. First,

$$\begin{aligned}\left\langle \mathrm{D}^+\mathbf{u},\mathrm{D}^+\mathbf{v}\right\rangle &= \sum_{i\in\mathbb{Z}}(u_{i+1}-u_i)(v_{i+1}-v_i)\\ &= \sum_{i\in\mathbb{Z}}(u_{i+1}-u_i)\left[\left(\xi_k(|i+1|)-\xi_k(|i|)\right)u_{i+1} + \xi_k(|i|)(u_{i+1}-u_i)\right]\\ &= \sum_{i\in\mathbb{Z}}\left(\xi_k(|i+1|)-\xi_k(|i|)\right)(u_{i+1}-u_i)u_{i+1} + \sum_{i\in\mathbb{Z}}\xi_k(|i|)(u_{i+1}-u_i)^2\\ &\geq \sum_{i\in\mathbb{Z}}\left(\xi_k(|i+1|)-\xi_k(|i|)\right)(u_{i+1}-u_i)u_{i+1}.\end{aligned}$$

Since

$$\left|\sum_{i\in\mathbb{Z}}\left(\xi_k(|i+1|)-\xi_k(|i|)\right)(u_{i+1}-u_i)u_{i+1}\right| \leq \sum_{i\in\mathbb{Z}}\frac{1}{k}|\xi'(s_i)|\cdot|u_{i+1}-u_i|\cdot|u_{i+1}|,$$

for some s_i between $|i|$ and $|i+1|$, and

$$\sum_{i\in\mathbb{Z}}|\xi'(s_i)|\,|u_{i+1}-u_i|\,|u_{i+1}| \leq C_0\sum_{i\in\mathbb{Z}}\left(|u_{i+1}|^2+|u_i||u_{i+1}|\right) \leq 4C_0\left\|\mathbf{u}\right\|^2.$$

Then it follows that for all $\mathbf{u}\in Q$ and $\mathbf{v}\in\ell^2$ defined componentwise as $v_i := \xi_k(|i|)u_i$, for $i\in\mathbb{Z}$,

$$\left\langle \mathrm{D}^+\mathbf{u},\mathrm{D}^+\mathbf{v}\right\rangle \geq -\frac{4C_0\|Q\|^2}{k}. \tag{22}$$

where $\|Q\| := \sup_{\mathbf{u}\in Q}\|\mathbf{u}\|$. On the other hand, by Assumption C2.2,

$$2\sum_{i\in\mathbb{Z}}\xi_k(|i|)u_i f(u_i) \leq -2\alpha\sum_{i\in\mathbb{Z}}\xi_k(|i|)|u_i|^2$$

and by Young's inequality

$$2\sum_{i\in\mathbb{Z}}\xi_k(|i|)g_i u_i \leq \alpha\sum_{i\in\mathbb{Z}}\xi_k(|i|)|u_i|^2 + \frac{1}{\alpha}\sum_{i\in\mathbb{Z}}\xi_k(|i|)|g_i|^2.$$

Thus

$$2\sum_{i\in\mathbb{Z}}\xi_k(|i|)u_i f(u_i) + 2\sum_{i\in\mathbb{Z}}\xi_k(|i|)g_i u_i \leq -\alpha\sum_{i\in\mathbb{Z}}\xi_k(|i|)|u_i|^2 + \frac{1}{\alpha}\sum_{|i|\geq k}|g_i|^2 \tag{23}$$

Using the estimates (22) and (23) in the equation (21) gives

$$\frac{d}{dt}\sum_{i\in\mathbb{Z}}\xi_k(|i|)|u_i|^2 + \alpha\sum_{i\in\mathbb{Z}}\xi_k(|i|)|u_i|^2 \leq \nu\frac{4C_0\|Q\|^2}{k} + \frac{1}{\alpha}\sum_{|i|\geq k}|g_i|^2 \tag{24}$$

Because $\mathbf{g}(\mathbf{t})$ is uniformly limited, for every $\varepsilon > 0$, there exists $K(\varepsilon)$ such that

$$\nu \frac{4C_0\|Q\|^2}{k} + \frac{1}{\alpha}\sum_{|i|\geq k} |g(t)_i|^2 \leq \varepsilon, \quad k \geq K(\varepsilon), \forall t \in \mathbb{R}.$$

The inequality (24) along with the relation above give

$$\frac{d}{dt}\sum_{i\in\mathbb{Z}} \xi_k(|i|)|u_i|^2 + \alpha \sum_{i\in\mathbb{Z}} \xi_k(|i|)|u_i|^2 \leq \varepsilon.$$

Then, Gronwall's lemma implies that

$$\sum_{i\in\mathbb{Z}} \xi_k(|i|)|u_i(t,\mathbf{u}_o)|^2 \leq e^{-\alpha t}\sum_{i\in\mathbb{Z}} \xi_k(|i|)|u_{o,i}|^2 + \frac{\varepsilon}{\alpha} \leq e^{-\alpha t}\|\mathbf{u}_o\|^2 + \frac{\varepsilon}{\alpha}.$$

Hence for every $\mathbf{u}_o \in Q$,

$$\sum_{i\in\mathbb{Z}} \xi_k(|i|)|u_i(t,\mathbf{u}_o)|^2 \leq e^{-\alpha t}\|Q\|^2 + \frac{\varepsilon}{\alpha}.$$

and therefore

$$\sum_{i\in\mathbb{Z}} \xi_k(|i|)\,|u_i(t,\mathbf{u}_o)|^2 \leq \frac{2\varepsilon}{\alpha}, \quad \text{for } t \geq T(\varepsilon) := \frac{1}{\alpha}\ln\frac{\alpha\|Q\|^2}{\varepsilon}.$$

In conclusion we can state that Th. 4.3 is proved.

5. Compact global attractors

Definition 5.1. *A family $\{I_y|\ y \in Y\}$ of compact subsets I_y of $\mathfrak{B}$ is said to be a compact global attractor for the cocycle $\langle \mathfrak{B}, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ if the following conditions are fulfilled:*

(i) *the set*

$$\mathcal{I} := \bigcup \{I_y|\ y \in Y\} \tag{25}$$

is precompact;

(ii) *the family of subsets $\{I_y|\ y \in Y\}$ is invariant, i.e., $\varphi(t, I_y, y) = I_{\sigma(t,y)}$ for any $(t, y) \in \mathbb{R}_+ \times Y$;*

(iii)

$$\lim_{t\to+\infty}\sup_{y\in Y} \beta(\varphi(t, M, y), \mathcal{I}) = 0 \tag{26}$$

for any compact subset M from $\mathfrak{B}$.

Definition 5.2. *A cocycle φ is said to be dissipative if there exists a bounded subset $K \subset \mathfrak{B}$ such that for any bounded subset $B \subset \mathfrak{B}$ there exists a positive number $L = L(B)$ such that $\varphi(t, B, Y) \subseteq K$ for any $t \geq L(B)$, where $\varphi(t, B, Y) := \{\varphi(t, u, y)|\ (u, y) \in B \times Y\}$.*

Theorem 5.3. [4, Ch.II] *Assume that the metric space Y is compact and the cocycle $\langle \mathfrak{B}, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ is dissipative and asymptotically compact.*

Then the cocycle φ has a compact global attractor.

Theorem 5.4. *Under the Conditions **(C1)**-**(C3)** the equation (10) (the cocycle φ generated by the equation (10)) has a compact global attractor $\{I_g|\ g \in H(f)\}$.*

Proof. This statement follows from Theorems 3.1, 4.3 and 5.3. $\square$

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7. Conflict of Interest

The authors declare that the have not conflict of interest.

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