Exam B- Unit V

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• Find the Fourier Series of the following periodic function

$[-\pi, \pi]$

$$f(x) = \begin{cases} e^{-x}, & -\pi \le x \le 0 \\ e^{x}, & 0 \le x \le \pi \end{cases}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} e^{-x}\,dx + \int_0^{\pi} e^{x}\,dx\right]$$

$$= \frac{1}{2\pi}\left[-e^{-x}\Big|_{-\pi}^{0} + e^{x}\Big|_0^{\pi}\right] = \frac{1}{2\pi}\left[-e^{0} - (-e^{-(-\pi)}) + e^{\pi} - e^{0}\right]$$

$$= \frac{1}{2\pi}\left[-1 + e^{\pi} + e^{\pi} - 1\right] = \frac{2e^{\pi} - 2}{2\pi} = \frac{2(e^{\pi}-1)}{2\pi} = \frac{e^{\pi}-1}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\left[\underset{(1)}{\int_{-\pi}^{0} e^{-x}\cos(nx)\,dx} + \underset{(2)}{\int_0^{\pi} e^{x}\cos(nx)\,dx}\right]$$

(1) $u = \cos(nx)$ $\quad v = -e^{-x}$ $\quad z = \sin(nx)$ $\quad w = -e^{-x}$

$du = -n\sin(nx)$ $\quad dv = e^{-x}dx$ $\quad dz = n\cos(nx)$ $\quad dw = e^{-x}dw$

$$-e^{-x}\cos(nx) - \int -e^{-x}(n\sin(nx))\,dx = -e^{-x}\cos(nx) - n\left(\sin(nx)e^{-x} - \int -e^{-x}(n\cos(nx))dx\right)$$

$$\int e^{-x}\cos(nx)\,dx = -e^{-x}\cos(nx) + n\sin(nx)e^{-x} - n^2\int e^{-x}\cos(nx)\,dx$$

$$(1+n^2)\int e^{-x}\cos(nx)\,dx = -e^{-x}\cos(nx) + n\sin(nx)e^{-x}$$

$$\int e^{-x}\cos(nx)\,dx = \frac{-e^{-x}\cos(nx) + n\sin(nx)e^{-x}}{1+n^2}$$

$$= \frac{e^{-x}(-\cos(nx) + n\sin(nx))}{1+n^2}$$

(2) $u = \cos(nx)$ $\quad du = -n\sin(nx)$ $\quad v = e^x$ $\quad dv = e^x dx$ $\quad z = \sin(nx)$ $\quad dz = n\cos(nx)dx$ $\quad w = e^x$ $\quad dw = e^x dx$

$$\int e^x \cos(nx)dx = e^x\cos(nx) - \int e^x(-n\sin(nx))dx$$

$$= e^x\cos(nx) + n\left(e^x\sin(nx) - n\int e^x\cos(nx)\right)$$

$$(1+n^2)\int e^x\cos(nx) = e^x(\cos(nx) + n\sin(nx))$$

$$\int e^x\cos(nx) = \frac{e^x(\cos(nx)+n\sin(nx))}{1+n^2}$$

$$a_n = \frac{1}{\pi}\left[\frac{e^{-x}(-\cos(nx)+n\sin(nx))}{1+n^2}\Bigg|_{-\pi}^{0} + \frac{e^x(\cos(nx)+n\sin(nx))}{1+n^2}\Bigg|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{-1}{1+n^2} - \left(\frac{e^{\pi}(-\cos(-n\pi)+(-n)\cancel{\sin(-n\pi)}^{0})}{1+n^2}\right)\right.$$

$$= \frac{1}{\pi}\left[\frac{e^{\pi}(-1)^n - 1}{1+n^2} + \frac{e^{\pi}(-1)^n - 1}{1+n^2}\right] = \frac{1}{\pi}\left(\frac{2(e^{\pi}(-1)^n - 1)}{1+n^2}\right)$$

$$a_n = \frac{2(e^{\pi}(-1)^n - 1)}{\pi(1+n^2)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{1}{\pi}\left[\underbrace{\int_{-\pi}^{0} e^{-x}\sin(nx)dx}_{(3)} + \underbrace{\int_{0}^{\pi} e^{x}\sin(nx)dx}_{(4)}\right]$$

(3) $u = \sin(nx)$ $\quad du = n\cos(nx)dx$ $\quad v = -e^{-x}$ $\quad dv = e^{-x}dx$ $\quad z = \cos(nx)$ $\quad dz = -n\sin(nx)dx$ $\quad w = -e^{-x}$ $\quad dw = e^{-x}dw$

$$\int e^{-x}\sin(nx)dx = -e^{-x}\sin(nx) - \int -e^{-x}n\cos(nx)dx$$

$$= -e^{-x}\sin(nx) + n\left(-e^{-x}\cos(nx) - \int -e^{-x}(-n)\sin(nx)dx\right)$$

$$\int e^{-x}\sin(nx)dx = -e^{-x}\sin(nx) - ne^{-x}\cos(nx) - n^2\int e^{-x}\sin(nx)dx$$

$$\int e^{-x}\sin(nx)dx = \frac{-e^{-x}(\sin(nx) + n\cos(nx))}{1+n^2}$$

(4)

$u = \sin(nx)$ $\quad du = n\cos(nx)\,dx$ $\quad v = e^x$ $\quad dv = e^x\,dx$ $\quad z = \cos(nx)$ $\quad dz = -n\sin(nx)\,dx$ $\quad w = e^x$ $\quad dw = e^x\,dx$

$$\int e^x \sin(nx)\,dx = e^x \sin(nx) - \int e^x\, n\cos(nx)\,dx$$

$$= e^x\sin(nx) - n\left(\cos(nx)e^x - \int e^x(-n)\sin(nx)\,dx\right)$$

$$\underline{\int e^x \sin(nx)\,dx} = e^x\sin(nx) - n e^x\cos(nx) - n^2 \underline{\int e^x \sin(nx)\,dx}$$

$$\int e^x \sin(nx)\,dx = \frac{e^x(\sin(nx) - n\cos(nx))}{1+n^2}$$

$$b_n = \frac{1}{\pi}\left[\left.\frac{-e^{-x}(\sin(nx) + n\cos(nx))}{1+n^2}\right|_{-\pi}^{0} + \left.\frac{e^x(\sin(nx) - n\cos(nx))}{1+n^2}\right|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{-n}{1+n^2} - \frac{(-e^{-(-\pi)})(\cancel{\sin(-n\pi)}^0 + n\cos(-n\pi))}{1+n^2} + \frac{e^{\pi}(\cancel{\sin(n\pi)}^0 - n(\cos(n\pi)))}{1+n^2} - \frac{(-n)}{1+n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{e^{\pi}(n(-1)^n) - n}{1+n^2} + \frac{e^{\pi}(-n(-1)^n + n)}{1+n^2}\right] = \frac{1}{\pi}\left[\frac{e^{\pi}(n(-1)^n - n(-1)^n) - n + n}{1+n^2}\right]$$

$$\underline{\underline{b_n = 0}}$$

$$a_n = \frac{2(e^{\pi}(-1)^n - 1)}{\pi(4+n^2)} \qquad a_0 = \frac{e^{\pi} - 1}{\pi}$$

$$\underline{\underline{f(x) = \frac{e^{\pi} - 1}{\pi} + \sum_{n=1}^{\infty}\left(\frac{2(e^{\pi}(-1)^n - 1)}{\pi(1+n^2)}\cos(nx)\right)}}$$

• Find the Fourier Series of the following periodic function $[-4,4]$

$$f(x) = \begin{cases} -x, & -4 \le x < 0 \\ 2, & 0 \le x \le 4 \end{cases}$$

$$a_0 = \frac{1}{2(4)} \int_{-4}^{4} f(x)dx = \frac{1}{8}\left[\int_{-4}^{0}(-x)dx + \int_0^4 2dx\right]$$

$$= \frac{1}{8}\left[\left(-\frac{x^2}{2}\right)\Big|_{-4}^{0} + 2x\Big|_0^4\right] = \frac{1}{8}\left[+\frac{(-4)^2}{2} + 2(4)\right] = \frac{1}{8}\left[\frac{+16}{2}+8\right]$$

$$= \frac{1}{8}[+8+8] = \underline{\underline{2}}$$

$$a_n = \frac{1}{4}\int_{-4}^{4} f(x)\cos\left(\frac{n\pi}{4}x\right)dx = \frac{1}{4}\left[\underbrace{\int_{-4}^{0}(-x)\cos\left(\frac{n\pi}{4}x\right)dx}_{1} + \underbrace{\int_0^4 2\cos\left(\frac{n\pi}{4}x\right)dx}_{2}\right]$$

1 $u = -x$ $\quad v = \frac{4}{n\pi}\sin\left(\frac{n\pi}{4}x\right)$

$du = -dx$ $\quad dv = \cos\left(\frac{n\pi}{4}x\right)dx$

$$-\frac{4}{n\pi}x\sin\left(\frac{n\pi}{4}x\right) - \int -\frac{4}{n\pi}\sin\left(\frac{n\pi}{4}x\right)dx = \underbrace{-\frac{4}{n\pi}x\sin\left(\frac{n\pi}{4}x\right) - \frac{16}{n^2\pi^2}\cos\left(\frac{n\pi}{4}x\right)}$$

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$$2\int\cos\left(\frac{n\pi}{4}x\right)dx = \frac{2\cdot 4}{n\pi}\sin\left(\frac{n\pi}{4}x\right) = \frac{8}{n\pi}\sin\left(\frac{n\pi}{4}x\right)$$

$$a_n = \frac{1}{4}\left[\frac{-4}{n\pi}\left(x\sin\left(\frac{n\pi}{4}x\right) + \frac{4}{n\pi}\cos\left(\frac{n\pi}{4}x\right)\right)\Big|_{-4}^{0} + \frac{8}{n\pi}\sin\left(\frac{n\pi x}{4}\right)\Big|_0^4\right]$$

$$= \frac{1}{4}\left[\frac{-4}{n\pi}\left(\frac{4}{n\pi} - \left(\cancel{-4\sin(-n\pi)}^{0} + \frac{4}{n\pi}\cos(-n\pi)\right) + \frac{8}{n\pi}\left(\cancel{\sin(n\pi)}^{0} - \cancel{\sin(0)}^{0}\right)\right)\right]$$

$$= \frac{1}{4}\left[\frac{-16}{n^2\pi^2}\left(1-(-1)^n\right)\right] = \frac{4}{n^2\pi^2}\left((-1)^n - 1\right) = \begin{cases} \frac{-8}{n^2\pi^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$$a_n = \begin{cases} \frac{-8}{n^2\pi^2}, & \text{if } n \text{ is odd} \Rightarrow \frac{-8}{(2k-1)^2\pi^2} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$$b_n = \frac{1}{4}\int_{-4}^{4} f(x)\sin\left(\frac{n\pi}{4}x\right)dx = \frac{1}{4}\left[\underbrace{\int_{-4}^{0}(-x)\sin\left(\frac{n\pi x}{4}\right)dx}_{3} + \underbrace{\int_{0}^{4}2\sin\left(\frac{n\pi}{4}x\right)dx}_{4}\right]$$

3 $u = -x \quad v = \frac{-4}{n\pi}\cos\left(\frac{n\pi}{4}x\right)$

$du = -dx \quad dv = \sin\left(\frac{\pi n}{4}x\right)dx$

$$\frac{4}{\pi n}x\cos\left(\frac{n\pi}{4}x\right) - \int \frac{-4}{n\pi}\cos\left(\frac{n\pi x}{4}\right)(-dx)$$

$$= \frac{4}{n\pi}x\cos\left(\frac{n\pi x}{4}\right) - \frac{16}{n^2\pi^2}\sin\left(\frac{n\pi}{4}x\right)$$

4 $2\int\sin\left(\frac{n\pi}{4}x\right)dx = \frac{-8}{n\pi}\cos\left(\frac{n\pi}{4}x\right)$

$$b_n = \frac{1}{4}\left[\frac{4}{n\pi}\left(x\cos\left(\frac{n\pi}{4}x\right) - \frac{4}{n\pi}\sin\left(\frac{n\pi}{4}x\right)\right)\Big|_{-4}^{0} + \left(\frac{-8}{n\pi}\right)\cos\left(\frac{n\pi}{4}x\right)\Big|_0^4\right]$$

$$b_n = \frac{1}{4}\left[\frac{4}{n\pi}\left(-\left(-4\cos(-n\pi) - \frac{4}{n\pi}\cancel{\sin(-n\pi)}^{0}\right)\right) - \frac{8}{n\pi}\left(\cos(n\pi) - 1\right)\right]$$

$$b_n = \frac{1}{4}\left[\frac{4}{n\pi}(4(-1)^n) - \frac{8}{n\pi}((-1)^n - 1)\right] = \frac{1}{4}\left(\frac{4}{n\pi}\right)\left(4(-1)^n - 2(-1)^n + 2\right)$$

$$b_n = \frac{1}{n\pi}\left(2(-1)^n + 2\right) = \frac{2}{n\pi}\left((-1)^n + 1\right) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \frac{4}{n\pi}, & \text{if } n \text{ is even} \Rightarrow \frac{4}{(2k)\pi} \end{cases}$$

$$f(x) = 2 + \sum_{n=1}^{\infty}\left[\frac{4}{n^2\pi^2}((-1)^n - 1)\cos\left(\frac{n\pi}{4}x\right) + \frac{2}{n\pi}((-1)^n + 1)\sin\left(\frac{n\pi}{4}x\right)\right]$$

$$f(x) = 2 + \sum_{k=1}^{\infty}\left[\frac{-8}{(2k-1)^2\pi^2}\cos\left(\frac{(2k-1)\pi}{4}x\right) + \frac{4}{(2k)\pi}\sin\left(\frac{(2k)\pi}{4}x\right)\right]$$