AZA, Písomný test AZA, A, 24.3.2017, 25 bodov

Meno a priezvisko:

Úloha	1	2	3	4	5	6	SPOLU
MaxBody	5	4	4	5	4	3	25
Body							

1. Rozhodnite o každom zo vzťahov ($\mathcal{O}, o, \Omega, \omega, \Theta$) medzi funkciami f, g, resp. F, G a svoje tvrdenie zdôvodnite. Ak napr. platí, že $f(n) = \mathcal{O}(g(n))$, potom do príslušného poľa tabuľky zapíšte ✓, inak zapíšte ×.

   (a) $f(n) = n + n^{\frac{1}{2}},\ g(n) = n + n^{\frac{1}{3}}$ (b) $F(n) = 2^{f(n)},\ G(n) = 2^{g(n)}$

	$\mathcal{O}$	o	Ω	ω	Θ
(a)					
(b)					

2. Usporiadajte funkcie podľa asymptotického rastu vzostupne. Svoje tvrdenie dokážte.

   $n^{\log_2 n}, \quad (\frac{10}{9})^{\sqrt{n}}, \quad n^{\ln n}, \quad (\frac{n+1}{n})^{n^{\frac{7}{5}}}$

3. Určte výpočtom asymptotický počet hviezdičiek (pomocou Θ notácie), ktoré vypíše *proc*0.

```
void proc0(int n)
  for (int i=n; i>0; i--) {
    for (int j=5; j<i+2; j++)
      for (int k=i+7; k>10; k--)
        printf("*");
}
```

---

4. Určte výpočtom presný počet hviezdičiek, ktoré vypíše *proc*1.

```
void proc1(int n) {
  if (n>0) {
    proc1(n-1);
    for (int i=0; i<n; i++)
      printf("*");
    proc1(n-1);
    for (int i=0; i<n; i++)
      printf("*");
    proc1(n-1);
  }
}
```

---

5. Určte výpočtom presný počet hviezdičiek, ktoré vypíše *proc*2.

```
void proc2(int n) {
  if (n<2) printf("*");
  else {
    for (int i=0; i<14; i++)
      proc2(n-2);
    for (i=5; i<10; i++)
      proc2(n-1);
  }
}
```

---

6. Použitím Master Theorem určte asymptoticky tesné hranice pre nasledujúce rekurencie :

   (a) $T(n) = 3\,T(n/2) + n^{\frac{3}{2}} \log^2 n$
   (b) $T(n) = 2\,T(n/5) + \sqrt{n}$
   (c) $T(n) = 4\,T(n/2) + (n \log n)^2$

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① a) $f(n) = n + n^{\frac{1}{2}}$ $g(n) = n + n^{\frac{1}{3}}$

$$\lim_{n\to\infty} \frac{n+n^{\frac{1}{2}}}{n+n^{\frac{1}{3}}} = \lim \frac{n\left(1+\frac{1}{n^{\frac{1}{2}}}\right)}{n\left(1+\frac{1}{n^{\frac{2}{3}}}\right)} = 1 \Rightarrow f(n) \sim g(n)$$

b)

$$\lim_{n\to\infty} \frac{2^{n+n^{\frac{1}{2}}}}{2^{n+n^{\frac{1}{3}}}} = \lim_{n\to\infty} \frac{2^{n^{\frac{1}{2}}}}{2^{n^{\frac{1}{3}}}} = \lim_{n\to\infty} 2^{n^{\frac{1}{2}}-n^{\frac{1}{3}}} =$$

$$= \cancel{\lim}\ 2^{\lim_{n\to\infty}\left(n^{\frac{1}{2}}-n^{\frac{1}{3}}\right)} = 2^{\lim_{n\to\infty} \underset{\infty}{\underset{\downarrow}{n^{\frac{1}{3}}}}\left(\underset{\infty}{\underset{\downarrow}{n^{\frac{1}{6}}-1}}\right)} = \infty$$

	O	o	Ω	ω	Θ
(a)	✓	×	✓	×	✓
(b)	×	×	✓	✓	×

② (2) $n^{\log_2 n} = e^{\log_2 n \cdot \ln n} = e^{\underset{>1}{\boxed{\frac{1}{\ln 2}}} \ln^2 n}$

(4) $\left(\frac{10}{9}\right)^{\sqrt{n}} = e^{\sqrt{n}\,\underset{>0}{\boxed{\ln\frac{10}{9}}}} = e^{n^{\frac{1}{2}} \ln\frac{10}{9}}$

(1) $n^{\ln n} = e^{\ln^2 n}$

(3) $\left(\frac{n+1}{n}\right)^{n^{\frac{7}{5}}} = \left(\left(\frac{n+1}{n}\right)^{n}\right)^{n^{\frac{2}{5}}} \rightarrow e^{n^{\frac{2}{5}}}$ zdola

$\frac{(n-3)(n-2)(2n-5)}{6}$

③

$$\sum_{i=1}^{n} \max\left(\sum_{j=5}^{i+1} \max\left(\sum_{k=11}^{i+7} 1, 0\right), 0\right) = \sum_{i=3}^{n} (i-3)^2 = \sum_{i=1}^{\overset{\|}{n-3}} i^2$$

$$\frac{(n-3)^3}{3} = \left[\frac{x^3}{3}\right]_0^{n-3} = \int_0^{n-3} x^2\,dx \le \sum_{i=1}^{n-3} i^2 \le \int_1^{n-2} x^2\,dx = \left[\frac{x^3}{3}\right]_1^{n-2} = \frac{(n-2)^3-1}{3} \le n^3$$

$\exists n_0$

$\frac{n^3}{4} \le$

$$\lim_{n\to\infty} F(n) = \lim_{n\to\infty} \frac{n^3}{3} = \infty \qquad \lim_{n\to\infty} \frac{\frac{(n+1)^3}{3}}{\frac{n^3}{3}} = 1$$

$$\sum_{i=1}^{n-3} i^2 = \Theta(n^3) \qquad \sum_{i=1}^{n-3} i^2 \sim \frac{n^3}{3}$$

④ $x_n = 3x_{n-1} + 2n \qquad x_0 = 0$

$x_n = 3^n y_n \qquad y_0 = 0$

$$y_n = y_{n-1} + \frac{2n}{3^n} = 2\sum_{i=1}^{n} i\left(\frac{1}{3}\right)^i = 2\sum_{i=0}^{n} i\left(\frac{1}{3}\right)^i =$$

$$= 2\left(x\frac{d}{dx}\right)\left(\frac{1-x^{n+1}}{1-x}\right)\Big|_{x=\frac{1}{3}} = \left[2x^n\left(\frac{x}{x-1}n - \frac{x}{(x-1)^2}\right) + 2\frac{x}{(x-1)^2}\right]_{x=\frac{1}{3}}$$

$$= 2\left(\frac{1}{3}\right)^n\left(\frac{\frac{1}{3}}{\frac{1}{3}-1}n - \frac{\frac{1}{3}}{\left(\frac{1}{3}-1\right)^2}\right) + 2\frac{\frac{1}{3}}{\left(\frac{1}{3}-1\right)^2} =$$

$$= 2\cdot\left(\frac{1}{3}\right)^n\left(-\frac{n}{2} - \frac{3}{4}\right) + \frac{3}{2} = \frac{3}{2} - \left(\frac{1}{3}\right)^n\left(n + \frac{3}{2}\right)$$

$$x_n = \frac{3^{n+1}}{2} - n - \frac{3}{2} = \frac{1}{2}\left(3^{n+1} - 2n - 3\right)$$

⑤ $x_n = 5x_{n-1} + 14x_{n-2} \qquad x_0 = x_1 = 1$

$\alpha^2 - 5\alpha - 14 = 0$

$(\alpha + 2)(\alpha - 7) = 0$

$x_n = c_1(-2)^n + c_2 7^n$

$1 = c_1 + c_2$

$1 = -2c_1 + 7c_2 \qquad 1 = (-2)(1-c_2) + 7c_2$

$1 = -2 + 9c_2$

$\frac{1}{3} = c_2 \qquad c_1 = \frac{2}{3}$

$$x_n = \frac{2}{3}(-2)^n + \frac{1}{3}7^n = \frac{1}{3}\left(7^n + (-1)^n \cdot 2^{n+1}\right)$$

⑥ a) $T(n) = 3T\left(\frac{n}{2}\right) + n^{\frac{3}{2}} \log^2 n$

$n^{\log_2 3}$ $\qquad n^{\frac{3}{2}} = n^{\log_2 2^{\frac{3}{2}}}$

$n^{1,58\ldots}$ $\qquad n^{\frac{3}{2}} \log^2 n = O\left(n^{\log_2 3 - \varepsilon}\right)$ $\qquad \varepsilon = 0{,}01$

$\Rightarrow T(n) = \Theta\left(n^{\log_2 3}\right)$

b) $T(n) = 2T\left(\frac{n}{5}\right) + \sqrt{n}$

$n^{\log_5 2}$ $\qquad \sqrt{n} = n^{\log_5 \sqrt{5}}$

$\sqrt{n} = \Omega\left(n^{\log_5 2 + \varepsilon}\right)$ $\qquad \log_5 \sqrt{5} \geq \log_5 2 + \varepsilon$

$\underbrace{\log_5 \underset{>1}{\boxed{\frac{\sqrt{5}}{2}}}}_{>0} \geq \varepsilon > 0$

$2\sqrt{\frac{n}{5}} \leq c\sqrt{n}$

$\frac{2}{\sqrt{5}} \leq c < 1 \quad \Rightarrow \quad T(n) = \Theta(\sqrt{n})$

c) $T(n) = 4T\left(\frac{n}{2}\right) + (n \log n)^2$

$n^{\log_2 4} = n^2$ $\qquad (n \log n)^2 = \Theta(n^2 \cdot \log^2 n) \Rightarrow$

$T(n) = \Theta(n^2 \log^3 n)$