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Network Analysis, Control Valve Placement and Optimal Control of Flow Velocity for Self-Cleaning Water Distribution Systems

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Abstract

In this paper, we consider the proactive control of flow velocities to maximise the self-cleaning capacity of the drinking water distribution systems under normal operations both through a change of the network topology and through an optimal control of pressure reducing valve (PRV) settings. Inspired by line outage flow distribution in electrical networks, we show how a fast network graph analysis of link closures can be used to estimate the potential changes in flow velocities, which are then used to determine the most favourable pipes for closure. Where closing of pipes cannot be used because of other conflicting objectives, we consider the optimal control of PRVs to maximise self-cleaning at peak demand periods. We formulate a novel optimisation problem to maximise the network operations for increased self-cleaning capacity, while satisfying hydraulic and regulatory pressure constraints at demand nodes. A new smooth objective function approximation for cleaning capacity of the network is proposed along with a scalable sequential convex programming method to solve the resulting valve optimization problems. We use a published benchmark network as a case study to show the efficacy of these new approaches.

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Keywords: Water distribution systems, discolouration, proactive management, self-cleaning network, optimal PRV control, water flow-redistribution matrix, control valve placement, multi-objective trade-offs, sequential convex programming

1. Introduction

In addition to microbiological and chemical quality controls, more recent regulations also stipulate the minimization of aesthetic water quality concerns, which are mainly caused by discolouration incidents and associated bad taste or odour in water [1]. Although the pathways to discolouration and its occurrence are unavoidable, the magnitude or severity of discolouration and its frequency can be controlled through the optimisation of design, maintenance and operations [2]. One of three approaches [3] is to reduce or prevent particulates entering the DWDS by mainly improving treatment processes, and by reducing external contamination in repairs and backflow from leaks. The second approach is to design self-cleaning networks with daily flow velocities that prevent residence and material accumulation. In some cases the redesigning of oversized looped DWDSs (i.e. larger diameter looped networks with high

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fire fighting demands) to have self-cleaning velocities has been successful [3]. Since the above two approaches are capital intensive, a conventional approach taken by utilities for controlling discolouration events is the use of periodic flushing through fire hydrants, air/water scouring and pigging to remove accumulated particulates [1].

By analysing flushing experiments in a real DWDS with RPM and coupled with simulations with realistic demand data, it was shown in [4] that a minimum self-cleaning velocity can be defined for pipes to reduce possible fouling by "cleaning" with the day-to-day maximum flows in the system during the morning demand peak. Ideally, the network diameter sizing and connectivity would be designed to have high velocities and unidirectional flow to mobilise particles and keep pipes clean; this is rarely the case. An alternative is to introduce more unidirectional flows (to reduce residence time) with higher velocity by changing the topology of the network through the closure of selected pipe sections between isolation valves. It was shown that this could work in real networks in experiments where selected valves were closed over a few months [5,6]. Although the reconfiguration may result in some pipes having a velocity below the self-cleaning velocity threshold, the aim is to get an overall lower fouling rate for the whole area under consideration [5].

In the present article, we investigate the self-cleaning capacity of a WDS and how it can be optimized through a change of network topology (i.e. an appropriate selection of pipes for closure) and through optimized pressure control for self-cleaning under diurnal high demand periods. We show how a network graph analysis of pipe closures can be used to estimate the potential changes in flow velocities, which are then used to determine valves for closure. Where closing of pipes cannot be used for other reasons, we consider optimal control of pressure control valves to maximise self-cleaning at peak demand periods. We formulate a novel optimisation problem to maximise the network operations for increased self-cleaning capacity, while satisfying regulatory pressure constraints at demand nodes. A sequential convex programming method [7] is used to solve the resulting optimization problems. The two approaches of topological optimization and pressure management can also be used together.

2. Problem formulation: increasing flow velocity through topological and operational optimization

The optimization problem we pose here is "find links to close from a set of candidates (i.e. links in a set N_v, which are pipe sections between isolation valves)" and "optimise PRV settings" such that the number of pipes that satisfy self-cleaning velocity thresholds in all otherwise open pipes are maximised. The pipes for which we aim to increase self-cleaning capacity may also be an indexed subset from the set all pipes ($N_c \subset N_p$); for example, we may want to concentrate the cleaning capacity to a chosen selection of pipes that are known to be vulnerable to particle accumulation and discolouration processes. These can be determined based on customer complaints or on the presence of vulnerable customers such as those undergoing kidney dialysis. The problem can be mathematically formulated as:

$$\begin{aligned} \underset{w_j \in N_v;\ \text{PRV settings}}{\text{maximize}} \quad & f(\cdot) := \sum_{t\in T} \sum_{j\in N_c} k(v_j^t) \\ \text{subject to:} \quad & \text{Regulatory and performance constraints for the WDS,} \\ & \text{Constraints on number of valves closed,} \\ & \text{Mass and energy balance of the system,} \end{aligned} \tag{1}$$

where the binary variable $w_j \in \{0, 1\}$ defines whether a link j is closed or not and the time indices t lie in a given period T; for example, a high demand period. The function $k(v)$ indicates whether the magnitude of a velocity in the i^{th} pipe is above a threshold $v_{i,min}$ for that pipe, i.e.

$$k(v_i) = \begin{cases} 1, & \text{if } |v_i| > v_{i,min}, \\ 0, & \text{otherwise.} \end{cases}$$

The simultaneous optimization of the network topology (i.e. link closures) and PRV settings to maximize self-cleaning capacity in (1) is a difficult mixed integer nonlinear programming (MINLP) problem. These class of optimization problems for a WDS are especially challenging because they combine discrete decision variables with the difficulties of handling non-convex nonlinear constraints [8]. In addition, the problem in (1) has an objective function is non-smooth around $\pm v_{min}$, resulting in unbounded gradients around $\pm v_{min}$.

To circumvent the difficulties in solving (1), we decouple the design and operational decisions to approximately solve it in two stages. In the first stage, we iteratively use efficient hydraulic simulation and graph analysis tools to identify the (isolation) links that will result in the most change in pipe velocities. In the case where we want to optimize PRV settings to maximize the cleaning potential of a chosen topology, we propose a smooth logistic function approximation for the threshold indicator $k(\cdot)$ and effectively use a sequential convex optimization approach for solving the resulting optimisation problem.

3. Flow velocity changes under link closures: a line outage distribution factor (LODF) approach

Determining the 'impact' of changes in topology when some sections are closed off is a ubiquitous problem in WDS analysis. For example, utilities use critical link analysis to measure network vulnerability. This consists of isolating sections of pipes in the network model to simulate failure, running hydraulic analysis on the model with the corresponding links closed, and recording how many customers would be affected by such a failure. A general limitation of such an approach is that the combination of possible failure scenarios grows exponentially with network size and is not a tractable tool [9]. Similarly, in security analysis of the power grid, the impact of line outages on the electrical network is studied to put in place appropriate contingency plans [10]. Because a security analysis is often needed to be run in a real time operational scenario, fast scalable algorithms that use linear sensitivity methods are employed to approximately quantify the impact of failures. Specifically, the line outage distribution factor (LODF) is used quantify the extra redistributed flow on all other lines (or edges of the network graph) when a given distribution line fails. The LODF is computed for all electricity distribution lines that can fail without explicitly simulating their failure using the nonlinear system model; this is done by simply solving a set of linear equations. Since the problem of quantifying the impact of multiple simultaneous outages is a difficult problem, LODFs can be iteratively evaluated to help identify the combination of failures that have the highest impact [10].

Assuming linear current flow on the electrical network edges, edge-to-edge graph analysis tools are proposed in [11] inspired by the LODF concept. For a linear flow network, Schaub et al [11] show that the current flow redistribution matrix does not depend on the flow injections (i.e. demands at nodes and external voltage or current sources) but merely on the topology of the network. Moreover, an explicit form for the matrix is derived, which is computed by solving a set of linear equations. In problem (1), typically an "isolation section" between valves is closed and so can be modelled as an outage of the corresponding pipe section. Using the language of [11], the equilibrium equations for the WDS we get after linearisation are (see the Appendix for notation and derivation details):

$$\begin{aligned} G^{-1}q - A_{12}h &= \bar{h}_{ext}, \\ A_{12}^T q &= d, \end{aligned} \tag{2}$$

where $\bar{h}_{ext} = A_{10}(h_0 - \bar{h}_0) + N(A_{12}\bar{h} + A_{10}\bar{h}_0) - A_{12}\bar{h}$ and all variables are as defined in the Appendix, and the bar notation (eg. $\bar{h}$) denotes operating head and flow values around which we linearize.

Let the weighted graph Laplacian of a given nominal network be defined as $L = A_{12}^T G A_{12}$, with G computed using the link flows q and heads h of the original network around a given operating point. Now assume that we close link j of the WDS and analyse the first order model (2). Applying the same derivation as in [11], the vector of changes in the link flows can be shown to be:

$$\delta_j q = \frac{G A_{12} L^{\dagger} b_j}{1 - g_j b_j^T L^{\dagger} b_j} q_j, \tag{3}$$

where $b_j = A_{12}^T(:, j)$ are the incidence vectors for the closed link, $L^{\dagger}$ is either the inverse or the Moore-Penrose pseudoinverse of the Laplacian and q_j is the flow on link j before its closure. Let $M = GA_{12}L^{\dagger}A_{12}^T$ and $\varepsilon_j = 1 - g_j b_j^T L^{\dagger} b_j$, which are defined as the *edge-to-edge trasfer matrix* and the *embededness* measures in [11]. We can then write the

Algorithm 1 LODF based algorithm for optimal link closures

Input: network data, threshold velocity v_{min},

```
set k = 1, and compute the WFRM in (4) using a single hydraulic simulation
while ∃ links to close AND k ≤ |N_v| do
    Compute v_new using (5)
    Rank links in N_v by the number of velocities in v_new greater than v_min (eg. using a Sorting algorithm)
    Close the highest ranked candidate link which satisfies feasibility w.r.t. all other constraints
    Update network topology
    Compute the WFRM in (4) using a single hydraulic simulation
    Set k to k + 1
end while
```

matrix of water flow redistribution for all edges as:

$$
\begin{aligned}
\Delta q &:= [\delta_1 q_1 \ldots \delta_{n_p} q_{n_p}] \\
&= [k_1 q_1 \ldots k_{n_p} q_{n_p}], \quad k_j = \frac{L^\dagger b_j}{1 - g_j b_j^T L^\dagger b_j} \\
\Delta q &= K\ \mathtt{diag}(q),
\end{aligned} \tag{4}
$$

where $K_{n_p \times n_p} := [k_1 \ldots k_{n_p}]$ is what we call the water flow redistribution matrix (WFRM) and $\mathtt{diag}(q)$ is diagonal matrix with the j^{th} diagonal element equal to q_j. The matrix of the changes in velocity Δv is constructed by simply dividing the change in flow in each pipe by its cross-sectional area. Column $j \in N_v$ of Δv estimates the change in velocity in all the links when the j^{th} link is closed. The new velocity vector when link j is closed can then be approximated by

$$v_{new} \approx v + \Delta v(:, j), \tag{5}$$

where v is the vector of velocities before closing link j. Note also that K can be computed by simply computing M and $\varepsilon = [\varepsilon_1, \ldots, \varepsilon_{n_p}]$ such that $K = M\ [\mathtt{diag}(\varepsilon)]^{-1}$. That is, K is computed by solving n_p linear equations ($Lw_j = b_j^T,\ j = 1, \ldots, n_p$), where only the right hand side b_j is varying. Therefore, in performing the computations $L^\dagger b_j^T$, the same factorization of L can be reused multiple times, reducing the computational cost.

A pseudocode of the method for deriving a new topology by optimally selecting isolation links to close off is shown in Algorithm 1. For clarity of notation, here we show the approach by computing velocity perturbations at peak flow time; by simply iterating over all peak demands and associated operating points in time, the velocities can be computed over any temporal range of interest. For example, it was shown in [4] that employing stochastic end-user models in hydraulic simulation [12] is more accurate in modelling flow velocities at detailed temporal and spatial scales. Ongoing work is investigating how such modelling can improve the performance of LODF approach in Algorithm 1.

4. Optimal Pressure control for self-cleaning velocities using sequential convex programming

Given a fixed network topology for a DWDS, here we consider the optimization of PRV settings to maximize the objective in (1). This problem is difficult to solve because the objective function is non-convex and non-differentiable. In addition, the hydraulic constraints are non-convex as shown in [7]. Here we first propose a smooth approximation of the objective function in (1) and then apply the sequential convex progamming (SCP) method we proposed in [7], since they allow us to solve a nonconvex optimization problem by considering a sequence of convex subproblems that can be solved accurately and efficiently.

To approximate the number of pipes that satisfy a minimum self-cleaning threshold velocity, i.e. to sum up the number of pipes with flow velocity magnitudes larger than the minimum velocity threshold set, we propose the use of the logistic function:

$$\psi(u) = (1 + e^{-\rho(u - v_{min})})^{-1} + (1 + e^{-\rho(-u - v_{min})})^{-1}, \tag{6}$$

where v_{min} is a threshold velocity, u is flow velocity in a pipe and ρ is a large enough positive number. The function (6) has the derivative $d\psi(\cdot)/dv = \rho(1 + e^{-\rho(v - v_{min})})^{-2} e^{-\rho(v - v_{min})} - \rho(1 + e^{-\rho(-v - v_{min})})^{-2} e^{-\rho(-v - v_{min})}$. Therefore, as we increase the value ρ, the approximation becomes closer to the step function $k(\cdot)$, while its gradient gets larger, becoming unbounded in the limit. A value of $\rho = 100$ is used here as depicted in Figure 1, which was found to be sufficiently step-like but still smooth enough at the threshold boundaries $\pm v_{min}$ not to cause numerical issues in gradient based optimization. Here we model PRV settings using a linear term η, which is a vector that represents additional head losses within links containing a PRV [7]. The objective function in (1) is then approximated by

$$\begin{aligned} \underset{\eta}{\text{maximize}} \quad & \tilde{f}(\cdot) := \sum_{t \in T} \sum_{j \in N_c} \psi(v_j^t) \\ \text{subject to:} \quad & \text{hydraulic constraints of the system, other performance constraints.} \end{aligned} \tag{7}$$

By linearizing the objective function and the nonlinear hydraulic constraints in (7), the strictly feasible SCP method can be used to iteratively solve a sequence of sparse linear programs, giving a local optima for (7). The optimization method for PRV control, outlined in detail in [7], has the key attribute of reliable convergence. To achieve this, the strictly feasible SCP method guarantees that each optimization step is strictly feasible with respect to the nonlinear hydraulic constraints, resulting in improved convergence properties. By using a null space algorithm for hydraulic analysis [13], the overhead required for hydraulic computations is also significantly reduced. In the next section, we demonstrate the efficacy of Algorithm 1 and the SCP method for optimising self-cleaning capacity of a network using an example DWDS model from literature.

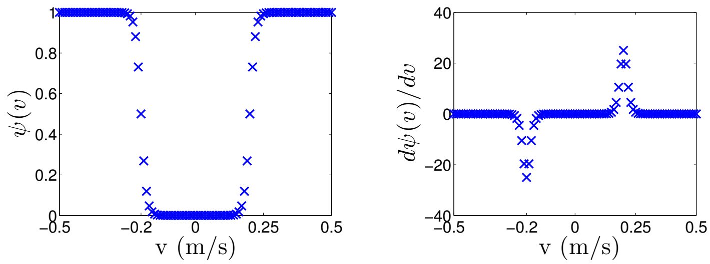


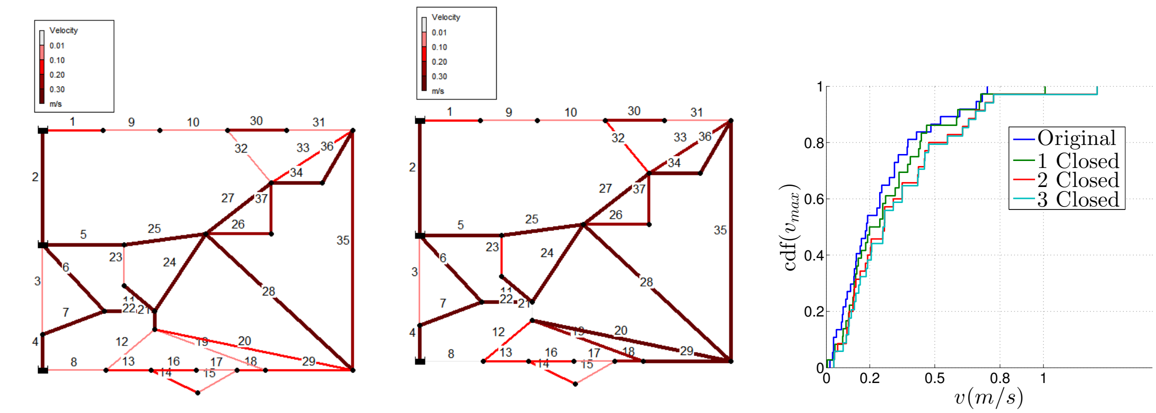
Figure 1: A smooth approximation of the threshold function $k_i(v; v_{min})$ using the smooth logistic function $\psi(v; \rho, v_{min})$: (a) $\psi(\cdot)$ (b) $d\psi(\cdot)/dv$.

5. Simulation Results and Discussion

The two approaches outlined for increasing self-cleaning capacity in a WDS were applied to the network model in Figure 2a. This case study network has 22 nodes, 37 pipes and 3 reservoirs. Details on the network pipe characteristics and nodal demands can be found in [14] and [15], respectively. The linear programs at each step of the SCP method were solved in the MATLAB environment with CPLEX [16]. In all simulations, feasible solutions are the ones that satisfy a 15m minimum pressure at all demand nodes.

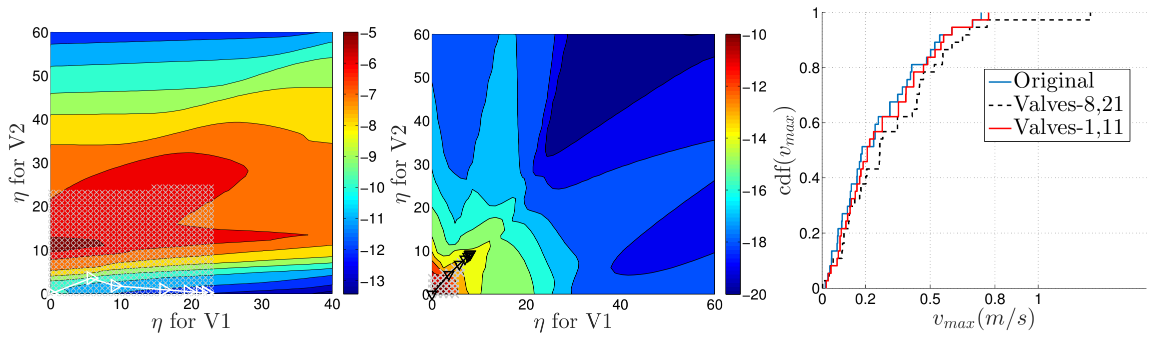
In Figure 2c, we show the empirical cumulative distribution function (CDF) of the (maximum) flow velocities of the given network over the whole pipe population. By applying the LODF inspired algorithm (Algorithm 1), links numbered 21, 8 and 17 were identified in sequence as having the most positive impact on velocities. Figure 2b shows how the velocities change when all three pipes are closed, where the line colour and thickness represent the magnitude of the flow velocities. Moreover, we show in Figure 2c how the CDF of the maximum velocities changes when closing these pipes one after the other. We note that, as the network is made less looped by closing the chosen pipes, the number of maximum flow velocities that are above the threshold (i.e. 0.2m/s we have set here) increase, which is in line with expectations when redundant loops are removed from the topology [1]. We also note that, the marginal increases in velocities are diminishing for more than two valves closed for this network. By using this marginal gain, one can find a suitable trade-off between the number of redundant loops to close off and the increase in self-cleaning capacity.

It is of course not always possible to close pipes in order to reduce redundant loops. For example, as a surrogate for resilience or to reduce head losses in supply [7,9], some network operators may favour to maintain the redundancy in supply routes. In such cases, we consider the use of pressure control optimised to increase self-cleaning capacity. In [15], the optimal placement of PRV's is considered in order to minimize average zonal pressure (AZP). For the



(a) (b) (c)

Figure 2: (a) Benchmark looped network described in [14], with maximum flow velocities at 8AM depicted in colour (b) Network as in (a) with pipes 21 and pipe 8 closed (c) A cumulative distribution of the maximum pipe flow velocities across the all pipes as we iteratively close 3 links with highest impact on flow velocity redistributions; pipes 21, 8 and 17 were closed.



(a) (b) (c)

Figure 3: The feasible search space for the PRV settings at 8AM (gray crosses) and iterates found by the SCP (arrows) when PRVs are placed on links (a) 1 and 11 (b) 21 and 8. The contours show the change in the objective function of (7), here we minimize $-\tilde{f}(\cdot)$, as we span the space of PRV settings. (c) A cumulative distribution of the maximum pipe flow velocities across all pipes when PRV settings are optimised to maximize self-cleaning capacity.

network in Figure 2, it is shown in [15] that the optimal placement for up to three PRVs are the set of links {1}, {1, 11} and {1, 11, 21}, respectively. However, if we consider PRV placement considering self-cleaning capacity, the optimal locations may be different. In [11], the LODF measure is shown to be an extremum of a partial link closure analysis. Therefore, pipes 21 and 8 could be the top two candidates for placing PRVs as they are the top two most impactful links on self-cleaning capacity when fully closed.

In Figure 3, we consider the optimization of two PRVs placed in the network for the new objective of self-cleaning capacity. We compare their performance to the optimization valves placed on the set of links {1, 11} as suggested for minimizing AZP and so leakage. We can see from the CDFs in Figure 3c that the optimal valve settings give rise to a higher increase in the number of flows that pass the self-cleaning threshold when they are placed on links 21 and 8, compared to optimised PRVs placed on links 1 and 11. From the plot of the feasibility space and the iterates from the SCP algorithm in Figure 3a, we note that only the PRV placed at link 1, and not the PRV placed at link 11, has an impact in controlling flow velocities. On the other hand, from Figure 3b, we see that both PRVs placed at links 21 and 8 act in tandem to increase the number of flow velocities that pass the threshold; pushing the CDF further to the bottom right. Although not shown here for the sake of brevity, it was observed that there is a clear trade-off between optimising average zone pressure and optimising self-cleaning capacity as defined in (1). An in-depth analysis of this trade-off is left as a topic for future investigation.

6. Conclusion

In this paper, we consider the proactive control of flow velocities to maximise the self-cleaning capacity of the DWDS under normal operations. This is done both through an optimal change of the network topology by isolating some links using isolation valves and through an optimal control of pressure control valves during peak demand hours. Inspired by line outage flow distribution in electrical networks, we have derived a linearisation of the hydraulics and have shown how a fast network graph analysis of link closures on the linear approximation can be used to estimate the potential changes in flow velocities. The results are then used to determine the most favourable links for closure. As a complimentary tool to further increase self-cleaning capacity and also for networks where disconnecting some supply routes cannot be used because of other conflicting objectives, we consider the optimal control of existing pressure reducing valves at peak demand periods. We formulate a novel optimisation problem to maximise the network operations for increased self-cleaning capacity, while satisfying hydraulic and regulatory pressure constraints at demand nodes. A new smooth objective function approximation is proposed along with a scalable sequential convex programming method to solve the resulting valve optimization problems. We have also shown that the placement and optimal control of pressure control valves for leakage minimisation and for optimising self-cleaning capacity can be different, showing a clear trade-off between the two objectives. We haved used a published benchmark network as a case study to show the efficacy of these new approaches.

Appendix: Linearized Model of Network Flow for Edge-to-Edge Analysis

In this article, we consider networks using demand-driven hydraulic analysis, where the demand is assumed known. For a network with n_p links connecting $n_n(< n_p)$ unknown head nodes, and n_0 known head nodes, we define the vector of unknown flows and pressure heads as $q = [q_1, \dots, q_{n_p}]^T$ and $h = [h_1, \dots, h_{n_n}]^T$, respectively. Let pipe p_j have flow q_j going from node i to node k, and with pressure heads h_i and h_k at nodes i and k, respectively. The frictional headloss (or flow resistance) across the pipe can then be represented as:

$$h_i - h_k = r_j|q_j|^{n-1}q_j, \tag{8}$$

where r_j, the resistance coefficient of the pipe, can be modelled as either independent of the flow or implicitly dependent on flow q_j and given as $r_j = \alpha L_j/(C_j^n D_j^m)$. The variables L_j, D_j and C_j denote the length, diameter and roughness coefficient of pipe j, respectively. The triplet α, n and m depend on the energy loss model used; Hazen-Williams (HW: $r_j = 10.670L_j/(C_j^{1.852}D_j^{4.871})$) and Darcy-Weisbach (DW) are two commonly used frictional head loss formulae [13]. In DW models, the dependence of the resistance coefficient on flow is implicit and therefore computed using iterative solvers [17, (2.10)]. With head loss equations defined for each pipe and the fixed heads and demands for each node taken into account, the steady-state fluid flows in a water network must satisfy the two hydraulic principles:

$$A_{12}^T q - d = 0, \tag{9}$$

$$A_{11}(q)q + A_{12}h + A_{10}h_0 = 0, \tag{10}$$

where the variables $h_0 \in \mathbb{R}^{n_0}$ and $d \in \mathbb{R}^{n_n}$ represent the known heads (eg. at a reservoir or tank) and demands at nodes, respectively. While (9) guarantees the conservation of flow at each junction node, (10) accounts for the frictional head loss across all links. Here, the matrices $A_{12}^T \in \mathbb{R}^{n_n \times n_p}$ and $A_{10}^T \in \mathbb{R}^{n_n \times n_0}$ are the node-to-edge incidence matrices for the n_n unknown head nodes and n_0 fixed head nodes, respectively. For example, each link is associated with an $n_n \times 1$ row vector in A_{12}: $A_{12}(j, i) = 1($ or $-1)$ if link j enters (or leaves) node i and $A_{12}(j, i) = 0$ otherwise. The square matrix $A_{11} \in \mathbb{R}^{n_p \times n_p}$ is a diagonal matrix with the elements

$$A_{11}(j, j) = r_j|q_j|^{n_j-1}, j = 1, \dots, n_p, \tag{11}$$

representing part of the loss formula in (8). In (10), the frictional headloss function is expressed as a function of the flows. Using a nonlinear transformation of the headloss in pipe p_j, (8) can be reformulated to:

$$q_j = r_j^{-1/n}|\Delta h_j|^{\frac{1-n}{n}}\Delta h_j, \quad \Delta h_j = h_i - h_k, \tag{12}$$

where pipe p_j is topologically represented as going from node i to node k. A matrix form of (12) is

$$q = \hat{A}_{11}(A_{12}h + A_{10}h_0), \tag{13}$$

where $\hat{A}_{11}(j, j) = r_j^{-1/n}|\Delta h_j|^{\frac{1-n}{n}}$, $j = 1, \ldots, n_p$.

It is evident from (12) that the flow in each pipe depends only on the head differential, and not on the pressure heads at the start and end nodes. For link j, we re-write the energy conservation equation (12) as $\phi(q_j, \Delta h_j) := r_j^{-1/n}|\Delta h_j|^{\frac{1-n}{n}}\Delta h_j - q_j = 0$. A first-order Taylor approximation of $\phi(\cdot)$ around the openrating point $\bar{w} := [\bar{h}_j,\ \bar{q}_j]^T$, gives:

$$q_j = \frac{1}{n} r_j^{-1/n}\Delta\bar{h}_j|^{\frac{1-n}{n}} \left(\Delta h_j + (n-1)\Delta\bar{h}_j\right). \tag{14}$$

Let $g_i(\Delta\bar{h}_j) := \frac{1}{n} r_j^{-1/n}|\Delta\bar{h}_j|^{\frac{1-n}{n}}$ represent the 'flow conductance' for pipe j and $G = \mathtt{diag}(g_i),\ i = 1, \ldots, n_p$; note that $G = N^{-1}\hat{A}_{11}(\Delta h)$. We can then re-write the matrix form of the linearized flow equation approximation (14) as:

$$q = G\left(A_{12}h + \bar{h}_{ext}\right), \tag{15}$$

where $\bar{h}_{ext} = A_{10}(h_0 - \bar{h}_0) + N(A_{12}\bar{h} + A_{10}\bar{h}_0) - A_{12}\bar{h}$ can be considered as an external fixed head source (or external voltage source for the analogues electrical network [11, Appx. A.1]). Note that we use the bar notation here because $\bar{h}_{ext}$ is a function of the operating point around which we linearize.

References

[1] J. Vreeburg, J. Boxall, Discolouration in potable water distribution systems: a review., Water research 41 (2007) 519–29.
[2] J. H. Vreeburg, Discolouration in drinking water systems: The role of particles clarified, Water Intelligence Online 9 (2010) 9781780401607.
[3] E. Blokker, P. Schaap, Particle accumulation rate of drinking water distribution systems determined by incoming turbidity, Procedia Engineering 119 (2015) 290–298.
[4] E. J. M. Blokker, J. H. G. Vreeburg, P. G. Schaap, J. C. van Dijk, The Self-Cleaning Velocity in Practice, in: Water Distribution Systems Analysis 2010, American Society of Civil Engineers, 2010, pp. 187–199. doi:10.1061/41203(425)19.
[5] M. Blokker, H. Vogelaar, K. Goos, J. Vreeburg, Using valve manipulation to manage discolouration risk in drinking water, Water Asset Management International 1 (2012) 7–10.
[6] E. M. Blokker, P. G. Schaap, J. H. Vreeburg, Comparing the fouling rate of a drinking water distribution system in two different configurations, Urban Water Management: Challenges and Opportunities, Exeter, UK (2011).
[7] R. Wright, E. Abraham, P. Parpas, I. Stoianov, Control of water distribution networks with dynamic dma topology using strictly feasible sequential convex programming, Water Resources Research (2015).
[8] F. Pecci, E. Abraham, I. Stoianov, Multiobjective Co-design Problems for Water Distribution Networks : a continuous relaxation approach, Structural and Multidisciplinary Optimization (2016) In Press.
[9] M. Herrera, E. Abraham, I. Stoianov, A graph-theoretic framework for assessing the resilience of sectorised water distribution networks, Water Resources Management 30 (2016) 1685–1699.
[10] A. J. Wood, B. F. Wollenberg, Power generation, operation, and control (1996).
[11] M. T. Schaub, J. Lehmann, S. N. Yaliraki, M. Barahona, Structure of complex networks: Quantifying edge-to-edge relations by failure-induced flow redistribution, Network Science 2 (2014) 66–89.
[12] E. J. M. Blokker, J. H. G. Vreeburg, J. C. Van Dijk, Simulating Residential Water Demand with a Stochastic End-Use Model, Journal of Water Resources Planning and Management 136 (2010) 19–26.
[13] E. Abraham, I. Stoianov, Constraint preconditioned inexact newton method for hydraulic simulation of large-scale water distribution networks, IEEE Transactions on Control of Network Systems PP (2016) 1–1.
[14] P. W. Jowitt, C. Xu, Optimal Valve Control in Water Distribution Networks, Journal of Water Resources Planning and Management 116 (1990) 455–472.
[15] F. Pecci, E. Abraham, I. Stoianov, Mathematical programming methods for pressure management in water distribution systems, Procedia Engineering 119 (2015) 937 – 946. Computing and Control for the Water Industry (CCWI2015) Sharing the best practice in water management.
[16] IBM, V12. 1: User's Manual for CPLEX, Technical Report, 2009. URL: ftp://public.dhe.ibm.com/software/websphere/ilog/docs/optimization/cplex/, [Online; accessed April 11, 2016].
[17] B. E. Larock, R. W. Jeppson, G. Z. Watters, Hydraulics of Pipeline Systems, 1999. doi:10.1201/9781420050318.