On the indefinite integral of the inverse function

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1 Framework

Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable, invertible, non-constant function such that

$$F(x) := \int f(x)dx$$

and $g := f^{-1}$. We wish to obtain an expression of

$$G(x) := \int g(x)dx$$

in terms of $F(x)$.

2 Derivation

The following derivations are straightforward from elementary calculus and have been made explicit for a clear understanding of the reader. We make use of the fundamental theorem of calculus and integration by parts.

$$\begin{aligned}
(f \circ g)(x) &= x \\
g'(x)f(g(x)) &= g'(x)x \\
\int g'(x)f(g(x))dx &= \int g'(x)xdx \\
\int F(g(x))'dx &= \int g'(x)xdx \\
F(g(x)) &= xg(x) - \int g(x)dx + c \\
F(g(x)) &= xg(x) - G(x) + c
\end{aligned}$$

and finally

$$G(x) = xg(x) - F(g(x)) + c \tag{1}$$

3 Examples

Equation 1 gives us a nice and simple formula for obtaining the integral of a function in terms of the integral of its inverse function. Of course, it makes sense to make use of it when the integral of its inverse is easier to compute. The following are a few expository examples.

1. $f(x) = e^x$, $g(x) = ln(x)$
   - $F(x) = e^x$
   - $G(x) = xln(x) - x + c$
2. $f(x) = sin(x)$, $g(x) = arcsin(x)$
   - $F(x) = -cos(x)$
   - $G(x) = x{*}arcsin(x){+}cos(arcsin(x)){+}c = x{*}arcsin(x){+}\sqrt{1-x^2}{+}c$
3. $f(x) = tan(x)$, $g(x) = arctan(x)$
   - $F(x) = -ln(cos(x))$
   - $G(x) = x * arctan(x) + ln(cos(arctan(x))) + c =$
     $x * arctan(x) + ln(\frac{1}{\sqrt{x^2+1}}) + c =$
     $x * arctan(x) - \frac{1}{2}ln(x^2 + 1) + c$
4. $f(x) = sinh(x)$, $g(x) = ln(x + \sqrt{x^2 + 1})$
   - $F(x) = cosh(x)$
   - $G(x) = xln(x + \sqrt{x^2 + 1}) - \frac{(\sqrt{x^2+1}+x)^2+1}{2\sqrt{x^2+1}+x)} + c =$
     $xln(x + \sqrt{x^2 + 1}) - \sqrt{x^2 + 1} + c$
5. $f(x) = \frac{e^x+1}{1-e^x}$, $g(x) = ln(\frac{x-1}{x+1})$
   - $F(x) = x - 2ln|1 - e^x|$
   - $G(x) = xln|\frac{x-1}{x+1}| - ln(\frac{x-1}{x+1}) + 2ln(1 - \frac{x-1}{x+1}) =$
     $(x - 1)ln(\frac{x-1}{x+1}) - 2ln(x + 1)$