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Optimized control of pressure reducing valves in water distribution networks with dynamic topology

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Abstract

A dynamic topology aggregates zones in water distribution networks (WDNs) for improved pressure management and resilience to failure. Based on a sequential convex programming (SCP) approach, we propose an optimization method for the control of pressure reducing valves (PRV) in WDNs with dynamic topology. By restricting the SCP iterations to the feasible search space, we show that reliable convergence of the method is achieved. Using an experimental study in a large operational network, the optimization of PRV settings with a dynamic topology is shown to result in pressure reductions of 3.7% compared to optimized PRVs in a closed DMA structure.

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Keywords: Dynamic Topology; Water Distribution Networks; Resilience; Optimization

1. Introduction

The subdivision of water distribution networks (WDN) into zones known as district metered areas (DMA), is a popular approach to leakage management adopted by water companies. The DMAs are formed by permanently closing isolation valves at the boundaries of each zone, therefore these closed valves are commonly referred to as boundary valves (BV). Each DMA commonly has just a single inlet, where the flow is monitored. Therefore, at times of low demand, such as at night, leakage can be estimated and bursts identified. This then helps water companies to better plan their pipe repair and replacement strategies. In the UK, where DMAs are ubiquitous, leakage has been reduced by up to 30% over the last 25 years [1]. Other countries are also adopting the DMA in order to efficiently tackle their leakage [2,3]. As well as leakage management, DMAs also facilitate the simple installation of pressure management schemes. By reducing pressure in the network, background leakage can be reduced [4]. Water companies must still ensure that customers receive water at adequate levels of pressure in order to ensure an acceptable level of customer service. Pressure management is typically implemented by installing pressure reducing valves (PRV). By utilizing DMAs, a single PRV can be installed at the DMA inlet, which reduces pressure and therefore leakage within

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the DMA. As pointed out in [5], a pressure controlled area should have a control valve at every inlet. The use of DMAs reduces the number of inlets to a zone and simplifies the pressure management process.

Whilst DMAs greatly assist water utilities to understand the behaviour of networks, reduce leakage, and implement simple forms of pressure management, the permanent closure of isolation valves has introduced a number of problems. First, by closing off supply routes to customers, fewer independent supply routes exist which results in the resilience of the network to failure being severely reduced. Second, frictional energy losses are also increased in a WDN with reduced redundancy. This results in the pressure management of WDNs with reduced redundancy becoming less efficient. Finally, the permanent closure of isolation valves has introduced water quality problems due to the build up of stagnant water at the dead-ends of DMAs [6]. In the event of network failure, boundary valves may need to be opened which can result in dirty water reaching customers.

Operating DMAs with a dynamic topology is a recently introduced approach that can successfully eliminate the disadvantages of DMAs whilst still retaining their benefits in leakage monitoring [7,8]. The approach combines several advancements in hydraulic monitoring, modelling, control and actuation. By replacing closed boundary valves with self-powered, remote controlled valves, it is possible to aggregate DMAs together into large pressure managed zones in order to take advantage of improvements in resilience (i.e. more independent supplies) and improved pressure management (i.e. less frictional energy losses). When leakage monitoring is to be carried out, the dynamic boundary valves automatically close in order to revert the WDN back to the original DMA structure.

In [7], an optimization method based on sequential convex programming (SCP) was proposed for the control of two PRVs and two BVs within a real WDN based in the UK that operates with a dynamic topology. This network was set up to experimentally investigate pioneering developments in WDN technology and management. In this paper, the convergence properties of the SCP method are improved by restricting iterations to the feasible space. A null space algorithm is also used to improve computational performance. Since the introduction of the case study network in [7], an additional PRV has been installed in order to further optimize pressure management in this network. We use this new optimization method to control three PRVs in this paper, whilst the BVs follow a position controlled schedule that aggregates zones in the day, and closes the network topology at night for leakage monitoring.

The remainder of this paper is organized as follows. Section 2 describes the experimental programme where the dynamic topology is located. Section 3 outlines the proposed SCP method for the control of PRVs in a WDN with a dynamic topology. Results of the SCP method are shown in section 4. Section 5 outlines some final conclusions and recommendations for future work.

2. Experimental Programme

The experimental programme was set up on a real, operational WDN serving approximately 8,000 properties in the UK for the exploration of pioneering approaches to water supply management. At present, the WDN is operating with a dynamic topology. An elevation plot of the case study together with details of the dynamic topology are shown in Figure 1. The experimental programme consists of two DMAs that were originally separated by three closed boundary valves. Each DMA has its own independent supply inlet. The network was identified as an area needing improvements in both pressure management and resilience to failure due to the high number of critical customers in the area. The hydraulic model of the case study consists of 2,434 pipes and 2,374 nodes.

The installation of a dynamic topology involved the replacement of two boundary valves (BV) with self-powered, remote controlled globe diaphragm valves. In addition, three PRVs have been installed in this network with the same technology as the boundary valves. Both DMA inlets to the network are pressure managed (PRV 1 and PRV 3 in Figure 1). The technology includes:

- Advanced control pilots for stem position, open/closed loop control, and flow modulation [9]. Boundary valves can be programmed to open in order to aggregate DMAs for improved pressure management and resilience to failure, and close down to revert the WDN back to the original DMA structure when leakage monitoring is carried out at night. PRVs can be programmed to regulate pressure according to a time varying pressure profile, or according to more advanced control such as flow modulation, where the outlet pressure is regulated according to a locally measured flow.

- Telecommunications for receiving hydraulic data and updating control settings.

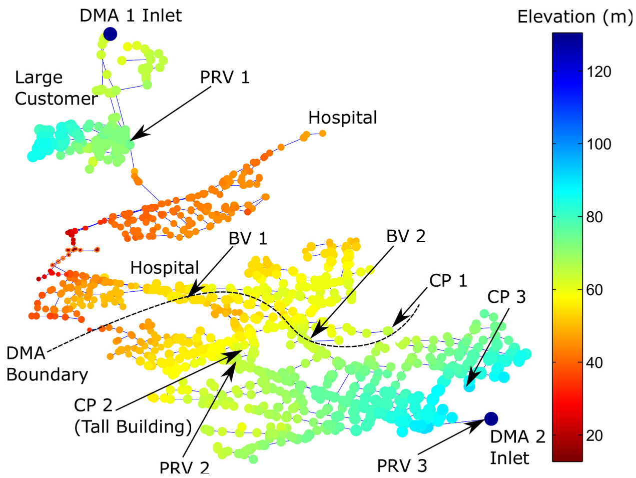


Fig. 1: Experimental programme elevation plot and details

- Insertion flow meter that uses the vortex shedding phenomenon to calculate flow [10]. The data is used to update customer demand profiles in a hydraulic model of the experimental programme in near real-time using the installed telecommunications. The flow signal can also be used to perform flow modulation control.
- Turbine for energy harvesting using the pressure differential of the valve [11]. The generation of power locally facilities more advanced control options in remote parts of a WDN where power may not be available.
- High speed (128S/s) pressure measurements at the valve inlet and outlet [12]. The more advanced forms of DMA management and control (i.e. a dynamic topology) could potentially introduce more hydraulic instabilities into a system. We measure pressure at high frequencies and transmit this in near real-time in order to instantly detect such behavior and guarantee the successful actuation of control.

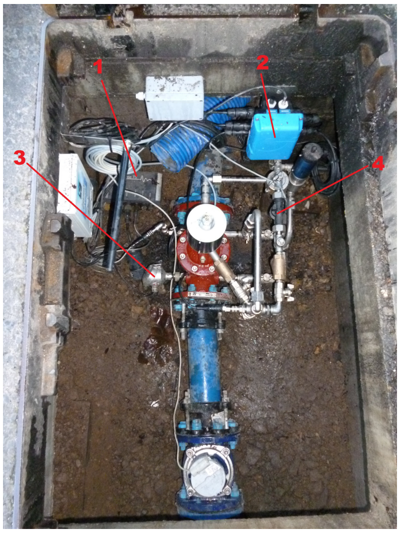
This technology is shown in a photograph of one of the BV installations in Figure 2a. This WDN represents a unique combination of control and data acquisition systems that can be used to explore pioneering techniques in WDN management. In the next section, an optimization method is outlined for the control of a WDN operating with a dynamic topology.

3. Optimization of PRV Control in a WDN with Dynamic Topology

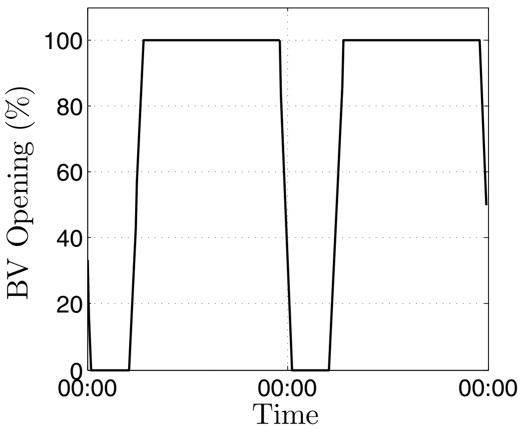
3.1. Background

The control of a dynamic topology with optimized PRV settings is a challenging problem due to the large size of the network model and nonlinear constraints associated with energy losses in the WDN. In [7], an off the shelf NLP solver was tested for the solution of the control optimization problem, however convergence was often slow and unreliable. It is therefore necessary to take into account the structure of the optimization problem in order to achieve acceptable computational speed and robustness.

Optimization of valve control has generally been studied with the intention of reducing either pressure or leakage in WDNs. In [13], a sequential linear programming method was used to minimize leakage for a small example network consisting of 25 nodes. The same network was used in [14] to demonstrate a sequential quadratic programming method. In addition, an objective function that minimizes the difference between target and actual nodal pressure was investigated in [14], which allowed minor pressure violations at nodes in order to achieve a greater overall reduction in network pressure. In [15], a nonlinear programming (NLP) problem was solved for the reduction of leakage by using a commercial NLP solver. In addition, the optimized PRV outlet pressure profile was correlated with the PRV flow to produce flow modulation curves. In [16] a valve placement and control problem was constructed as a polynomial



(a) BV Technology



(b) BV Schedule

Fig. 2: (a) One of the boundary valves with (1) InfraSense pressure monitoring (128s/s), (2) position control pilot, (3) insertion flow meter and (4) energy harvesting and (b) the boundary valve open-loop stem position schedule

optimization problem. This was undertaken by firstly approximating the Hazen-Williams equation with a quadratic relationship, and secondly, by assigning two variables representing bidirectional flow to each pipe. A combined valve placement and control problem was also solved in [17] using genetic algorithms, which are particularly useful when multiple objective functions are evaluated, but are not appropriate for real-time control applications due to the computational time required to produce a solution.

The approach to control optimization in this paper uses sequential convex programming (SCP). SCP methods are useful for real-time control applications because optimal solutions are often found to large scale optimization problems with fast and reliable convergence [18]. This is important for the control of dynamic topology in the experimental programme, which is currently being operated with near-real time data acquisition and control, and therefore requires an optimization solver for its control that is fast and reliable.

The SCP method presented in this paper extends the work in [7] by improving the reliability of convergence. Although SCP methods often find an optimal solution to NLP problems, the iterations can sometimes get stuck in the search space [19]. With the introduction of PRV 3 in the experimental programme (Figure 1) and boundary valves that follow a stem position schedule (i.e. open during the day, closed at night for leakage monitoring), convergence problems were sometimes experienced that did not occur in [7]. In order to avoid convergence problems, the iterations of the SCP method are restricted to the feasible search space in this paper. The rest of this chapter outlines this new approach to solving the PRV control optimization problem.

3.2. Problem Formulation

Optimized PRV settings are calculated by solving a series of NLPs that represent steady states of the WDN model at different points in time. The simulation of multiple steady states is known as an extended period simulation, where customer demand and reservoir levels vary throughout a period of time. Using the data acquisition equipment outlined in section 2, customer demand and reservoir levels are updated in near real-time. In addition, the boundary valves vary according to the time of day (as shown in Figure 2b), therefore the resistance of these particular valves in the hydraulic model and optimization also vary. Details of the resistance of valves is generally provided by the valve manufacturer. For each steady state simulation of the WDN, the customer demand, reservoir level and boundary valve resistance is constant.

A WDN is represented as a graph consisting a set of nodes N_n, where $|N_n| = n_n$, a set of reservoirs N_o, where $|N_o| = n_o$, a set of links N_p, where $|N_p| = n_p$, and a set of PRVs N_v, where $|N_v| = n_v$ and $N_v \subseteq N_p$. The optimization problem for minimizing the AZP (denoted as function f) in a WDN for a single steady state simulation is as follows:

$$\min_{q,h,\eta} f(q; h; \eta) := \sum_{j=1}^{n_n} \beta_j h_j, \tag{1a}$$

$$\text{Subject to: } A_{11}(q)q + A_{12}h + A_{10}h_0 + A_{13}\eta = 0 \tag{1b}$$

$$A_{12}^T q - d_j = 0 \tag{1c}$$

$$-q_i \leq 0, \ \forall i \in N_v \tag{1d}$$

$$-\eta_i \leq 0, \ \forall i \in N_v \tag{1e}$$

$$\underline{h}_j - h_j \leq 0, \ \forall j \in N_n \tag{1f}$$

where $h \in \mathbb{R}^{n_n}$ are the nodal piezometric heads, $q \in \mathbb{R}^{n_p}$ are the pipe flows, $\eta \in \mathbb{R}^{n_v}$ are the PRV settings, $A_{12} \in \mathbb{R}^{n_p \times n_n}$, $A_{13} \in \mathbb{R}^{n_p \times n_v}$, and $A_{10} \in \mathbb{R}^{n_p \times n_o}$ are incidence matrices describing the relationship between links and nodes, valves and reservoirs respectively, $d \in \mathbb{R}^{n_n}$ are water consumptions (assumed known), $h_0 \in \mathbb{R}^{n_o}$ are known heads, $\underline{h}_j \in \mathbb{R}^{n_n}$ are the minimum service level pressures. The square matrix $A_{11} \in \mathbb{R}^{n_p \times n_p}$ is a diagonal matrix with the elements

$$A_{11}(i, i) = r_i |q_i|^{n_i - 1}, i = 1, \ldots, n_p, \tag{2}$$

where $r \in \mathbb{R}^{n_p}$ is the vector of frictional resistance factors of the pipes and $n \in \mathbb{R}^{n_p}$ are constants related to the frictional head loss. In this article, the Hazen-Williams friction formula is used to model frictional losses; $n_i = 1.85$ and $r_i = \frac{10.675 L_i}{c_i^n D_i^{4.87}}$, where L_i, D_i and c_i denote the length, diameter and roughness coefficient of pipe i, respectively. For valves, $n_i = 2$ and an empirical value for r_i is generally supplied by the valve manufacturer.

The coefficients $\beta \in \mathbb{R}^{n_n}$ in (1a) relate the AZP to the nodal pressure by weighting each node proportionately to the length of all connected links. It is calculated as follows:

$$\beta = \frac{\texttt{abs}(A_{12}^T)L}{2 \sum_{i=1}^{n_p} L_i} \tag{3}$$

The equality constraints in (1b) and (1c) represent energy and mass conservation respectively. The modelling of PRVs in (1b) is undertaken by adding an additional linear head loss to a link containing a PRV. More details of this approach can be found in [7]. The inequalities in (1d) and (1e) come from the fact that PRVs must regulate pressure in a single flow direction. Finally, the inequality in (1f) is a constraint placed on the nodal heads which ensures customers receive an adequate water pressure. We denote a solution point in the search space as $x := [q^T \ h^T \ \eta^T]^T$.

The term $r_i q_i |q_i|^{n_i - 1}$ describing frictional energy losses in pipes in (1b) results in a nonlinear program (NLP) that is also non smooth at $q_i = 0$.

3.3. Strictly Feasible Sequential Convex Programming

Sequential convex programming (SCP) is an optimization method for non-convex optimization problems that possess a convex substructure. SCP finds solutions to nonconvex problems by solving a sequence of convex approximations of the original problem. It will often find a good local optimum with reliable convergence [20]. Two convex approximations can be formed from the problem formulation defined in (1). We denote these Subproblem A and Subproblem B.

Subproblem A is formed by solving (1) without (1a) and (1d) - (1f). This represents the system of hydraulic equations that is convex and has a unique solution for q and h for a given set of PRV settings (η) [7]). This system of equations is denoted $g(\eta)$ and can be written in matrix form as follows:

$$g(\eta) := \begin{bmatrix} A_{11}(q^s) & A_{12} \\ A_{12}^T & 0 \end{bmatrix} \begin{bmatrix} q_s \\ h_s \end{bmatrix} + \begin{bmatrix} A_{10}h_0 + A_{13}\eta \\ -d \end{bmatrix} \tag{4}$$

where we denote q_s and h_s as the flow and head calculated in a hydraulic simulation. In this paper, subproblem A, defined in (4), is solved using a null space algorithm [21]. The iterations of the null space algorithm start with an initial solution to (1c) by using, for example, a least square method. A null space is then generated and further Newton iterations are carried out in this space until (1b) is also satisfied. The method offers strong improvements in computational speed in comparison to other hydraulic solvers particularly when $n_p - n_n$ is small, which is generally the case for most real networks [22]. The other advantage of using the null space algorithm is its ability to handle links with a zero flow. Typically, methods such as the nodal Newton-Raphson method do not accurately calculate zero flows which can result in poor convergence when used in optimization problems. This is because when $q_i = 0$, $A_{11}^{-1}(q)$ becomes singular. The optimization of WDNs with a dynamic topology will always have zero flows because the boundary valves generally close at night resulting in a zero flow between interconnected DMAs.

Subproblem B is formed by applying the convex approximation used in [18] to (1a) - (1f). This results in the following program, which is linear is objective function and constraints and is therefore convex [23]:

$$\min_{q,h,\eta} f(q;h;\eta) := \sum_{j=1}^{n_n} \beta_j h_j, \tag{5a}$$

$$\text{Subject to: } NA_{11}(q^s)(q - q^s) + A_{11}(q^s)q^s + A_{12}h + A_{10}h_0 + A_{13}\eta = 0 \tag{5b}$$

$$A_{12}^T q - d_j = 0 \tag{5c}$$

$$-q_i \leq 0, \ \forall i \in N_v \tag{5d}$$

$$-\eta_i \leq 0, \ \forall i \in N_v \tag{5e}$$

$$\underline{h}_j - h_j \leq 0, \ \forall j \in N_n \tag{5f}$$

where $N = \texttt{diag}(n_i),\ i = 1, \ldots, n_p$. In [7], the SCP method proposed would iterate between subproblem A and subproblem B in order to converge to a solution. The method started at iteration $k = 1$ with a solution to subproblem A given $\eta^1 = 0$. Subproblem B was then solved to determine the next iteration of PRV settings, η^{k+1}, and this was used again in subproblem A. This process was repeated until the difference between q and q_s was sufficiently small.

In this paper, we propose the introduction of strictly feasible iterations in the SCP process, i.e. strictly feasible sequential convex programming (SFSCP). By ensuring that each iteration of the optimization process is strictly feasible, a more reliable convergence can be achieved. The result of this is shown in Figure 3, which shows a 3D plot of the search space and objective function contour when controlling 3 PRVs, where (1b), (1c) (1d), (1e) are satisfied. The surface that encompasses the axis intercept $(0, 0, 0)$ shows where (1f) is also satisfied. The red line shows iterations of the SCP method of [7] applied to the optimization problem in (1). It is seen that steps outside of the feasibility space are taken, and the optimization method gets stuck in a loop. The blue line shows the SFSCP method proposed in this paper applied to (1). It is seen that iterations are restricted to the feasible space where (1b)-(1f) are satisfied. This results in reliable convergence to a local optimal point.

The restriction of iterations to the feasible space is undertaken as follows:

1. For the first iteration, set $k = 0$ and the valves to be fully open $\eta^0 = 0$
2. Calculate q^s by solving subproblem A with input η^k
3. Solve subproblem B to get the next iterate of PRV settings η^{k+1}
4. Calculate a search direction:
$$d\eta^k = \eta^{k+1} - \eta^k$$
5. Solve $g(\eta^k + \alpha d\eta^k)$ in (4) with $\alpha = 1$ initially to get q^s and h^s.
6. IF the point $x := [q^s;\ h^s;\ \eta^k + \alpha d\eta^k]$ reduces the objection function f in (1a) AND satisfies (1b)-(1f), go to step 7, ELSE set $\alpha = \alpha/2$ and return to step 5.
7. IF $(f^k - f^{k+1}) < tol$ STOP, ELSE, set $k = k + 1$ and go to step 2

For this study, tol has been set to 10^{-6}. In the next section, we present the results of optimized control over a 24 hour period.

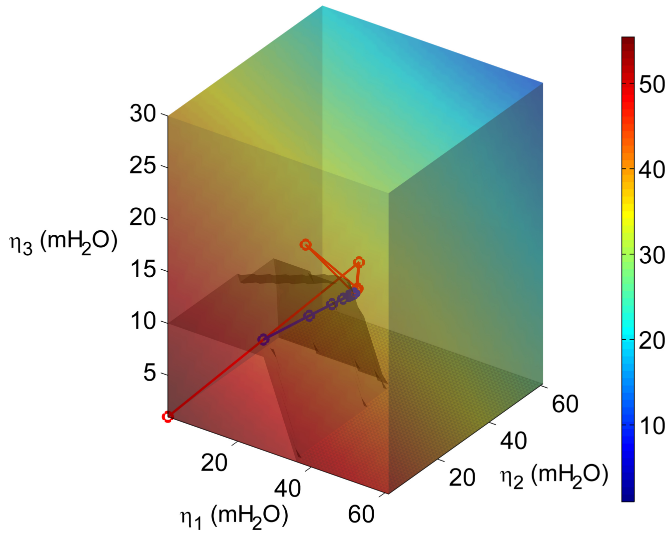
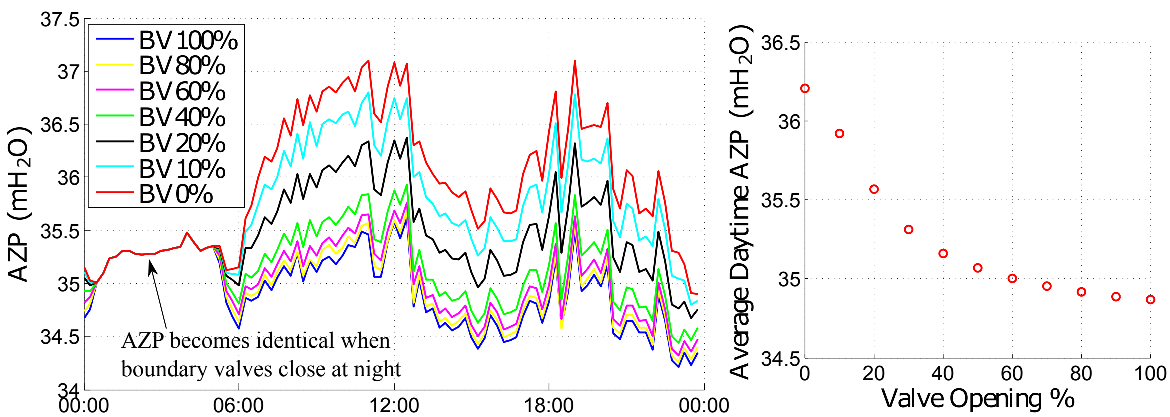


Fig. 3: Search space of AZP showing the convergence properties of SCP (red line) and SFSCP (blue line)



(a) AZP over 24 hours

(b) Average daytime AZP

Fig. 4: AZP results when the SFSCP method proposed in section 3 is applied to the experimental programme network for different levels of boundary valve opening %

4. Results

We simulate and optimize three PRVs in the experimental programme shown in Figure 1 for operations over a 24 hour period. The two boundary valves (BV) follow a predefined opening/closing schedule (open 07:00 - 00:30 and fully shut 02:30 - 05:30 as shown in Figure 2b). We discretized the 24 hour period into 15 minute intervals, which results in 96 separate NLPs to solve. This process is then repeated for 11 different levels of BV % opening. In total, 1056 NLPs are solved reliably and efficiently, with the optimization methods converging in 9 iterations on average.

The AZP throughout the 24 hour period at different % BV openings are shown in Figure 4a. It is seen that when the boundary valves are allowed to open further, the optimally controlled AZP reduces. The difference between opening 0% (i.e. a closed DMA topology) and 100% opening is 3.7%, which is an AZP reduction of approximately $1.5mH_2O$. At night, when the boundary valves close down during the operation of a dynamic topology, the AZP becomes equivalent in all simulations. In order to simplify results, the average daytime AZP (07:00 - 00:00) is plotted in Figure 4b for different BV % openings.

5. Conclusion

In this paper, we formulate a NLP that represents a PRV control optimization problem to minimize average zone pressure for networks with a dynamic topology. We show that the convergence of an SCP method can sometimes become unstable, and demonstrate improved reliability of convergence using a novel method that restricts iterations to the feasible search space. We solve the NLP over 1,000 times successfully, and demonstrate the reduction in AZP of up to 3.7% when a dynamic topology is in operation compared to an optimally controlled closed network topology. We present an experimental programme where a WDN is currently operating with a dynamic topology. This network represents a unique experimental site set up with the objective of exploring pioneering approach to WDN management. The hydraulic model was used as the test network for the optimization problem.

Future work should focus on how to incorporate the control optimization of PRVs with boundary valve control. This is a challenging problem because the boundary valves allow flow in two directions, which complicates the problem formulation. Other work in the area of dynamic topologies is exploring how the resilience and water quality is affected under different network configurations.

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