Unit I - Exam A

Talavera Rivera Luis Fernando

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1.- Find the second solution using the order reduction method.

$y'' \underbrace{-\frac{1}{x}}_{P(x)} y' - 4x^2 y = 0$ First solution is $y_1(x) = e^{-x^2}$

$\int P(x) = -\int \frac{1}{x}\,dx = -\ln|x| + c$

$w(x) = \int \frac{e^{-\int P(x)}}{y_1^2}\,dx$

$u = 2x^2 \quad du = 4x\,dx$

$= \int \frac{e^{-(-\ln|x|)}}{e^{-2x^2}}\,dx = \int x e^{2x^2}\,dx = \frac{1}{4}\int 4xe^{2x^2}\,dx$

$= \frac{1}{4}\int e^u\,du = \frac{1}{4}e^u = \frac{1}{4}e^{2x^2}$

$y_2 = y_1 w = \frac{1}{4}e^{2x^2}e^{-x^2} = \frac{1}{4}e^{x^2}$ $\quad y_2 = \frac{1}{4}e^{x^2}$

$y_G = C_1 e^{-x^2} + C_2 e^{x^2}$

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2- Find the general solution of the given Diff. Eq.

$y'' + 4y' + 4y = 2e^{-2x} + \sin(x)$

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$m^2 + 4m + 4 = (m+2)^2 \Rightarrow m_1 = m_2 = -2$

$y_c = C_1 e^{-2x} + C_2 x e^{-2x}$

$y_{p1} = Ae^{-2x} \Rightarrow$ In complementary solution
$y_{p1} = Axe^{-2x} \to$ Still in complementary solution
$y_{p1} = Ax^2e^{-2x} \to$ Not in complementary solution

$y_{p2} = B\cos(x) + C\sin(x)$

$y'_{p1} = -2Ax^2e^{-2x} + 2Axe^{-2x} = (-2Ax^2 + 2Ax)e^{-2x}$

$y''_{p1} = -2(-2Ax^2 + 2Ax)e^{-2x} + (-4Ax + 2A)e^{-2x}$

$y''_{p1} = (+4Ax^2 - 4Ax - 4Ax + 2A)e^{-2x} = (+4Ax^2 - 8Ax + 2A)e^{-2x}$

$y''_{p1} + 4y' + 4y = e^{-2x}(\underline{+4Ax^2} - \underline{8Ax} + 2A - \underline{8Ax^2} + \underline{8Ax} \ \underline{+4Ax^2})$

$= 2Ae^{-2x} = 2e^{-2x} \Rightarrow \underline{A = 1}$

$y'_{p2} = -B\sin(x) + C\cos(x)$

$y''_{p2} = -B\cos(x) - C\sin(x)$

$y''_{p2} + 4y'_{p2} + 4y_{p2} = \underline{-B\cos(x)} \ \underline{-C\sin(x)} + \underline{4C\cos(x)} - \underline{4B\sin(x)}$
$+ \underline{4B\cos(x)} + \underline{4C\sin(x)}$

$3B + 4C = 0$
$-4B + 3C = 1$

$B = -4/25 \quad C = 3/25$

$\begin{vmatrix} 3 & 4 \\ -4 & 3 \end{vmatrix} = 9 + 16 = 25 \quad \begin{vmatrix} 0 & 4 \\ 1 & 3 \end{vmatrix} = -4 \quad \begin{vmatrix} 3 & 0 \\ -4 & 1 \end{vmatrix} = 3$

$y_p = x^2 e^{-2x} - \frac{4}{25}\cos(x) + \frac{3}{25}\sin(x)$

$\rightarrow \boxed{y_G = c_1 e^{-2x} + c_2 x e^{-2x} + x^2 e^{-2x} - \frac{4}{25}\cos(x) + \frac{3}{25}\sin(x)}$

3.- Find the general solution of the given Diff. Eq.

$y'' + 4y = \csc(2x) \leftarrow f(x)$

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Final answer at next page

$m^2 + 4 = 0 \Rightarrow m^2 = -4 \Rightarrow m = \sqrt{-4} = \pm 2i$

$y_c = C_1\cos(2x) + C_2\sin(2x)$

$y_1 = \cos(2x) \qquad y_2 = \sin(2x)$

$y_1' = -2\sin(2x) \qquad y_2' = 2\cos(2x)$

$\Delta = \begin{vmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{vmatrix} = 2\cos^2(2x) + 2\sin^2(2x) = 2(\underbrace{\sin^2(2x) + \cos^2(2x)}_{1}) = 2$

$\Delta_1 = \begin{vmatrix} 0 & \sin(2x) \\ \csc(2x) & 2\cos(2x) \end{vmatrix} = -\csc(2x)\sin(2x) = -\frac{\sin(2x)}{\sin(2x)} = -1$

$\Delta_2 = \begin{vmatrix} \cos(2x) & 0 \\ -2\sin(2x) & \csc(2x) \end{vmatrix} = \cos(2x)\csc(2x) = ctg(2x)$

$w_1(x) = \int \frac{\Delta_1}{\Delta}dx = \int \frac{-1}{2}dx = -\frac{1}{2}x \qquad u = 2x \quad du = 2dx$

$w_2(x) = \int \frac{\Delta_2}{\Delta}dx = \int \frac{ctg(2x)}{2}dx = \frac{1}{4}\int ctg(2x)\,2dx = \frac{1}{4}\int ctg(u)du$

$= \frac{1}{4}\ln|\sin u| = \frac{1}{4}\ln|\sin(2x)|$

$$y_p = -\frac{1}{2}x\cos(2x) + \frac{1}{4}\ln|\sin(2x)|\sin(2x)$$

$$\boxed{y_G = C_1\cos(2x) + C_2\sin(2x) - \frac{1}{2}x\cos(2x) + \frac{1}{4}\ln|\sin(2x)|\sin(2x)} \leftarrow$$

4- Find the general solution of the given Diff. Eq.

$x^2y'' - 3xy' + 4y = x^2$

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$m^2 - m + 4 = (m-2)^2 = m_{1,2} = 2$

$y_c = C_1x^2 + C_2\ln|x|x^2$

$y_1 = x^2$ $\quad y_2 = \ln|x|x^2$

$y_1' = 2x$ $\quad y_2' = x + 2x\ln|x|$

$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 1 \leftarrow f(x)$

$$\Delta = \begin{vmatrix} x^2 & \ln|x|x^2 \\ 2x & x+2x\ln|x| \end{vmatrix} = x^3 + 2x^3\ln|x| - 2x^3\ln|x| = x^3$$

$$\Delta_1 = \begin{vmatrix} 0 & \ln|x|x^2 \\ 1 & x+2x\ln|x| \end{vmatrix} = -x^2\ln|x|$$

$$\Delta_2 = \begin{vmatrix} x^2 & 0 \\ 2x & 1 \end{vmatrix} = x^2$$

$u = \ln|x| \quad du = \frac{1}{x}dx$

$$w_1(x) = \int\frac{\Delta_1}{\Delta}dx = \int\frac{-x^2\ln|x|}{x^3}dx = -\int\frac{1}{x}\ln|x|dx = -\int u\,du$$

$$= -\frac{u^2}{2} = \frac{-\ln|x|^2}{2} = \frac{-\ln^2|x|}{2}$$

$$w_2(x) = \int\frac{\Delta_2}{\Delta}dx = \int\frac{x^2}{x^3}dx = \int\frac{1}{x}dx = \ln|x|$$

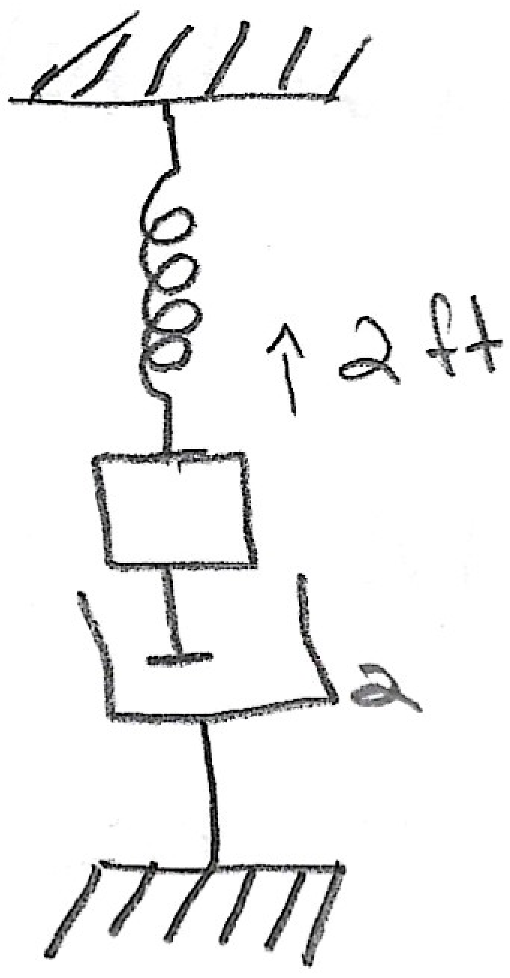
$$y_p = \frac{-\ln^2|x|x^2}{2} + \ln|x|\ln|x|x^2 = \frac{-\ln^2|x|x^2}{2}\ln^2|x|x^2$$

$$= \frac{\ln^2|x|x^2}{2}$$

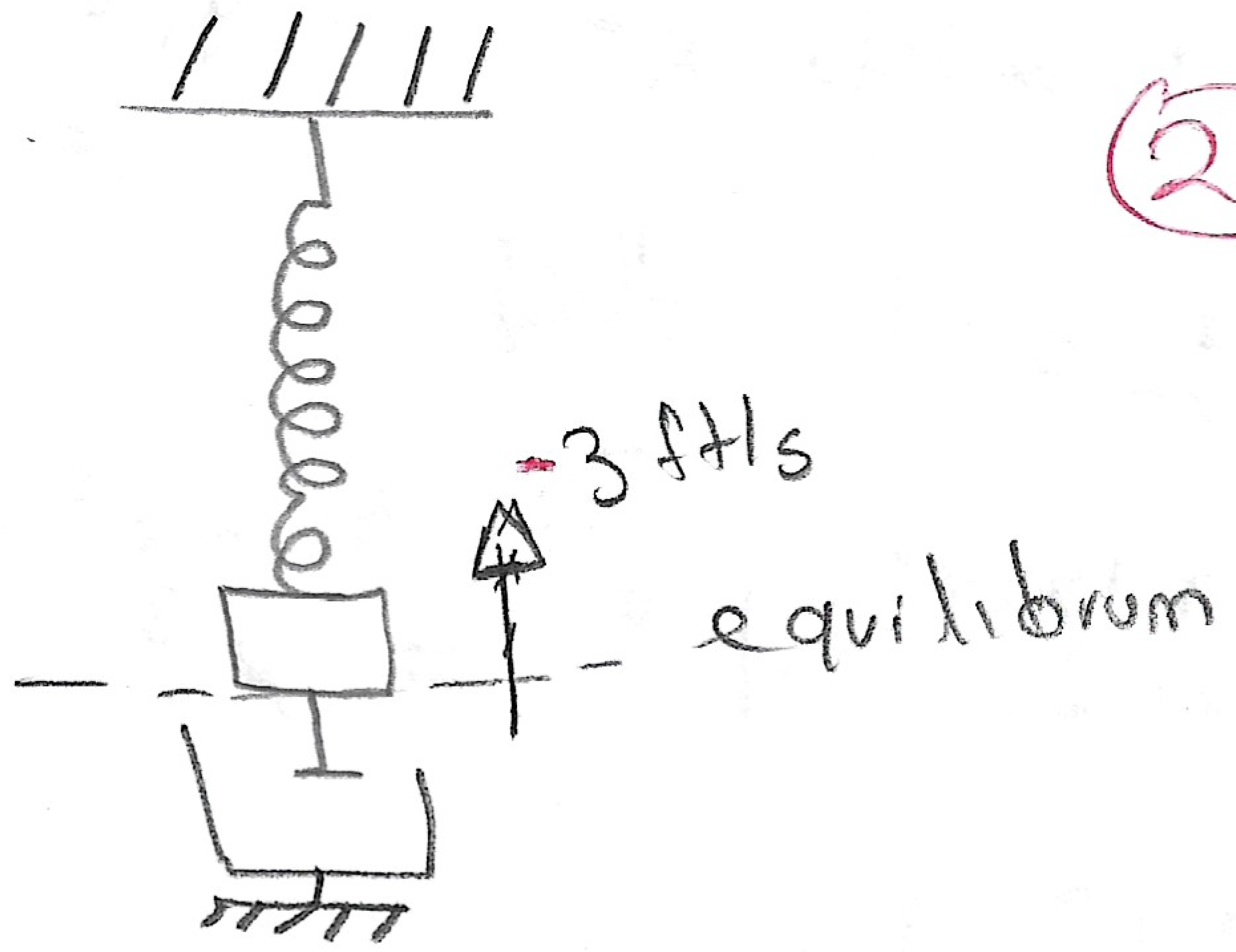
$$\boxed{y_G = C_1x^2 + C_2\ln|x|x^2 + \frac{\ln^2|x|x^2}{2}}$$

5. A mass weighting 8 pounds stretches a spring 2 feet. Assuming that a damping force numerically equal 2 times the instantaneous velocity acts on the system, determine the equation of motion if the same mass is initially released from the equilibrium position with an upward velocity of 3 ft/s.

a. Find the equation of motion
b. Classify the motion as under, over or critically damped.



$W = 8\ lb$



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$W = ks$

$8\ lb = k(2ft)$

$k = 4\ \frac{lb}{ft}$

$m = \frac{8\ lb}{32\ ft/s} = \frac{1}{4}$ slug

$\beta = 2$

$x(t) = c_1 e^{-4t} + c_2 t e^{-4t}$

$x(0) = c_1 e^{-4(0)} + c_2 0 e^{-4(0)} = 0$

$\underline{c_1 = 0}$

$x'(t) = -4c_1 e^{-4t} + c_2(-4t+1)e^{-4t}$

$x'(0) = -4c_1 e^{-4(0)} + c_2(-4(0)+1)e^{-4(0)}$

$= \underline{c_2 = -3}$

$mx'' = -kx - \beta x'$

$mx'' + \beta x' + kx = 0$

$\frac{1}{4}x'' + 2x' + 4x = 0$

$p = \frac{-2 \pm \sqrt{4 - 4(\frac{1}{4})4}}{2(1/4)} = \frac{-2}{\frac{1}{2}} = -4 = p_1 = p_2$

a. $x(t) = -3te^{-4t}$

b.- It is known that the motion is critically damped by the shape of the general solution.