LDS tasks

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11 июня 2025 г.

Содержание

1 Almost periodic motions of LDS

1.1 $\sup_{\|u\|=1}(\Lambda u, u)$

Лемма 1.1. $\sup_{\|u\|=1}(\Lambda u, u) = 0$.

Доказательство.

$$\begin{aligned}\sup_{\|u\|=1}(\Lambda u, u) &= \sup_{\|u\|=1} \sum_i u_{i+1}u_i - 2u_i^2 + u_{i-1}u_i \\ &= \sup_{\|u\|=1} \sum_i u_{i+1}u_i - 2\sum_i u_i^2 + \sum_i u_{i-1}u_i \\ &= \sup_{\|u\|=1} 2\sum_i u_{i+1}u_i - 2\sum_i u_i^2 \\ &= \sup_{\|u\|=1} -\left(\sum_i u_i^2 + \sum_i u_{i+1}^2 - 2\sum_i u_{i+1}u_i\right) \\ &= \sup_{\|u\|=1} -\sum_i (u_i - u_{i+1})^2 \leqslant 0.\end{aligned}$$

We provide an example that achieves the supremum. Let $k \in \mathbb{N}^*$ and consider the sequence

$$u_i^k = \begin{cases} 1/\sqrt{k}, & 0 < i \leqslant k, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\|u^k\| = 1$ and $(\Lambda u^k, u^k) = -2/k$. So $\sup_{\|u\|=1}(\Lambda u, u) \geqslant \sup_{k\in\mathbb{N}^*} -2/k = 0$. □

1.2 Lagrange stability criterion for $f \in C(\mathbb{R}, \ell^2)$

Теорема 1.2. *[Shc72][Bro75] The function $f \in C(\mathbb{R}, \mathfrak{L})$ is Lagrange stable if and only if $f : \mathbb{R} \to \mathfrak{L}$ is equicontinuous and set $f(\mathbb{R}) := \{f(t) | t \in \mathbb{R}\}$ is relatively compact in $\mathfrak{L}$.*

Лемма 1.3. *Let (M, d) be a metric space. A subset $A \subset M$ is totally bounded if and only if*

$$\forall \varepsilon > 0 \ \exists \text{ compact } K \subset M : \quad \sup_{x\in A} d(x, K) < \varepsilon.$$

Лемма 1.4. *$f \in C(\mathbb{R}, \ell^2)$is Lagrange stable if and only if the set*

1. *functions $\{f_i\}$ are equicontinuous, e.g. $\forall\varepsilon > 0 \exists\delta > 0$ such that $|f_i(t_1) - f_i(t_2)| < \varepsilon, \forall i \in \mathbb{Z}, |t_1 - t_2| < \delta$ where $t_1, t_2 \in \mathbb{R}$;*
2. *functions $\{f_i\}$ are uniformly bounded, e.g. $\exists M > 0$ such that $|f_i(t)| < M, \forall i \in \mathbb{Z}, \forall t \in \mathbb{R}$;*
3. *$\forall\varepsilon > 0 \exists n_\varepsilon \in \mathbb{N} : \sum_{|i|\geqslant n_\varepsilon} |f_i(t)|^2 < \varepsilon, \forall t \in \mathbb{R}$*

Доказательство. We use well-established equivalence between totally boundeness and relative compactness in metric spaces. A set $\Sigma_f := \{f^h \mid h \in \mathbb{R}\}$ is totally bounded if and only if $\forall\varepsilon > 0\exists$ compact set $C : d(f, C) < \varepsilon, \forall f \in \Sigma_f$. $\Longrightarrow$. If $f \in C(\mathbb{R}, \ell^2)$ is Lagrange stable, then by the Theorem 1.2, f is equicontinuous and $f(\mathbb{R})$ is relatively compact in ℓ^2. Remains to be shown that the second and third conditions are result of relative compactness. Since $f(\mathbb{R})$ is relatively compact, it is totally bounded. And totally bounded set in a metric space is also bounded so the second condition is satisfied.

To show that the third condition holds, we must use the fact that the set $f(\mathbb{R})$ is totally bounded. For any given $\varepsilon > 0$, since $f(\mathbb{R})$ is totally bounded, it can be covered by a finite number of balls of radius $\frac{\varepsilon}{2}$. This means there exists a finite set of points (an $\frac{\varepsilon}{2}$-net) $\{y_1, y_2, \ldots, y_k\}$ in ℓ^2 such that for any $t \in \mathbb{R}$, there is some y_j in this set for which

$$d(f(t), y_j) = \left(\sum_{i=-\infty}^{\infty} |f_i(t) - y_{j,i}|^2 \right)^{1/2} < \frac{\varepsilon}{2}.$$

Since each y_j is a point in ℓ^2, the tail of its series must converge to zero. That is, for each $j \in \{1, \ldots, k\}$,

$$\lim_{n \to \infty} \sum_{|i| \geqslant n} |y_{j,i}|^2 = 0.$$

Because there are only a finite number of points y_j, we can find a single natural number n_ε large enough such that for all $j \in \{1, \ldots, k\}$ simultaneously, the tails are uniformly small:

$$\sum_{|i| \geqslant n_\varepsilon} |y_{j,i}|^2 < \left(\frac{\varepsilon}{2}\right)^2.$$

Now, for any $t \in \mathbb{R}$, we choose the corresponding y_j that is within $\frac{\varepsilon}{2}$ of $f(t)$. Using the triangle inequality on the ℓ^2 norm for the tail of the sequence, we get:

$$\left(\sum_{|i| \geqslant n_\varepsilon} |f_i(t)|^2 \right)^{1/2} \leqslant \left(\sum_{|i| \geqslant n_\varepsilon} |f_i(t) - y_{j,i}|^2 \right)^{1/2} + \left(\sum_{|i| \geqslant n_\varepsilon} |y_{j,i}|^2 \right)^{1/2}.$$

The first term on the right is bounded by the total distance between $f(t)$ and y_j:

$$\left(\sum_{|i| \geqslant n_\varepsilon} |f_i(t) - y_{j,i}|^2 \right)^{1/2} \leqslant \left(\sum_{i=-\infty}^{\infty} |f_i(t) - y_{j,i}|^2 \right)^{1/2} < \frac{\varepsilon}{2}.$$

The second term on the right is bounded by our choice of n_ε:

$$\left(\sum_{|i| \geqslant n_\varepsilon} |y_{j,i}|^2 \right)^{1/2} < \frac{\varepsilon}{2}.$$

Combining these inequalities, we have for any $t \in \mathbb{R}$:

$$\left(\sum_{|i| \geqslant n_\varepsilon} |f_i(t)|^2 \right)^{1/2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since the choice of n_ε only depended on ε and not on the specific value of t, this condition holds uniformly for all $t \in \mathbb{R}$. This completes the proof of the forward direction.

$\Longleftarrow$. We need to show that $\forall \varepsilon > 0$ there exists an ε-close a compact set C. Then, for $n \in \mathbb{N}$ consider the projection

$$P_n \colon C(\mathbb{R}, \ell^2) \to C(\mathbb{R}, \ell^2), \qquad [P_n(f)]_m = \begin{cases} f_m(t), & m \leqslant n, \\ 0, & m > n. \end{cases}$$

For the given Σ_f and $\varepsilon > 0$, choose n_ε such that

$$\sum_{n=n_\varepsilon}^{\infty} |f_n^h|^2 < \frac{\varepsilon}{2} \quad \forall h \in \mathbb{R}.$$

Then let $C = P_{n_\varepsilon}(\Sigma_f)$. C is a compact set because it suffices Arzelà–Ascoli theorem; indeed f^h hence $P_{n_\varepsilon}(\Sigma_f)$ is uniformly bounded and equicontinuous. By choice of n_ε, we have

$$d(f^h, C) = \| f^h - P_{n_\varepsilon}(f^h)\|_2 < \varepsilon \quad \forall h \in \mathbb{R},$$

So the criterion is satisfied, hence Σ_f is totally bounded. $\square$

1.3 Example of Lipschitz continuous function

Пример 1.1. Let $F \in C(\mathbb{R}, \mathbb{R})$ be a function satisfying the Lipschitz condition: $(F(x_1) - F(x_2))(x_1 - x_2) \leqslant -\alpha |x_1 - x_2|^2 \quad \alpha > 0 \forall x_1, x_2 \in \mathbb{R}$. To prove that the function $\Phi : \ell^2 \to \ell^2$ defined by $[\Phi u]_i = F(u_i) \forall i \in \mathbb{Z}$ satisfies the condition: $\langle \Phi(u_1) - \Phi(u_2), u_1 - u_2 \rangle_{\ell^2} \leqslant -\alpha \|u_1 - u_2\|_{\ell^2}^2$

Доказательство.

$$\begin{aligned} \langle \Phi(u_1) - \Phi(u_2), u_1 - u_2 \rangle_{\ell^2} &= \sum_{i \in \mathbb{Z}} (F(u_{1,i}) - F(u_{2,i}))(u_{1,i} - u_{2,i}) \\ &\leqslant -\alpha \sum_{i \in \mathbb{Z}} |u_{1,i} - u_{2,i}|^2 = -\alpha \|u_1 - u_2\|_{\ell^2}^2. \end{aligned}$$

□

Список литературы

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[Bro75] Idel Usherovich Bronstein. «Расширения минимальных групп преобразований». В: (1975).