AZA B 6.4.2018

1) $$\lim_{n\to\infty}\frac{2^n+\sqrt{n}}{2^{n-1}+n}=\lim_{n\to\infty}\frac{2^{n-1}\left(2+\frac{\sqrt{n}}{2^{n-1}}\right)}{2^{n-1}\left(1+\frac{n}{2^{n-1}}\right)}=2$$

$$\lim_{n\to\infty}\frac{2^{(2^n+\sqrt{n})}}{2^{(2^{n-1}+n)}}=\lim 2^{(2^{n-1}+\sqrt{n}-n)}=\infty$$

$$\lim_{n\to\infty}(2^{n-1}+\sqrt{n}-n)=\lim_{n\to\infty}2^{n-1}\underbrace{\left(1+\frac{\sqrt{n}}{2^{n-1}}-\frac{n}{2^{n-1}}\right)}_{\to 1}=\infty$$

2) $$x_n=\frac{1}{2}x_{n-1}+n2^n \qquad x_0=0$$

$$x_n=\left(\frac{1}{2}\right)^n y_n \qquad 0=\left(\frac{1}{2}\right)^0 y_0 \qquad y_0=0$$

$$y_n=y_{n-1}+\frac{n2^n}{\left(\frac{1}{2}\right)^n}=y_{n-1}+n\cdot 2^{2n}=y_{n-1}+n\cdot 4^n$$

$$=\sum_{i=0}^{n} i\cdot 4^i=\left(x\frac{d}{dx}\right)\left(\frac{x^{n+1}-1}{x-1}\right)\Big|_{x=4}=$$

$$=\left[x^n\left(\frac{x}{x-1}n-\frac{x}{(x-1)^2}\right)+\frac{x}{(x-1)^2}\right]_{x=4}=4^n\left(\frac{4}{3}n-\frac{4}{9}\right)+\frac{4}{9}$$

$$x_n=\frac{1}{2^n}\left[2^{2n}\left(\frac{4}{3}n-\frac{4}{9}\right)+\frac{4}{9}\right]=\frac{1}{2^n}\left(2^{2n+2}\left(\frac{n}{3}-\frac{1}{9}\right)+\frac{2^2}{9}\right)$$

$$=2^{n+2}\left(\frac{n}{3}-\frac{1}{9}\right)+\frac{1}{9}\cdot\left(\frac{1}{2}\right)^{n-2}=\frac{1}{9}\left[2^{n+2}(3n-1)+\frac{1}{2^{n-2}}\right]$$

3) $j=i^2, 2i^2, 3i^2, \dots, ki^2$ kde $k=\left\lfloor\frac{n}{i^2}\right\rfloor$

$$\sum_{i=1}^{\lfloor\sqrt{n}\rfloor}\left\lfloor\frac{n}{i^2}\right\rfloor\geq\sum_{i=1}^{\lfloor\sqrt{n}\rfloor}\left(\frac{n}{i^2}-1\right)\geq n\left(\sum_{i=1}^{\lfloor\sqrt{n}\rfloor}\frac{1}{i^2}\right)-\sqrt{n}\geq n\int_1^{\lfloor\sqrt{n}\rfloor+1}x^{-2}\,dx-\sqrt{n}$$

$$=n\left[-\frac{1}{x}\right]_1^{\lfloor\sqrt{n}\rfloor+1}-\sqrt{n}=n\left(1-\frac{1}{\lfloor\sqrt{n}\rfloor+1}\right)-\sqrt{n}\geq n-2\sqrt{n}$$

$$=\Theta(n)$$

4) $x_n = (n+1)\, x_{n-1} + 3(n+1) \qquad x_0 = 0$

$x_n = (n+1)!\, y_n \qquad y_0 = 0$

$y_n = y_{n-1} + \frac{3(n+1)}{(n+1)!} = y_{n-1} + \frac{3}{n!} = 3\sum_{i=1}^{n} \frac{1}{i!} =$

$= 3\sum_{i=0}^{n} \frac{1^i}{i!} - 3 \qquad \lim_{n \to \infty} y_n = 3(e-1)$

$x_n = (n+1)!\, 3(e-1)$

5)

$x_n = 2x_{n-1} + 24\, x_{n-2} \qquad x_0 = x_1 = 1$

$a^2 - 2a - 24 = 0$

$(a+4)(a-6) = 0$

$x_n = c_1(-4)^n + c_2 6^n$

$1 = c_1 + c_2$

$1 = -4c_1 + 6c_2$

$5 = 10c_2 \qquad c_2 = \frac{1}{2} \qquad c_1 = \frac{1}{2}$

$x_n = \frac{1}{2}\left((-4)^n + 6^n\right)$

6) a) $T(n) = 3T\left(\frac{n}{4}\right) + n^{\frac{4}{5}} / \log n$

$n^{\log_4 3} \doteq n^{0{,}792} < n^{0{,}8}$

$n^{\frac{4}{5}} / \log n = \Omega\left(n^{\log_4 3 + \varepsilon}\right)$ pre $\varepsilon = 0{,}001$

$$3\left(\frac{n}{4}\right)^{\frac{4}{5}} / \log\frac{n}{4} \leq c\, n^{\frac{4}{5}} / \log n$$

$$\frac{3}{4^{\frac{4}{5}}} \cdot \underbrace{\left(\frac{\log n}{\log n - \log 4}\right)}_{\to 1} \leq c$$

$$\frac{3}{4^{\frac{4}{5}}} \doteq \frac{3}{3{,}03} \doteq 0{,}990 \qquad c = 0{,}999$$

$$\underset{MT3}{\Longrightarrow} T(n) = \Theta(n^{\frac{4}{5}}/\log n)$$

b) $T(n) = 4T\left(\frac{n}{5}\right) + n^{\frac{4}{5}}$

$$n^{\log_5 4} \doteq n^{0{,}86} > n^{\frac{4}{5}}$$

$$n^{\frac{4}{5}} = O(n^{\log_5 4 - \varepsilon}) \quad \text{pre } \varepsilon = 0{,}01$$

$$\underset{MT1}{\Longrightarrow} \quad T(n) = \Theta(n^{\log_5 4})$$

c) $T(n) = 16\, T\left(\frac{n}{4}\right) + n^2 \log n$

$$n^{\log_4 16} = n^2$$

$$n^2 \log n = \Theta(n^{\log_4 16} \cdot \log n) \underset{}{\overset{MT2}{\Longrightarrow}}$$

$$T(n) = \Theta(n^2 \log^2 n)$$