②

a) Find the auto covariance and autocorrelation functions for this process when $\theta = 0,8$

First, the autocovariance function $\gamma(h)$ is defined as

$$\gamma(h) = Cov\ (X_t, X_{t+h})$$

For a process MA, the autocovariance is

$$\gamma(h) = \begin{cases} \sigma_z^2\ (1+\theta^2) & ; \quad h = 0 \\ \sigma_z^2\ \theta & ; \quad h = 2 \\ 0 & ; \quad \text{another case} \end{cases}$$

Since $\sigma_z^2 = 1$ and $\theta = 0,8$

$$\gamma(0) = 1\ (1 + 0,8^2)$$
$$= 1 + 0,64$$
$$= 1,64$$

$$\gamma(2) = 1\ (0,8)$$
$$= 0,8$$

$\gamma(h) = 0$ for $h \neq 0$ and $h \neq 2$

The autocorrelation function $\rho(h)$ is defined as

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} \quad , \quad \forall\ h \in \mathbb{Z}$$

For values of $\gamma(h)$

$$\rho(0) = \frac{\gamma(0)}{\gamma(0)} = 1 \quad ; \quad \rho(2) = \frac{\gamma(2)}{\gamma(0)} = \frac{0,8}{1,64} \approx 0,4878$$

$\rho(h) = 0$ for $h \neq 0$ and $h \neq 2$

b) Compute the mean and variance

$$\overline{X}_n = \frac{X_1 + X_2 + X_3 + X_4}{4} \quad \text{when} \quad \theta = 0{,}8$$

Since $\{Z_t\} \sim WN(0, \sigma^2)$

$$\mu_X = E(X_t) = E(Z_t + \theta Z_{t-2}) = 0$$

Therefore, the mean of $\overline{X}_n$ is

$$E(\overline{X}_n) = \frac{E(X_1) + E(X_2) + E(X_3) + E(X_4)}{4}$$

$$= 0$$

Next, the variance

$$Var(\overline{X}_n) = Var\left(\frac{X_1 + X_2 + X_3 + X_4}{4}\right)$$

Since X_t is a process MA, the covariances between X_t and X_{t+h} are 0. Then, there is only covariances between X_t and X_{t+2}

So,

$$Var(\overline{X}_n) = \frac{1}{16}\Big(Var(X_1) + Var(X_2) + Var(X_3) + Var(X_4) + 2\,Cov(X_2, X_4)\Big)$$

by properties of variance

$$Var(\overline{X}_n) = \frac{1}{16}\left(1,64 + 1,64 + 1,64 + 1,64 + 2(0,8) + 2(0,8)\right)$$

$$= 0,61$$

c) Repeat (b) when $\theta = -0,8$ and compare your answer with the result obtained in (b).

For $\theta = -0,8$ the autocovariance

$$\gamma(0) = 1\left(1 + (-0,8)^2\right)$$

$$= 1,64$$

$$\gamma(2) = 1 \cdot (-0,8)$$

$$= -0,8$$

$$\gamma(h) = 0 \text{ for } h \neq 0 \text{ and } h \neq 2$$

The mean

$$E(X_t) = E(Z_t - 0,8\, Z_{t-2})$$

$$= \underbrace{E(Z_t)}_{0} - 0,8\, \underbrace{E(Z_{t-2})}_{0}$$

$$= 0$$

The variance

$$Var(X_t) = \gamma(0) = Var(Z_t - 0,8\, Z_{t-2})$$

$$= 1 + (-0,8)^2$$

$$= 1,64$$

The mean of $\overline{X}_n$

$$E(\overline{X}_n) = \frac{1}{4}\left(E(X_1) + E(X_2) + E(X_3) + E(X_4)\right)$$

$$= 0$$

The variance of $\overline{X}_n$

$$Var(\overline{X}_n) = \frac{1}{16}\left(Var(X_1) + Var(X_2) + Var(X_3) + Var(X_4)\right)$$

$$= \frac{1}{16}\left(1{,}64 + 1{,}64 + 1{,}64 + 1{,}64\right)$$

$$= 0{,}41$$