ALMOST PERIODIC SOLUTIONS OF LATTICE DYNAMICAL SYSTEMS WITH MONOTONE NONLINEARITY

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Abstract. The aim of this paper is studying the problem of almost periodicity of almost periodic lattice dynamical systems of the form $u_i' = \nu(u_{i-1} - 2u_i + u_{i+1}) - \lambda u_i + F(u_i) + f_i(t)$ $(i \in \mathbb{Z},\ \lambda > 0)$. We prove the existence a unique almost periodic solution of this system if the nonlinearity F is monotone.

1. Introduction

Denote by $\mathbb{R} := (-\infty, \infty)$, $\mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}$ and ℓ_2 the Hilbert space of all two-sided sequences $\xi = (\xi_i)_{i\in\mathbb{Z}}$ $(\xi_i \in \mathbb{R})$ with

$$\sum_{i\in\mathbb{Z}} |\xi_i|^2 < +\infty$$

and equipped with the scalar product

$$\langle \xi, \eta \rangle := \sum_{i\in\mathbb{Z}} \xi_i \eta_i .$$

Let (X, ρ) be a complete metric space with the distance ρ, $C(\mathbb{R}, X)$ be the space of all continuous functions $f : \mathbb{R} \to X$ equipped with the distance

$$d(f_1, f_2) := \sup_{L>0} \min\{\max_{|t|\le L} \rho(f_1(t), f_2(t)), L^{-1}\}. \tag{1}$$

The metric space $(C(\mathbb{R}, X), d)$ is complete and the distance d, defined by (1), generates on the space $C(\mathbb{R}, X)$ the compact-open topology.

Let $h \in \mathbb{R}$, $f \in C(\mathbb{R}, X)$, $f^h(t) := f(t + h)$ for any $t \in \mathbb{R}$ and $\sigma : \mathbb{R} \times C(\mathbb{R}, X) \to C(\mathbb{R}, X)$ be a mapping defined by $\sigma(h, f) := f^h$ for any $(h, f) \in \mathbb{R} \times C(\mathbb{R}, X)$. Then [2, Ch.I] the triplet $(C(\mathbb{R}, X), \mathbb{R}, \sigma)$ is a shift dynamical system (or Bebutov's dynamical system) on he space $C(\mathbb{R}, X)$. By $H(f)$ the closure in the space $C(\mathbb{R}, X)$ of $\{f^h |\ h \in \mathbb{R}\}$ is denoted.

Recall that a subset $A \subset \mathbb{R}$ is called relatively dense if there exits a positive number l such that

$$A \bigcap [a, a + l] \neq \emptyset \tag{2}$$

for any $a \in \mathbb{R}$.

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A function $f \in C(\mathbb{R}, X)$ is called almost periodic [3, 8], if for any positive number ε the set

$$\mathcal{F}(f,\varepsilon) := \{\tau \in \mathbb{R} |\ \rho(f(t+\tau), f(t)) < \varepsilon \ \text{ for any } t \in \mathbb{R}\} \tag{3}$$

is relatively dense.

In this paper we study the problem of existence at least one almost periodic solution of the systems

$$u_i' = \nu(u_{i-1} - 2u_i + u_{i+1}) - \lambda u_i + F(u_i) + f_i(t) \ (i \in \mathbb{Z}), \tag{4}$$

where $\lambda > 0$, $F \in C(\mathbb{R}, \mathbb{R})$ and $f \in C(\mathbb{R}, \ell_2)$ ($f(t) := (f_i(t))_{i\in\mathbb{Z}}$ for any $t \in \mathbb{R}$) is an almost periodic function.

The system (4) can be considered as a discrete (see, for example, [1, 7] and the bibliography therein) analogue of a reaction-diffusion equation in $\mathbb{R}$:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial^2 x} - \lambda u + F(u) + f(t,x),$$

where grid points are spaced h distance apart and $\nu = D/h^2$.

This study continues the authors works [6] devoted to the study the problem of existence of compact global attractor for (4) and the work [5] dedicated to the study the invariant sections of monotone nonautonomous dynamical systems and their applications to differen classes of evolution equations. The invariant sections play a very important role in the study the problem of existence of almost periodic (respectively, almost automorphic, recurrent and Poisson stable) solutions of differential equations.

The paper is organized as follows. In the second section we show that under some conditions the equation (4) generates a cocycle which plays a very important role in the study of the asymptotic properties of the equation (4). In the third section we prove that under some conditions the existence of a compact global attractor for the equation (4). The fourth section is dedicated to the study the invariant sections of the cocycle generated by the equation (4). In the fifth section we study the structure of the compact global attractor for the equation (4). Namely, we show that the equation (4) is convergent, i.e., it admits a compact global attractor $\mathcal{I} = \{I_y |\ y \in Y\}$ such that every set I_y consists of a single point. Finally, we are analyzing an example of the equation of the form (4) which illustrate our general results.

2. Cocycles

Consider a non-autonomous system

$$u_i' = \nu(u_{i-1} - 2u_i + u_{i+1}) - \lambda u_i + F(u_i) + f_i(t) \ (i \in \mathbb{Z}). \tag{5}$$

Below we use the following conditions.

Condition (**C1**). The function $f \in C(\mathbb{R}, \mathfrak{B})$ is almost periodic.

Condition (**C2**). The function $F \in C(\mathbb{R}, \mathbb{R})$ is Lipschitz continuous on bounded sets and $F(0) = 0$.

Denote by $\widetilde{F} : \ell_2 \to \ell_2$ the Nemytskii operator generated by F, i.e., $\widetilde{F}(\xi)_i := F(\xi_i)$ for any $i \in \mathbb{Z}$.

Condition (**C3**). The function F is monotone, i.e., there exists a number $\alpha \geq 0$ such that

(6) $$(x_1 - x_2)(F(x_1) - F(x_2)) \leq -\alpha |x_1 - x_2|^2$$

for any $x_1, x_2 \in \mathbb{R}$.

Lemma 2.1. *The following statements hold:*

(i) *if the function f satisfies the Conditions (C1), (C3) and $F(0) = 0$, then*

(7) $$F(s)s \leq -\alpha s^2$$

for any $s \in \mathbb{R}$;

(ii) *if the function F satisfies the Condition (C3), then the Nemytskii operator $\widetilde{F}$ generated by F possesses the following property*

(8) $$\langle u_1 - u_2, \widetilde{F}(u_1) - \widetilde{F}(u_2) \rangle \leq -\alpha \|u_1 - u_2\|^2$$

for any $u_1, u_2 \in \ell_2$.

Proof. □

Definition 2.2. *A function $F \in C(Y \times \mathfrak{B}, \mathfrak{B})$ is said to be Lipschitzian on bounded subsets from $\mathfrak{B}$ if for any bounded subset $B \subset \mathfrak{B}$ there exists a positive constant L_B such that*

(9) $$|F(y, v_1) - F(y, v_2)| \leq L_B |v_1 - v_2|$$

for any $v_1, v_2 \in B \subset \mathfrak{B}$.

Definition 2.3. *The smallest constant L (respectively L_B) with the property (**??**) is called Lipshchitz constant of function F (notation $Lip(F)$ (respectively, $Lip_B(F)$)).*

Let $B \subset \mathfrak{B}$, denote by $CL(Y \times B, \mathfrak{B})$ the Banach space of any Lipschitzian functions $F \in C(Y \times B, \mathfrak{B})$ equipped with the norm

$$||F||_{CL} := \max_{y \in Y} |F(y, 0)| + Lip_B(F).$$

Lemma 2.4. [1] *Under the Condition (**C2**) it is well defined the mapping $\widetilde{F} : \ell_2 \to \ell_2$ and*

$$\|\widetilde{F}(\xi) - \widetilde{F}(\eta)\| \leq Lip_B(F) \|\xi - \eta\|$$

for any $\xi, \eta \in \ell_2$, where $\| \cdot \|^2 := \langle \cdot, \cdot \rangle$ and $\| \cdot \|$ is the norm on the space ℓ_2.

For any $u = (u_i)_{i \in \mathbb{Z}}$, the discrete Laplace operator Λ is defined [7, Ch.III] from ℓ_2 to ℓ_2 component wise by $\Lambda(u)_i = u_{i-1} - 2u_i + u_{i+1}$ ($i \in \mathbb{Z}$). Define the bounded linear operators D^+ and D^- from ℓ_2 to ℓ_2 by $(D^+ u)_i = u_{i+1} - u_i$, $(D^- u)_i = u_{i-1} - u_i$ ($i \in \mathbb{Z}$).

Note that $\Lambda = D^+ D^- = D^- D^+$ and $\langle D^- u, v \rangle = \langle u, D^+ v \rangle$ for any $u, v \in \ell_2$ and, consequently, $\langle \Lambda u, u \rangle = -|D^+ u|^2 \leq 0$. Since Λ is a bounded linear operator acting on the space ℓ_2, it generates a uniformly continuous semi-group on ℓ_2.

Under the Conditions (**C1**) and (**C2**) the system of differential equations (5) can be written in the form of an ordinary differential equation

$$u' = \nu\Lambda u + \Phi(u) + f(t) \tag{10}$$

in the Banach space $\mathfrak{B} = \ell_2$, where $\Phi(u) := -\lambda u + \widetilde{F}(u)$ and $\Lambda(u)_i := u_{i-1} - 2u_i + u_{i+1}$ for any $u = (u_i)_{i\in\mathbb{Z}} \in \ell_2$. Along with the equation (10) we consider also it H-class, i.e., the family equations

$$u' = \nu\Lambda u + \Phi(u) + g(t), \tag{11}$$

where $g \in H(f)$.

The family of the equations (11) can be rewritten as follows

$$u' = F(\sigma(t,g),u) \quad (g \in H(f)), \tag{12}$$

where $F : H(f) \times \ell_2 \to \ell_2$ is defined by $F(g,u) := \nu\Lambda u + \Phi(u) + g(0)$. It easy to see that $F(\sigma(t,g),u) = \nu\Lambda u + \Phi(u) + g(t)$ for any $(t,u,g) \in \mathbb{R} \times \mathfrak{B} \times H(f)$.

Let $(Y,\mathbb{R},\sigma)$ be a dynamical system on the metric space Y.

Theorem 2.5. [6] *Under the Conditions (**C1**)-(**C3**) the following statements hold:*

- (i) *for any $(v,g) \in \ell_2 \times H(f)$ there exists a unique solution $\varphi(t,v,g)$ of the equation (11) passing through the point v at the initial moment $t = 0$ and defined on the semi-axis $\mathbb{R}_+ := [0,+\infty)$;*
- (ii) *$\varphi(0,v,g) = v$ for any $(v,g) \in \ell_2 \times H(f)$;*
- (iii) *$\varphi(t+\tau,v,g) = \varphi(t,\varphi(\tau,v,g),g^\tau)$ for any $t,\tau \in \mathbb{R}_+$, $v \in \ell_2$ and $g \in H(f)$;*
- (iv) *the mapping $\varphi : \mathbb{R}_+ \times \ell_2 \times H(f) \to \ell_2$ $((t,v,g) \to \varphi(t,v,g))$ for any $(t,v,g) \in \mathbb{R}_+ \times \ell_2 \times H(f)$ is continuous.*

Let Y be a complete metric space and $(Y,\mathbb{R},\sigma)$ be a dynamical system on Y.

Definition 2.6. *Recall* [2, Ch.I] *that $\langle \mathfrak{B},\varphi,(Y,\mathbb{R},\sigma)\rangle$ is said to be a cocycle over $(Y,\mathbb{R},\sigma)$ with the fiber $\mathfrak{B}$ if φ is a continuous mapping acting from $\mathbb{R}_+ \times \mathfrak{B} \times Y$ to $\mathfrak{B}$ and satisfying the following conditions:*

- (i) *$\varphi(0,u,y) = v$ for any $(v,y) \in \mathfrak{B} \times Y$;*
- (ii) *$\varphi(t+\tau,u,y) = \varphi(t,\varphi(\tau,u,t),\sigma(\tau,y))$ for any $t,\tau \in \mathbb{R}_+$ and $(u,y) \in \mathfrak{B} \times Y$.*

Corollary 2.7. *Under the conditions of Theorem 2.5 the equation (10) (respectively, the family of equations (11)) generates a cocycle $\langle \ell_2,\varphi,(H(f),\mathbb{R},\sigma)\rangle$ over the shift dynamical system $(H(f),\mathbb{R},\sigma)$ with the fiber ℓ_2.*

Theorem 2.8. *Under the conditions (C1)-(C3) the cocycle $\langle \ell_2,\varphi,(H(f),\mathbb{R},\sigma)\rangle$ generated by the equation (10) possesses the following property:*

$$\|\varphi(t,v_1,g) - \varphi(t,v_2,g)\| \le e^{-(\lambda+\alpha)t}\|v_1 - v_2\| \tag{13}$$

for any $v_1, v_2 \in \ell_2$, $t \ge 0$ and $g \in H(f)$.

Proof. □

3. Compact global attractors

Definition 3.1. *A family $\{I_y|\ y \in Y\}$ of compact subsets I_y of $\mathfrak{B}$ is said to be a compact global attractor for the cocycle $\langle \mathfrak{B}, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ if the following conditions are fulfilled:*

(i) *the set*
$$\mathcal{I} := \bigcup \{I_y|\ y \in Y\}$$
is precompact;
(ii) *the family of subsets $\{I_y|\ y \in Y\}$ is invariant, i.e., $\varphi(t, I_y, y) = I_{\sigma(t,y)}$ for any $(t, y) \in \mathbb{R}_+ \times Y$;*
(iii)
$$\lim_{t\to+\infty} \sup_{y\in Y} \beta(\varphi(t, M, y), \mathcal{I}) = 0$$

for any compact subset M from $\mathfrak{B}$.

Definition 3.2. *A cocycle φ is said to be dissipative if there exists a bounded subset $K \subset \mathfrak{B}$ such that for any bounded subset $B \subset \mathfrak{B}$ there exists a positive number $L = L(B)$ such that $\varphi(t, B, Y) \subseteq K$ for any $t \geq L(B)$, where $\varphi(t, B, Y) := \{\varphi(t, u, y)|\ (u, y) \in B \times Y\}$.*

Theorem 3.3. [4, Ch.II] *Assume that the metric space Y is compact and the cocycle $\langle \mathfrak{B}, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ is dissipative and asymptotically compact.*

Then the cocycle φ has a compact global attractor.

Theorem 3.4. *Under the Conditions (**C1**)-(**C3**) the equation (10) (the cocycle φ generated by the equation (10)) has a compact global attractor $\{I_g|\ g \in H(f)\}$.*

Proof. □

4. Invariant sections of monotone nonautonomous dynamical systems

Below we prove that under some conditions a nonautonomous dynamical system admits an invariant continuous section.

Let $(Y, \mathbb{R}, \sigma)$ be a two-sided dynamical system, $(X, \mathbb{R}_+, \pi)$ be a semi-group dynamical system and $h : X \to Y$ be a homomorphism of $(X, \mathbb{R}_+, \pi)$ onto $(Y, \mathbb{R}, \sigma)$.

Let $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ be a nonautonomous dynamical system.

Recall that a point $x \in X$ (respectively, a motion $\pi(t, x)$) is said to be almost periodic, if the continuous mapping $\pi(\cdot, x) : \mathbb{T} \to X$ is almost periodic.

Definition 4.1. *A mapping $\gamma : Y \mapsto X$ is called a continuous invariant section if the following conditions are fulfilled:*

(i) *$h(\gamma(y)) = y$ for all $y \in Y$;*
(ii) *$\gamma(\sigma(t, y)) = \pi(t, \gamma(y))$ for any $y \in Y$ and $t \in R_+$;*
(iii) *γ is continuous.*

Lemma 4.2. *Let $(X,\mathbb{T}_1,\pi)$ and $(Y,\mathbb{T}_2,\sigma)$ be two dynamical systems, $\mathbb{T}_1\subseteq\mathbb{T}_2$ and $\gamma : Y\mapsto X$ be a continuous invariant section. If the point $x\in X$ (respectively, the motion $\pi(t,x)$) is almost periodic, then the point $y=h(x)$ (respectively, the motion $\sigma(y,t)=\sigma(t,h(x))=h(\pi(t,x))$ for any $t\in\mathbb{T}_1$) is also almost periodic.*

Proof. □

Remark 4.3. *A continuous section $\gamma\in\Gamma(Y,X)$ is invariant if and only if $\gamma\in\Gamma(Y,X)$ is a stationary point of the semigroup $\{S^t \mid t\in\mathbb{R}_+\}$, where $S^t:\Gamma(Y,X)\to\Gamma(Y,X)$ is defined by the equality $(S^t\gamma)(y):=\pi(t,\gamma(\sigma(-t,y)))$ for all $y\in Y$ and $t\in\mathbb{R}_+$.*

Theorem 4.4. *Let $\langle\ell_2,\varphi,(H(f),\mathbb{R},\sigma)\rangle$be the cocycle generated by the equation (10). Under the condition (C1)-(C3) there exists a unique invariant section $\nu:H(f)\to\ell_2$ of the cocycle φ and*

$$\|\varphi(t,v,g)-\nu(\sigma(t,g))\|\le e^{-\alpha t}\|v-\nu(g)\| \tag{14}$$

for any $t\in\mathbb{R}_+$ and $v\in\ell_2$.

Proof. □

Corollary 4.5. *Under the Conditions (C!)-(C3) the equation (10) has a unique almost periodic solution.*

5. Convergent nonautonomous lattice dynamical systems

Let $\langle W,\varphi,(Y,\mathbb{T},\sigma)\rangle$ (or shortly φ) be a cocycle over dynamical system $(Y,\mathbb{T},\sigma)$ with the fiber W.

Definition 5.1. *A cocycle $\langle W,\varphi,(Y,\mathbb{T},\sigma)\rangle$ wit the compact base space Y is said to be convergent if the following conditions are fulfilled:*

(i) *the cocycle φ admits a compact global attractor $\mathcal{I}=\{I_y|\ y\in Y\}$;*
(ii) *for any $y\in Y$ the set I_y consists of a single point $\{w_y\}$, i.e., $I_y=\{w_y\}$.*

Theorem 5.2. *Under the Conditions (**C1**)-(**C3**) the equation (10) (the cocycle φ generated by the equation (10)) is convergent, i.e., it has a compact global attractor $\mathcal{I}=\{I_g|\ g\in H(f)\}$ such that for any $g\in H(f)$ the set I_g consists of a single point.*

Proof. □

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7. Conflict of Interest

The authors declare that the have not conflict of interest.

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