Full Hamiltonian:

$$\mathcal{H} = \mathcal{H}_0 + k \sum_{i<j} \frac{1}{r_{ij}},$$

Schrödinger eq. $\mathcal{H}|\Psi\rangle = \mathcal{E}|\Psi\rangle$ in the eigenbasis of $\mathcal{H}_0$:

$$\langle\alpha|\mathcal{H}|\Psi\rangle = \mathcal{E}\langle\alpha|\Psi\rangle$$

$$\sum_\beta \underbrace{\langle\alpha|\mathcal{H}|\beta\rangle}_{\mathcal{H}_{\alpha\beta}}\underbrace{\langle\beta|\Psi\rangle}_{c_\beta} = \mathcal{E}\underbrace{\langle\alpha|\Psi\rangle}_{c_\alpha}$$

$$\sum_\beta \mathcal{H}_{\alpha\beta} c_\beta = c_\alpha, \qquad \mathcal{H}_{\alpha\beta} = \langle\alpha|\mathcal{H}|\beta\rangle$$

$$\mathcal{H}_{\alpha\beta} = \mathcal{E}^{(0)}_\alpha \delta_{\alpha\beta} + k \sum_{i<j}^{1,n_e} \langle\alpha|\tilde{r}_{ij}^{-1}|\beta\rangle$$

The interaction matrix elements:

$$\langle\alpha|\tilde{r}_{ij}^{-1}|\beta\rangle = \delta_{\alpha^*\beta^*}\langle n_{\rho i}, m_i, n_{zi}; n_{\rho j}, m_j, n_{zj}|\tilde{r}_{ij}^{-1}|n'_{\rho i}, m'_i, n'_{zi}; n'_{\rho j}, m'_j, n'_{zj}\rangle$$

where $\alpha^* = \alpha/\{n_{\rho i}, m_i, n_{zi}; n_{\rho j}, m_j, n_{zj}\}$, $\beta^* = \beta/\{n'_{\rho i}, m'_i, n'_{zi}; n'_{\rho j}, m'_j, n'_{zj}\}$

and $|n_{\rho i}, m_i, n_{zi}; n_{\rho j}, m_j, n_{zj}\rangle = |\psi_{n_{\rho i}, m_i, n_{zi}}\rangle|\psi_{n_{\rho j}, m_j, n_{zj}}\rangle$.

Here:

$$\langle n_{\rho i}, m_i, n_{zi}; n_{\rho j}, m_j, n_{zj}|\tilde{r}_{ij}^{-1}|n'_{\rho i}, m'_i, n'_{zi}; n'_{\rho j}, m'_j, n'_{zj}\rangle =$$

$$= \int_0^{+\infty}\tilde{\rho}_i d\tilde{\rho}_i \int_0^{+\infty}\tilde{\rho}_j d\tilde{\rho}_j \int_0^{2\pi} d\varphi_i \int_0^{2\pi} d\varphi_j \int_{-\infty}^{+\infty} d\tilde{z}_i \int_{-\infty}^{+\infty} d\tilde{z}_j \, f_{n_{\rho i} m_i}(\tilde{\rho}_i) f_{n_{\rho j} m_j}(\tilde{\rho}_j) f_{n'_{\rho i} m'_i}(\tilde{\rho}_i) f_{n'_{\rho j} m'_j}(\tilde{\rho}_j)$$

$$g_{n_{zi}}(\tilde{z}_i)\, g_{n_{zj}}(\tilde{z}_j)\, g_{n'_{zi}}(\tilde{z}_i)\, g_{n'_{zj}}(\tilde{z}_j) \frac{e^{i(m'_i - m_i)\varphi_i} e^{i(m'_j - m_j)\varphi_j}}{4\pi^2(\tilde{\rho}_{ij}^2 + \tilde{z}_{ij}^2)^{1/2}}$$

$$\langle i, j | r_{ij}^{-1} | i', j' \rangle = \int_0^\infty \tilde{\rho}_i d\tilde{\rho}_i \int_0^\infty \tilde{\rho}_j d\tilde{\rho}_j \int_{-\infty}^{+\infty} d\tilde{z}_i \int_{-\infty}^{+\infty} d\tilde{z}_j \times$$

$$f_i(\tilde{\rho}_i) f_j(\tilde{\rho}_j) f_{i'}(\tilde{\rho}_i) f_{j'}(\tilde{\rho}_j) g_i(\tilde{z}_i) g_j(\tilde{z}_j) g_{i'}(\tilde{z}_i) g_{j'}(\tilde{z}_j) \cdot \langle m_i m_j | \tilde{r}_{ij}^{-1} | m_i' m_j' \rangle,$$

where:

$$\langle m_i m_j | \tilde{r}_{ij}^{-1} | m_i' m_j' \rangle = \frac{1}{4\pi^2} \int_0^{2\pi} d\varphi_i \int_0^{2\pi} d\varphi_j \frac{e^{i(m_i'-m_i)\varphi_i} e^{i(m_j'-m_j)\varphi_j}}{(\tilde{\rho}_{ij}^2 + \tilde{z}_{ij}^2)^{1/2}}$$

since:

$$\tilde{r}_{ij} = (\tilde{\rho}_{ij}^2 + \tilde{z}_{ij}^2)^{\frac{1}{2}} = \left[\tilde{\rho}_i^2 + \tilde{\rho}_j^2 - 2\tilde{\rho}_i\tilde{\rho}_j \cos(\varphi_i - \varphi_j) + (\tilde{z}_i - \tilde{z}_j)^2\right]^{1/2}$$

$$= \left[\tilde{r}_i^2 + \tilde{r}_j^2 - 2\tilde{z}_i\tilde{z}_j - 2\tilde{\rho}_i\tilde{\rho}_j \cos(\varphi_i - \varphi_j)\right]^{1/2}$$

$$= \sqrt{\tilde{r}_i^2 + \tilde{r}_j^2 - 2\tilde{z}_i\tilde{z}_j} \cdot \sqrt{1 - \frac{2\tilde{\rho}_i\tilde{\rho}_j}{\tilde{r}_i^2 + \tilde{r}_j^2 - 2\tilde{z}_i\tilde{z}_j} \cos(\varphi_i - \varphi_j)},$$

it is:

$$\langle m_i m_j | \tilde{r}_{ij}^{-1} | m_i' m_j' \rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{i(m_i' + m_j' - m_i - m_j)\varphi_j} d\varphi_j \times$$

$$\times \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i(m_i' - m_i)\varphi_{ij}}}{r_{ij}} d\varphi_{ij}$$

$$= \delta_{m_i + m_j, m_i' + m_j'} \cdot \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i(m_i' - m_i)\varphi_{ij}}}{r_{ij}} d\varphi_{ij}$$

$$= \frac{\delta_{m_i + m_j, m_i' + m_j'}}{(\tilde{r}_i^2 + \tilde{r}_j^2 - 2\tilde{z}_i\tilde{z}_j)^{1/2}} I_{m_i' - m_i}\left(\frac{2\tilde{\rho}_i\tilde{\rho}_j}{\tilde{r}_i^2 + \tilde{r}_j^2 - 2\tilde{z}_i\tilde{z}_j}\right),$$

where:

$$I_m(x) = \int_0^{2\pi} \frac{e^{im\varphi}}{\sqrt{1 - x\cos\varphi}} d\varphi.$$

$\underline{m_1, n_{e1}, n_{z1}, m_2, n_{e2}, n_{z2}}$

a) $m_1, m_2 = -m, \dots, m$; $n_{e_1} = n_{e_2} = n_{z_1} = n_{z_2} = 0$

$$\begin{array}{c}
2m+1\left\{\begin{array}{c}\\ \\ \\ \end{array}\right. \\
\\
2m+1\left\{\begin{array}{c}\\ \\ \\ \end{array}\right.
\end{array}
\left.\begin{array}{c|c}
m_1 & m_2 \\ \hline
-m & -m \\
\vdots & \vdots \\
m & -m \\ \hline
\vdots & \vdots \\ \hline
-m & m \\
\vdots & \vdots \\
m & m
\end{array}\right\} (2m+1)^2$$

$$\boxed{i = 1 + (m_1 + m) + (2m+1)(m_2 + m)}$$

b) $m_1, m_2 = -m, \dots, m$; $n_{e_1}, n_{e_2} = 0, \dots, n_{e\max}$; $n_{z_1} = n_{z_2} = 0$

$$\boxed{i = 1 + (m_1 + m) + (2m+1)(m_2 + m) + (2m+1)^2 [n_{e_1} + (n_{e\max} + 1) n_{e_2}]}$$

$$\begin{array}{c}
n_{e\max}+1\left\{\begin{array}{c}\\ \\ \\ \end{array}\right. \\
\\
n_{e\max}+1\left\{\begin{array}{c}\\ \\ \\ \end{array}\right.
\end{array}
\left.\begin{array}{c|c}
n_1 & n_2 \\ \hline
0 & 0 \\
\vdots & \vdots \\
n_{e\max} & 0 \\ \hline
\vdots & \vdots \\ \hline
0 & n_{e\max} \\
\vdots & \vdots \\
n_{e\max} & n_{e\max}
\end{array}\right\} (n_{e\max}+1)^2$$

c) $m_1, m_2 = -m, \dots, m$; $n_{e_1}, n_{e_2} = 0, \dots, n_{e\max}$; $n_{z_1}, n_{z_2} = 0, \dots, n_{z\max}$

$$i = 1 + (m_1 + m) + (2m+1)(m_2 + m) + (2m+1)^2 [n_{e_1} + (n_{e\max} + 1) n_{e_2}] +$$

$$+ (2m+1)^2 (n_{e\max} + 1)^2 \cdot [n_{z_1} + (n_{z\max} + 1) n_{z_2}]$$