A look at Fibonacci through linear algebra

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1 Framework

Let $F_0 = F_1 = 0$, and consider the following recursion

$$F_{n+2} = F_{n+1} + F_n$$

$\forall n \in \mathbb{N}$.

The sequence that arises -known as the Fibonacci sequence- has proven to be relevant in several areas of mathematics. In this article we analyze it using the tools of linear algebra, getting a closed form of the $n-th$ Fibonacci number, namely

$$F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}$$

for all $n \in \mathbb{N}$, and showing that

$$\lim_{n\to\infty} \frac{F_{n+2}}{F_{n+1}} = \varphi$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

2 Derivation

One can restate Fibonacci's recurrence as the following linear system:

$$\begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$$

where

$$T := \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

In order to compute $\lim_{n\to\infty} \frac{F_{n+2}}{F_{n+1}}$, it suffices to compute T^n for all $n \in \mathbb{N}$, and then take the limit of its entries. We do so by diagonalizing the matrix according to the following

Theorem 1 (Diagonalizable Matrix Theorem) *A matrix $T \in \mathbb{R}^{n\times n}$ is diagonalizable if there exists an invertible matrix P such that*

$$T = PDP^{-1},$$

where D is a diagonal matrix containing the eigenvalues of T.

We now proceed to calculate the eigenvalues and eigenvectors of $T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

Eigenvalues of T

$$T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

The characteristic polynomial is

$$\det(T - \lambda I) = \lambda^2 - \lambda - 1 = 0,$$

with solutions

$$\lambda_1 = \frac{1+\sqrt{5}}{2} = \varphi, \quad \lambda_2 = \frac{1-\sqrt{5}}{2} = (-\varphi)^{-1}.$$

Associated eigenvectors
For λ_1:

$$(T - \lambda_1 I)\mathbf{v}_1 = 0 \implies \mathbf{v}_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}.$$

For λ_2:

$$(T - \lambda_2 I)\mathbf{v}_2 = 0 \implies \mathbf{v}_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}.$$

Change of basis matrix P and its inverse P^{-1}

$$P = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}, \quad P^{-1} = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}, \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},$$

We can then easily compute T^n by calculating the matrix product

$$T^n = PD^nP^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$$

and thus

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = PD^nP^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

By explicit computation. we get

$$T^n = PD^nP^{-1} = \frac{1}{\sqrt{5}}\begin{bmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} & -\lambda_1\lambda_2^n + \lambda_2\lambda_1^n \\ \lambda_1^n - \lambda_2^n & -\lambda_2\lambda_1^n + \lambda_1\lambda_2^n \end{bmatrix}$$
$$= \frac{1}{\sqrt{5}}\begin{bmatrix} \varphi^{n+1} - (-\varphi)^{-(n+1)} & -\varphi(-\varphi)^{-n} + (-\varphi)^{-1}\varphi^n \\ \varphi^n - (-\varphi)^{-n} & -(-\varphi)^{-1}\varphi^n + \varphi(-\varphi)^{-n} \end{bmatrix}$$

so finally

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = T^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} \varphi^{n+1} - (-\varphi)^{-(n+1)} & -\varphi(-\varphi)^{-n} + (-\varphi)^{-1}\varphi^n \\ \varphi^n - (-\varphi)^{-n} & -(-\varphi)^{-1}\varphi^n + \varphi(-\varphi)^{-n} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

which yields Binet's formula

$$F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}$$

for all $n \in \mathbb{N}$.

From this and the fact that $\lim_{n\to\infty}(-\varphi)^{-n} = 0$, it is immediate that

$$\lim_{n\to\infty} \frac{F_{n+2}}{F_{n+1}} = \varphi$$

3 A problem and an elegant solution

It never seizes to amaze me that $\frac{\varphi^n-(-\varphi)^{-n}}{\sqrt{5}}$ is an integer for all natural numbers (since they are defined by the Fibonacci recursion, with base case in the integers). But how do we actually compute this? We still have to calculate φ^n and $(-\varphi)^{-n}$, which don't seem easy since both φ and $(-\varphi)^{-1}$ are irrational numbers. Let's take a closer look at Binet's formula.

Recall $F_n = \frac{\varphi^n-(-\varphi)^{-n}}{\sqrt{5}}$. Note that $\varphi > 1$ -since $\sqrt{5} > 2$- and thus one is tempted to say that $F_n \approx \frac{\varphi^n}{\sqrt{5}}$. This leaves us with a somewhat bad taste, since we just computed an exact formula and now we are going back to approximations. However, I claim that this approximation can get us a better exact formula, if we look at it closely.

We can ask ourselves: how far from F_n is $\frac{\varphi^n}{\sqrt{5}}$? If it is not too far, the we can make our claim stronger by finding the error term. In particular, if $|F_n - \frac{\varphi^n}{\sqrt{5}}| < 1$, since we know that F_n is an integer, we could simply calculate $\frac{\varphi^n}{\sqrt{5}}$ and round it to the nearest integer to get a better and simpler exact formula. Let's calculate:

$$|F_n - \frac{\varphi^n}{\sqrt{5}}| = |\frac{(-\varphi)^{-n}}{\sqrt{5}}|$$

and

$$|\frac{(-\varphi)^{-n}}{\sqrt{5}}| < 1 \iff |(-\varphi)^{-n}| < \sqrt{5}$$

which clearly holds since $\varphi^{-1} < 1$.

Thus, we can safely conclude that

$$F_n = \begin{cases} \lfloor \frac{\varphi^n}{\sqrt{5}} \rfloor, & \text{if } n \text{ is even} \\ \lceil \frac{\varphi^n}{\sqrt{5}} \rceil, & \text{if } n \text{ is odd} \end{cases}$$

which in any case corresponds to rounding $\frac{\varphi^n}{\sqrt{5}}$ to the nearest integer. This is a simpler, more elegant, and computationally cheaper exact formula to calculate F_n.