AZA A 6.4.2018

1)
$$\lim_{n\to\infty} \frac{n^2+\sqrt{n}}{n^2+\log n} = \lim_{n\to\infty} \frac{n^2\left(1+\frac{1}{n^{\frac{3}{2}}}\right)}{n^2\left(1+\frac{\log n}{n^2}\right)} = 1$$

$$\lim_{n\to\infty} \frac{2^{n^2+\sqrt{n}}}{2^{n^2+\log n}} = \lim_{n\to\infty} 2^{\sqrt{n}-\log n} = \infty$$

$$\lim_{n\to\infty} (\sqrt{n}-\log n) = \lim_{n\to\infty} \sqrt{n}\,\underbrace{\left(1-\frac{\log n}{\sqrt{n}}\right)}_{\searrow 1} = \lim_{n\to\infty}\sqrt{n} = \infty$$

2)
$$x_n = 3x_{n-1} + 2n \qquad x_0 = 1$$

$$x_n = 3^n y_n \qquad 1 = 3^0 \cdot y_0 \qquad y_0 = 1$$

$$y_n = y_{n-1} + \frac{2n}{3^n} = 1 + 2\sum_{i=1}^{n} i\left(\frac{1}{3}\right)^i = 1 + 2\sum_{i=0}^{n} i\left(\frac{1}{3}\right)^i =$$

$$= 1 + 2\left(x\frac{d}{dx}\right)\left(\frac{x^{n+1}-1}{x-1}\right)\Big|_{x=\frac{1}{3}} = 1+2\left[x^n\left(\frac{x}{x-1}n - \frac{x}{(x-1)^2}\right) + \frac{x}{(x-1)^2}\right]_{x=\frac{1}{3}} =$$

$$= 1 + 2\left[\left(\frac{1}{3}\right)^n\left(\frac{\frac{1}{3}}{-\frac{2}{3}}n - \frac{\frac{1}{3}}{\frac{4}{9}}\right) + \frac{\frac{1}{3}}{\frac{4}{9}}\right] = 1 + 2\left(\frac{1}{3}\right)^n\left(-\frac{n}{2}-\frac{3}{4}\right)+\frac{3}{2} =$$

$$= 1 + \left(\frac{1}{3}\right)^n\left(-n-\frac{3}{2}\right) + \frac{3}{2} = \frac{5}{2} - \left(\frac{1}{3}\right)^n\left(n+\frac{3}{2}\right)$$

$$x_n = \frac{5}{2}\cdot 3^n - n - \frac{3}{2} = \frac{1}{2}\left(5\cdot 3^n - 2n - 3\right)$$

3) $j = i, 2i, 3i, \dots, ki$, kde $k = \left\lfloor \frac{n}{i} \right\rfloor$

$$\sum_{i=1}^{n}\left\lfloor\frac{n}{i}\right\rfloor \le \sum_{i=1}^{n}\frac{n}{i} = n\sum_{i=1}^{n}\frac{1}{i} = n + n\sum_{i=2}^{n}\frac{1}{i} \le n + n\int_1^n \frac{1}{x}\,dx =$$

$$= n + n\left[\ln x\right]_1^n = (n + n\ln n) = \Theta(n\ln n)$$

4) $x_n = n\, x_{n-1} + 2n \qquad x_0 = 0$

$x_n = n!\, y_n \qquad y_0 = 0$

$y_n = y_{n-1} + \frac{2n}{n!} = y_{n-1} + \frac{2}{(n-1)!} = 2\sum_{i=0}^{n-1} \frac{1}{i!}$

$\lim_{n\to\infty} y_n = 2e \qquad x_n \to 2e\,n!$

5) $x_n = 2x_{n-1} + 8x_{n-2} \qquad x_0 = x_1 = 1$

$a^2 - 2a - 8 = 0$

$(a+2)(a-4) = 0$

$x_n = c_1(-2)^n + c_2 4^n$

$1 = c_1 + c_2$

$1 = -2c_1 + 4c_2$

$3 = 6c_2 \quad c_2 = \frac{1}{2} \quad c_1 = \frac{1}{2}$

$x_n = \frac{1}{2}\left(4^n + (-2)^n\right)$

6) a) $T(n) = 7T\left(\frac{n}{5}\right) + n^{\frac{6}{5}} \log^2 n$

$n^{\log_5 7} > n^{\frac{6}{5}} = n^{\log_5 (5^{\frac{6}{5}})}$

$\log_5 7 \doteq 1{,}209$

$7 > 5^{\frac{6}{5}} \doteq 6{,}899$

$n^{\frac{6}{5}} \log^2 n = O\left(n^{\log_5 7 - \varepsilon}\right)$ pre $\varepsilon = 0{,}001$

$\xRightarrow{MT1} T(n) = \Theta\left(n^{\log_5 7}\right)$

b) $T(n) = 2T\left(\frac{n}{3}\right) + n^{\frac{2}{3}} / \log n$

$n^{\log_3 2} < n^{\log_3 (3^{\frac{2}{3}})}$ $\quad 2 < 3^{\frac{2}{3}} = 2{,}08$

$n^{\frac{2}{3}} / \log n = \Omega\left(n^{\log_3 2 + \varepsilon}\right)$ pre $\varepsilon = 0{,}01$

$\log_3 2 \doteq 0{,}63$

$$2 \frac{\left(\frac{n}{3}\right)^{\frac{2}{3}}}{\log \frac{n}{3}} \leq c \frac{n^{\frac{2}{3}}}{\log n}$$

$$\frac{2}{3^{\frac{2}{3}}} \frac{\log n}{\log \frac{n}{3}} \leq c \qquad \frac{2}{2{,}08} \cdot \left(\underbrace{\frac{\log n}{\log n - \log 3}}_{\to 1}\right) \leq c$$

$$\frac{2}{2{,}08} \doteq 0{,}96 \qquad c = 0{,}99$$

$\xRightarrow{MT3}$ $T(n) = n^{\frac{2}{3}} / \log n$

c) $T(n) = 8T\left(\frac{n}{2}\right) + n^3 \log^3 n$

$n^{\log_2 8} = n^3$

$n^3 \log^3 n = \Theta\left(n^{\log_2 8} \cdot \log^3 n\right) \xRightarrow{MT2}$

$T(n) = \Theta\left(n^3 \log^4 n\right)$