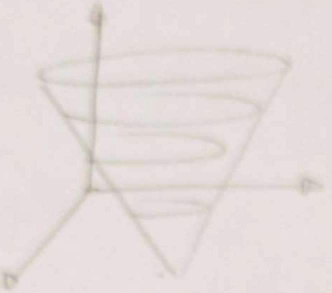
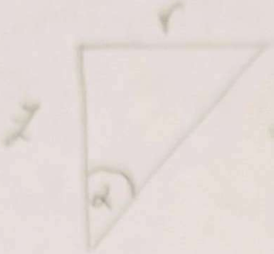
1. Trayectoria más corta de un cono invertido de vértice α en coordenadas cilíndricas.





$$\tan\alpha = \frac{r}{z}$$

$$\rightarrow \quad z = r\cot\alpha$$
$$r = z\tan\alpha$$

$$\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial \dot{q}}\right) - \frac{\partial f}{\partial q} = 0 \qquad ds^2 = dr^2 + r^2 d\theta + dz^2$$

$$I = \int_A^B ds^2 = \int_A^B \underbrace{dr^2 + r^2 d\theta + dz^2}_{\text{Función a minimizar}}$$

Reemplazando:

$$ds^2 = dz^2\tan^2\alpha + z^2\tan^2\alpha\, d\theta^2 + dz^2$$
$$= dz^2(\tan^2\alpha + 1) + z^2\tan^2\alpha\, d\theta^2$$
$$= dz^2\sec^2\alpha + z^2\tan^2\alpha\, d\theta^2$$

$$ds^2 = d\theta^2\left[\left(\frac{dz}{d\theta}\right)^2\sec^2\alpha + z^2\tan^2\alpha\right]$$

$$ds = d\theta\sqrt{\dot{z}^2\sec^2\alpha + z^2\tan^2\alpha}$$

$$\frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial\dot{z}}\right)-\frac{\partial f}{\partial z}=0$$

Para mayor facilidad se tomará el cuadrado de la Longitud.

$$\frac{\partial f}{\partial\dot{z}}=2\sec^2(\alpha)\,\dot{z}\Rightarrow\frac{\partial}{\partial\theta}\left(2\sec^2(\alpha)\,\dot{z}\right)=2\sec^2(\alpha)\,\ddot{z}$$

$$\frac{\partial f}{\partial z}=2z\tan^2\alpha$$

$$\Rightarrow 2\sec^2(\alpha)\,\ddot{z}-2z\tan^2\alpha=0$$

$$\ddot{z}\,\sec^2\alpha-z\tan^2\alpha=0$$

$$\ddot{z}=\frac{z\tan^2\alpha}{\sec^2\alpha}=z\,\frac{\tan^2\alpha}{1+\tan^2\alpha}$$

$$\frac{d\dot{z}}{d\theta}=z\,\frac{\tan^2\alpha}{1+\tan^2\alpha}\Rightarrow\int\frac{dz}{z}=\int\frac{\tan^2\alpha}{1+\tan^2\alpha}\,d\theta$$

$$\Rightarrow \ln|z|=\frac{\tan^2\alpha}{1+\tan^2\alpha}\,\theta+K \qquad \frac{tg^2(\alpha)}{1+tg^2(\alpha)}=A$$

$$\dot{z}(\theta)=Ke^{\left[\frac{tg^2(\alpha)}{1+tg^2(\alpha)}\right]\theta}$$

$$dz=Ke^{A\theta}\cdot d\theta$$

$$z=\frac{Ke^{A\theta}}{A}+K_2\Rightarrow \text{función mínima.}$$

Solución 2

$$I = \int_0^1 \left[(y')^2 + 12xy \right] dx$$

donde $y(x)$ tiene $y(0) = 0$ y $y(1) = 1$

se tiene entonces $L(x, y, \dot{y}) = \dot{y}^2 + 12xy$

Por euler-lagrange tenemos:

$$\frac{d}{dx}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$$

$$\frac{\partial L}{\partial y} = 12x \quad y \quad \frac{\partial L}{\partial \dot{y}} = 2\dot{y}$$

$$\Rightarrow \frac{d}{dx}(2\dot{y}) = 2\ddot{y}$$

$\Rightarrow$ queda $2\ddot{y} - 12x = 0$

$$2\ddot{y} = 12x$$

$$\int \ddot{y} = \int 6x$$

$$\int \dot{y} = \int 3x^2 + C_1$$

$$y(x) = x^3 + C_1 x + C_2$$

como $y(0) = 0 \Rightarrow \boxed{C_2 = 0}$

$y(1) = 1 \Rightarrow 1 = y(1) = 1 + C_1(1)$

$$\Rightarrow \boxed{C_1 = 0}$$

$\therefore \quad y(x) = x^3$

Ahora como queremos calcular el valor mínimo de la integral. Sustituimos

$$\Rightarrow \quad I = \int_0^1 \left[\left((x^3)' \right)^2 + 12x(x^3) \right] dx$$

$$= \int_0^1 \left[(3x^2)^2 + 12x^4 \right] dx$$

$$= \int_0^1 \left[9x^4 + 12x^4 \right] dx$$

$$= \int_0^1 21x^4 \, dx = \frac{21x^5}{5} \Big|_0^1$$

$$\boxed{I_{min} = \frac{21}{5}}$$

3. Encontrar la geodesica entre $P_1(a,0,0)$ y $P_2(-a,0,\pi)$ sobre la superficie $x^2+y^2-a^2=0$

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2 \qquad x^2+y^2=a^2=r^2$$

$$ds^2 = a^2 d\theta^2 + dz^2$$

$$ds = \sqrt{d\theta^2\left(a^2+(dz/d\theta)^2\right)} = d\theta\sqrt{a^2+\dot{z}^2}$$

$$I = \int \sqrt{a^2+\dot{z}^2}\, d\theta$$

$$\frac{d}{d\theta}\left(\frac{\partial f}{\partial \dot{z}}\right) - \frac{\partial f}{\partial z} = 0 \Rightarrow \frac{\partial f}{\partial z} = 0$$

$$\frac{\partial f}{\partial \dot{z}} = \frac{1}{2\sqrt{a^2+\dot{z}^2}} \cdot 2\dot{z} = \frac{\dot{z}}{\sqrt{a^2+\dot{z}^2}}$$

Como $\frac{\partial}{\partial\theta}\left[\frac{\dot{z}}{\sqrt{a^2+\dot{z}^2}}\right] = 0$, entonces $\dot{z}/\sqrt{a^2+\dot{z}^2}$ es cte y $\dot{z}$ es una cte.

$$\frac{\dot{z}}{\sqrt{a^2+\dot{z}^2}} = c \Rightarrow \dot{z}^2 = c^2(a^2+\dot{z}^2) = (ac)^2 + (\dot{z}c)^2$$

$$\dot{z}^2(1-c^2) = (ac)^2$$

$$\dot{z} = \frac{ac}{\sqrt{1-c^2}}$$

$$\dot{z} = \frac{ac}{\sqrt{1-c^2}} \Rightarrow \frac{dz}{d\theta} = \frac{ac}{\sqrt{1-c^2}}$$

$$\Rightarrow \int dz = \int \frac{ac}{\sqrt{1-c^2}} d\theta \Rightarrow z = \frac{ac}{\sqrt{1-c^2}} \theta + K$$

Las condiciones son $z(0) = 0$ y $z(\pi) = \pi$

$$z(0) = 0 = K \quad ; \quad z(\pi) = \pi = \frac{\pi a c}{\sqrt{1-c^2}}$$

$$\Longrightarrow 1 - c^2 = a^2 c^2$$

$$1 = a^2c^2 + c^2 = c^2(a^2+1)$$

$$\sqrt{\frac{1}{a^2+1}} = c$$

$$\frac{a}{\sqrt{1 - \frac{1}{a^2+1}}} \cdot \sqrt{\frac{1}{a^2+1}}\, \theta = z(\theta)$$

$$\frac{\cancel{a}\, \theta}{\cancel{a}} = z(\theta) \Rightarrow \theta = z(\theta).$$

$$I = \int_0^{\pi} \sqrt{a^2 + \dot{z}^2}\, d\theta = \int_0^{\pi} \sqrt{a^2+1}\, d\theta = \sqrt{a^2+1}\, \pi$$

Solución 4.

Un cuerpo se deja caer desde una altura h y alcanza el suelo en un tiempo T. La ecuación de movimiento podría ser de la forma

$$y = h - g_1 t \qquad y = h - \frac{1}{2} g_2 t^2 \qquad y = h - \frac{1}{4} g_3 t^3$$

Demostrar que la forma correcta es aquella que produce el valor de mínima acción.

a) trayectoria 1 $y = h - g_1 t$ (1)

el lagrangiano de acción para cada caso está dado por:

$$L = T - V = \frac{1}{2} m v^2 - mgy$$

Para trayectoria 1 se tiene

$$\dot{y} = -g_1 \quad \text{por tanto}$$

$$S_1 = \int_0^{t_f} \frac{1}{2} m (-g_1)^2 - mg(h - g_1 t)\, dt$$

$$S_1 = \frac{1}{2} m g_1^2 t - mg\left(ht - \frac{1}{2} g_1 t^2\right)\Big|_0^{t_f}$$

$$= \frac{1}{2} m g_1^2 t_f - mg\left(ht_f - \frac{1}{2} g_1 t_f^2\right)$$

ahora bien debemos tener en cuenta las condiciones iniciales

$$0 = \dot{y}(0) = -g_1 \Rightarrow g_1 = 0$$

$$\Rightarrow S_1 = -mght \qquad \therefore S_1 = -mgh\left(\frac{h}{g_1}\right)$$

por (1) para $y = 0$ $\Rightarrow \frac{h}{g_1} = t$

para trayectoria 2

$$y = h - \frac{1}{2} g_2 t^2 \quad (2) \quad \Rightarrow \dot{y} = -\frac{1}{2} g_2 \cdot 2t$$

$$\Rightarrow L = \frac{1}{2} m (-g_2 t)^2 - mg(h - \frac{1}{2} g_2 t^2)$$

$$\Rightarrow S_2 = \int_0^{t_f} \frac{1}{2} m g_2^2 t^2 - mgh + \frac{1}{2} mg t^2 g_2$$

$$= \frac{1}{6} m g_2^2 t^3 - mght + \frac{1}{6} m g g_2 t^3 \Big|_0^{t_f}$$

Los condiciones iniciales nos dan:

$$\ddot{y}(0) = -g = -g_2 \quad \therefore \quad g_2 = g.$$

$$\Rightarrow S_2 = \frac{m g_2 t_f^3}{6} (g_2 + g) - mght_f$$

$$= \frac{2mg_2^2}{6} t_f^3 - mght_f.$$

para $y = 0$ se tiene $0 = h - \frac{1}{2} g_2 t^2$

$$t = \sqrt{\frac{2h}{g_2}}$$

$$S_2 = \frac{2mg_2^2}{6} \left(\frac{2h}{g_2}\right)^{3/2} - mgh \left(\frac{2h}{g_2}\right)^{1/2}$$

Para trayectoria 3

$$\dot{y}_3 = -\frac{3}{4} g_3 t^2$$

$$\Rightarrow L = \frac{1}{2} m \left(-\frac{3}{4} g_3 t^2\right)^2 - mg\left(h - \frac{1}{4} g_3 t^3\right)$$

$$S_3 = \int_0^{t_f} \left(\frac{9}{32} m g_3^2 t^4 - mg\left(h - \frac{1}{4} g_3 t^3\right)\right) dt$$

$$= \frac{9}{160} m g_3^2 t_f^5 - mgh t_f + \frac{1}{16} m g g_3 t_f^4$$

pero ya que se tiene

$$-g = \ddot{y}_3(t) = -\frac{6}{4} g_3 t \quad \therefore \quad g_3 = \frac{2}{3} \frac{g}{t_f}$$

y para $y = 0$

$$0 = h - \frac{1}{4} g_3 t_f^3 \qquad t_f = \left(\frac{4h}{g_3}\right)^{1/3}$$

$$\Rightarrow g_3 = \left(\frac{2}{3} \frac{g}{(4h)^{1/3}}\right)^{3/2}$$

queda por concluir que S_2 es el extremo

veamos que se había concluido:

$$S_1 = -mgh\left(\frac{h}{g_1}\right) \quad \text{con} \quad g_1 = 0$$

$$\Rightarrow \boxed{\lim_{g_1 \to 0} S_1 = -\infty}$$

sin importa h, m S_1 es $-\infty$

ahora bien si $h = 10$ m $g = 9.8\ m/s^2$ $m = 1$ kg

se tendría que: para la trayectoria 2 como $g_2 = g$.

$$\Rightarrow \quad S_2 = \frac{2(1)(9.8)^2}{6}\left(\frac{2(10)}{9.8}\right)^{3/2} - (9.8)\left(\frac{2\times 10}{9.8}\right)^{1/2}$$

$$\boxed{S_2 \approx -46.6\overline{6}}$$

y para la trayectoria 3.

se tendría $$g_3 = \left(\frac{2}{3}\cdot\frac{9.8}{(4(10))^{1/3}}\right)^{3/2}$$

$$g_3 \approx 2.64 \quad \text{y} \quad t_f = \left(\frac{4(10)}{2.64}\right)^{1/3} \approx 2.47.$$

$$\Rightarrow \quad S_3 = \frac{9}{160}(1)(2.64)^2(2.47)^5 - (1)(9.8)(10)(2.47)$$

$$+ \frac{1}{16}(9.8)(2.64)(2.47)^4$$

$$\boxed{S_3 \approx -145.83}$$

$\therefore$ S_2 es un extremo de todas las acciones por tanto se cumple con la forma correcta pues al crear el lagrangiano de la caida libre se tiene:

$$L = \frac{1}{2} m(\dot{y})^2 - mgy$$ y por E. Lagrange Euler

$$\Rightarrow \frac{d}{dt}\left(\underbrace{\frac{\partial L}{\partial \dot{y}}}_{m\dot{y}}\right) - \underbrace{\frac{\partial L}{\partial y}}_{-mg} = 0$$

como resultado $\Rightarrow \cancel{m}\ddot{y} + \cancel{m}g = 0 \Rightarrow \int \ddot{y} = \int -g$

$$\Rightarrow \int \dot{y} = \int -gt + C_1 \Rightarrow y(t) = -\frac{gt^2}{2} + C_1 t + C_2$$

para $t=0$ $y=h \Rightarrow C_2 = h$ y

para $\dot{y}(0) = 0 \Rightarrow C_1 = 0$

$$\therefore \boxed{y(t) = -\frac{gt^2}{2} + h}$$

que se corrobora con el calculo de la acción de la misma trayectoria.

$\therefore$ la trayectoria 2 genera el extremo de la acción.

Lagrangiano de una función. $L = T - V$ $\quad \frac{dx}{dt}$

Si $L = \frac{m^2 \dot{x}^4}{12} + m\dot{x}^2 f(x) - f^2(x)$, cuál es la ec del MAU

tenemos $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$ $\quad \frac{\partial f(t)}{\partial t} \quad \frac{\partial x}{\partial x}\cdot\frac{dx}{dt}$

$* \frac{\partial L}{\partial \dot{x}} = \frac{m^2}{12} 4\dot{x}^3 + m f(x) + m\dot{x}\frac{\partial f(x)}{\partial \dot{x}}\cdot\frac{\partial x}{\partial \dot{x}} - 2f(x)\frac{\partial x}{\partial \dot{x}}$

$\frac{\partial L}{\partial \dot{x}} = \frac{m^2}{3}\dot{x}^3 + 2m\dot{x}f(x)$

$* \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{m^2}{3} 3\dot{x}^2\ddot{x} + 2m\ddot{x}f(x) + 2m\dot{x}f'(x)\dot{x} = m^2\dot{x}^2\ddot{x} + 2m\ddot{x}f(x) + 2m\dot{x}^2 f'(x)$

$* \frac{\partial L}{\partial x} = m\dot{x}^2 f'(x) - 2f(x)f'(x)$

$\Rightarrow m^2\dot{x}^2\ddot{x} + 2m\ddot{x}f(x) + 2m\dot{x}^2 f'(x) - m\dot{x}^2 f'(x) + 2f(x)f'(x) = 0$

$\ddot{x}(m^2\dot{x}^2 + 2mf(x)) + 2m\dot{x}^2 f'(x) - m\dot{x}^2 f'(x) + 2f(x)f'(x) = 0$

$\ddot{x}m(m\dot{x}^2 + 2f(x)) + f'(x)\left[2m\dot{x}^2 - m\dot{x}^2 + 2f(x)\right] = 0$

$$\ddot{x}m = -\frac{f'(x)\left[2m\dot{x}^2 - m\dot{x}^2 + 2f(x)\right]}{m\dot{x}^2 + 2f(x)}$$

Solución 6.

demostrar que si $\mathcal{L} = \frac{1}{2} g_{ab}(q_c) \dot{q}^a \dot{q}^b$ (1)

$$\Rightarrow \ddot{q}^a + \Gamma^a_{bc} \dot{q}^b \dot{q}^c = 0.$$

donde $\Gamma^a_{bc} = \frac{1}{2} g^{ad} \left(\frac{\partial g_{bd}}{\partial q^c} + \frac{\partial g_{cd}}{\partial q^b} - \frac{\partial g_{bc}}{\partial q^d} \right)$

a partir del lagrangiano (1) usamos las ecuaciones de euler lagrange

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^a}\right) - \frac{\partial \mathcal{L}}{\partial q^a} = 0$$

$$\Rightarrow \frac{\partial}{\partial \dot{q}^a}\left(\frac{1}{2} g_{cb}(q_c) \dot{q}^c \dot{q}^b\right) = \frac{1}{2} g_{cb} \left[\frac{\partial \dot{q}^c}{\partial \dot{q}^a} \dot{q}^b + \dot{q}^c \frac{\partial \dot{q}^b}{\partial \dot{q}^a} \right]$$

$$= \frac{1}{2} g_{cb} \left[\delta^c_a \dot{q}^b + \dot{q}^c \delta^b_a \right] = \frac{1}{2} g_{cb} \delta^c_a \dot{q}^b + \frac{1}{2} g_{cb} \delta^b_a \dot{q}^c$$

$$= \frac{1}{2} g_{ab} \dot{q}^b + \frac{1}{2} g_{ca} \dot{q}^c \quad c \text{ es un indice libre}$$

$$= \frac{1}{2} g_{ab} \dot{q}^b + \frac{1}{2} g_{ba} \dot{q}^b$$

usando la hipotesis $g_{ab} = g_{ba}$

$$= \frac{1}{2} g_{ab} \dot{q}^b + \frac{1}{2} g_{ab} \dot{q}^b = g_{ab} \dot{q}^b$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^a}\right) = \frac{d}{dt}(g_{ab}\dot{q}^b) = \frac{\partial g_{ab}}{\partial q^c}\dot{q}^c\dot{q}^b + g_{ab}\ddot{q}^b$$

y

$$\frac{\partial \mathcal{L}}{\partial q^a} = \frac{\partial}{\partial q^a}\left(\frac{1}{2} g_{bc}\dot{q}^b\dot{q}^c\right)$$

$$= \frac{1}{2}\frac{\partial g_{bc}}{\partial q^a}\dot{q}^b\dot{q}^c$$

sustituyendo en E. Lagrange euler.

~~indices mudos~~ factor común.

$$\frac{\partial g_{ab}}{\partial q^c}\dot{q}^c\dot{q}^b + g_{ab}\ddot{q}^b - \frac{1}{2}\frac{\partial g_{bc}}{\partial q^a}\dot{q}^b\dot{q}^c = 0$$

$$g_{ab}\ddot{q}^b + \dot{q}^c\dot{q}^b\left(\frac{\partial g_{ab}}{\partial q^c} - \frac{1}{2}\frac{\partial g_{bc}}{\partial q^a}\right) = 0$$

multiplicando ambos lados por izquierda por g^{da}

$$g^{da}g_{ab}\ddot{q}^b + \dot{q}^c\dot{q}^b g^{da}\left(\frac{\partial g_{ab}}{\partial q^c} - \frac{1}{2}\frac{\partial g_{bc}}{\partial q^a}\right) = 0$$

$$\delta^d_b\ddot{q}^b + \frac{1}{2}\dot{q}^c\dot{q}^b g^{da}\left(2\frac{\partial g_{ab}}{\partial q^c} - \frac{\partial g_{bc}}{\partial q^a}\right) = 0$$

Como b y c no forman parte de la suma de einstein son mudos

$$\Rightarrow \ddot{q}^d + \frac{1}{2}\dot{q}^c\dot{q}^b g^{da}\left(\frac{\partial g_{ab}}{\partial q^c} + \frac{\partial g_{ac}}{\partial q^b} - \frac{\partial g_{bc}}{\partial q^a}\right)$$

intercambiando los indises a por d

$$\ddot{q}^a + \frac{1}{2} g^{ad}\left(\frac{\partial \overset{g_{bd}}{g_{db}}}{\partial q^c} + \frac{\partial \overset{g_{cd}}{g_{dc}}}{\partial q^b} - \frac{\partial \overset{g_{bc}}{g_{cb}}}{\partial q^d}\right)\dot{q}^b \dot{q}^c = 0$$

Por la propiedad $g_{ab} = g_{bc}$

$$\Rightarrow \ddot{q}^a + \frac{1}{2} g^{ad}\left(\frac{\partial g_{bd}}{\partial q^c} + \frac{\partial g_{cd}}{\partial q^b} - \frac{\partial g_{bc}}{\partial q^d}\right)\dot{q}^b \dot{q}^c = 0$$

$$\boxed{\ddot{q}^a + \Gamma^a_{bc}\,\dot{q}^b \dot{q}^c = 0}$$

con $\Gamma^a_{bc} = \frac{1}{2} g^{ad}\left(\frac{\partial g_{bd}}{\partial q^c} + \frac{\partial g_{cd}}{\partial q^b} - \frac{\partial g_{bc}}{\partial q^d}\right)$