The autonomous reaction-diffusion LDS

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Outline

Let $(\mathfrak{X}, d)$ be a complete metric space, and let $\mathfrak{P}_{cc}(\mathfrak{X})$ denote the collection of all *non-empty compact subsets* of $\mathfrak{X}$. The distance between two points $x, y \in \mathfrak{X}$ is given by

$$d(x, y) = d(y, x).$$

And between two sets $A, B \in \mathfrak{P}_{cc}(\mathfrak{X})$ is given by

$$\operatorname{dist}(A, B) := \sup_{a \in A} \operatorname{dist}(a, B) = \sup_{a \in A} \inf_{b \in B} d(a, b)$$

Definition

Let ℓ^2 be the Hilbert space of real-valued square summable bi-infinite sequences $u = (u_i)_{i\in\mathbb{Z}}$ with norm and inner product

$$\|u\| := \left(\sum_{i\in\mathbb{Z}} u_i^2\right)^{1/2}, \quad \langle u, v\rangle := \sum_{i\in\mathbb{Z}} u_i v_i \quad \text{for} \quad u, v \in \ell^2.$$

Note that it is well-known that ℓ^2 is a complete metric space.

Definition

An **autonomous semi-dynamical system** on a metric space $(\mathfrak{X}, \mathfrak{d})$ is given by a mapping $\varphi\colon \mathbb{R}^+ \times \mathfrak{X} \to \mathfrak{X}$, which satisfies the properties:

- i *initial condition:* $\varphi(0, x_0) = x_0$ for all $x_0 \in \mathfrak{X}$,
- ii *semi-group under composition:* $\varphi(s+t, x_0) = \varphi(s, \varphi(t, x_0))$ for all $s, t \in \mathbb{R}^+$, $x_0 \in \mathfrak{X}$,
- iii *continuity:* the mapping $(t, x) \mapsto \varphi(t, x)$ is continuous at all points $(t_0, x_0) \in \mathbb{R}^+ \times \mathfrak{X}$.

Definition

A semi-dynamical system $\{\varphi(t)\}_{t\geq 0}$ on a complete metric space $(\mathfrak{X}, \mathfrak{d})$ is said to be **asymptotically compact** if, for every sequence $\{t_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}^+$ with $t_k \to \infty$ as $k \to \infty$ and every bounded sequence $\{x_k\}_{k\in\mathbb{N}}$ in $\mathfrak{X}$, the sequence $\{\varphi(t_k, x_k)\}_{k\in\mathbb{N}}$ has a convergent subsequence.

Definition

A set $B_0 \subset \mathfrak{X}$ is said to be **absorbing** for a dynamical system $(\mathfrak{X}, \varphi)$ if for any bounded set B in $\mathfrak{X}$ there exists $t_0 = t_0(B)$ such that $\varphi(t, B) \subset B_0$ for every $t \geq t_0$.

A bounded closed set $A_1 \subseteq X$ is called a **global attractor** for a dynamical system $(\mathfrak{X}, \varphi)$, if

1. A_1 is an invariant set, i.e., $\varphi(t, A_1) = A_1$ for any $t > 0$;
2. the set A_1 uniformly attracts all trajectories starting in bounded sets,
   i.e., for any bounded set B from $\mathfrak{X}$
   $$\limsup_{t \to \infty} \{\mathrm{dist}(\varphi(t, y), A_1) : y \in B\} = 0.$$

Theorem

Let $\{\varphi(t)\}_{t\geq 0}$ be an autonomous semi-dynamical system on a complete metric space $(\mathfrak{X}, \mathfrak{d})$ which is asymptotically compact and has a closed and bounded absorbing set $Q \subset \mathfrak{X}$. Then φ has an attractor A, which is contained in Q and is given by

$$A = \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} \varphi(s, Q)}.$$

Asymptotic tails property: autonomous systems Let $\varphi = (\varphi_i)_{i \in \mathbb{Z}}$ be an autonomous semi-dynamical system on the Hilbert space $(\ell^2, ||\cdot||)$ and let B be a positively invariant, closed and bounded subset of ℓ^2, which is φ-positive invariant. Then φ is said to satisfy an asymptotic tails property in B if for every $\epsilon > 0$ there exist $T(\epsilon) > 0$ and $I(\epsilon) \in \mathbb{N}$ such that

$$\sum_{|i|>I(\epsilon)} |\varphi_i(t, x_0)|^2 \leq \epsilon \quad \forall x_0 \in B \text{ and } t \geq T(\epsilon).$$

Lemma (2.5)

Let Assumption about asymptotic tails property hold. Then the semi-dynamical system ϕ is asymptotically compact in B.

Lattice dynamical systems (LDS) are essentially infinite dimensional systems of ordinary differential equations (ODEs). In particular, they can be formulated as ordinary differential equations on a Hilbert or Banach space of bi-infinite sequences. LDS may arise from discretisation of continuum models or as infinite dimensional counterparts of finite ODE models. A classical lattice dynamical system is based on a **reaction-diffusion equation**

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} - \lambda u + f(u) + g(x),$$

where λ and ν are positive constants, on a one-dimensional domain $\mathbb{R}$. It is obtained by using a central difference quotient to discretise the Laplacian. Setting the stepsize scaled to equal 1 leads to the infinite dimensional system of ordinary differential equations

$$\frac{du_i}{dt} = \nu(u_{i-1} - 2u_i + u_{i+1}) - \lambda u_i + f(u_i) + g_i \quad i \in \mathbb{Z},$$

Consider the autonomous LDS

$$\frac{du_i}{dt} = \nu(u_{i-1} - 2u_i + u_{i+1}) + f(u_i) + g_i, \quad i \in \mathbb{Z} \quad (0)$$

in the space ℓ^2, which will be investigated here under the following assumptions.

Assumption 1

The function $f:\mathbb{R}\to\mathbb{R}$ is a continuously differentiable function satisfying

$$f(s)s \leq -\alpha s^2 \quad \forall\, s \in \mathbb{R},$$

for some $\alpha > 0$ with bounded derivative f'.

Assumption 2

Let $\mathbf{g} = (g_i)_{i\in\mathbb{Z}} \in \ell^2$.

Definition

Define the operator $\Lambda : \ell^2 \to \ell^2$ by

$$(\Lambda \mathbf{u})_i = u_{i-1} - 2u_i + u_{i+1}, \quad i \in \mathbb{Z}$$

and the operators $\mathbf{D}^+, \mathbf{D}^- : \ell^2 \to \ell^2$ by

$$(\mathbf{D}^+ \mathbf{u})_i = u_{i+1} - u_i, \quad (\mathbf{D}^- \mathbf{u})_i = u_{i-1} - u_i, \quad i \in \mathbb{Z}.$$

It is straightforward to check that

$$-\Lambda = \mathbf{D}^+\mathbf{D}^- = \mathbf{D}^-\mathbf{D}^+ \quad \text{and} \quad \langle \mathbf{D}^-\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{D}^+\mathbf{v} \rangle \quad \forall \mathbf{u}, \mathbf{v} \in \ell^2,$$

and hence $\langle \Lambda \mathbf{u}, \mathbf{u} \rangle = -\|\mathbf{D}^+\mathbf{u}\|^2 \leq 0$ for any $\mathbf{u} \in \ell^2$. Λ operator is a linear, bounded and nonpositive operator.

The lattice system (0) can be written as an ODE

$$\frac{du}{dt} = \nu \Lambda u + F(u) + g \quad (*)$$

on ℓ^2, where $g = (g_i)_{i \in \mathbb{Z}} \in \ell^2$, $F : \ell^2 \to \ell^2$ is given component wise by $F_i(u) := f(u_i)$ for some continuously differentiable globally Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$. It follows that the function on the right side of the infinite dimensional ODE $(*)$ maps ℓ^2 into itself and is globally Lipschitz on ℓ^2.

Theorem 1.2 (Global). Let the function $f(t,x) : \mathbb{R} \times \mathfrak{B} \to \mathfrak{B}$, continuous with respect to t, satisfy the following conditions for $t \in [a,b]$, $x \in \mathfrak{B}$:

$$\|f(t,x)\| \leq M_1 + M_0\|x\|, \tag{1}$$

$$\|f(t,x_2) - f(t,x_1)\| \leq M_2\|x_2 - x_1\|, \tag{2}$$

where M_0, M_1, and M_2 are constants. Then, for any $x_0 \in \mathfrak{B}$ and $t_0 \in [a,b]$, the differential equation $\frac{dx}{dt} = f(t,x)$ has a unique solution $x = \varphi(t)$ on the entire interval $[a,b]$, satisfying the initial condition $\varphi(t_0) = x_0$.

Under Theorem 1.2 conditions, equation $(*)$ generates a semigroup dynamical system (ℓ^2, φ) in the space ℓ^2, where $\varphi(t,x) := u(t,x)$ and $u(t,x)$ is the solution of equation $(*)$ with the initial condition $u(0,x) = x$ $(x \in \ell^2)$.

LDS Problem

Theorem

Suppose that Assumptions are satisfied. The autonomous semi-dynamical system $\{\varphi(t)\}_{t\geq 0}$ generated by the ODE $(*)$ on ℓ^2 has a global attractor $\mathcal{A}$ in ℓ^2.

$$\frac{du}{dt} = \nu\Lambda u + F(u) + g \quad (*)$$

Taking the inner product of equation $(*)$ with $\mathbf{u}$ gives

$$\frac{d}{dt}\|\mathbf{u}\|^2 = 2\nu\langle\Lambda\mathbf{u},\mathbf{u}\rangle + 2\langle F(\mathbf{u}),\mathbf{u}\rangle + 2\langle\mathbf{g},\mathbf{u}\rangle$$

$$\leq 2\sum_{i\in\mathbb{Z}} u_i f(u_i) + 2\sum_{i\in\mathbb{Z}} g_i u_i \leq -\alpha\|\mathbf{u}\|^2 + \frac{\|\mathbf{g}\|^2}{\alpha},$$

Since Assumption 1 gives $2\sum_{i\in\mathbb{Z}} u_i f(u_i) \leq -2\alpha\|\mathbf{u}\|^2$ and Young's inequality gives $2\sum_{i\in\mathbb{Z}} g_i u_i \leq \alpha\|\mathbf{u}\|^2 + \|\mathbf{g}\|^2/\alpha$. Hence, Gronwall's lemma implies that

$$\|\mathbf{u}(t)\|^2 \leq \|\mathbf{u}_0\|^2 e^{-\alpha t} + \frac{\|\mathbf{g}\|^2}{\alpha^2}\left(1 - e^{-\alpha t}\right), \quad t \geq 0 \quad (3.7)$$

Define the closed ball Q in ℓ^2 by

$$Q := \left\{\mathbf{u} \in \ell^2 : \|\mathbf{u}\|^2 \leq R^2 := 1 + \frac{\|\mathbf{g}\|^2}{\alpha^2}\right\}.$$

The estimate then implies that Q is an absorbing set for the autonomous semi-dynamical system φ. In fact, for any $\mathbf{u_0} \in B$, which is a bounded set in ℓ^2, it is straightforward to check that

$$\varphi(t, B) \subset Q \quad \forall t \geq \frac{2}{\alpha} \ln \|B\|,$$

where $\|B\| := \sup_{\mathbf{u} \in B} \|\mathbf{u}\|$.
Moreover, for every $\mathbf{u_0} \in Q$, by estimate (3.7) we have

$$\|\varphi(t, \mathbf{u_0})\|^2 \leq R^2 e^{-\alpha t} + R^2(1 - e^{-\alpha t}) = R^2,$$

In order to establish the asymptotic compactness property in Assumption 2 for the autonomous semi-dynamical system $\{\varphi(t)\}_{t\geq 0}$ in ℓ^2, the asymptotic tails property needs to be shown to hold. Consider a smooth function $\xi : \mathbb{R}^+ \to [0,1]$ satisfying

$$\xi(s) = \begin{cases} 0, & 0 \leq s \leq 1, \\ \in [0,1], & 1 \leq s \leq 2, \\ 1, & s \geq 2, \end{cases}$$

and note that there exists a constant C_0 such that $|\xi'(s)| \leq C_0$ for all $s \geq 0$. Notice that this function ξ or a similar function will be used repeatedly throughout this book. Then for a fixed $k \in \mathbb{N}$ (its value will be specified later), define

$$\xi_k(s) = \xi\left(\frac{s}{k}\right) \quad \text{for all} \quad s \in \mathbb{R}.$$

Given $\mathbf{u} \in \ell^2$, define $\mathbf{v} \in \ell^2$ component-wise as

$$v_i := \xi_k(|i|)u_i \quad i \in \mathbb{Z}.$$

Taking the inner product of equation $(*)$ with $\mathbf{v}$ gives

$$\frac{d}{dt}\langle \mathbf{u}, \mathbf{v}\rangle + \nu\langle D^+\mathbf{u}, D^+\mathbf{v}\rangle = \langle F(\mathbf{u}), \mathbf{v}\rangle + \langle \mathbf{g}, \mathbf{v}\rangle.$$

that is

$$\frac{d}{dt}\sum_{i\in\mathbb{Z}}\xi_k(|i|)|u_i|^2 + 2\nu\langle D^+\mathbf{u}, D^+\mathbf{v}\rangle = 2\sum_{i\in\mathbb{Z}}\xi_k(|i|)u_i f(u_i) + 2\sum_{i\in\mathbb{Z}}\xi_k(|i|)g_i u$$

(3.9)

Each term in equation will now be estimated. First,

$$\langle D^+\mathbf{u}, D^+\mathbf{v}\rangle = \sum_{i\in\mathbb{Z}}(u_{i+1}-u_i)(v_{i+1}-v_i)$$

$$= \sum_{i\in\mathbb{Z}}(u_{i+1}-u_i)\left[(\xi_k(|i+1|)-\xi_k(|i|))u_{i+1}+\xi_k(|i|)(u_{i+1}-u_i)\right]$$

$$= \sum_{i\in\mathbb{Z}}(\xi_k(|i+1|)-\xi_k(|i|))(u_{i+1}-u_i)u_{i+1}+\sum_{i\in\mathbb{Z}}\xi_k(|i|)(u_{i+1}-u_i)^2$$

$$\geq \sum_{i\in\mathbb{Z}}(\xi_k(|i+1|)-\xi_k(|i|))(u_{i+1}-u_i)u_{i+1}.$$

Since

$$\sum_{i\in\mathbb{Z}}\left(\xi_k(|i+1|)-\xi_k(|i|)\right)(u_{i+1}-u_i)u_{i+1} \le \sum_{i\in\mathbb{Z}}\frac{1}{k}|\xi'(s_i)|\cdot|u_{i+1}-u_i|\cdot|u_i|+|u$$

for some s_i between $|i|$ and $|i+1|$, and

$$\sum_{i\in\mathbb{Z}}|\xi'(s_i)||u_{i+1}-u_i||u_{i+1}| \le C_0\sum_{i\in\mathbb{Z}}\left(|u_{i+1}|^2+|u_i||u_{i+1}|\right) \le 4C_0\|\mathbf{u}\|^2.$$

Then it follows that for all $\mathbf{u}\in Q$ and $\mathbf{v}\in\ell^2$ defined component-wise as $v_i := \xi_k(|i|)u_i$ for $i\in\mathbb{Z}$,

$$\langle D^+\mathbf{u}, D^+\mathbf{v}\rangle \ge -\frac{4C_0\|Q\|^2}{k} \quad (3.10)$$

where $\|Q\| := \sup_{\mathbf{u}\in Q}\|\mathbf{u}\|$.

On the other hand, by Assumption 1,

$$2\sum_{i\in\mathbb{Z}}\xi_k(|i|)u_if(u_i)\leq -2\alpha\sum_{i\in\mathbb{Z}}\xi_k(|i|)|u_i|^2$$

and by Young's inequality

$$2\sum_{i\in\mathbb{Z}}\xi_k(|i|)g_iu_i\leq \alpha\sum_{i\in\mathbb{Z}}\xi_k(|i|)|u_i|^2+\frac{1}{\alpha}\sum_{i\in\mathbb{Z}}\xi_k(|i|)|g_i|^2.$$

Thus,

$$2\sum_{i\in\mathbb{Z}}\xi_k(|i|)u_if(u_i)+2\sum_{i\in\mathbb{Z}}\xi_k(|i|)g_iu_i$$

$\leq -\alpha\sum_{i\in\mathbb{Z}}\xi_k(|i|)|u_i|^2+\frac{1}{\alpha}\sum_{|i|\geq k}|g_i|^2 \quad (3.11).$

Using the estimates (3.10) and (3.11) in equation (3.9) gives

$$\frac{d}{dt}\sum_{i\in\mathbb{Z}}\xi_k(|i|)|u_i|^2+\alpha\sum_{i\in\mathbb{Z}}\xi_k(|i|)|u_i|^2\leq \nu\frac{4C_0\|Q\|^2}{k}+\frac{1}{\alpha}\sum_{|i|\geq k}|g_i|^2 \quad (3.12).$$

The inequality (3.12) along with the relation above give

$$\frac{d}{dt}\sum_{i\in\mathbb{Z}}\xi_k(|i|)|u_i|^2+\alpha\sum_{i\in\mathbb{Z}}\xi_k(|i|)|u_i|^2\le\epsilon.$$

Then, Gronwall's lemma implies that

$$\sum_{i\in\mathbb{Z}}\xi_k(|i|)|u_i(t,\mathbf{u_0})|^2\le e^{-\alpha t}\sum_{i\in\mathbb{Z}}\xi_k(|i|)|u_{0,i}|^2+\frac{\epsilon}{\alpha}\le e^{-\alpha t}\|\mathbf{u_0}\|^2+\frac{\epsilon}{\alpha}.$$

Hence for every $\mathbf{u_0}\in Q$,

$$\sum_{i\in\mathbb{Z}}\xi_k(|i|)|u_i(t,\mathbf{u_0})|^2\le e^{-\alpha t}\|Q\|^2+\frac{\epsilon}{\alpha},$$

and therefore

$$\sum_{i\in\mathbb{Z}}\xi_k(|i|)|u_i(t,\mathbf{u_0})|^2\le\frac{2\epsilon}{\alpha},\quad \text{for}\quad t\ge T(\epsilon):=\frac{1}{\alpha}\ln\frac{\alpha\|Q\|^2}{\epsilon}.$$

This is the desired asymptotic tails property. The asymptotic compactness in ℓ^2 of φ in the absorbing set Q then follows from Lemma 2.5. This completes the proof of the Theorem.