On the Stability and Motion of a Cube Balanced on a Cylinder

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This paper deals with different approaches of analyzing the motion and stability to a classical problem first posed to students in early classes of analytical mechanics. Inspired equally by A. Beléndez work on the presentation of an exact solution for the non-linear pendulum [1] and by my old esteemed analytical mechanics professor Dr. Alexandre de Resende Camara at UERJ (University of the State of Rio de Janeiro), who first presented me with the problem. I have decided to further explore the problem, both for self-teaching and educational purposes.

I first present the problem and the generic approach used to describe the system's stability using energetics. I thoroughly explore the stability of the cube, and the precision of different Taylor Expansions used to solve the non-linear equation yielded. I explore the limits of the ratio between the half-side of the cube and radius of the cylinder, to yield both no stability and continuous stability for a certain angle range.

Secondly, I present a derivation of the differential equation describing the oscillations of the cube on top of the cylinder using Lagrangian mechanics, and perform a numerical analysis on the movement, since the equation cannot be solved analytically.

Keywords: Cube Balanced on Cylinder, Stability, Numerical Solution, Critical Angle.

This problem is perhaps one of the classic examples presented to students when first learning about the stability of mechanical systems by analysis of the potential energy. It is possible to calculate the critical angles for which the system is stable and for which the cube should not topple off the cylinder. Much like the pendulum, a result can be achieved with a small angle approximation, though it can be further developed and explored with numerical techniques.

The problem is posed as such: A cylinder with radius R is fixed about its horizontal axis. A Cube of mass m and side $2b$ is balanced on top of the cylinder with its center exactly above the cylinder's axis. The cube is positioned such that four of its sides are parallel to the cylinder's axis. The contact surface of the cube with the cylinder does not slip, disregarding the effects of friction on the motion of the cube. As shown in the Figure 1.

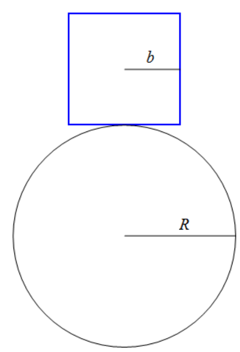


Figure 1: Rest State of a Cube Balanced on a Cylinder

By tilting the cube an angle θ, the figure would look like:

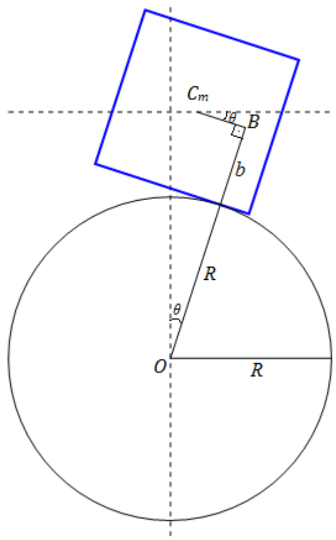


Figure 2: Tilted State of a Cube Balanced on a Cylinder.

It can be noted that the segment $\overline{C_m B}$ has a length equal to the length traveled by the cube on the surface of the cylinder, with has a value of $R\theta$. Also notice that both θ's represented in the figure are identical. From this we can write a function relating the position of the cube's center of mass, with the angle by which it has been tilted.

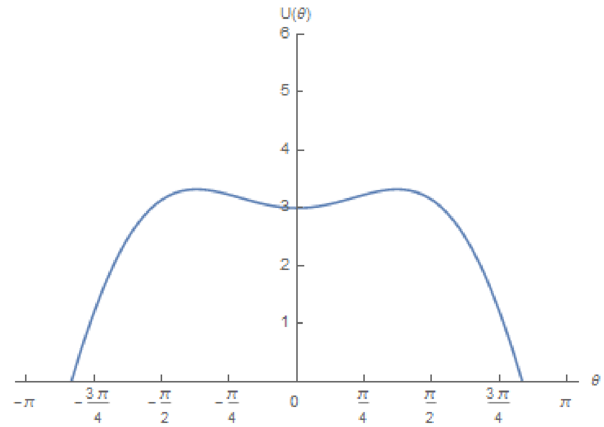
$$\overrightarrow{x_{cm}}(\theta) = (R+b)Sin(\theta) - R\theta Cos(\theta)\hat{\boldsymbol{x}} + (R+b)Cos(\theta) + R\theta Sin(\theta)\hat{\boldsymbol{y}} \tag{1}$$

On the Stability of the Cube:

The gravitational potential energy of the cube's center of mass is given as $U_g = mg\overrightarrow{x_{cm}} \cdot \hat{\boldsymbol{y}}$, since it only depends on the cube's height above the ground, so:

$$U_g(\theta) = mg[(R+b)Cos(\theta) + R\theta Sin(\theta)]$$

Which yields the plot:



Plot 1: Gravitational Potential Energy of the cube's center of mass depending on the angle. A clear valley can be seen formed in the middle. This plot is obtained with the parameters $R = 2, b = 1, m = 1,$ and $g = 1$. This will be the main example used from now on.

It can be clearly seen that there is a region of stability at the whereabouts of $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. By taking the first derivative of $U(\theta)$ with respect to θ and setting it to 0, in order to find the two maximums shown in the plot, we get:

$$\frac{dU_g}{d\theta} = mg[bSin(\theta) - R\theta Cos(\theta)] = 0$$

$$bSin(\theta) = R\theta Cos(\theta)$$

$$Tan(\theta) = \frac{R}{b}\theta \tag{2}$$

Besides the trivial solution of $\theta = 0$, the equation cannot be solved analytically. But we can approach the problem by using Taylor Expansion on $Tan(\theta)$ to different degrees, transforming the problem into an easy-to-solve polynomial.

For example, by choosing an expansion to the third degree, we get $Tan(\theta) \approx \theta + \frac{\theta^3}{3}$, so

$$\left(1 - \frac{R}{b}\right)\theta + \frac{\theta^3}{3} = 0$$

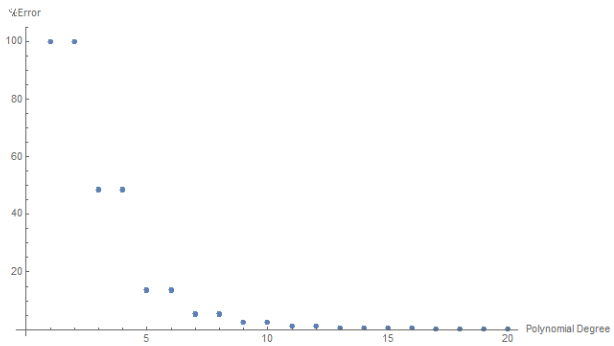
For which the non-trivial roots of the equation are:

$$\theta_{crit} = \pm\sqrt{3\left(\frac{R}{b} - 1\right)}$$

It is easy to see that there will be at least 2 real solutions for the equations, and that those will be symmetric, meaning one can tilt the cube clockwise or counter-clockwise by the same angle.

For the given example of $R = 2$ *and* $b = 1$, this would mean the critical angle is approximately 1.73205 rad. By comparing this result to the result found numerically of 1.16556 rad, the percent error is of about 48%.

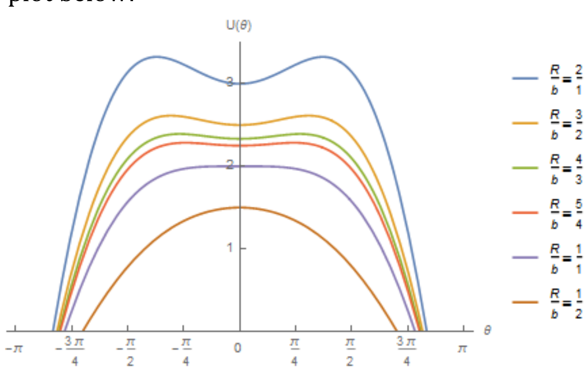
By expanding $Tan(\theta)$ into polynomials of higher degrees, the error behaves as shown in the following plot:



Plot 2: Percent Error of the result obtained by expanding $Tan(\theta)$ into polynomials of higher degrees. After the expansion to the 7th degree, the error quickly becomes less than 10%. This is obtained for the example mentioned above.

As it happens, the equation will always yield 2 non-trivial real solutions, which are symmetrical to each other.

However, we can observe from equation (2), that once $b \geq R$, the only solution for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, is $\theta = 0$, meaning there is no critical angle or stability, and any slight nudge on the cube would cause it to fall off the cylinder. This can be more clearly seen on the plot below:



Plot 3: Plots of Different $U(\theta)$ depending on the ratio $\frac{R}{b}$. After $\frac{R}{b} = 1$, the function forms a plateau, meaning there is no valley to provide stability. The value of b was fixed and R was changed in order to produce the different ratios.

Similarly, from knowing the critical angle of the system, we can derive the maximum amount of kinetic energy that can be given to the block at $\theta = 0$ such that it never falls over, by simply plugging in θ_{crit}:

$$E_{Max} = mg[(R + b)Cos(\theta_{crit}) + R\theta_{crit}Sin(\theta_{crit})]$$

By assuming an initial kinetic energy of 0, and that the initial θ cannot exceed an arc length greater then b (if it did, the cube would start balancing on its edge rather than on its side), we can impose that the situations where $R\theta_{crit} \geq b$ is stable at all angles.

Using this approach on equation (2) we can conclude that $\frac{R}{b}\theta_{crit} \geq 1$, and therefore $Tan(\theta_{crit}) \geq 1$. By assuming an interval of $0 \leq \theta_{crit} \leq \frac{\pi}{2}$, the inequality is only valid if $\theta_{crit} \geq \frac{\pi}{4}$. In can then be concluded that:

$$b \leq \frac{\pi}{4}R$$

Meaning the system will be stable at all possible starting angles, given that half the side of the cube is no greater than 78.5% the radius of the cylinder, and that the cube does not, at any point, balance on its edge.

On the Motion of the Cube

By simple deduction, it can be seen that the motion of the cube on top of the cylinder, would be of an oscillatory nature. The cube would oscillate atop the cylinder back and forth and, as long as the amplitude of the oscillation does not exceed the critical angle of the system, the oscillation should remain constant.

We can approach the problem using Lagrangian Mechanics [2]. In that context, the function $\mathcal{L} \equiv T - U$, where T is the kinetic energy of the system, and U the potential energy, must satisfy the equation:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{\partial \mathcal{L}}{\partial \theta}$$

As previously stated the gravitation potential energy is given as:

$$U = mg[(R + b)Cos(\theta) + R\theta Sin(\theta)] \quad (3)$$

In order to find the kinetic energy at a given point we need to consider both the translational and rotational motion of the cube. The translational kinetic energy is given as $K_{eT} = \frac{mv_{cm}{}^2}{2}$, where $v_{cm}{}^2 = \overrightarrow{\dot{x}_{cm}}^2$

From equation (1) we can derive that:

$$\overrightarrow{v_{cm}} = \overrightarrow{\dot{x}_{cm}} = \dot{\theta}[bCos(\theta) + R\theta Sin(\theta)]\hat{\boldsymbol{x}} + \dot{\theta}[-bSin(\theta) + R\theta Cos(\theta)]\hat{\boldsymbol{y}}$$

$$v_{cm}{}^2 = \overrightarrow{\dot{x}_{cm}}^2 = \dot{\theta}^2[bCos(\theta) - R\theta Sin(\theta)]^2 + \dot{\theta}^2[-bSin(\theta) + R\theta Cos(\theta)]^2$$

$$v_{cm}{}^2 = \dot{\theta}^2(b^2 + R^2\theta^2)$$

So:

$$K_{eT} = \frac{m\dot{\theta}^2(b^2 + R^2\theta^2)}{2}$$

(4)

Knowing the Moment of Inertia of a cube with side $2b$ is $I = \frac{2}{3}mb^2$, we have that:

$$K_{eR} = \frac{mb^2\dot{\theta}^2}{3}$$

(5)

So using equations (4) and (5) we have that the total kinetic energy is:

$$T = \frac{m\dot{\theta}^2(b^2 + R^2\theta^2)}{2} + \frac{mb^2\dot{\theta}^2}{3}$$

$$T = \frac{1}{2}m\dot{\theta}^2\left(\frac{5}{3}b^2 + R^2\theta^2\right)$$

(6)

So, using equations (3) and (6) we find that:

$$\mathcal{L} = \frac{1}{2}m\dot{\theta}^2\left(\frac{5}{3}b^2 + R^2\theta^2\right) - mg[(R + b)Cos(\theta) + R\theta Sin(\theta)]$$

Thus:

$$\frac{\partial \mathcal{L}}{\partial \theta} = m\dot{\theta}^2R^2\theta - mg[-bSin(\theta) + R\theta Cos(\theta)]$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m\dot{\theta}\left(\frac{5}{3}b^2 + R^2\theta^2\right)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = m\ddot{\theta}\left(\frac{5}{3}b^2 + R^2\theta^2\right) + 2m\dot{\theta}^2R^2\theta$$

So the final equation to describe the motion of the cube, after some algebraic manipulation, is:

$$\ddot{\theta}\left(\frac{5}{3}b^2 + R^2\theta^2\right) + \dot{\theta}^2R^2\theta - g[bSin(\theta) - R\theta Cos(\theta)] = 0$$

(7)

As a first approach, we can make an approximation for small oscillations by taking only the first order terms, where $Sin(\theta) \approx \theta$, $Cos(\theta) \approx 1$, $\dot{\theta} \approx 0$, and $\theta^2 \approx 0$.

So:

$$\frac{5}{3}b^2\ddot{\theta} = g\theta(b - R)$$

By defining:

$$\omega^2 = \frac{3g(R - b)}{5b^2}$$

$$\theta(0) = \theta_0$$

$$\dot{\theta}(0) = \omega_0$$

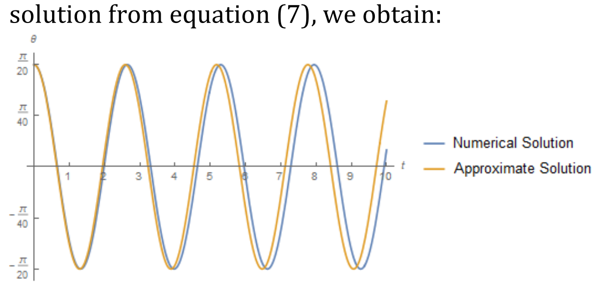
The equation can be rewritten as:

$$\ddot{\theta} + \theta\omega^2 = 0$$

Which yields the solution

$$\theta(t) = \theta_0 Cos(\omega t) + \frac{\omega_0}{\omega}Sin(\omega t)$$

By comparing the plot of the newly found approximation, to that of the numerically found solution from equation (7), we obtain:



Plot 4: Comparison of Numerical and Approximate solutions for the oscillatory movement of the cube. The parameters used were $R = 2, b = 1, g = 9.81, \theta_0 = \frac{\pi}{20}$, and $\omega_0 = 0$.(Done with Mathematica®)

Conclusion

Many useful results can be drawn from the analysis of the stability of the cube, such as the critical angles for a given system, and the limits of the size of the cube for zero or total stability with respect to the cylinder.

As there is no simple analytical solution to be found for the differential equation describing the motion of the cube (7), the only way to foresee its behavior is through numerical calculations.

References

[1] A. Belendez, C. Pascual, D.I. Mendez, T. Belendez, and C. Neipp. *Exact Solution for the*

Nonlinear Pendulum. Revista Brasileira de Ensino de Fisica, v. 29, n. 4, p. 645-648, (2007)

[2] David Morin. *Chapter 6: The Lagrangian Method.* Draft Version.

Appendix:

Mathematica code used

https://github.com/andrerg01/mathematica-codes