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ISC 4220
Continuous Algorithms
ODEs

Q1. Initial Value Problem (from Exam 2015)

(i) Is the ODE linear or nonlinear? What is the order of the ODE?

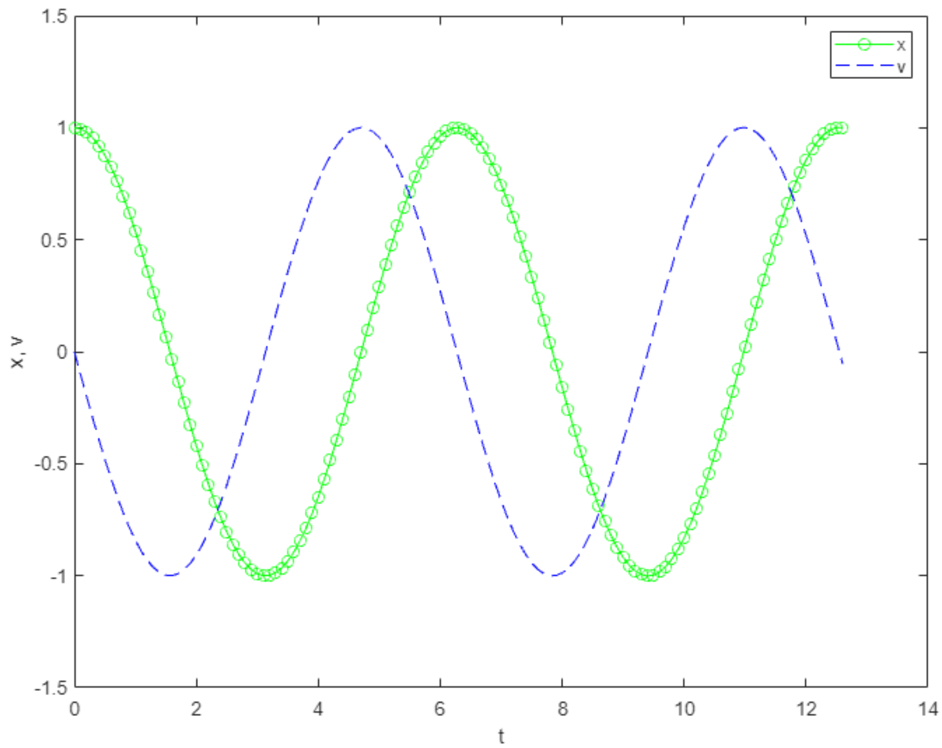
The ODE is linear, and it is a second order ODE.

(ii) Rewrite the ODE as a system of two first order ODEs. Also rewrite the initial conditions, as required.

System of 1st order ODEs:		Conditions:
dx/dt = x' =	v	x(0) = 1
dv/dt = x'' =	-(μ/m)v -(k/m)x	v(0) = 0

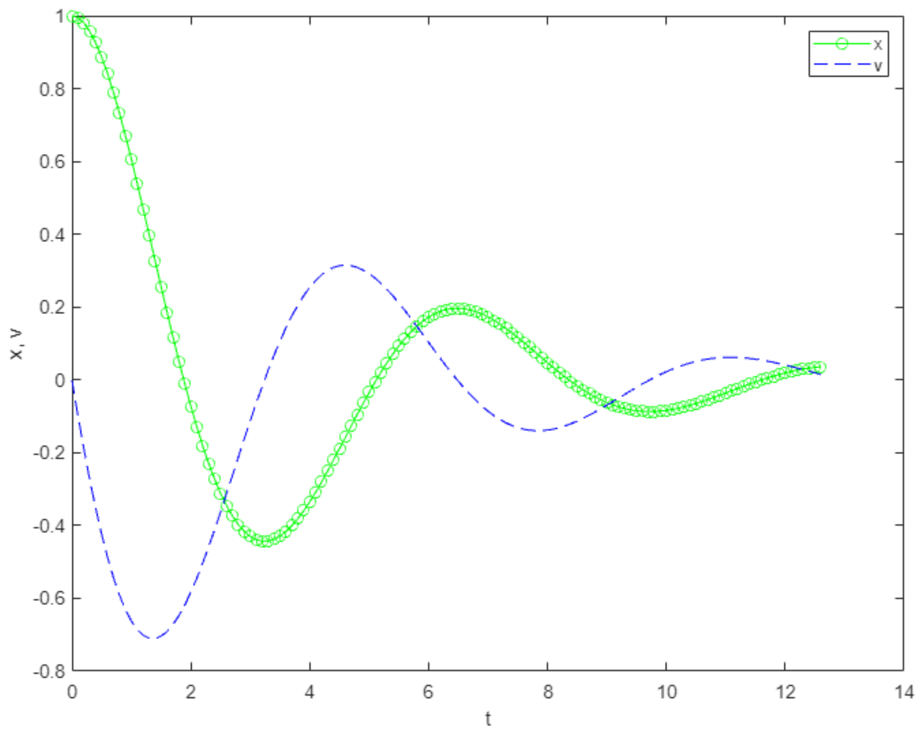
(iii) Use midpoint method with step-size h = 0.1, and solve for x(t) between t = 0 and t = 4π with, m = k = 1, and

(a) μ = 0.0



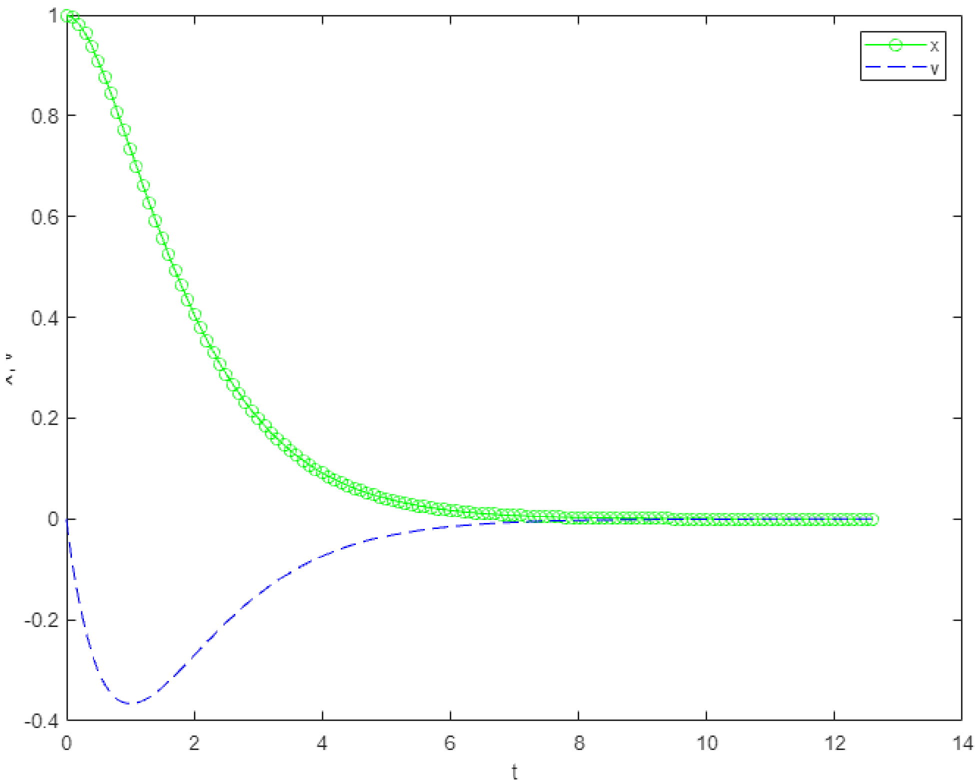
When μ = 0, -(μ/m)v -(k/m)x becomes -(k/m)x. Because of this, the line for v resembles the line of x. So this particular value of μ isn't very useful to us, since a good approximation can't be made from these results.

(b) μ = 0.5



With this value of μ we do get a useful approximation of x and v. Notice that the oscillations slowly converge into a single point, we can say that this model is somewhat stable.

(c) μ = 2



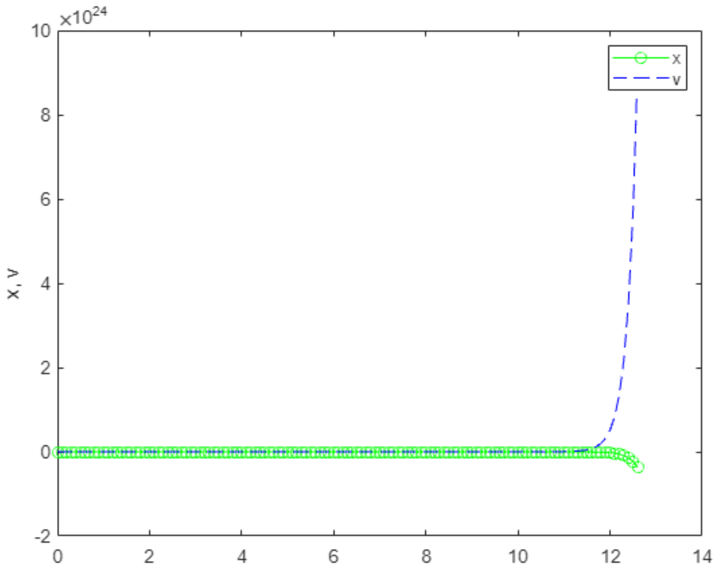
With this value of μ we also get a useful approximation of x. Notice that the x and v start to converge much faster and with less oscillations than before. we can say that this model is stable.

(iv) Use the test equation method to show that the region of stability for the midpoint method is given by:

Test equation for midpoint method:
$y(n+1) = y(n-1) + 2h\lambda y(n)$

(v) Compute x(t) as in part (iii) with μ = 25. Interpret your observations using the stability criterion above.

As we can see in the graph below, when μ = 25, we exit the region of stability for the midpoint method of this particular problem, and thus divergence occurs.



Q2. IVP (from Exam 2013)

(i) Explain why (or why not) the following labels apply to this problem: first ordered, homogeneous, linear, implicit, initial value problem.

first ordered: The highest derivative of y in this ode is the first (y').

homogeneous: y'=ty+t , the ode is homogeneous.

linear: y'-ty = t, the ode is linear.

implicit: y'=ty+t, y' can be isolated. The ode is NOT implicit.

initial value problem: Since we are given y(t = 0) = 0, this is an IVP.

(ii) This problem can be rewritten as y′ = f(t,y) = t(1 + y). Carry out a single step of the backward Euler method with step-size h = 0.1 to find y(t = 0.1).

Backward euler:
y1 = y0 + hf(t1,y1)

y1 = y0 +h[t1 (1+y1)]

$y1 = 0 + 0.1[0.1(1+y1)]$

$y1 = 0.01(1+y1)$

$y1 = 0.01 + 0.01*y1$ → solve for y1

$y1 = 1/99$

$y1 = 0.01$

(iii) The exact solution to the problem is $y(t) = \exp(t^2/2) - 1$. Verify that this satisfies the ODE and the initial condition.

y(0) using $y(t) = \exp(t^2/2) - 1 = 0$ → Initial condition satisfied

y(0.1) using $y(t) = \exp(t^2/2) - 1 = 0.005$
if $y = \exp(t^2/2) - 1$, then $y' = t(\exp(t^2/2))$

y'(0.1) using $y' = t(\exp(t^2/2)) = 0.1005$
$y' = ty + t$ → $y' = 0.1*0.005 + 0.1 = 0.1005$ → ODE satisfied

(iv) Find the local truncation error for y(t = 0.1) computed by the backward Euler above.

computed y1 = 0.01
actual y1 = 0.1
Local Trunc Error = actual − computed = 0.1 - 0.01 = 0.09

(v) True or False: Step-size required for the accuracy of a numerical method for solving ODEs is always smaller than that required for stability.

False. Normally the step-size required for accuracy is smaller than the step-size required for stability (hsta > hacc). However, for a stiff problem, the opposite is true (hacc > hsta) .

Q3. Stiff Equations

1. Set δ = 0.1. Use the Matlab program ode45 to solve the ODE with a relative error of 10^{-4}. Hint: use opts = odeset('RelTol',1.0e-4). Find the amount of time it takes to solve the problem.

```
Elapsed time is 0.136856 seconds.
```

2. Try to repeat the above with δ = 0.0001 or δ = 0.00001. Describe and explain what you observe. You may have to zoom in after t = 1/δ to figure out. Find the amount of time it takes to solve the problem.

```
Elapsed time is 0.239607 seconds.
```

The time nearly doubled from before, this is because “tspan” increases as δ decreases. The greater the “tspan” the more operations need to be computed, and thus the time elapsed is also increased.

3. Repeat with the Matlab program ode23s. Find the amount of time it takes to solve the problem.
When δ = 0.1:

```
Elapsed time is 0.093414 seconds.
```

When δ = 0.0001:

```
Elapsed time is 0.110273 seconds.
```

Q4. Boundary Value Problem (from Exam 2014)

(i) Verify the exact solution.

check if: y(t = 0) = 0
y(0) = 0 +(e/(e^2 −1))(e^−0−e^0) = 0

check if: y(t = 1) = 0
y(1) = 1 +(e/(e^2 −1))(e^−1−e^1) = 0

Exact solutions are accurate.

(ii) Rewrite the second order ODE as a system of two first order IVPs.

System of 1st order ODEs:
dy/dt = y’ = v

dv’/dt = v’’ = y - t

Conditions:
y(0) = y(1) = 0

(iii) Use Matlab’s intrinsic IVP solver ode45 to solve the IVPs with a relative tolerance of 10−4.

Using Shooting method : v0 = 0.15

(iv) Plot the exact and computed solutions over the domain 0 ≤t ≤1.
Exact = orange line
Computed = Blue line