Financial Derivatives

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Outline

Motivation

Derivatives: Were introduced by the ancient cultures in the trade and hedge of commodities (agriculture products) through the study of the astronomy after predict the rain and dry seasons.

Can be defined as a financial instrument whose value depends from the values of other underlying variables.

Today there are trade in standardizied markets and in OTC (over the count) markets

Type of investors

1. Hedge Instruments
2. Speculative
3. Arbitrage

Type of derivatives instruments [Hull, 2003]

1. Forwards and Futures
2. Options
3. Swaps

Most important Trade Centers

1. CBOE Chicago board options exchange
2. CME Chicago mercantile Exchange
3. LME London Metal Exchange
4. Bolsa mercantil de Colombia

Derivative Instruments

A forward contract (called a futures contract if traded on an exchange) is an agree- ment between two parties that one party will purchase an asset from the counterparty on a certain date in the future for a predetermined price.

The holder of a forward contract is obligated to buy the underlying asset at the for- ward price (also called delivery price) K on the expiration date of the contract.

Let S_T denote the asset price at expiry T.Since the holder pays K dollars to buy an asset worth S_T , the terminal payoff to the holder (long position) is seen to be $S_T - K$. The seller (short position) of the forward faces the terminal payoff $K - S_T$, which is negative to that of the holder (by the zero-sum nature of the forward contract).

Specifications

- The Asset
- Contract Size
- Delivery Arrangements
  - Delivery Months
  - Price Quotes
  - Daily price movements limits
- Position Limits

Forwards vs Futures

Forward	**Futures**
Traded on OTC market	Traded on an exchange
Non standardized	Standardizierd contract
Usually one specified deliver date	Range of delivery dates
Settled at end of contract	Settled daily
Delivery or final cash settlement	Closed out prior to maturity

Values and Prices of Forward Contracts : We would like to consider the pricing formulas for forward contracts under three separate cases of dividend behaviors of the underlying asset, namely, no dividend, known discrete dividends and known continuous dividend yields.

* Nondividend Paying Asset: Let $F(S, T)$ denote, respectively, the value and the price of a forward contract with current asset value S and time to maturity T.

$$F(S, T) = S - Ke^{-rT} \tag{1}$$

Where r is the free risk rate, K is the delivery price in the contract and F is the value of a long forward contract today.

* Discrete Dividend Paying Asset: Let D denote the present value of all dividends paid by the asset within the life of the forward. To find the value of the forward contract, we modify the above second portfolio to contain one unit of the asset plus borrowing of D dollars.

$$F(S,T) = S - D - Ke^{-rT} \tag{2}$$

* Continuous Dividend Paying Asset: The dividend is paid continuously throughout the whole time period and the dividend amount over a differential time interval dt is $qS\ dt$, where S is the spot asset price. Under this dividend behavior, we choose the second portfolio to contain e^{-qT} units of asset with all dividends being reinvested to acquire additional units of asset.

$$F(S,T) = Se^{-qT} - Ke^{-rT} \tag{3}$$

* Interest Rate Parity Relation: Forward contracts on foreign currencies, the value of the underly- ing asset S is the price in the domestic currency of one unit of the foreign currency. The foreign currency considered as an asset can earn interest at the foreign riskless rate r_f .

$$F(S, T) = Se^{(r-r_f)T} \tag{4}$$

* Cost of Carry and Convenience Yield: For commodities like grain and livestock, there may be additional costs to hold the assets such as storage, insurance, spoilage, etc.Suppose we let U denote the present value of all additional costs that will be incurred during the life of the forward contract.

$$F(S, T) = (S + U)e^{-rT} \quad F(S, T) = Se^{(r+u)T} \tag{5}$$

Relation between Forward and Futures Prices:
Forward contracts and futures are much alike, except that the former are traded over- the-counter and the latter are traded in exchanges. Since the exchanges would like to organize trading such that contract defaults are minimized, an investor who buy a futures in an exchange is requested to deposit funds in a margin account to safe- guard against the possibility of default (the futures agreement is not honored at maturity).

This process is called marking to market the account. Therefore, the payment required on the maturity date to buy the underlying asset is simply the spot price at that time. However, for a forward contract traded outside the exchanges, no money changes hands initially or during the life-time of the contract.

A swap is a financial contract between two counterparties who agree to exchange one cash flow stream for another according to some prearranged rules. Two important types of swaps are considered in this section: interest rate swaps and currency swaps.

Interest rate swaps have the effect of transforming a floating-rate loan into a fixed- rate loan or vice versa.

Currency swap can be used to transform a loan in one currency into a loan in another currency.

Interest Rate Swaps: The most common form of an interest rate swap is a fixed-for-floating swap, where a series of payments, calculated by applying a fixed rate of interest to a notional principal amount, are exchanged for a stream of payments calculated using a floating rate of interest.

Floatings Rates

- LIBOR (London Interbank offer rate)
- Eurodollar US dollar deposit outside USA.
- DTF (Deposito a término Fijo)

$$V_{swap} = B_{fl} - B_{fix} \tag{6}$$

Where B_{fl} is the floating rate and B_{fix} is the fix rate. If it is the other perspective the difference is in the oppsite way.

Currency Swaps: A currency swap is used to transform a loan in one currency into a loan of another currency.

$$V_{swap} = B_D - S_0 B_F \tag{7}$$

Where B_F is the value in foering currency, B_D is the value of the Bond in local currency and S_0 is the spot exchange rate expressen in unist of the domestic currency.

An option gives the holder the right (but not the obligation) to buy or sell an asset by a certain date for a pre- determined price, is classified either as a call option or a put option. A call (or put) option is a contract which gives its holder the right to buy (or sell) a prescribed asset, known as the underlying asset, by a certain date (expiration date) for a predetermined price.

If the option can only be exercised on the expiration date, then the option is called a European option. Otherwise, if the exercise is allowed at any time prior to the expiration date, then the option is called an American option.

The counterparty to the holder of the option contract is called the option writer. The holder and writer are said to be, respectively, in the long and short positions of the option contract.

Factors

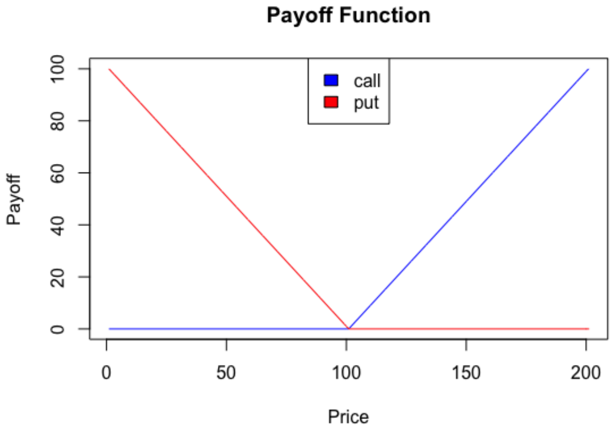
- Current Spot Price of the underlying S_0
- Strike price K
- Time to expiration T
- The volatility of the underlying σ
- The risk free rate r

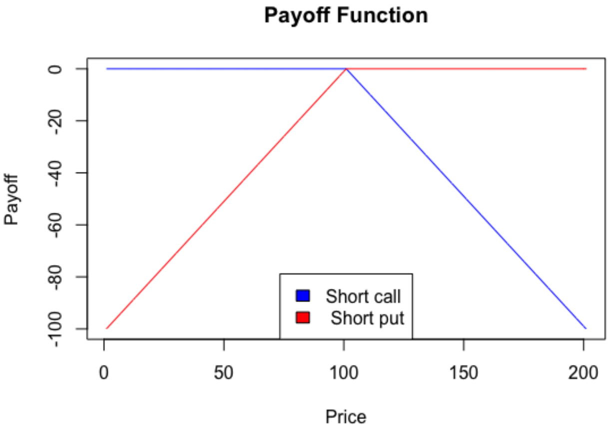
Pay-off of the option

- Call $V_{call} = max\{S - K, 0\}$ Put $V_{put} = max\{K - S, 0\}$

Moneyness

- in-the-money if $S > K$ for call or $S < K$ for put
- out of the money if if $K > S$ for call or $K < S$ for put
- At the money if $S = K$


Payoff Function
call
put
Payoff
Price
0
20
40
60
80
100
0
50
100
150
200

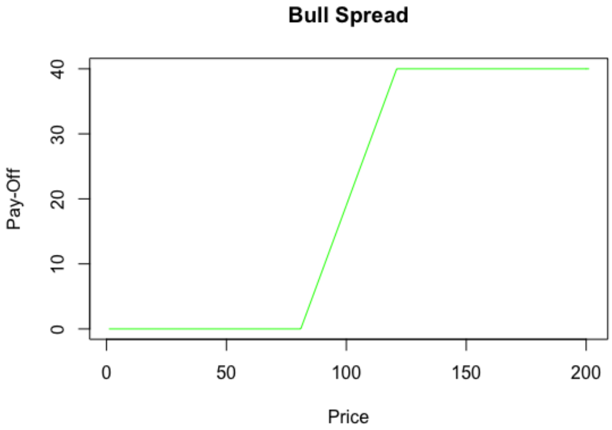

Payoff Function
Payoff
0
-20
-40
-60
-80
-100
0
50
100
150
200
Price
Short call
Short put

Trading Strategies :A spread strategy refers to a portfolio which consists of options of the same type (that is, two or more calls, or two or more puts) with some options in the long position and others in the short position in order to achieve a certain level of hedging effect.

- Bull (Long Strategy)
- Bear (Short Strategy)
- Straddle
- Strangle
- Butterfly

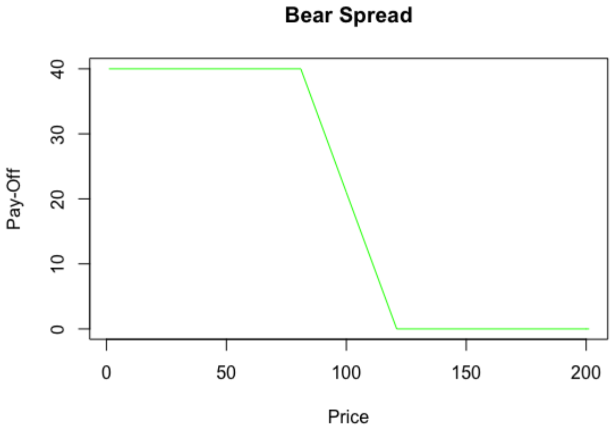
Bull (Long Strategy) :Is a long position because is created by buying a call option on an underlying with a certain strike price and selling an option on the same stock with a higher strike price.

Event	**Call**	**S-Call**	**Pay-off**
$S < K_1$	0	0	0
$K_1 < S < K_2$	$S - K_1$	0	$S - K_1$
$S > K_2$	$S - K_1$	$K_2 - S$	$K_2 - K_1$


Bull Spread
Pay-Off
0
10
20
30
40
0
50
100
150
200
Price

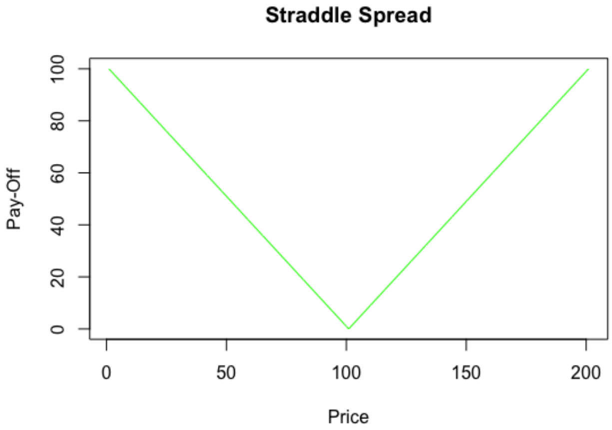
Bear (Short Strategy) :Is a short position because is created by buying a put option on an underlying with a certain strike price and selling an option on the same stock with a lower strike price.

Event	**Put**	**S-Put**	**Pay-off**
$S < K_1$	$K_2 - S$	$S - K_1$	$K_2 - K_1$
$K_1 < S < K_2$	$K_2 - S$	0	$K_2 - S$
$S > K_2$	0	0	0


Bear Spread
Pay-Off
0
10
20
30
40
0
50
100
150
200
Price

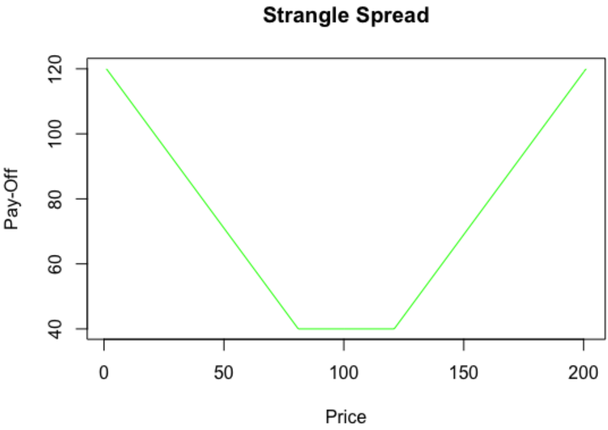
Straddle :Involves buying a call option on an underlying with a certain strike price and buying a put option on the same stock.

Event	**Put**	**Call**	**Pay-off**
$S < K$	$K - S$	0	$K - S$
$S > K$	0	$S - K$	$S - K$


Straddle Spread
Pay-Off
0
20
40
60
80
100
0
50
100
150
200
Price

Strangle :Involves buying a call option on an underlying with a certain strike price and buying a put option on the same stock with lower strike price.

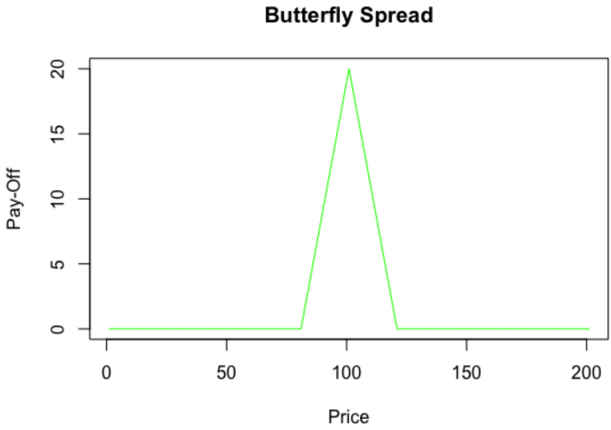
Event	**Put**	**Call**	**Pay-off**
$S < K_1$	$K_1 - S$	0	$K_1 - S$
$K_1 < S < K_2$	0	0	0
$S > K_2$	0	$S - K_2$	$S - K_2$



Strangle Spread
Pay-Off
120
100
80
60
40
0
50
100
150
200
Price

Butterfly :Uses four options wich two of them are short positions with the same strike price, and two long call with different strike price.

Event	**Call** K_1	**2S-Call** K_2	**Call** K_3	**Pay-off**
$S < K_1$	0	0	0	0
$K_1 < S < K_2$	$S - K_1$	0	0	$S - K_1$
$K_2 < S < K_3$	$S - K_1$	$2(K_2 - S)$	0	$2K_2 - K_1 - S$
$S > K_3$	$S - K_1$	$2(K_2 - S)$	$S - K_3$	$2K_2 - K_3 - K_1$


Butterfly Spread
Pay-Off
0
5
10
15
20
0
50
100
150
200
Price

Binomial Model

Definition: Let Ω be a nonempty set, and let $\mathcal{F}$ be a collection of subsets on Ω. We say that $\mathcal{F}$ is a σ [Shreve, 2004] :

- The empty set $\emptyset$ bleongs to $\mathcal{F}$
- whenever a set $\mathcal{A}$ belongs to $\mathcal{F}$, $\mathcal{A}^c$ also belongs to $\mathcal{F}$.
- whenever a sequence of sets $\mathcal{A}_1, \mathcal{A}_2, ...\mathcal{A}_n$ belongs to $\mathcal{F}$, their union $\cup$ de $\mathcal{A}_1, \mathcal{A}_2, ...\mathcal{A}_n$ also belongs to $\mathcal{F}$.

Definition Let Ω be a nonempty set with a collection of subsets on $\mathcal{F}$. A probability meassure $\mathbb{P}$ is a function that, to every set $A in \mathcal{F}$ assigns a number $[0,1]$ called the probability of A and written $\mathbb{P}(A)$ [Shreve, 2004]:

- $\mathbb{P}(\Omega) = 1$
- whenever $A_1, A_2, ... A_n$ is a sequence of disjoint sets in $\mathcal{F}$
- $$\mathbb{P}\Big(\cup_{n=1}^{\infty} \mathcal{A}_n\Big) = \sum_{n=1}^{\infty} \mathbb{P}(\mathcal{A}_n)$$

A probabilty space is $(\Omega, \mathcal{F}, \mathbb{P})$

Teorema: Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probablity space. For A, B $(A_n)_{n\in\mathbb{N}}$ y $(B_n)_{n\in\mathbb{N}} \in \mathcal{F}$

- $\mathbb{P}(\emptyset) = 0$
- $P(A^c) = 1 - P(A)$
- Si $A \subseteq B$ entonces $P(A) \leq P(B)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Example: We toss a coin infinitely many times and let Ω denote the set of possible outcomes. We assume the probability of head on each toss is $p > 0$, the probability of a tail is $q = 1 - p > 0$ and the different tosses are independent. Construct the outcomes for $\mathcal{F}$ from $0 to 2$.

Definition: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A random variable is a real valued function X defined on Ω with the property that for every Borel subset B of $\mathbb{R}$, the subset of Ω given by

$$\{X \in B\} = \{\omega \in \Omega; X(\omega) \in B\}$$

is in the σ algebra $\mathcal{F}$.

Definition: Let X be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The distribution measure of X is the probability measure μ_x that assigns to each Borel subset B of $\mathbb{R}$ the mass $\mu_X(B) = P\{X \in B\}$

Definition Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space and X a random variable [**?**].

a If X is a discrete random variable $x_1, x_2, \ldots$ the expected valur of X exits if $\sum_{k=1}^{\infty} |x_k| = P(\{X = x_k\}) < \infty$. Then

$$\mathbb{E}(X) = \sum_{x} x\mathbb{P}(x) \tag{8}$$

b If X is a continuos randmo variable with density function $f(x)$, the expected value for X is $\int_{-\infty}^{\infty} |x| f(x) dx < \infty$

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x) dx \tag{9}$$

c If $E[X] < \infty$, then $Var[X] := E([X - E(X)]^2)$ and $\sigma = \sqrt{Var(X)}$

Expected Value: Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probabilty space and X a random vaiable

- $E(a) = a$ for all $a \in \mathbb{R}$.
- si $X_1 \leq X_2$, then $E(X_1) \leq E(X_2)$
- $E(aX_1 + bX_2) = aE(X_1) + bE(X_2)$ for all $a, b \in \mathbb{R}$
- $|E(X)| \leq E(|X|)$

Variance Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probabilty space and X a random variable.

- $Var(X) = E(X^2) - (E(X))^2$
- $Var(aX + b) = a^2 Var(X)$ para $a, b \in \mathbb{R}$

Binomial Distribution: Let x a discrete random variable.
$x \sim B(n, p)$

$$P(X = x) = \binom{n}{x} p^x q^{n-x} \tag{10}$$

With p and q as probability in $(0, 1)$ and n as number of attemps

Normal Distribution Let x a continuos random variable $x \sim N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

With μ as mean and σ^2 as variance

Key Features

Arbitrage: One of the fundamental concepts in the theory of option pricing is the absence of arbitrage opportunities, is called the no arbitrage principle.
A complet market should have at least three conditions to full the no arbitrage princip

a Depth in transactions

b Liquidity

c The same information for all participants

More precisely, an arbitrage opportunity can be defined as a self-financing trading strategy requiring no initial investment, having zero probability of negative value at expiration, and yet having some possibility of a positive terminal payoff.

Self- Financing Strategy: Suppose an investor holds a portfolio of securities, such as a combination of options, stocks and bonds. As time passes, the value of the portfolio changes because the prices of the securities change. Besides, the trading strategy of the investor affects the portfolio value by changing the proportions of the securities held in the portfo- lio, say, and adding or withdrawing funds from the portfolio. An investment strategy is said to be self-financing if no extra funds are added or withdrawn from the ini- tial investment. The cost of acquiring more units of one security in the portfolio is completely financed by the sale of some units of other securities within the same portfolio.

Neutral Risk $\mathbb{Q}$: A probability measssure in $[0, 1]$ is equivalent to $\mathbb{P}$, but with the effects from a financial random variable in a complet market.

In other words, an investor does not control the market and he can only meassure the tendence and the variance to take a position and make a strategy without the certain to get benefits from the market.

No arbitrage opportunities exist if and only if there exists a risk neutral probability measure Q.

Example: The probabilty of the movement of an asset up is $\widetilde{p}$ or down $\widetilde{q} = 1 - \widetilde{p}$. When the asset foes up the magnitud is $U = e^{\sigma\sqrt{t}}$ and when the asset goes down the magnitud is $D = e^{-\sigma\sqrt{t}}$. Using the risk free rate r, a risk neutral meassure is:

$$\widetilde{p} = \frac{(1+r) - D}{U - D} \tag{11}$$

It is only true for 13 when

$$D < (1+r) < U \tag{12}$$

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$$\widetilde{p} = \frac{(1+r) - D}{U - D} \tag{13}$$

It is only true for 13 when

$$D < (1+r) < U \tag{14}$$

Martingale Property Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, Let T be a fixed positive number, and let $\mathcal{F}(t)$, $0 \leq t \leq T$ be a filtration of sub σ algebra of $\mathcal{F}$. Consider an adapted stochastic process $M(t), \quad 0 \leq s \leq t \leq T$.[Shreve, 2004]

i If $\mathbb{E}[M(t)|\mathcal{F}(s)] = M(s)$ for all $0 \leq s \leq t \leq T$ then $M(t)$ is a Martingale and has no tendency to rise or fall.

ii If $\mathbb{E}[M(t)|\mathcal{F}(s)] \geq M(s)$ for all $0 \leq s \leq t \leq T$ then $M(t)$ is a submartingale and it may have tendency to rise.

iii If $\mathbb{E}[M(t)|\mathcal{F}(s)] \leq M(s)$ for all $0 \leq s \leq t \leq T$ then $M(t)$ is a supermartingale and it may have a tendency to fall.

The future value is the same as the present value when exist a risk neutral meassure $\mathbb{Q}$ and under a risk free rate it is also a martingale.

$$\frac{S_n}{(1+r)^n} = \mathbb{E}_n^{\mathbb{Q}}\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right] \tag{15}$$

Using 14 the probability meassure $\widetilde{q} = (1 - \widetilde{p})$ adn $D = 1/U$.

$$\mathbb{E}_n^{\mathbb{Q}}\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right] = \mathbb{E}_n^{\mathbb{Q}}\left[\frac{S_n}{(1+r)^{n+1}}\frac{S_{n+1}}{S_n}\right] \tag{16}$$

$$= \frac{S_n}{(1+r)^n}\mathbb{E}_n^{\mathbb{Q}}\left[\frac{1}{(1+r)}\frac{S_{n+1}}{S_n}\right] \tag{17}$$

$$\frac{S_n}{(1+r)^n}\frac{1}{(1+r)}\mathbb{E}_n^{\mathbb{Q}}\left[\frac{S_{n+1}}{S_n}\right] \tag{18}$$

$$\frac{S_n}{(1+r)^n}\frac{\widetilde{p}U + \widetilde{q}D}{1+r} \tag{19}$$

$$\frac{S_n}{(1+r)^n} \tag{20}$$

Filtration Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which is defined a stocastic process. A filtration is a collector of σ algebra $\mathcal{F}(t)$

i For $0 \leq s \leq t$ every set in $\mathcal{F}(s)$ is also in $\mathcal{F}(t)$. In other words there is at least as much information available at the later time $\mathcal{F}(t)$ as there is at the earlier time $\mathcal{F}(s)$.

ii For each $t \geq 0$ a stocastic process at te time t is $\mathcal{F}(t)$ measureable. In other words, the information available at the time t is sufficient to evaluate the stocastic process.

iii For $0 \leq t < u$, the increment $X(u) - X(t)$ is independent of $\mathcal{F}(t)$.

Stock prices are assumed to follow this simple model: The initial stock price during the period under study is denoted S_0.

- The stock price moves up by a factor U if the coin comes out heads H.
- Down by a factor of D if it comes out tails T

$$\Omega = \{U, D\}^n = \{(\omega_1, \omega_2, ..., \omega_n\}; \omega_i = U \quad o \quad \omega_i = D$$

$$X_i = \Omega \to \mathbb{R} \quad \omega \to \omega_i$$

$$H_i : \Omega \to \mathbb{R} \quad \omega \to \#\{j \leq i : \omega_j = U\}$$

$$T_i : \Omega \to \mathbb{R} \quad \omega \to \#\{j \leq i : \omega_j = D\}$$

The stock price at the moment S_i is calculated by the multiplication of the results of the random variable with S_0.

$$S_i = S_0 \Pi_{j=1}^{i} X_j = S_0 \, U^{H_i} D^{T_i} \tag{21}$$

The expected value is the sum from each of the results multiplied by its probability.

$$\mathbb{E}_{\mathbb{Q}}\left(\frac{1}{(1+R)^j} S_j | \mathcal{F}_i \right) = \frac{1}{(1+R)^i} S_i \quad para \quad i < j$$

Where the filtration $\mathcal{F}_i \ i \geq 0$ is a martingale for $0 \leq i \leq j \leq n$

$$S_j = S_i \Pi_{k=i+1}^{j} X_k$$

Let $\mathcal{F}_i$ adapted, $\Pi_{k=i+1}^{j} X_k$ and independent from $\mathcal{F}_i$

$$\mathbb{E}_{\mathbb{Q}}(S_j|\mathcal{F}_i) = S_i \mathbb{E}_{\mathbb{Q}}(\Pi_{k=i+1}^{j} X_k|\mathcal{F}_i)$$

$$= S_i \mathbb{E}_{\mathbb{Q}}(\Pi_{k=i+1}^{j} X_k) = S_i[U\widetilde{p} + D\widetilde{q}]^{j-i}$$

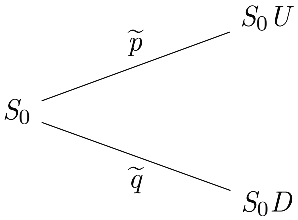
$$= S_i(1+R)^{j-i}$$

Using 21 and 10 you can get the expected value of S_n for any $\mathcal{F}_n$ in a discrete time $(0, n)$ [Cox et al., 1979].

$$E_{\mathbb{Q}}[S_n|\mathcal{F}_n] = \sum_{x=0}^{n} \binom{n}{x} \widetilde{p}^x (1-\widetilde{p})^{n-x} S_0 U^x D^{n-x} \qquad (22)$$

The risk neutral probability is the same as 13.

Binomial Model



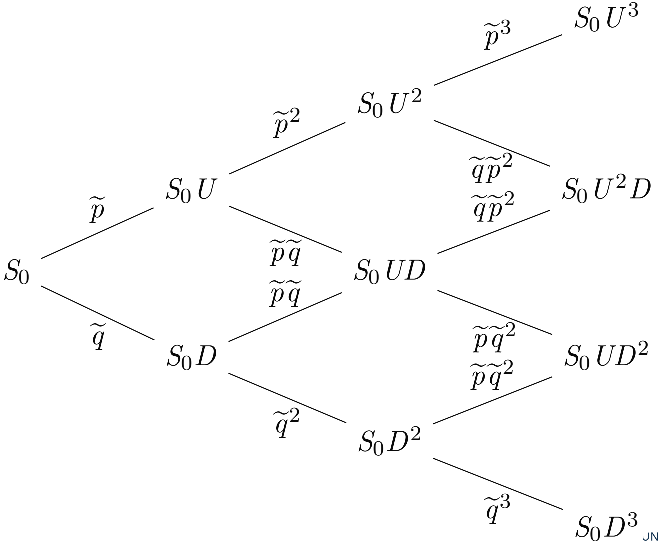
$$S_0$$

$$\widetilde{q} = 1 - \widetilde{p}$$

$$U = e^{\sigma\sqrt{dt}}$$

$$D = e^{-\sigma\sqrt{dt}}$$

$$
\begin{tikzcd}
 & & S_0\,U^2 \\
 & S_0\,U \arrow[ur, "\widetilde{p}^2"] \arrow[dr, "\widetilde{p}\widetilde{q}"'] & \\
S_0 \arrow[ur, "\widetilde{p}"] \arrow[dr, "\widetilde{q}"'] & & S_0\,UD \\
 & S_0\,D \arrow[ur, "\widetilde{p}\widetilde{q}"] \arrow[dr, "\widetilde{q}^2"'] & \\
 & & S_0\,D^2
\end{tikzcd}
$$


$S_0 U^3$
$\widetilde{p}^3$
$S_0 U^2$
$\widetilde{p}^2$
$\widetilde{q}\widetilde{p}^2$
$S_0 U$
$S_0 U^2 D$
$\widetilde{q}\widetilde{p}^2$
$\widetilde{p}$
$\widetilde{p}\widetilde{q}$
S_0
$S_0 UD$
$\widetilde{p}\widetilde{q}$
$\widetilde{q}$
$\widetilde{p}\widetilde{q}^2$
$S_0 D$
$S_0 UD^2$
$\widetilde{p}\widetilde{q}^2$
$\widetilde{q}^2$
$S_0 D^2$
$\widetilde{q}^3$
$S_0 D^3$

Modelo Cox Ross Rubinstein The problem now is to find the exact formula or method wich transform the asset price and the time of the expiration into the value of an option. [Cox et al., 1979]. Using the price function and the option payoff there is a function to get the current option value for an european/amercian option.

$$S_n = S_0 U^x D^{n-x} \tag{23}$$

For each option there is a payoff function.

- Option Call (buy) $V(S, n) = max(S - K, 0)$.
- Option Put (sell) $V(S, n) = max(K - S, 0)$

There is an underlying prce S and a strike price K.

$$V(S, n) = \left(\sum_{x=0}^{n} \binom{n}{x} \widetilde{p}^x (1-\widetilde{p})^{n-x} max(S_0 U^x D^{n-x} - K, 0) \right) / (1+r)^n \tag{24}$$

$$V(S, n) = \left(\sum_{x=0}^{n} \binom{n}{x} \widetilde{p}^x (1-\widetilde{p})^{n-x} max(K - S_0 U^x D^{n-x}, 0) \right) / (1+r)^n \tag{25}$$

One Step Model

$$S_0$$

$$\begin{tikzcd}[column sep=large]
 & V_H = S_0 U - K \\
V(S,n) \arrow[ur, dash, "\widetilde{p}"] \arrow[dr, dash, "\widetilde{q}"'] & \\
 & V_T = S_0 D - K
\end{tikzcd}$$

K=strike

One Step Model:Suppose a derivative security pays off the amount V_1 at the time 1, where V_1 is a $\mathcal{F}_1$ measurable random variable. We construct a self- financing portfolio.

- Sell the security for V_0 at time 0
- Buy Δ shares of stock at time 0
- Invest $V_0 - \Delta S_0$ in the money market, at risk free rate r

The wealth at time 1 is

$$V_1 = (1 + r) V_0 + \Delta(S(1) - (1 + r)S_0) \tag{26}$$

It could be negative

$$\begin{aligned} V_1(H) &= (1+r)V_0 + \Delta(S_1(H) - (1+r)S_0) \\ V_1(T) &= (1+r)V_0 + \Delta(S_1(T) - (1+r)S_0) \end{aligned}$$

Finding Δ

$$\Delta = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)},$$

$$V_1(H) = (1+r)\,V_0 + \Delta(S_1(H) - (1+r)S_0) \tag{27}$$

$$V_1(T) = (1+r)\,V_0 + \Delta(S_1(T) - (1+r)S_0) \tag{28}$$

Finding Δ

$$\Delta = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}, \tag{29}$$

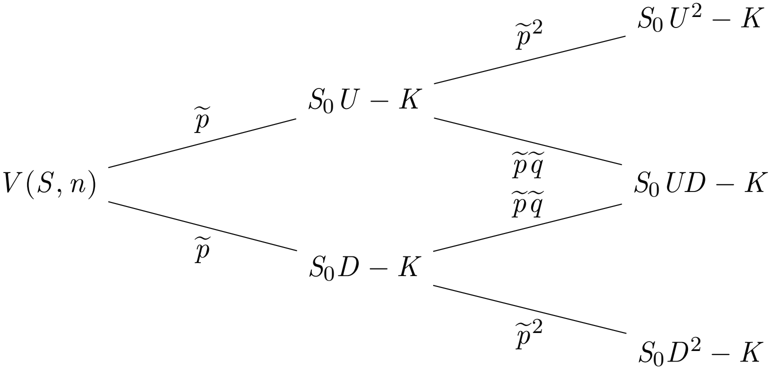
Plug 29 in 27

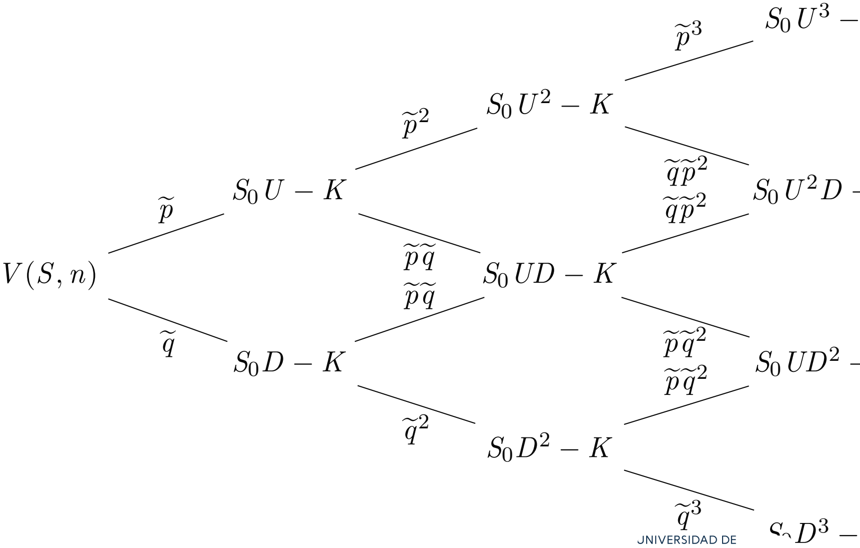
$$
\begin{aligned}
(1+r)V_0 &= V_1(H) - \Delta(S_1(H) - (1+r)S_0) \\
&= V_1(H) - \tfrac{V_1(H)-V_1(T)}{(U-D)S_0}(\Delta(S_1(H) - (1+r)S_0) \\
& \quad (U-1-r)S_0 \\
&= \tfrac{1}{U-D}[(U-D)V_1(H) - (V_1(H) - V_1(T))(U-1-r)] \\
&= \tfrac{(1+r)-D}{U-D} V_1(H) + \tfrac{U-(1+r)}{U-D} V_1(T)
\end{aligned}
$$

$$
\widetilde{p} = \frac{(1+r)-D}{(U-D)}, \quad \widetilde{q} = \frac{U-(1+r)}{(U-D)}
$$

$$
V_0 = \frac{1}{1+r}[\widetilde{p}\,V(H) + \widetilde{q}\,V(T)] \tag{30}
$$

Cox Ross Rubinstein




$V(S,n)$
$\widetilde{p}$
$\widetilde{q}$
$S_0 U - K$
$S_0 D - K$
$\widetilde{p}^2$
$\widetilde{p}\widetilde{q}$
$\widetilde{p}\widetilde{q}$
$\widetilde{q}^2$
$S_0 U^2 - K$
$S_0 UD - K$
$S_0 D^2 - K$
$\widetilde{p}^3$
$\widetilde{q}\widetilde{p}^2$
$\widetilde{q}\widetilde{p}^2$
$\widetilde{p}\widetilde{q}^2$
$\widetilde{p}\widetilde{q}^2$
$\widetilde{q}^3$
$S_0 U^3 -$
$S_0 U^2 D -$
$S_0 UD^2 -$
$S_0 D^3 -$

Definition Let $(\Omega, \mathcal{F}, P)$ be a probability function, for each $\omega \in \Omega$ suppose there is a function $W(t)$ that satisfies $W(0) = 0$ and that depends on ω. The following three properties are equivalent

- For all $0 = t_0 < t_1 < \cdots < t_m$, the increments

  $$W(t_1) = W(t_1) - W(t_0), ..., W(t_n) = W(t_n) - W(t_{n-1})$$

  are independent and each of these increments is normally distributed with mean and variance.
- For all $0 = t_0 < t_1 < \cdots < t_m$ the random variables $W(t_1), W(t_2), ..., W(t_m)$ are jointly normally distributed with means equal to zero and covariance matrix

Martingale Property Browian Motion is a Martingale

$$
\begin{aligned}
\mathbb{E}[W(s)|\mathcal{F}(s)] &= W(s) \\
&= \mathbb{E}[(W(t) - W(s)) + W(s)|\mathcal{F}(s)] \\
&= \mathbb{E}[(W(t) - W(s))|\mathcal{F}(s)] + \mathbb{E}[W(s)|\mathcal{F}(s)] \\
&= \mathbb{E}[(W(t) - W(s))] + W(s) \\
&= W(s)
\end{aligned}
$$

$$\mathbb{E}[W(t_1) - W(t_0)] = 0 \tag{31}$$

$$Var[W(t_1) - W(t_0)] = t_1 - t_0 \tag{32}$$

Standardized Brownian Motion

The distribution of a Browinan Motios is $W(t) \sim (0, t)$ with a standardized process Z the value for $W(t)$ is.

$$W(t) = Z\sqrt{t} = \frac{W(t) - 0}{\sqrt{t}} \tag{33}$$

Where $Z = \frac{x-\mu}{\sigma}$.

For the Geometric Brownian Motion, we use the increments of the Brownian Motion $dW(t) = \sqrt{dt}Z$

Black, Scholes and Merton model

Black and Scholes (1973) revolutionalized the pricing theory of options by showing how to hedge continuously the exposure on the short position of an option. Consider the writer of a European call option on a risky asset. He or she is exposed to the risk of unlimited liability if the asset price rises above the strike price. To protect the writer's short position in the call option, he or she should consider purchasing a certain amount of the underlying asset so that the loss in the short position in the call option is offset by the long position in the asset. In this way, the writer is adopting the hedging procedure.

The riskless hedging principle gives the derive for the governing partial differential equation for the price of a European call option. In their seminal paper [Black and Scholes, 1973] made the following assumptions on the financial market.

- Trading takes place continuously in time.
- The riskless interest rate r is known and constant over time.
- The asset pays no dividend.
- There are no transaction costs in buying or selling the asset or the option, and no taxes.
- The assets are perfectly divisible.
- There are no penalties to short selling and the full use of proceeds is permitted.
- There are no riskless arbitrage opportunities.

Geometric Brownian Motions is: Let μ and $\sigma > 0$ adapted, a GMB is described by:

$$S(t) = S(0)e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t} \tag{34}$$

Define

$$f(t, x) = S(0)e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma x}$$

so

$$S(t) = f(t, W(t))$$

Using Taylor series

$$f_t = (\mu - \frac{1}{2}\sigma^2)f, f_x = \sigma f, f_{xx} = \sigma^2 f$$

Aplying Ito Formula $dW(t)dW(t) = dt$

$$
\begin{aligned}
dS(t) &= df(t, W(t)) \\
&= f_t dt + f_x dW + \tfrac{1}{2} f_{xx} dW dW \\
&= (\mu - \tfrac{1}{2}) f dt + \sigma f dW + \tfrac{1}{2}\sigma^2 f dt \\
&= \mu S(t) dt + \sigma S(t) dW(t)
\end{aligned}
$$

Geometric Brownian motion in differential form is

$$dS(t) = \mu S(t) dt + \sigma S(t) dW(t) \tag{35}$$

The Integral form is:

$$S(t) = S(0) exp\left\{ \int_0^t \sigma(s) dW(s) + \int_0^t (\mu(s) - \frac{1}{2}\sigma^2(s)) ds \right\} \tag{36}$$

Pricing an European Call

$$V(S,t) = SN(d1) - Ke^{-r(T-t)}N(d2) \tag{37}$$

We calcualted the expected value of the current price of an option.

$$\mathbb{E}\left[e^{-rT}(S(T)-K)^{+}\right] = S(0)N(d_1(T,S(0))) - Ke^{-rT}N(d_2(T,S(0)))$$

Pricing an European Call

$$V(S,t) = SN(d1) - Ke^{-r(T-t)}N(d2) \tag{38}$$

We calcualted the expected value of the current price of an option.

$$\mathbb{E}\left[e^{-rT}(S(T)-K)^{+}\right] = S(0)N(d_1(T,S(0))) - Ke^{-rT}N(d_2(T,S(0)))$$

Where

$$d_{1,2}(T,S(0)) = \frac{1}{\sigma\sqrt{T}}\left[\log\frac{S(0)}{K} + \left(r \pm \frac{\sigma^2}{2}\right)T\right],$$

N is the Normal CDF

$$N(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{1}{2}z^2}\,dz = \frac{1}{\sqrt{2\pi}}\int_{-y}^{\infty} e^{-\frac{1}{2}z^2}\,dz.$$

$$
\begin{aligned}
& \mathbb{E}\left[e^{-rT}(S_T-K)^+\right] \\
= \;& e^{-rT}\int_{\frac{1}{\sigma}\left[\ln\frac{K}{S_0}-(r-\frac{1}{2}\sigma^2)\,T\right]}^{\infty}\left(S_0\,e^{(r-\frac{1}{2}\sigma^2)\,T+\sigma x}-K\right)\frac{e^{-\frac{x^2}{2T}}}{\sqrt{2\pi\,T}}\,dx \\
= \;& e^{-rT}\int_{\frac{1}{\sigma\sqrt{T}}\left[\ln\frac{K}{S_0}-(r-\frac{1}{2}\sigma^2)\,T\right]}^{\infty}\left(S_0\,e^{(r-\frac{1}{2}\sigma^2)\,T+\sigma\sqrt{T}y}-K\right)\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\,dy \\
= \;& S_0\,e^{-\frac{1}{2}\sigma^2\,T}\int_{\frac{1}{\sigma\sqrt{T}}\left[\ln\frac{K}{S_0}-(r-\frac{1}{2}\sigma^2)\,T\right]}^{\infty}\frac{1}{\sqrt{2\pi}}\,e^{-\frac{y^2}{2}+\sigma\sqrt{T}y}\,dy \\
& -Ke^{-rT}\int_{\frac{1}{\sigma\sqrt{T}}\left[\ln\frac{K}{S_0}-(r-\frac{1}{2}\sigma^2)\,T\right]}^{\infty}\frac{1}{\sqrt{2\pi}}\,e^{-\frac{y^2}{2}}\,dy \\
= \;& S_0\int_{\frac{1}{\sigma\sqrt{T}}\left[\ln\frac{K}{S_0}-(r-\frac{1}{2}\sigma^2)\,T\right]-\sigma\sqrt{T}}^{\infty}\frac{1}{\sqrt{2\pi}}\,e^{-\frac{\xi^2}{2}}\,d\xi \\
& -Ke^{-rT}N\left(\frac{1}{\sigma\sqrt{T}}\left(\ln\frac{S_0}{K}+\left(r-\frac{1}{2}\sigma^2\right)T\right)\right) \\
= \;& S_0N(d_1(T,S_0))-Ke^{-rT}N(d_2(T,S_0)).
\end{aligned}
$$

Partial Differential Equation Black and Scholes

The governing equation for a European put option can be derived similarly and the same Black–Scholes equation is obtained. Let V (S, t) denote the price of a deriv- ative security with dependence on S and t, it can be shown that V is governed by

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S} rS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} S^2\sigma^2 - rV = 0 \tag{39}$$

These solutions gives the fundamental steps to evaluate any financial instrument

Remarks

- The original derivation of the governing partial differential equation focuses on the financial notion of riskless hedging but misses the precise analysis of the dynamic change in the value of the hedged portfolio.
- The ability to construct a perfectly hedged portfolio relies on the assumption of continuous trading and continuous asset price path. These two and other as- sumptions in the Black–Scholes pricing model have been critically examined by later works in derivative pricing theory.
- The interest rate is widely recognized to be fluctuating over time in an ir- regular manner rather than being constant.
- The Black–Scholes pricing approach assumes continuous hedging at all times. In the real world of trading with transaction costs.

Put-Call Parity In general, after a result is obtained for a European call option, the corresponding result for a European put option can be derived by using the Put-Call parity.

Put-call parity states the relation between the prices of a pair of call and put options. For a pair of European put and call options on the same underlying asset and with the same expiration date and strike price.

$$
\begin{aligned}
P(S,t) &= Ke^{-r(T-t)} - Se^{-q(T-t)} + C(S,t) \\
&= Ke^{-r(T-t)} - Se^{-q(T-t)} \\
&\quad +(Se^{-q(T-t)}N(d1) - Ke^{-r(T-t)}N(d2) \\
&= Ke^{-r(T-t)}(1 - N(d2)) - Se^{-q(T-t)}(1 - N(d1)) \\
&= Ke^{-r(T-t)}N(-d2) - Se^{-q(T-t)}N(-d1)
\end{aligned}
$$

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