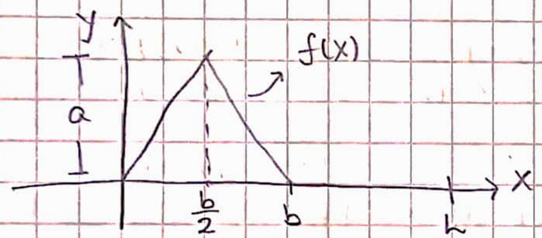
Cuerda fija en los dos extremos

$$\nabla^2 u = \frac{1}{c^2} u_{tt}$$

$$f(x) = \begin{cases} \frac{a}{b}x & 0 \leq x < b \\ -\frac{a}{b}x + 2a & b \leq x < 2b \\ 0 & 2b \leq x \leq L \end{cases}$$

$u(0,t) = 0$
$u(L,t) = 0$

$u(x,0) = f(x)$
$u_t(x,0) = 0$



$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

Separación variables: $u(x,t) = X(x)\,T(t)$

$$T\frac{\partial^2 X}{\partial x^2} = \frac{1}{c^2} X \frac{\partial^2 T}{\partial t^2}$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{c^2 T}\frac{\partial^2 T}{\partial t^2} = \lambda$$

Esto conduce a:

$$\boxed{\frac{\partial^2 X}{\partial x^2} = \lambda X}$$

$$\boxed{\frac{\partial^2 T}{\partial t^2} = c^2 \lambda T}$$

Caso 1) las condiciones de frontera $u(0,t) = u(L,t) = 0$ imponen que $\lambda = -\mu^2 < 0$, lo que conduce a:

$$\boxed{\begin{aligned} X(x) &= A_1 \operatorname{sen}(\mu x) \\ T(t) &= B_1 \cos(c\mu t) \end{aligned}} \rightarrow u_t(x,0) = 0$$

$$\boxed{\mu = \frac{n\pi}{L}}$$

Por ende, la solución general será:

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(c\frac{n\pi}{L}t\right)\operatorname{sen}\left(\frac{n\pi}{L}x\right)$$

La condición $u(x,0) = f(x)$ nos permite encontrar los coeficientes A_n. Donde se ve involucrada una serie de Fourier:

$$A_n = \frac{2}{L}\int_0^L \operatorname{sen}\left(\frac{n\pi}{L}x\right) f(x)\, dx$$

Para $f(x) = \begin{cases} \frac{a}{b}x & 0 \leq x < b \\ -\frac{a}{b}x + 2a & b \leq x < 2b \\ 0 & 2b \leq x \leq L \end{cases}$ tenemos: $0 < b < L$

$$A_n = \frac{2}{L}\left\{ \underbrace{\int_0^b \operatorname{sen}\left(\frac{n\pi}{L}x\right)\frac{a}{b}x\, dx}_{①} + \underbrace{\int_b^{2b} \operatorname{sen}\left(\frac{n\pi}{L}x\right)\left[-\frac{a}{b}x + 2a\right] dx}_{②} \right\}$$

① $I = \int \operatorname{sen}\left(\frac{n\pi}{L}x\right) x\, dx$

	D	I
+	x	$\operatorname{sen}\left(\frac{n\pi}{L}x\right)$
−	1	$-\frac{\cos\left(\frac{n\pi}{L}x\right)}{\frac{n\pi}{L}}$
	0	$-\frac{\operatorname{sen}\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^2}$

$$I = -\frac{L}{n\pi}\cos\left(\frac{n\pi}{L}x\right)x + \frac{L^2}{(n\pi)^2}\operatorname{sen}\left(\frac{n\pi}{L}x\right)$$

$$\Rightarrow \int_0^b x\operatorname{sen}\left(\frac{n\pi}{L}x\right)dx = \frac{L}{n\pi}\left(\frac{L}{n\pi}\operatorname{sen}\left(\frac{n\pi}{L}x\right) - x\cos\left(\frac{n\pi}{L}x\right)\right)\Big|_0^b$$

$$= \frac{L}{n\pi}\left(\frac{L}{n\pi}\operatorname{sen}\left(\frac{n\pi}{L}b\right) - b\cos\left(\frac{n\pi}{L}b\right)\right) \quad ①$$

② $$I = -\frac{a}{b}\int_b^{2b} x\operatorname{sen}\left(\frac{n\pi}{L}x\right)dx + 2a\int_b^{2b}\operatorname{sen}\left(\frac{n\pi}{L}x\right)dx$$

$$= -\frac{L}{n\pi}\frac{a}{b}\left[\frac{L}{n\pi}\operatorname{sen}\left(\frac{n\pi}{L}x\right) - x\cos\left(\frac{n\pi}{L}x\right)\right]_b^{2b} - 2a\frac{\cos\left(\frac{n\pi}{L}x\right)}{\frac{n\pi}{L}}\Big|_b^{2b}$$

$$= -\frac{L}{n\pi}\frac{a}{b}\left[\frac{L}{n\pi}\operatorname{sen}\left(\frac{2bn\pi}{L}\right) - \cancel{2b\cos\left(\frac{n\pi}{L}2b\right)} - \frac{L}{n\pi}\operatorname{sen}\left(\frac{n\pi}{L}b\right) + \underbrace{b\cos\left(\frac{n\pi}{L}b\right)}\right]$$

$$-\frac{L}{n\pi}2a\left[\cancel{\cos\left(\frac{n\pi}{L}2b\right)} - \underbrace{\cos\left(\frac{n\pi}{L}b\right)}\right]$$

$$② = -\left(\frac{L}{n\pi}\right)^2 \frac{a}{b}\, sen\left(\frac{n\pi}{L}2b\right) + \left(\frac{L}{n\pi}\right)^2 \frac{a}{b}\, sen\left(\frac{n\pi}{L}b\right) + \frac{L}{n\pi}\, a \cos\left(\frac{n\pi}{L}b\right)$$

$$\Rightarrow A_n = \frac{2}{L}\left\{\frac{a}{b}① + ②\right\} = \frac{2}{L}\left\{ \left(\frac{L}{n\pi}\right)^2 \frac{a}{b}\, sen\left(\frac{n\pi}{L}b\right) - \cancel{\frac{L}{n\pi}\, a \cos\left(\frac{n\pi}{L}b\right)} \right.$$

$$- \left(\frac{L}{n\pi}\right)^2 \frac{a}{b}\, sen\left(\frac{n\pi}{L}2b\right) + \left(\frac{L}{n\pi}\right)^2 \frac{a}{b}\, sen\left(\frac{n\pi}{L}b\right)$$

$$\left. + \cancel{\frac{L}{n\pi}\, a \cos\left(\frac{n\pi}{L}b\right)} \right\}$$

$$\Rightarrow A_n = \frac{2}{L}\left\{ -\left(\frac{L}{n\pi}\right)^2 \frac{a}{b}\, sen\left(\frac{n\pi}{L}2b\right) + 2\left(\frac{L}{n\pi}\right)^2 \frac{a}{b}\, sen\left(\frac{n\pi}{L}b\right) \right\}$$

$$A_n = \frac{2a}{b}\frac{L}{n^2\pi^2}\left[- sen\left(\frac{n\pi}{L}2b\right) + 2\, sen\left(\frac{n\pi}{L}b\right) \right]$$