The Hamiltonian for a circular 2D quantum dot (QD) containing n_e electrons reads

$$H = \frac{1}{2m^*}\sum_{i=1}^{n_e}(\mathbf{p}_i - e\mathbf{A}_i)^2 + \tfrac{1}{2}m^*\omega_0^2\sum_{i=1}^{n_e} r_i^2 + \frac{e^2}{4\pi\varepsilon_0\varepsilon_r}\sum_{i<j}^{1,n_e}\frac{1}{|\mathbf{r}_i - \mathbf{r}_j|}, \tag{1}$$

where e, m^*, ε_0 and ε_r are the unit charge, effective electron mass, vacuum and relative dielectric constants of a semiconductor, respectively. $\hbar\omega_0$ is the energy scale of confinement in the xy-plane.

For the perpendicular magnetic field we choose the vector potential with a gauge $\mathbf{A}_i = \frac{1}{2}\mathbf{B}\times\mathbf{r}_i = \frac{1}{2}B(-y_i, x_i, 0)$. Then

$$H = \frac{1}{2m^*}\sum_{i=1}^{n_e}\mathbf{p}_i^2 + \tfrac{1}{2}m^*\Omega^2\sum_{i=1}^{n_e} r_i^2 + \frac{e^2}{4\pi\varepsilon_0\varepsilon_r}\sum_{i<j}^{1,n_e}\frac{1}{|\mathbf{r}_i - \mathbf{r}_j|} - \omega_L L_z, \tag{2}$$

where $L_z = \sum_{i=1}^{n_e} l_{zi}$ is the (z-projection of) total angular momentum, $\omega_L = eB/2m^*$ is the Larmor frequency and $\Omega^2 = \omega_0^2 + \omega_L^2$.

It is convenient to use the scaled coordinates $\tilde{\mathbf{r}}_i = \mathbf{r}_i/l_0$, $\tilde{\mathbf{p}}_i = \mathbf{p}_i l_0/\hbar$, where $l_0 = (\hbar/m^*\omega_0)^{1/2}$ is the characteristic length of the confinement potential. In these variables the Hamiltonian takes the form (in units of $\hbar\omega_0$)

$$\mathcal{H} \equiv \frac{H}{\hbar\omega_0} = \tfrac{1}{2}\sum_{i=1}^{n_e}(\tilde{\mathbf{p}}_i^2 + \tilde{\Omega}^2\,\tilde{r}_i^2) + k\sum_{i<j}^{1,n_e}\frac{1}{\tilde{r}_{ij}} - \tilde{\omega}_L M, \tag{3}$$

where $\tilde{r}_{ij} = |\tilde{\mathbf{r}}_i - \tilde{\mathbf{r}}_j|$ and $\tilde{\Omega} = \Omega/\omega_0$, $\tilde{\omega}_L = \omega_L/\omega_0$, $k = e^2/(4\pi\varepsilon_0\varepsilon_r\hbar\omega_0 l_0)$, $M = L_z/\hbar$.

In the approximation of non-interacting electrons ($k = 0$) the total Hamiltonian can be written as the sum

$$\mathcal{H}_0 = \sum_{i=1}^{n_e} h_i \tag{4}$$

of the single-electron Hamiltonians

$$h_i = \tfrac{1}{2}\,(\tilde{\mathbf{p}}_i^2 + \tilde{\Omega}^2\,\tilde{r}_i^2) - \tilde{\omega}_L m_i, \tag{5}$$

where $m_i = l_{zi}/\hbar$. The eigenenergies of (5) are the Fock-Darwin levels

$$\epsilon_{n_i,m_i} = \tilde{\Omega}\,(2n_i + |m_i| + 1) - \tilde{\omega}_L m_i, \tag{6}$$

where n_i and m_i are the radial and magnetic quantum numbers of ith electron, respectively. The corresponding eigenstates are

$$\psi_{n_i,m_i}(\tilde{\mathbf{r}}_i) = f_{n_i,m_i}(\tilde{r}_i)\frac{e^{\mathrm{i}m_i\varphi_i}}{\sqrt{2\pi}}, \tag{7}$$

where

$$f_{n_i,m_i}(\tilde{r}_i) = \sqrt{\frac{2\,\tilde{\Omega}\,n_i!}{(n_i+|m_i|)!}}\; x_i^{|m_i|}\, e^{-\frac{1}{2}x_i^2}\, L_{n_i}^{|m_i|}(x_i^2), \quad x_i = \tilde{\Omega}^{1/2}\tilde{r}_i. \tag{8}$$

Then, the eigenenergies and eigenstates of h_0 are

$$\mathcal{E}_\alpha^{(0)} = \sum_{i=1}^{n_e} \epsilon_{n_i,m_i}, \quad \Psi_\alpha^{(0)}(\tilde{\mathbf{r}}_1, \dots, \tilde{\mathbf{r}}_{n_e}) = \prod_{i=1}^{n_e} \psi_{n_i,m_i}(\tilde{\mathbf{r}}_i), \tag{9}$$

respectively, where $\alpha = \{n_1, m_1, \dots, n_{n_e}, m_{n_e}\}$.

The eigenvalue problem of the full Hamiltonian $\mathcal{H} = \mathcal{H}_0 + k\sum_{i<j}\tilde{r}_{ij}^{-1}$,

$$\mathcal{H}\,|\Psi\rangle = \mathcal{E}\,|\Psi\rangle, \tag{10}$$

can be solved using representation in the eigenbasis of $\mathcal{H}_0$. Then Eq. (10) transformes to the set of linear equations

$$\sum_\beta \mathcal{H}_{\alpha\beta}\, c_\beta = \mathcal{E}\, c_\alpha, \tag{11}$$

where $c_\alpha \equiv \langle\Psi_\alpha^{(0)}|\Psi\rangle$ and the Hamiltonian matrix elements are

$$\mathcal{H}_{\alpha\beta} \equiv \langle\Psi_\alpha^{(0)}|\mathcal{H}|\Psi_\beta^{(0)}\rangle = \mathcal{E}_\alpha^{(0)}\delta_{\alpha\beta} + k\sum_{i<j}^{1,n_e}\langle\Psi_\alpha^{(0)}|\tilde{r}_{ij}^{-1}|\Psi_\beta^{(0)}\rangle. \tag{12}$$

The eigenenergies $\mathcal{E}$ and the values for coefficients c_α follow after solving the secular problem (11). i.e. after diagonalization of the matrix (12). The corresponding eigenstates $|\Psi\rangle$ are then

$$|\Psi\rangle = \sum_\alpha c_\alpha|\Psi_\alpha^{(0)}\rangle. \tag{13}$$

The interaction matrix elements are

$$\langle\Psi_\alpha^{(0)}|\tilde{r}_{ij}^{-1}|\Psi_\beta^{(0)}\rangle = \delta_{\alpha^*\beta^*}\,\langle n_i, m_i; n_j, m_j|\tilde{r}_{ij}^{-1}|n_i', m_i'; n_j', m_j'\rangle, \tag{14}$$

where $\alpha^* = \alpha/\{n_i, m_i; n_j, m_j\}$, $\beta^* = \beta/\{n'_i, m'_i; n'_j, m'_j\}$ and $|n_i, m_i; n_j, m_j\rangle \equiv |\psi_{n_i,m_i}\rangle|\psi_{n_j,m_j}\rangle$. Here

$$\langle n_i, m_i; n_j, m_j|\tilde{r}_{ij}^{-1}|n'_i, m'_i; n'_j, m'_j\rangle = \sqrt{\Omega}\int_0^\infty x_i\,\mathrm{d}x_i\int_0^\infty x_j\,\mathrm{d}x_j\,\times \\ f_{n_i,m_i}(x_i) f_{n_j,m_j}(x_j) f_{n'_i,m'_i}(x_i) f_{n'_j,m'_j}(x_j)\,\langle m_i, m_j|x_{ij}^{-1}|m'_i, m'_j\rangle, \quad (15)$$

where $f_{n_i,m_i}(x_i) = \Omega^{-1/2} f_{n_i,m_i}(\tilde{r}_i)$ and the angular parts of interaction matrix elements

$$\langle m_i, m_j|x_{ij}^{-1}|m'_i, m'_j\rangle = \frac{1}{4\pi^2}\int_0^{2\pi}\mathrm{d}\varphi_i\int_0^{2\pi}\mathrm{d}\varphi_j\frac{e^{\mathrm{i}(m'_i-m_i)\varphi_i}e^{\mathrm{i}(m'_j-m_j)\varphi_j}}{\sqrt{x_i^2+x_j^2-2x_ix_j\cos(\varphi_i-\varphi_j)}}. \quad (16)$$

If $\varphi_{ij} = \varphi_i - \varphi_j$, then $\mathrm{d}\varphi_i\mathrm{d}\varphi_j = \mathrm{d}\varphi_{ij}\mathrm{d}\varphi_j$ and

$$\begin{aligned}
\langle m_i, m_j|x_{ij}^{-1}|m'_i, m'_j\rangle &= \frac{1}{2\pi}\int_0^{2\pi} e^{\mathrm{i}(m'_i+m'_j-m_i-m_j)\varphi_j}\,\mathrm{d}\varphi_j \\
&\quad \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{\mathrm{i}(m'_i-m_i)\varphi_{ij}}}{\sqrt{x_i^2+x_j^2-2x_ix_j\cos\varphi_{ij}}}\,\mathrm{d}\varphi_{ij} \\
&= \delta_{m_i+m_j,m'_i+m'_j}\,\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{\mathrm{i}(m'_i-m_i)\varphi}}{\sqrt{x_i^2+x_j^2-2x_ix_j\cos\varphi}}\,\mathrm{d}\varphi \\
&= \frac{\delta_{m_i+m_j,m'_i+m'_j}}{\sqrt{x_i^2+x_j^2}}\,\mathcal{I}_{m'_i-m_i}\left(\frac{2x_ix_j}{x_i^2+x_j^2}\right), \qquad (17)
\end{aligned}$$

where

$$\mathcal{I}_m(x) = \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{\mathrm{i}m\varphi}}{\sqrt{1-x\cos\varphi}}\,\mathrm{d}\varphi. \quad (18)$$

The solution for this integral can be given in the form of series

$$\mathcal{I}_m(x) = \sum_{k=0}^{\infty}\frac{(4k+2|m|-1)!!}{k!\,(k+|m|)!}\left(\frac{x}{4}\right)^{2k+|m|}, \quad |x|<1. \quad (19)$$

This result can be obtained using expansion $(1-x)^{-1/2} = \sum_{n=0}^{\infty}\frac{(2n-1)!!}{n!\,2^n}x^n$, which converges for $|x|<1$. Then

$$\mathcal{I}_m(x) = \frac{1}{2\pi}\sum_{n=0}^{\infty}\frac{(2n-1)!!}{n!\,2^n}\,x^n\int_0^{2\pi}e^{\mathrm{i}m\varphi}\cos^n\varphi\,\mathrm{d}\varphi. \quad (20)$$

Since

$$\int_0^{2\pi} e^{\mathrm{i}m\varphi} \cos^n \varphi \, \mathrm{d}\varphi = \frac{2\pi}{2^n} \begin{cases} \binom{n}{\frac{m+n}{2}}, & \text{for } m+n = \text{even and } |m| \leq n \\ 0, & \text{else} \end{cases} \tag{21}$$

it is

$$\mathcal{I}_m(x) = \sum_{n=|m|}^{\infty\,*} \frac{(2n-1)!!}{n!\, 4^n} \binom{n}{\frac{m+n}{2}} x^n, \tag{22}$$

where $*$ denotes that the sum is taken over n such that $m+n$ are even. These values are given by $n = 2k - m$ (k-integer). Then, using the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, we have

$$\mathcal{I}_m(x) = \sum_{k=\frac{m+|m|}{2}}^{\infty} \frac{(4k-2m-1)!!}{k!\,(k-m)!} \left(\frac{x}{4}\right)^{2k-m}. \tag{23}$$

Finally, if we translate the index $k \to k + (m+|m|)/2$, we obtain Eq. (19).