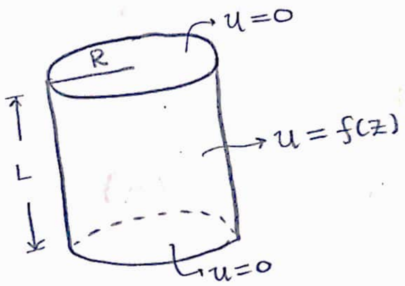
Taller de laplacianos en varios sabores

Estudiante: Juan Andrés Guarín Rojas

1)



Ecuación de calor en estado estacionario:

$$\nabla^2 u = 0$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Condiciones Frontera
$$\begin{cases} u(\rho,\varphi,0) = 0 \\ u(\rho,\varphi,L) = 0 \\ u(R,\varphi,z) = f(z) \end{cases}$$

condición de continuidad: $u(\rho,\varphi,z) = u(\rho,\varphi+2\pi,z)$

Separación variables:

$$u(\rho,\varphi,z) = P(\rho)\Phi(\varphi)Z(z),$$

$$\Phi(\varphi)Z(z)\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial P(\rho)}{\partial\rho}\right) + \frac{1}{\rho^2}P(\rho)Z(z)\frac{\partial^2\Phi(\varphi)}{\partial\varphi^2} + P(\rho)\Phi(\varphi)\frac{\partial^2 Z(z)}{\partial z^2} = 0$$

$$\rightarrow \frac{1}{P}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial P}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\varphi^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = 0$$

$$\frac{1}{P}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial P}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\varphi^2} = \lambda = -\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}$$

Esto conduce a una primera EDO:
$$\boxed{Z'' = -\lambda Z} \quad (1)$$

El otro lado, multiplicando ρ^2, tenemos:

$$\frac{\rho}{P}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial P}{\partial\rho}\right) + \frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\varphi^2} = \lambda\rho^2$$

$$\frac{\rho}{P}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial P}{\partial\rho}\right) - \lambda\rho^2 = \lambda_1 = -\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\varphi^2}$$

Lo que conduce a:
$$\boxed{\Phi'' = -\lambda_1\Phi} \quad (2)$$

y
$$\boxed{\frac{\rho}{P}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial P}{\partial\rho}\right) - \lambda\rho^2 - \lambda_1 = 0} \quad (3)$$

a) Analizando las condiciones de frontera
$$u(\rho,\varphi,0) = P(\rho)\Phi(\varphi)Z(0) = 0 \rightarrow Z(0) = 0$$
$$u(\rho,\varphi,L) = P(\rho)\Phi(\varphi)Z(L) = 0 \rightarrow Z(L) = 0$$

Como tenemos condiciones de frontera periódicas para Z, la única solución de (1) es para:

$$\lambda = \mu^2 > 0 \rightarrow Z(z) = C_1\cos(\mu z) + C_2\,\mathrm{sen}(\mu z)$$

$$\rightarrow Z(0) = 0 \wedge Z(L) = 0 \Rightarrow \boxed{Z_n(z) = C_2\,\mathrm{sen}\left(\frac{n\pi}{L}z\right)} \quad \boxed{\mu_n = \frac{n\pi}{L}}$$