# Sampling of Continuous Time Signals

## Contents

## Introduction

Discrete time Signals mostly occur as representations of continuous time signals. This is partly due to the fact that processing of continuous time signals is often carried out by discrete time processing of sequences obtained by sampling.

## Periodic Sampling (mathematical view)

A Sequence of Samples x[n] is obtained from a continuous signal $x_c(t)$ according to the relation

$$x[n] = x_c(nT) \quad -\infty < n < +\infty$$

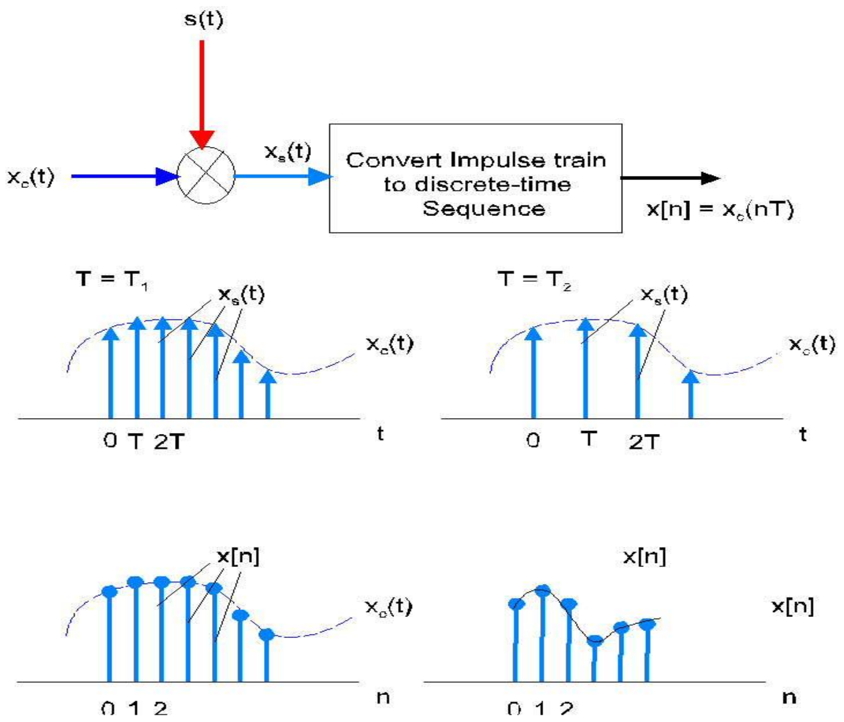



**Figure 1 Sampling with a periodic impulse train s(t) (Oppenheim Schafer 1989)**

s(t) is the periodic impulse train of δ(t) (with period T)

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

where

$$\delta(t) = \begin{cases} \infty, t = 0 \\ 0, t \neq 0 \end{cases}$$

As depicted in Figure 1, x[n] contains no explicit information about the sampling rate. x[n] is an Array of Samples which is processed by a Signal processor. Note that $X_e(t)$ is quantized in time by the impulse train, but there is (in this ideal system) no quantization of the Amplitude.

$$x_s\ (t) =\ x_c(t)s(t) =\ x_c(t) \sum_{n=-\infty}^{\infty} \delta(\mathrm{t}-\mathrm{nT}) = \sum_{n=-\infty}^{\infty} x_c\ (\mathrm{nT}\ )\ \delta(\mathrm{t}-\mathrm{nT})$$

$$X_s\ (s) = \sum_{n=-\infty}^{\infty} x_c\ (\mathrm{nT}\ )\ e^{-snT}$$

From Fourier transform follows

$$\mathrm{S}(\mathrm{j}\Omega) =\ \mathrm{F}(\mathrm{s}(\mathrm{t})) = \frac{2\pi}{\mathrm{T}} \sum_{\mathrm{k}=-\infty}^{\mathrm{k}=+\infty} \delta(\Omega - \mathrm{k}\Omega_{\mathrm{S}})$$

$$\boldsymbol{X(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\,(j\Omega - jk\Omega_s\,)}$$

X(jΩ) has periodical serializations of the (band limited) Fourier spectrum of $x_c(t)$, with the period $\Omega_s$ .

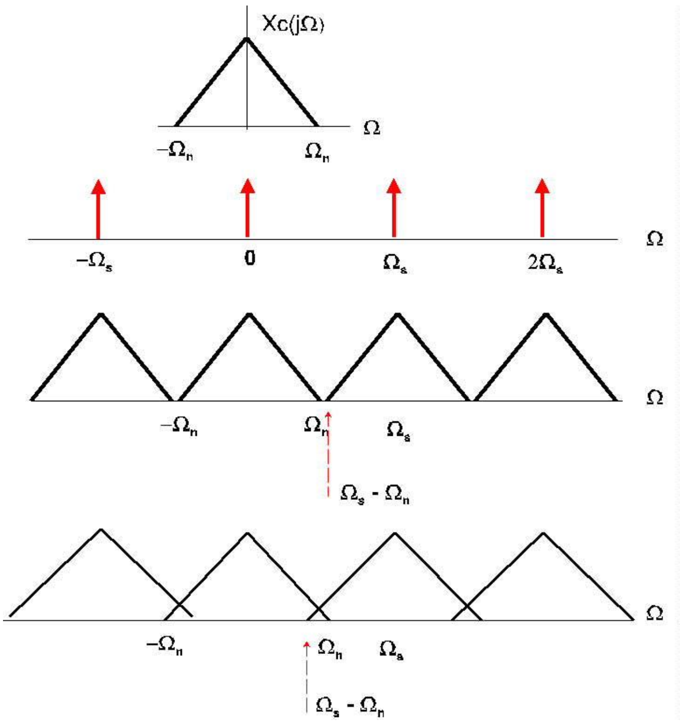



**Figure 2 Effect in the frequency Domain of Sampling (Oppenheim Schafer 1989)**

To inhibit aliasing, the periodic representations of $x_s(j\Omega)$ must not overlap, so

$\Omega_n < \ \Omega_s - \ \Omega_n \Leftrightarrow \Omega_s > 2\ \Omega_n$ (Shannon theorem)

## Exact recovery of continuous time Signals

The periodic Fourier spectrum of the impulse Sequence $X_s(t)$ can be exact recovered via an ideal low pass filter.

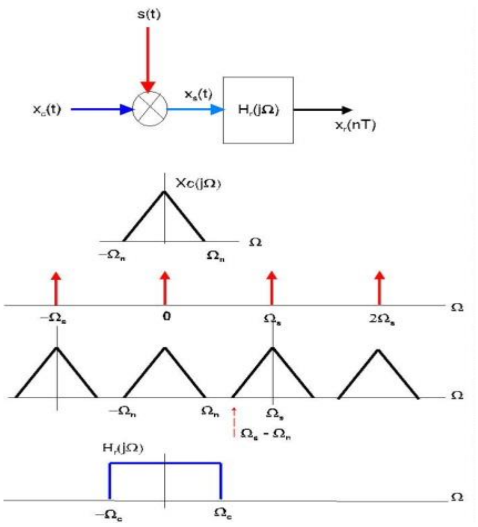



Figure 3 Exact recovery of continues time signals (Oppenheim Schafer 1989)

An ideal low pass filter cuts all periodic serial functions of $X_c(j\Omega)$.

Impulse response of an ideal band limited filter

$$\Omega_g = \frac{1}{2} * \frac{2\pi}{T} = \frac{\pi}{T}$$

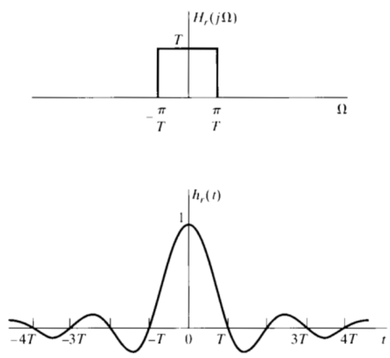


Figure 4 Impulse response of an ideal band limited filter is not causal! (Oppenheim Schafer 1989)

$$h_r\,(t) = \frac{1}{2\pi}\int_{\frac{-\pi}{T}}^{\frac{\pi}{T}} T * e^{j\Omega t}\; d\Omega = \frac{T}{2\pi}\,\frac{1}{jt}\,\left(e^{jt\frac{\pi}{T}} - e^{-jt\frac{\pi}{T}}\right) = \frac{T}{\pi}\,\frac{1}{t}\sin\left(\frac{t\pi}{T}\right) = \frac{\sin\left(\frac{t\pi}{T}\right)}{\frac{t\pi}{T}}$$

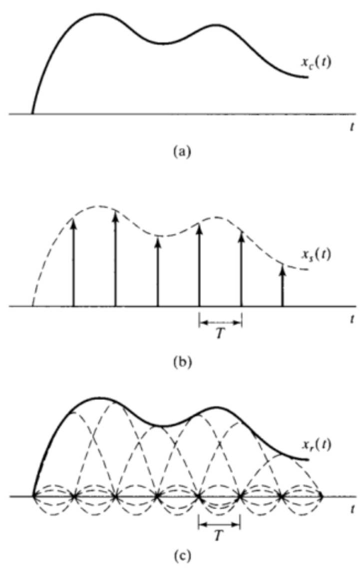



Figure 5 signal recovery from an ideal sampling signal (Oppenheim Schafer 1989)

In discrete time world there is no real frequency available. Only the digital frequency exists which is the physical frequency referred to the sampling rate.

$$\omega = \Omega T$$

| | |
|---|---|
| $\Omega$ | Physical signal frequency |
| T | sampling period |
| $\Omega_s$ | Sampling frequency |
| $\omega$ | Digital frequency |

The digital frequency has no dimension and reach from 0 to π. If Ω is the half sampling frequency then

$$\omega_{max} = \Omega_{max} T = \Omega_{max} \frac{2\pi}{\Omega_s} = \frac{\Omega_s}{2} \frac{2\pi}{\Omega_s} = \pi \text{ (aliasing border)}$$

Note:

$$e^{j\omega n} = e^{j(\omega + 2k\pi)n}$$

So the digital frequency is periodic, with the maximal frequency = π.

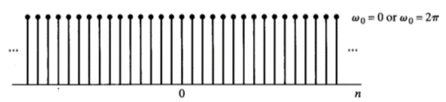



Figure 6 ω = 0 or 2π

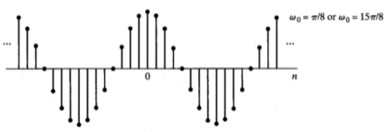



Figure 7 ω = π/8 or 7π/8

# The effect of Aliasing

## Frequency view

If the sampling frequency $f_s < 2f_c$ aliasing effects are recognized. The following figure shows the effect if the signal is a simple cos function.

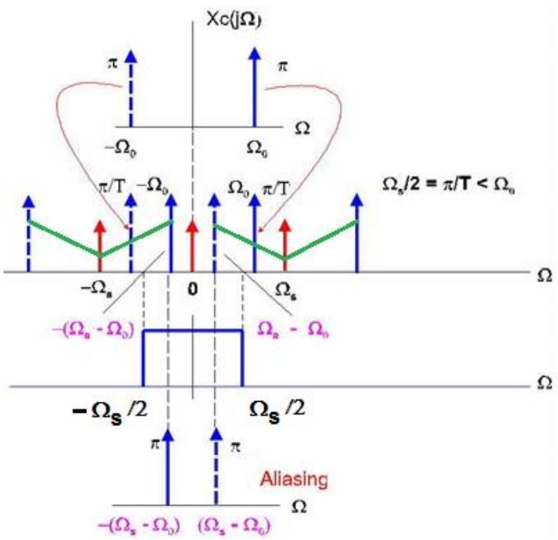
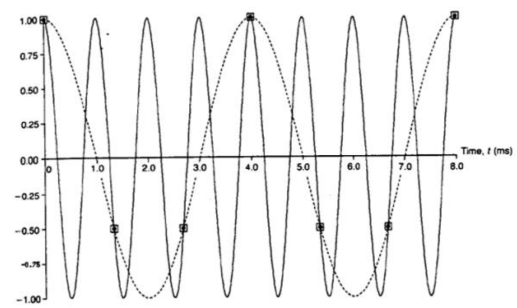



Figure 8 Aliasing Effekt (Oppenheim Schafer 1989)

A wrong frequency $\Omega_s$ - $\Omega_0$ instead of $\Omega_0$ occurs if the Shannon theorem is violated.

## Time View

*Abtastung von zwei Sinussignalen $f_1$ = 1kHz und $f_2$ = 250Hz mit einer Abtastfrequenz $f_A$ = 750 Hz ( $T_A$ = 1.33ms) liefert die gleichen Abtastwerte*



## Analog to digital conversion

The multiplication of an input signal with the impulse train is a good idea for mathematical analysis but there is no possibility for realization.

A realizable circuit is a sample and hold which generate the following signal

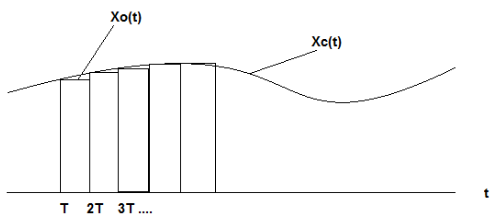



Figure 9 Input and output signal of a Sample and hold

- Transfer function of the Sample & Hold

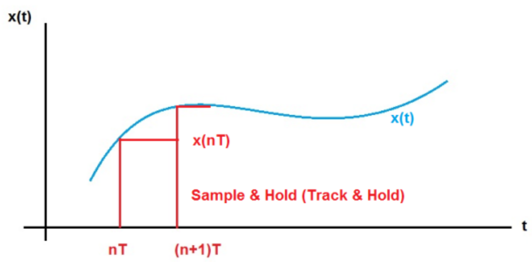



Abbildung 10 Abtasten eines analogen Signals

(1) Look at the sample signal between nT and (n+1)T
(2) Sum of all intervals

➔

$$X(s) = \int_{nT}^{(n+1)T} x(nT) * e^{-st}\, dt = x(nT) * \int_{nT}^{(n+1)T} e^{-st}\, dt = x(nT)\left(-\frac{1}{s}\right) e^{-st} \begin{Bmatrix} (n+1)T \\ nT \end{Bmatrix} = x(nT)\left(-\frac{1}{s}\right) * \left(e^{-(n+1)sT} - e^{-snT}\right) = x(nT)\,\frac{1}{s}\, e^{snT}\,(1 - e^{-sT}) = x(nT) * e^{-snT}\,\frac{1 - e^{-sT}}{s}$$

$$X(s) = \sum_{n=-\infty}^{n=+\infty} x(nT) * e^{-snT} * \frac{1 - e^{-sT}}{s}$$

The result match with the result of the impulse train except of Sample & Hold transfer term.

$G(s)_{S\&H} = \frac{1 - e^{-sT}}{s}$.

In the Frequenz domain

$$G(j\omega)_{S\&H} = \frac{1- e^{-j\omega T}}{j\omega} = e^{-j\omega\frac{T}{2}}\frac{e^{+j\omega\frac{T}{2}} - e^{-j\omega\frac{T}{2}}}{j\omega} = \frac{2}{\omega}e^{-j\omega\frac{T}{2}}\frac{e^{+j\omega\frac{T}{2}} - e^{-j\omega\frac{T}{2}}}{j2}$$
$$= \frac{2}{\omega}e^{-j\omega\frac{T}{2}} * \sin\left(\omega\frac{T}{2}\right) = \frac{2}{\omega} * \sin\left(\omega\frac{T}{2}\right) * e^{-j\omega\frac{T}{2}}$$

The group delay of the Sample&Hold $T_G = T/2$.

➔

$$X(s) = \sum_{n=-\infty}^{n=+\infty} x(nT) * e^{-snT} * \frac{1- e^{-sT}}{s} = \sum_{n=-\infty}^{n=+\infty}\{x(nT) * e^{-snT}\} * \frac{2}{\omega} * \sin\left(\omega\frac{T}{2}\right) * e^{-j\omega\frac{T}{2}}$$

Mit

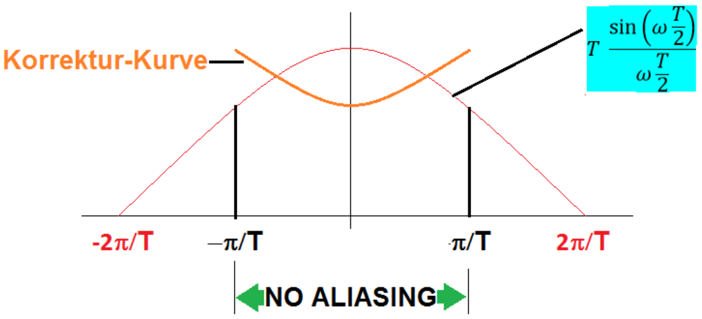
s→ jω && $2\pi * \delta(\omega - \omega_0) = F\{e^{j\omega_0 t}\}$

➔

$$X(j\omega) = \sum_{n=-\infty}^{n=+\infty}\{x(nT) * 2\pi * \delta(\omega - n\omega_0)\} * \frac{2}{\omega} * \sin\left(\omega\frac{T}{2}\right) * e^{-j\omega\frac{T}{2}}$$
$$= \sum_{n=-\infty}^{n=+\infty}\{x(nT) * 2\pi\ \delta(\omega - n\omega_0)\} * T\frac{\sin\left(\omega\frac{T}{2}\right)}{\omega\frac{T}{2}}$$

Ideal S&H

Additional term to describe a real S&H




The difference between the real S&H and the ideal S&H is the blue Term:

$$G\_CORR\_S\&H = T\frac{\sin\left(\omega\frac{T}{2}\right)}{\omega\frac{T}{2}}!$$

# Real implementation of digital signal processing

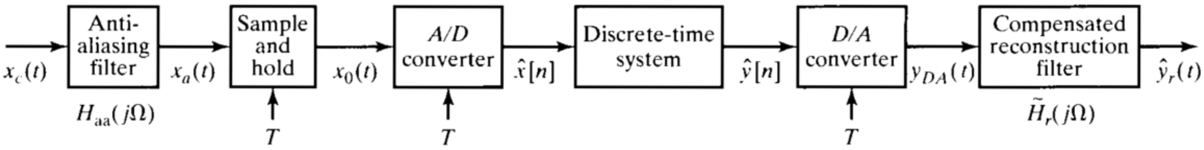



**Figure 11 Implementation of a digital signal processing system (Openheim Schafer 1986)**

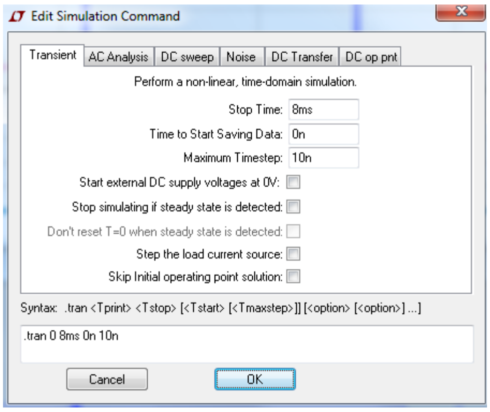
→ The real sample and hold is identical with the impulse train modulation followed by linear filtering with the zero order hold system.

# Examples

## Normal operation fs > 2f

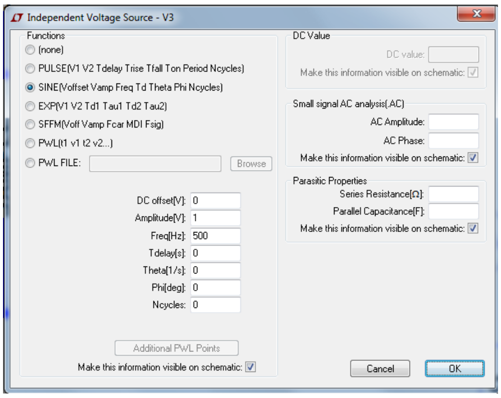
- Sampling frequency  $f_s$ = 10kHz
- Input Signal  $U_e$ = 1V f = 500Hz sinus

Simulation settings

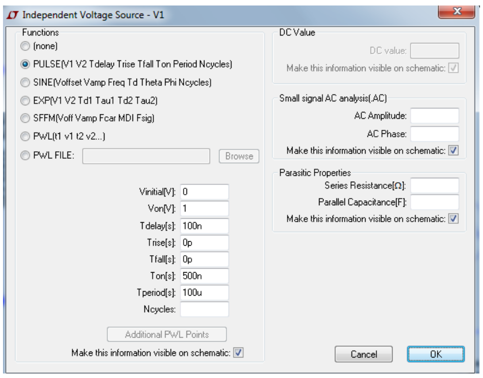



Voltage sources

- Input signal $x_c(t)$




- Sampling train s(t)



- Ideal Switch

.model SW SW(  Vt=0.5 Vh=0)

- Circuit

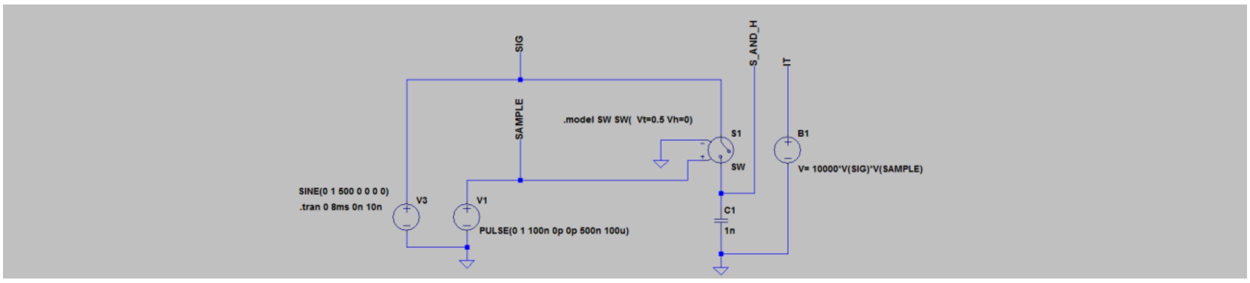
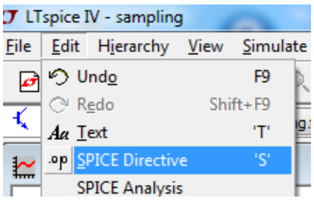
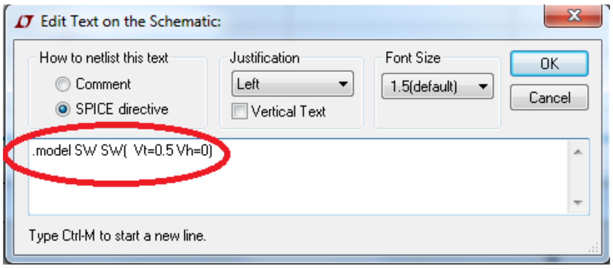



Figure 12 Ideal sampling circuit

The switch need a Spice Directive where you insert the model parameters of the switch.




Timing diagramm

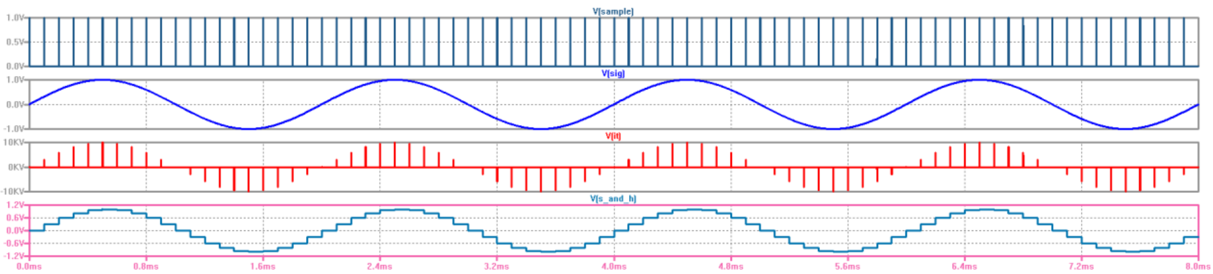



Figure 13 Sampling of continuous signals

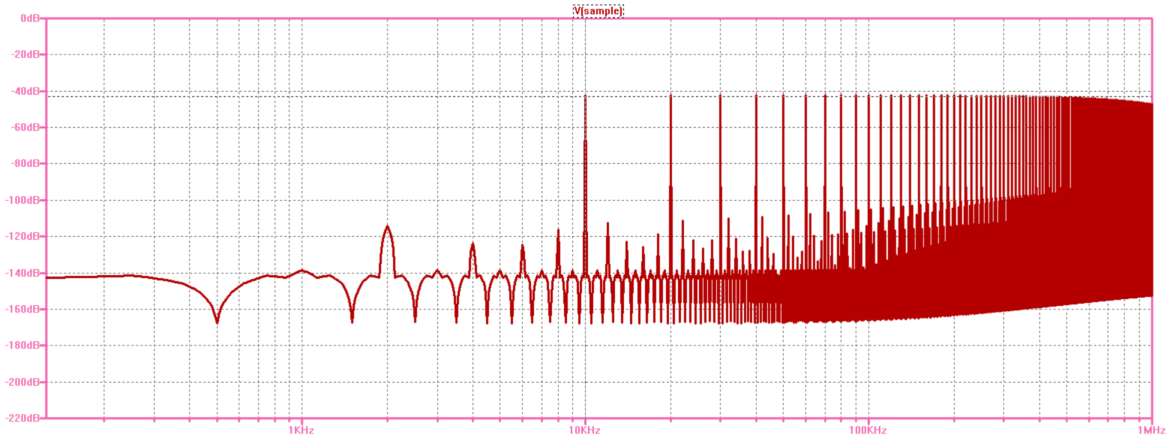



Figure 14 Spectrum of the sample train s(t)

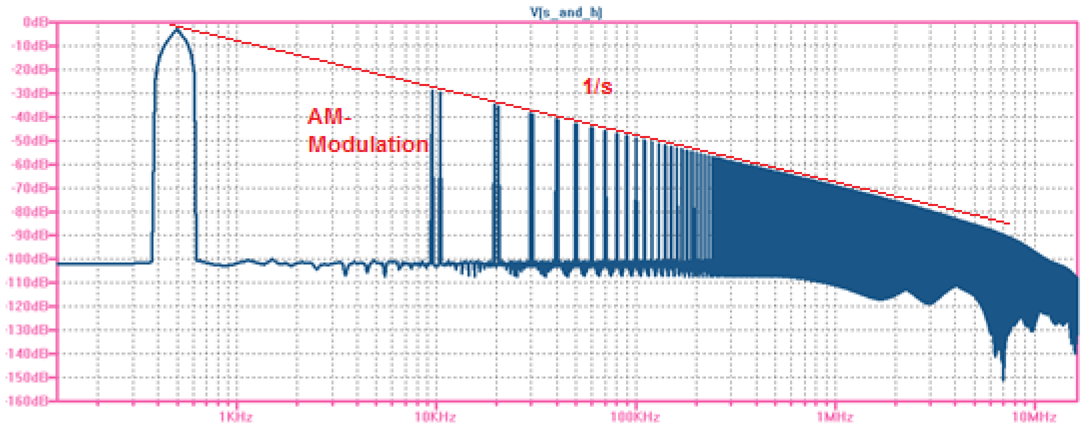



Figure 15 Spectrum of the sample and hold signal s_and_h

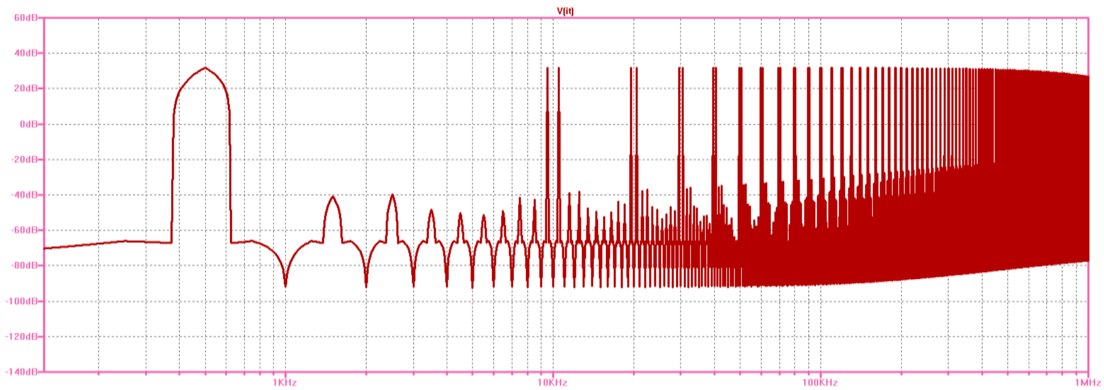



Figure 16 Spectrum of the ideal sampled input signal $x_s$ (t)

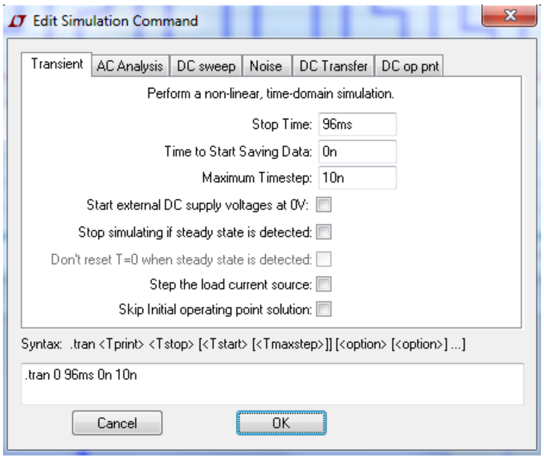
The spectrum of the ideal sampled input signal is very similar to the spectrum of the sample and hold. This spectrum has constant amplitude at each multiple sampling frequency.

You can interpret the switch as a multiplication of the sampling signal with the input signal, so you see an AM spectrum at each “carrier”. The ramp of s(t) is not ideal zero and due to calculation errors a spectrum between the carrier lines appear.

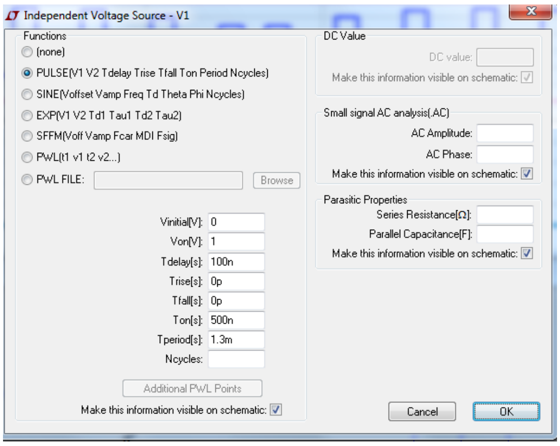
# Aliasing Error

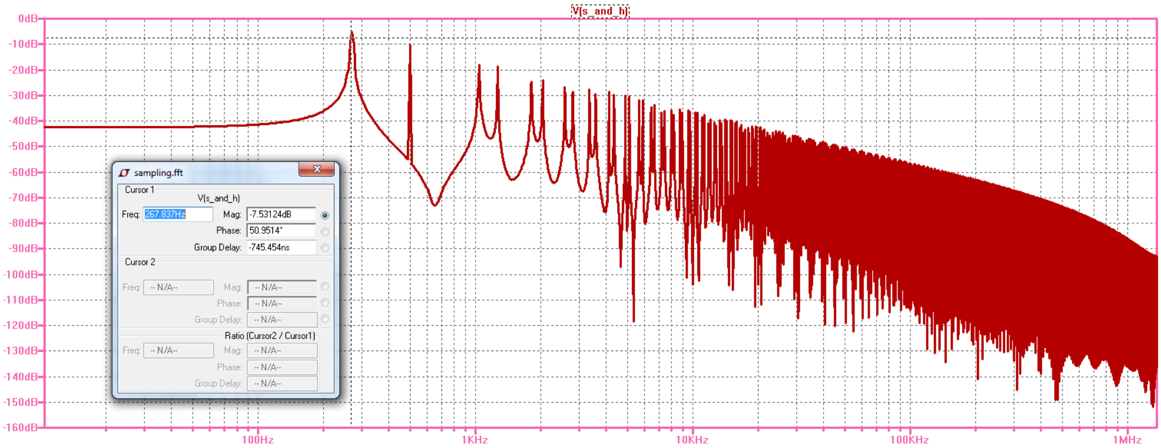
- Sampling frequency 770 Hz < 2*f
- Input Signal $U_e$ = 1V f = 500Hz sinus

Simulation settings




- Sampling train s(t) $f_s$ = 770 Hz , T = 1.3ms






The main peak is located at $f_s - f_0 = 770 - 500 = 270$ Hz