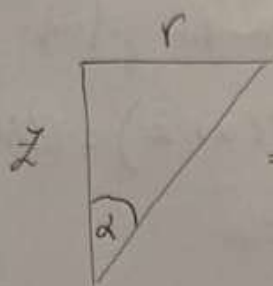
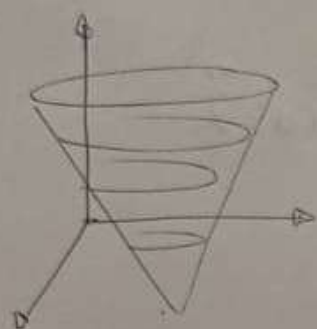


1. Trayectoria más corta de un cono invertido de vértice  $x$  en coordenadas cilíndricas.



$$\tan \alpha = \frac{r}{z}$$

$$\Rightarrow z = r \cot \alpha$$

$$r = z \tan \alpha$$

$$\frac{\partial}{\partial t} \left( \frac{\partial f}{\partial \dot{q}} \right) - \frac{\partial f}{\partial q} = 0$$

$$ds^2 = dr^2 + r^2 d\theta + dz^2$$

$$I = \int_A^B ds^2 = \int_A^B \underbrace{dr^2 + r^2 d\theta + dz^2}_{\text{Función a minimizar}}$$

Reemplazando:

$$\begin{aligned} ds^2 &= dz^2 \tan^2 \alpha + z^2 \tan^2 \alpha d\theta^2 + dz^2 \\ &= dz^2 (\tan^2 \alpha + 1) + z^2 \tan^2 \alpha d\theta^2 \\ &= dz^2 \sec^2 \alpha + z^2 \tan^2 \alpha d\theta^2 \end{aligned}$$

$$ds^2 = d\theta^2 \left[ \left( \frac{dz}{d\theta} \right)^2 \sec^2 \alpha + z^2 \tan^2 \alpha \right]$$

$$ds = d\theta \sqrt{\dot{z}^2 \sec^2 \alpha + z^2 \tan^2 \alpha}$$

$$\frac{\partial}{\partial \theta} \left( \frac{\partial F}{\partial \dot{z}} \right) - \frac{\partial F}{\partial z} = 0$$

Para mayor facilidad se tomará el cuadrado de la longitud.

$$\frac{\partial F}{\partial \dot{z}} = 2 \sec^2(\alpha) \dot{z} \rightarrow \frac{\partial}{\partial \theta} (2 \sec^2(\alpha) \dot{z}) = 2 \sec^2(\alpha) \ddot{z}$$

$$\frac{\partial F}{\partial z} = 2 z \tan^2 \alpha$$

$$\Rightarrow 2 \sec^2(\alpha) \ddot{z} - 2 z \tan^2 \alpha = 0$$

$$\ddot{z} \sec^2 \alpha - z \tan^2 \alpha = 0$$

$$\ddot{z} = \frac{z \tan^2 \alpha}{\sec^2 \alpha} = \frac{z \tan^2 \alpha}{1 + \tan^2 \alpha}$$

$$\frac{d\dot{z}}{d\theta} = \frac{z \tan^2 \alpha}{1 + \tan^2 \alpha} \Rightarrow \int \frac{d\dot{z}}{\dot{z}} = \int \frac{\tan^2 \alpha}{1 + \tan^2 \alpha} d\theta$$

$$\Rightarrow \ln |\dot{z}| = \frac{\tan^2 \alpha}{1 + \tan^2 \alpha} \theta + K$$

$$\frac{\tan^2(\alpha)}{1 + \tan^2(\alpha)} = A$$

$$\dot{z}(\theta) = K e^{\left[ \frac{\tan^2(\alpha)}{1 + \tan^2(\alpha)} \right] \theta}$$

$$d\dot{z} = K e^{A\theta} d\theta$$

$$z = \frac{K e^{A\theta}}{A} + K_2 \Rightarrow \text{función mínima.}$$



## Solución 2

$$I = \int_0^1 \left[ (\dot{y})^2 + 12xy \right] dx$$

donde  $y(x)$  tiene  $y(0) = 0$  y  $y(1) = 1$

se tiene entonces  $L(x, y, \dot{y}) = \dot{y}^2 + 12xy$

por euler-lagrange tenemos:

$$\frac{d}{dx} \left( \frac{\partial L}{\partial \dot{y}} \right) - \frac{\partial L}{\partial y} = 0$$

$$\frac{\partial L}{\partial y} = 12x \quad , \quad \frac{\partial L}{\partial \dot{y}} = 2\dot{y}$$

$$\Rightarrow \frac{d}{dx} (2\dot{y}) = 12x$$

$$\Rightarrow \text{queda } 2\ddot{y} - 12x = 0$$

$$2\ddot{y} = 12x$$

$$\int \ddot{y} = \int 6x$$

$$\int \dot{y} = \int 3x^2 + C_1$$

$$y(x) = x^3 + C_1 x + C_2$$

como  $y(0) = 0 \Rightarrow C_2 = 0$

$y(1) = 1 \Rightarrow 1 = y(1) = 1 + C_1(1)$

$\Rightarrow C_1 = 0$

$$\therefore y(x) = x^3$$

Ahora como queremos calcular el valor mínimo de la integral. Sustituimos

$$\Rightarrow I = \int_0^1 \left[ \left( (x^3)' \right)^2 + 12x(x^3) \right] dx$$

$$= \int_0^1 \left[ (3x^2)^2 + 12x^4 \right] dx$$

$$= \int_0^1 \left[ 9x^4 + 12x^4 \right] dx$$

$$= \int_0^1 21x^4 dx = \frac{21x^5}{5} \Big|_0^1$$

$$\boxed{I_{\min} = \frac{21}{5}}$$

3. Encontrar la geodésica entre  $P_1(a, 0, 0)$  y  $P_2(-a, 0, \pi)$  sobre la superficie  $x^2 + y^2 - a^2 = 0$ .

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2$$

$$x^2 + y^2 = a^2 = r^2$$

$$ds^2 = a^2 d\theta^2 + dz^2$$

$$ds = \sqrt{d\theta^2 (a^2 + (dz/d\theta)^2)} = d\theta \sqrt{a^2 + \dot{z}^2}$$

$$I = \int \sqrt{a^2 + \dot{z}^2} d\theta$$

$$\frac{d}{d\theta} \left( \frac{\partial f}{\partial \dot{z}} \right) - \frac{\partial f}{\partial z} = 0 \Rightarrow \frac{\partial f}{\partial z} = 0$$

$$\frac{\partial f}{\partial \dot{z}} = \frac{1}{2 \sqrt{a^2 + \dot{z}^2}} \cdot 2\dot{z} = \frac{\dot{z}}{\sqrt{a^2 + \dot{z}^2}}$$

Como  $\frac{\partial}{\partial \theta} \left[ \frac{\dot{z}}{\sqrt{a^2 + \dot{z}^2}} \right] = 0$ , entonces  $\dot{z}/\sqrt{a^2 + \dot{z}^2}$  es cte y  $\dot{z}$  es una cte.

$$\frac{\dot{z}}{\sqrt{a^2 + \dot{z}^2}} = C \Rightarrow \dot{z}^2 = C^2 (a^2 + \dot{z}^2) = (aC)^2 + (\dot{z}C)^2$$
$$\dot{z}^2 (1 - C^2) = (aC)^2$$
$$\dot{z} = \frac{aC}{\sqrt{1 - C^2}}$$



$$\dot{z} = \frac{ac}{\sqrt{1-c^2}} \Rightarrow \frac{dz}{d\theta} = \frac{ac}{\sqrt{1-c^2}}$$

$$\Rightarrow \int dz = \int \frac{ac}{\sqrt{1-c^2}} d\theta \Rightarrow z = \frac{ac}{\sqrt{1-c^2}} \theta + K$$

Las condiciones son  $z(0) = 0$  y  $z(\pi) = \pi$

$$z(0) = 0 = K ; \quad z(\pi) = \pi = \frac{\pi ac}{\sqrt{1-c^2}}$$

$$\Rightarrow 1 - c^2 = a^2 c^2$$

$$1 = a^2 c^2 + c^2 = c^2 (a^2 + 1)$$

$$\sqrt{\frac{1}{a^2 + 1}} = c$$

$$\frac{a}{\sqrt{1 - \frac{1}{a^2 + 1}}} \cdot \sqrt{\frac{1}{a^2 + 1}} \theta = z(\theta)$$

$$\frac{a \theta}{a} = z(\theta) \Rightarrow \theta = z(\theta)$$

$$I = \int_0^\pi \sqrt{a^2 + \dot{z}^2} d\theta = \int_0^\pi \sqrt{a^2 + 1} d\theta = \sqrt{a^2 + 1} \pi$$

## Solución 4.

Un cuerpo se deja caer desde una altura  $h$  y alcanza el suelo en un tiempo  $T$ . la ecuación de movimiento podría ser de la forma

$$y = h - g_1 t \quad y = h - \frac{1}{2} g_2 t^2 \quad y = h - \frac{1}{4} g_3 t^3$$

demostrar que la forma correcta es aquella que produce el valor de mínima acción.

a) trayectoria  $\downarrow$   $y = h - g_1 t$  (1)

el lagrangiano de acción para cada caso está dado por:

$$L = T - U = \frac{1}{2} m v^2 - mgy$$

Para trayectoria  $\downarrow$  se tiene

$$\ddot{y} = -g_1 \quad \text{por tanto}$$

$$S_1 = \int_0^{t_f} \left( \frac{1}{2} m (-g_1)^2 - mg(h - g_1 t) \right) dt$$

$$S_1 = \left( \frac{1}{2} m g_1^2 t - mg(h t - \frac{1}{2} g_1 t^2) \right) \Big|_0^{t_f}$$

$$= \frac{1}{2} m g_1^2 t_f - mg(h t_f - \frac{1}{2} g_1 t_f^2)$$

ahora bien debemos tener en cuenta las condiciones iniciales

$$0 = \dot{y}(0) = -g_1 \quad \Rightarrow \quad g_1 = 0$$

$$\Rightarrow S_1 = -mgh t$$

$$\therefore S_1 = -mgh \left( \frac{h}{g_2} \right)$$

por (1) para  $y=0 \Rightarrow \frac{h}{g_2} = t$

para trayectoria 2

$$y = h - \frac{1}{2} g_2 t^2 \quad (2) \quad \Rightarrow \dot{y} = -\frac{1}{2} g_2 \cdot 2t$$

$$\Rightarrow L = \frac{1}{2} m (-g_2 t)^2 - mg \left( h - \frac{1}{2} g_2 t^2 \right)$$

$$\Rightarrow J_2 = \int_0^{t_f} \frac{1}{2} m g_2^2 t^2 - mgh + \frac{1}{2} m g t^2 g_2$$

$$= \frac{1}{6} m g_2^2 t^3 - mgh t + \frac{1}{6} m g g_2 t^3 \Big|_0^{t_f}$$

Las condiciones iniciales nos dan:

$$\ddot{y}(0) = -g = -g_2 \quad \therefore g_2 = g.$$

$$\Rightarrow S_2 = \frac{m g_2 t_f^3}{6} (g_2 + g) - mgh t_f$$

$$= \frac{2m g_2^2}{6} t_f^3 - mgh t_f.$$

para  $y = 0$  se tiene  $0 = h - \frac{1}{2} g_2 t^2$

$$t = \sqrt{\frac{2h}{g_2}}$$

$$S_2 = \frac{2m g_2^2}{6} \left( \frac{2h}{g_2} \right)^{3/2} - mgh \left( \frac{2h}{g_2} \right)^{1/2}$$



Para trayectoria 2.

$$\ddot{y}_3 = -\frac{3}{4} g_3 t^2.$$

$$\Rightarrow L = \frac{1}{2} m \left( -\frac{3}{4} g_3 t^2 \right)^2 - m g \left( h - \frac{1}{4} g_3 t^3 \right)$$

$$S_3 = \int_0^{t_f} \left( \frac{9}{32} m g_3^2 t^4 - m g \left( h - \frac{1}{4} g_3 t^3 \right) \right) dt$$

$$= \frac{9}{160} m g_3^2 t_f^5 - m g h t_f + \frac{1}{16} m g g_3 t_f^4$$

pero ya que se tiene

$$-g = \ddot{y}_3(t) = -\frac{6}{4} g_3 t \quad \therefore g_3 = \frac{2}{3} \frac{g}{t_f}$$

y para  $y=0$

$$0 = h - \frac{1}{4} g_3 t_f^3 \quad t_f = \left( \frac{4h}{g_3} \right)^{1/3}$$

$$\Rightarrow g_3 = \left( \frac{2}{3} \frac{g}{(4h)^{1/3}} \right)^{3/2}$$

queda por concluir que  $S_2$  es el extremo  
veamos que se habia concluido!

$$S_1 = -m g h \left( \frac{h}{g} \right) \quad \text{con } g_1 = 0$$

$$\Rightarrow \left( \lim_{g_1 \rightarrow 0} S_1 = -\infty \right)$$

sin importa  $h, m$   $S_1$  es  $-\infty$

ahora bien si  $h = 10 m$   $g = 9.8 m/s^2$   $m = 1 kg$

se tendría que: para la trayectoria 2 como  $g_2 = g$ .

$$\Rightarrow S_2 = \frac{1}{6} (1) (9.8)^2 \left( \frac{2(10)}{9.8} \right)^{3/2} - (1)(9.8) \left( \frac{2 \times 10}{9.8} \right)^{1/2}$$

$$S_2 \approx -16.66$$

y para la trayectoria 3.

$$se\ tendr\u00eda \quad g_3 = \left( \frac{2}{3} \cdot \frac{9.8}{(4(10))^{1/3}} \right)^{3/2}$$

$$g_3 \approx 2.64 \quad y \quad t_c = \left( \frac{4(10)}{2.64} \right)^{1/3} \approx 2.47$$

$$\Rightarrow S_3 = \frac{1}{6} (1) (2.64)^2 (2.47)^3 - (1)(9.8)(10)(2.47) + \frac{1}{6} (9.8) (2.64) (2.47)^4$$

$$S_3 \approx -14.583$$

$\therefore S_2$  es un extremo de todas las acciones por tanto se cumple con la forma correcta pues al crear el lagrangiano de la caída libre se tiene:



$$L = \frac{1}{2} m(\dot{y})^2 - mgy \quad \text{y por E. Lagrange Euler}$$

$$\Rightarrow \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}} \right) - \frac{\partial L}{\partial y} = 0$$

$\downarrow$   $m\dot{y}$                        $\downarrow$   $-mg$

Como resultado  $\Rightarrow m\ddot{y} + mg = 0 \Rightarrow \ddot{y} = -g$

$$\Rightarrow \int \ddot{y} = \int -g dt + C_1 \Rightarrow \dot{y}(t) = -\frac{gt}{1} + C_1$$

$$\Rightarrow \int \dot{y} = \int -gt + C_1 dt \Rightarrow y(t) = -\frac{gt^2}{2} + C_1 t + C_2$$

Para  $t=0$   $y=h \Rightarrow C_2 = h$  y

para  $\dot{y}(0) = 0 \Rightarrow C_1 = 0$

$$\boxed{y(t) = -\frac{gt^2}{2} + h}$$

que se corrobora con el cálculo de la acción de la misma trayectoria.

o. su trayectoria genera el extremo de la acción.

Lagrangiano di una función.  $L = T - V$

gi:  $L = \frac{m^2 \dot{x}^4}{12} + m \dot{x}^2 f(x) - f^2(x)$ , anal es lo es  
de mas

tenemos  $\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0$

\*  $\frac{\partial L}{\partial \dot{x}} = \frac{m^2}{12} 4 \dot{x}^3 + m f(x) + m \dot{x} \frac{\partial f(x)}{\partial \dot{x}} \cdot \frac{\partial x}{\partial \dot{x}} - 2f(x) \frac{\partial x}{\partial \dot{x}}$

$\frac{\partial L}{\partial \dot{x}} = \frac{m^2}{3} \dot{x}^3 + 2m \dot{x} f(x)$

\*  $\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = \frac{m^2}{3} 3 \dot{x}^2 \ddot{x} + 2m \ddot{x} f(x) + 2m \dot{x} f'(x) \dot{x} = m^2 \dot{x}^2 \ddot{x} + 2m \ddot{x} f(x) + 2m \dot{x}^2 f'(x)$

\*  $\frac{\partial L}{\partial x} = m \ddot{x} f'(x) - 2f(x) f'(x)$

$\Rightarrow m^2 \dot{x}^2 \ddot{x} + 2m \ddot{x} f(x) + 2m \dot{x}^2 f'(x) - m \ddot{x} f'(x) + 2f(x) f'(x) = 0$

$\ddot{x} (m^2 \dot{x}^2 + 2m f(x)) + 2m \dot{x}^2 f'(x) - m \dot{x}^2 f'(x) + 2f(x) f'(x) = 0$

$\ddot{x} m (m \dot{x}^2 + 2 f(x)) + f'(x) [2m \dot{x}^2 - m \dot{x}^2 + 2f(x)] = 0$

$\ddot{x} m = - \frac{f'(x) [2m \dot{x}^2 - m \dot{x}^2 + 2f(x)]}{m \dot{x}^2 + 2 f(x)}$



## Solución 6.

demostrar que si  $L = \frac{1}{2} g_{ab}(q_c) \dot{q}^a \dot{q}^b$  (1)

$$\Rightarrow \ddot{q}^a + \Gamma_{bc}^a \dot{q}^b \dot{q}^c = 0$$

donde

$$\Gamma_{bc}^a = -\frac{1}{2} g^{ad} \left( \frac{\partial g_{bd}}{\partial q^c} + \frac{\partial g_{cd}}{\partial q^b} - \frac{\partial g_{bc}}{\partial q^d} \right)$$

a partir del lagrangiano (1) obtenemos las ecuaciones de Euler Lagrange

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^a} \right) - \frac{\partial L}{\partial q^a} = 0$$

$$\Rightarrow \frac{\partial}{\partial \dot{q}^a} \left( \frac{1}{2} g_{ab}(q_c) \dot{q}^a \dot{q}^b \right) = \frac{1}{2} g_{ab} \left[ \frac{\partial \dot{q}^c}{\partial \dot{q}^a} \dot{q}^b + \dot{q}^c \frac{\partial \dot{q}^b}{\partial \dot{q}^a} \right]$$

$$= \frac{1}{2} g_{cb} \left[ \delta_a^c \dot{q}^b + \dot{q}^c \delta_a^b \right] = \frac{1}{2} g_{cb} \delta_a^c \dot{q}^b + \frac{1}{2} g_{cb} \delta_a^b \dot{q}^c$$

$$= \frac{1}{2} g_{ab} \dot{q}^b + \frac{1}{2} g_{ca} \dot{q}^c \quad \text{c es un índice libre}$$

$$= \frac{1}{2} g_{ab} \dot{q}^b + \frac{1}{2} g_{ba} \dot{q}^b$$

usando la hipótesis  $g_{ab} = g_{ba}$

$$= \frac{1}{2} g_{ab} \dot{q}^b + \frac{1}{2} g_{ab} \dot{q}^b = g_{ab} \dot{q}^b$$

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}^a} \right) = \frac{d}{dt} \left( g_{ab} \dot{q}^b \right) = \frac{\partial g_{ab}}{\partial q^c} \dot{q}^c \dot{q}^b + g_{ab} \ddot{q}^b$$

y

$$\frac{\partial \mathcal{L}}{\partial \dot{q}^a} = \frac{\partial}{\partial \dot{q}^a} \left( \frac{1}{2} g_{bc} \dot{q}^b \dot{q}^c \right)$$

$$= \frac{1}{2} \frac{\partial g_{bc}}{\partial \dot{q}^a} \dot{q}^b \dot{q}^c$$

Substituyendo en E. Lagrange euler.

$$\frac{\partial g_{ab}}{\partial q^c} \dot{q}^c \dot{q}^b + g_{ab} \ddot{q}^b - \frac{1}{2} \frac{\partial g_{bc}}{\partial \dot{q}^a} \dot{q}^b \dot{q}^c = 0$$

~~ind. g. mat.~~  
factor común.

$$g_{ab} \ddot{q}^b + \dot{q}^c \dot{q}^b \left( \frac{\partial g_{ab}}{\partial q^c} - \frac{1}{2} \frac{\partial g_{bc}}{\partial \dot{q}^a} \right) = 0$$

multiplicando ambos lados por izquierda por  $g^{da}$

$$g^{da} g_{ab} \ddot{q}^b + \dot{q}^c \dot{q}^b g^{da} \left( \frac{\partial g_{ab}}{\partial q^c} - \frac{1}{2} \frac{\partial g_{bc}}{\partial \dot{q}^a} \right) = 0$$

$$\delta_b^d \ddot{q}^b + \frac{1}{2} \dot{q}^c \dot{q}^b g^{da} \left( 2 \frac{\partial g_{ab}}{\partial q^c} - \frac{\partial g_{bc}}{\partial \dot{q}^a} \right) = 0$$

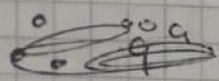
Como  $b$  y  $c$  no dependen de la derivada de  $q^a$  son constantes

$$\Rightarrow \ddot{q}^d + \frac{1}{2} \dot{q}^c \dot{q}^b g^{da} \left( \frac{\partial g_{ab}}{\partial q^c} + \frac{\partial g_{bc}}{\partial q^a} - \frac{\partial g_{ac}}{\partial q^b} \right)$$



intercambiando los índices a por d

$$\ddot{q}^a + \frac{1}{2} g^{ad} \left( \frac{\partial g_{db}}{\partial q^c} + \frac{\partial g_{dc}}{\partial q^b} - \frac{\partial g_{cb}}{\partial q^d} \right) \dot{q}^b \dot{q}^c = 0$$

 Por la propiedad  $g_{ab} = g_{ba}$

$$\Rightarrow \ddot{q}^a + \frac{1}{2} g^{ad} \left( \frac{\partial g_{bd}}{\partial q^c} + \frac{\partial g_{cd}}{\partial q^b} - \frac{\partial g_{bc}}{\partial q^d} \right) \dot{q}^b \dot{q}^c = 0$$

$$\ddot{q}^a + \Gamma_{bc}^a \dot{q}^b \dot{q}^c = 0$$

$$\text{con } \Gamma_{bc}^a = \frac{1}{2} g^{ad} \left( \frac{\partial g_{bd}}{\partial q^c} + \frac{\partial g_{cd}}{\partial q^b} - \frac{\partial g_{bc}}{\partial q^d} \right)$$