LDS tasks

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Содержание

1 Almost periodic motions of LDS

1.1 $\sup_{\|u\|=1}(\Lambda u, u)$

Лемма 1.1. $\sup_{\|u\|=1}(\Lambda u, u) < 0$.

Доказательство.

$$
\begin{aligned}
\sup_{\|u\|=1}(\Lambda u, u) &= \sup_{\|u\|=1} \sum_i u_{i+1}u_i - 2u_i^2 + u_{i-1}u_i \\
&= \sup_{\|u\|=1} \sum_i u_{i+1}u_i - 2\sum_i u_i^2 + \sum_i u_{i-1}u_i \\
&= \sup_{\|u\|=1} 2\sum_i u_{i+1}u_i - 2\sum_i u_i^2 \\
&= \sup_{\|u\|=1} -\left(\sum_i u_i^2 + \sum_i u_{i+1}^2 - 2\sum_i u_{i+1}u_i\right) \\
&= \sup_{\|u\|=1} -\sum_i (u_i - u_{i+1})^2 < 0.
\end{aligned}
$$

□

1.2 Lagrange stability criterion for $f \in C(\mathbb{R}, \ell^2)$

Лемма 1.2. *Let (M, d) be a metric space. A subset $A \subset M$ is totally bounded if and only if*

$$\forall \varepsilon > 0 \ \exists \text{ compact } K \subset M : \quad \sup_{x \in A} d(x, K) < \varepsilon.$$

Лемма 1.3. *$f(\mathbb{R}, \ell^2)$is Largange stable if and only if the set*

1. *functions $\{f_i\}$ are equicontinuous, e.g. $\forall\varepsilon > 0 \exists\delta > 0$ such that $|f_i(t_1) - f_i(t_2)| < \varepsilon, \forall i \in \mathbb{Z}, |t_1 - t_2| < \delta$ where $t_1, t_2 \in \mathbb{R}$;*
2. *functions $\{f_i\}$ are uniformly bounded, e.g. $\exists M > 0$ such that $|f_i(t)| < M, \forall i \in \mathbb{Z}, \forall t \in \mathbb{R}$;*
3. *$\forall\varepsilon > 0 \exists n_\varepsilon \in \mathbb{N} : \sum_{|i| \geqslant n_\varepsilon} |f_i(t)|^2 < \varepsilon, \forall t \in \mathbb{R}$*

Доказательство. A set $\Sigma_f := \{f^h \mid h \in \mathbb{R}\}$ is totally bounded if and only if $\forall\varepsilon > 0 \exists$ compact set $C : d(f, C) < \varepsilon, \forall f \in \Sigma_f$.

Then, for $n \in \mathbb{N}$ consider the projection

$$P_n : C(\mathbb{R}, \ell^2) \to C(\mathbb{R}, \ell^2), \qquad [P_n(f)]_m = \begin{cases} f_m(t), & m \leqslant n, \\ 0, & m > n. \end{cases}$$

For the given Σ_f and $\varepsilon > 0$, choose n_ε such that

$$\sum_{n=n_\varepsilon}^{\infty} |f_n^h|^2 < \frac{\varepsilon}{2} \quad \forall h \in \mathbb{R}.$$

Then let $C = P_{n_\varepsilon}(\Sigma_f)$. C is a compact set because it suffices Arzelà–Ascoli theorem; indeed f^h hence $P_{n_\varepsilon}(\Sigma_f)$ is uniformly bounded and equicontinuous. By choice of n_ε, we have

$$d(f^h, C) = \| f^h - P_{n_\varepsilon}(f^h)\|_2 < \varepsilon \quad \forall h \in \mathbb{R},$$

So the criterion is satisfied, hence Σ_f is totally bounded. □

1.3 Example of Lipschitz continuous function

Пример 1.1. Let $F \in C(\mathbb{R}, \mathbb{R})$ be a function satisfying the Lipschitz condition: $(F(x_1) - F(x_2))(x_1 - x_2) \leqslant -\alpha|x_1 - x_2|^2 \quad \alpha > 0 \forall x_1, x_2 \in \mathbb{R}$. To prove that the function $\Phi : \ell^2 \to \ell^2$ defined by $[\Phi u]_i = F(u_i) \forall i \in \mathbb{Z}$ satisfies the condition: $\langle \Phi(u_1) - \Phi(u_2), u_1 - u_2 \rangle_{\ell^2} \leqslant -\alpha \|u_1 - u_2\|_{\ell^2}^2$

Доказательство.

$$\begin{aligned} \langle \Phi(u_1) - \Phi(u_2), u_1 - u_2 \rangle_{\ell^2} &= \sum_{i \in \mathbb{Z}} (F(u_{1,i}) - F(u_{2,i}))(u_{1,i} - u_{2,i}) \\ &\leqslant -\alpha \sum_{i \in \mathbb{Z}} |u_{1,i} - u_{2,i}|^2 = -\alpha \|u_1 - u_2\|_{\ell^2}^2. \end{aligned}$$

□