CS 131 – Problem Set 10

Problems must be submitted by Monday December 8, 2025 at 11:59pm, on Gradescope.

Problem 1. (4 parts with 5 points each, 20 points total) In 2040, Boston University admits 2,000 undergraduate computer science first-years due to an explosion in the major's popularity. To accommodate these CS first-years, each of the 800 rooms in Warren Towers has been expanded into a luxurious quad to hold four students each. All of the students in this class cohort are forced to be housed in Warren Towers.
NOTE: You can assume that the students are distinguishable.

To align interests between students, BU decides to group Warren Towers' rooms by major. This means that each CS first-year is placed in a Warren quad with three other CS first-years as their roommates; **a room has either four CS first-years or no CS first-years at all**.

a) Let a room that houses four CS first-years be called a CS room. How many CS rooms would BU need to accommodate the 2,000 CS first-years? How many ways can BU distribute the CS rooms among the 800 total rooms in Warren Towers?

b) How many ways can the CS first-years be distributed among the CS rooms?

c) To ensure a smooth on-boarding of students, the CS department instates a mandatory CS-specific orientation. 1,556 of the 2,000 students attend. Show that at least a group of four students from the same CS room attended this orientation.

d) To adjust to the rise in CS majors, CS131 chooses to raise its number of seats to 600. 560 first-years enroll in the course. What is the **minimum number** of CS rooms that have **more than one** of its residents taking CS131?

Problem 2. (20 points) Exactly 131 BU students are taking the Red Line late at night. An empty train with six cars, named A, B, C, D, E, F arrives. Students enter the train. An MBTA inspector writes down the number of students in car A, B, C, D, E, and F, in that order. How many different sequences of 6 numbers can the inspector get? Your answer needs to be **justified** or else you will get no points. (Same rules as problem 3)

Hint: this problem is similar to Problem 1 on Lab 10, but different, because 0 students in a car is also possible. There are many different ways to solve it.

Problem 3. (6 parts at 5 points each, 30 points total) This problem refers to "Additional Exercises" in our zyBook.

Your answers need to be justified; a numerical answer without an explanation, even if correct, will be worth nothing. Your explanations should be in paragraph form, with both proper English writing and proper mathematical explanations.

You may use exponentials, factorials, $C(n, r)$, and $P(n, r)$ in your answers. There is no need to calculate things out. For example, it is perfectly okay to leave your answer in the form $2^5 \cdot 7! \cdot P(10, 3)$ (again, with an explanation for how you got it).

- 10.5.6(b)
  Suppose a network has 40 computers of which five fail.

  Suppose that three of the computers in the network have a copy of a particular file. How many sets of failures wipe out all the copies of the file? That is, how many 5-subsets contain the three computers that have the file?

- 10.6.4(b)
  20 different comic books will be distributed to five kids. How many ways are there to distribute the comic books if they are divided evenly so that 4 go to each kid?

- 10.7.3(c)
  12 players of a basketball team travel to an away game.

  From these 12 players, the coach must select her starting line-up. She will select a player for each of the five positions: center, power forward, small forward, shooting guard and point guard. However, there are only three of the 12 players who can play center. Otherwise, there are no restrictions. How many ways are there for her to select the starting line-up?

- 10.8.2(b)
  A 5-card hand is drawn from a deck of standard playing cards. How many 5-card hands have at least two cards with the same rank?

- 10.9.5
  A round-robin tournament is one where each player plays each of the other players exactly once. Prove that if no person loses all their games, then there must be two players with the same number of wins.

- 11.3.5(a)
  A family lines up for a photograph. In each of the following situations, how many ways are there for the family to line up so that the mother is next to at least one of her daughters? Suppose the family consists of a mother, a father, two daughters, and two sons.

Problem 4. (5 parts with 6 points each, 30 points total) Digital images are ultimately stored as bitstrings (strings of 0s and 1s). One common way to represent a color images is as follows :

- The image consists of a grid of pixels.
- Each pixel has 3 color components (red, green, blue)
- Each of theses components is stored using 8 bits.

We will use this representation throughout the problem. However, images take up a lot of space. This motivates techniques for reducing file size, such as compression, which we will explore in the questions below.

a) iPhone 16 has a screen resolution of 1179x2556 pixels. Using the representation above without any compression, how many bits does it take up?

b) How many different images can be represented with this many bits?

c) You are hoping to reduce the amount of storage needed for the screen image. That means that you will represent each image by a shorter bit string, using some sort of compression.

Here's the idea behind compression (We transform a long bitstring into a shorter one, send or store the shorter version, and later decompress it to get the original):

$$\begin{array}{c} \boxed{111011101010100010101011100100} \\ \downarrow \\ \text{compress} \\ \downarrow \\ \boxed{1111010} \\ \downarrow \\ \text{transit} \\ \downarrow \\ \boxed{1111010} \\ \downarrow \\ \text{decompress} \\ \downarrow \\ \boxed{111011101010100010101011100100} \end{array}$$

For the decompression to always recover the exact original, the compression function must be injective (no two inputs map to the same output). Such compression is called "lossless."

Let's think about what happens when you compress a string of n bits to some shorter string.

How many bitstrings of length less than n are there?

d) Consider any compression function that takes in n-bit inputs and always produces an output of length less than n. Prove that it is **not** lossless.

e) Suppose you decided that you are okay with lossy compression. That is, you no longer require the compression function to be injective. Because there are now many inputs for every output of the compression, the decompression function will pick just one of the possible pictures that could have been compressed to this output.

Prove that if your compression function saves at least 300 bits, then there exists a compressed image that can be decompressed in at least as many possible ways as the current estimate for the number of particles in the visible universe ($\approx 2^{265} \approx 10^{80}$).

Hint: if you are compressing an n-bit input and saving at least 300 bits, then your compression function can output any string of length $0, 1, \ldots, n-300$.