Stochastic Control Project
Optimal Portfolio Liquidation with Limit Orders

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1 Introduction

The execution of large orders in financial markets has long been a challenging problem for both practitioners and researchers. Traditional frameworks, such as the work of Almgren and Chriss [1], address the trade-off between market impact and price risk by optimizing execution schedules based on market orders. While these approaches provide valuable insights, they typically neglect the benefits—and the additional risks—associated with using limit orders.

Recent developments in electronic trading have shifted attention towards strategies that incorporate limit orders. Unlike market orders, limit orders allow traders to potentially capture better prices and reduce explicit execution costs; however, they introduce *non-execution risk*, as orders may not be filled if posted too far from the current market levels. In this context, Obizhaeva and Wang [2] were among the first to analyze execution strategies that interact directly with the limit order book, emphasizing liquidity consumption. Other works, such as those by Avellaneda and Stoikov [3], have modeled market making using similar principles, highlighting the role of execution intensity and order-book dynamics.

Guéant, Lehalle, and Fernandez-Tapia [4] extend this line of research by proposing a unified framework that integrates both trade scheduling and limit order placement. Their model captures the dynamics of a trader who must liquidate a large portfolio using passive limit orders. In their approach, the reference price is modeled as a Brownian motion with drift, and the execution probability of a limit order decays exponentially with the distance from this "fair" price. This setting allows the authors to formulate the problem as a Hamilton-Jacobi-Bellman (HJB) equation, which, through a suitable change of variables, is reduced to a system of linear ordinary differential equations. The closed-form expressions for the optimal quotes derived from this reduction offer clear insights into how various market parameters—such as risk aversion, volatility, and order arrival intensity—affect the optimal execution strategy.

The main contributions of this project are twofold. First, we rigorously replicate the theoretical and numerical results presented in [4]. This involves solving the system of ODEs using a backward Euler scheme and verifying the analytical expressions for the optimal ask quotes and trading curves across different market regimes (including asymptotic time horizons, the no-drift/no-volatility limit, and scenarios with high liquidation costs). Second, we extend the original analysis by performing an study of the impact of liquidation cost b on the average final inventory, and by comparing the optimal strategy with a set of benchmark strategies. In particular, we introduce alternative quoting rules (such as a *safe* strategy, a *greedy* strategy, and a *random* strategy) and analyze their performance in terms of both trading speed (i.e., the evolution of inventory over time) and the final cash outcome. For clarity and ease of interpretation, our sensitivity analyses and performance comparisons are presented using plots rather than tables—even though the numerical values coincide with those in the original study.

In summary, this report builds upon the foundational work of Almgren and Chriss [1] and Obizhaeva and Wang [2], while leveraging the innovative contributions of Guéant, Lehalle, and Fernandez-Tapia [4]. By replicating and extending their model, we provide both theoretical insights and practical guidelines for optimal portfolio liquidation using limit orders in modern electronic markets.

2 Model Setup

In this section, we describe the mathematical framework for optimal portfolio liquidation using limit orders, as introduced in [4]. The model is designed to capture both the price risk and the non-execution risk that arise when a trader liquidates a large position via limit orders.

2.1 Price Dynamics and Order Flow

The reference (or "fair") price process S_t is assumed to follow a Brownian motion with drift:

$$dS_t = \mu\, dt + \sigma\, dW_t, \tag{1}$$

where μ is the drift, σ is the volatility, and W_t is a standard Brownian motion (sBM). Note that in this framework, the price could, theoretically, be negative. However, due to the small time frames that are covered in high-frequency trading, and by setting and initial price large enough, we could assume the positivity of prices under a sBM. At time t, the trader posts a limit order at the ask price

$$S_t^a = S_t + \delta_t^a, \tag{2}$$

where δ_t^a is the additional premium (measured in ticks) above the reference price. Note that in our simulations, we will set the lifetime of a limit order as one time tick. Therefore, for each time step the trader will cancel out the previous limit order (if not already filled) and will post a new one. The execution of the limit order is modeled via a point process with intensity

$$\lambda^a(\delta_t^a) = A \exp\left(-k\, \delta_t^a\right), \tag{3}$$

with parameters $A > 0$ and $k > 0$. This formulation implies that orders placed further from the current price are less likely to be executed.

When an order is executed, the trader sells one unit of the asset, which in theory could be a batch, but the authors prefer to assume one unit. It is important to note that this model relies on the uniqueness of the amount of shares sold at each time t. Now, let q_t denote the trader's inventory at time t, with

$$q_t = q_0 - N_t^a, \tag{4}$$

where q_0 is the initial inventory and N_t^a is the cumulative number of executed orders. The evolution of the cash process X_t is given by

$$dX_t = (S_t + \delta_t^a)\, dN_t^a. \tag{5}$$

This means that the cash of the trader is only updated whenever a share is sold.

2.2 The Optimization Problem

The trader's goal is to liquidate the entire position over a fixed time horizon T while managing both price risk and non-execution risk and maximizing the generated cash. At the terminal time T, any unsold shares incur a penalty cost b per unit. The problem is formulated using a Constant Absolute Risk Aversion (CARA) utility function. Specifically, the trader seeks to maximize over the set of possible quotes the expected utility of terminal wealth:

$$\sup_{(\delta_t^a)_{t\in[0,T]}} \mathbb{E}\left[-\exp\left(-\gamma\left(X_T + q_T(S_T - b)\right)\right)\right], \tag{6}$$

where $\gamma > 0$ represents the risk aversion parameter.

2.3 Hamilton-Jacobi-Bellman Equation

Applying the dynamic programming principle, the value function $u(t, x, q, s)$ (representing the maximum expected utility given the state (x, q, s)) satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned} &\partial_t u(t,x,q,s) + \mu\, \partial_s u(t,x,q,s) + \frac{1}{2}\sigma^2\, \partial_{ss} u(t,x,q,s) \\ &\qquad + \sup_{\delta^a} \lambda^a(\delta^a)\left[u\left(t, x + S_t + \delta^a,\, q-1,\, s\right) - u(t,x,q,s)\right] = 0, \end{aligned} \tag{7}$$

with the terminal condition

$$u(T,x,q,s) = -\exp\left(-\gamma\left(x + q(s-b)\right)\right), \tag{8}$$

and the boundary condition $u(t, x, 0, s) = -\exp(-\gamma x)$.

Exploiting the CARA structure of the utility function, [4] introduces a change of variables to factor out the mark-to-market value. They assume that the value function can be written as

$$u(t, x, q, s) = -\exp\left(-\gamma(x + qs)\right)\, w_q(t)^{-\frac{\gamma}{k}}, \tag{9}$$

where the functions $w_q(t)$ (with $q = 0, 1, 2, \ldots$) satisfy a system of linear ordinary differential equations (ODEs).

Substituting (9) into the HJB equation (7) and performing algebraic manipulations (the details of which are provided in the Appendix of [4]), the problem reduces to solving the following system for $w_q(t)$:

$$\dot{w}_q(t) = \left(\alpha q^2 - \beta q\right) w_q(t) - \eta\, w_{q-1}(t), \tag{10}$$

with the terminal conditions

$$w_q(T) = e^{-kqb}, \quad \text{for } q \geq 1, \qquad \text{and} \qquad w_0(t) = 1, \quad \forall t. \tag{11}$$

The parameters in (10) are defined as:

$$\alpha = \frac{1}{2} k\, \gamma\, \sigma^2, \quad \beta = k\, \mu, \quad \eta = A\left(1 + \frac{\gamma}{k}\right)^{-\left(1+\frac{k}{\gamma}\right)}. \tag{12}$$

2.4 Optimal Ask Quote

Once the system (10) is solved (typically via numerical schemes such as the backward Euler method), the optimal ask quote is obtained by a verification theorem. The resulting optimal control is given by:

$$\delta^{a*}(t, q) = \frac{1}{k} \ln\left(\frac{w_q(t)}{w_{q-1}(t)}\right) + \frac{1}{\gamma} \ln\left(1 + \frac{\gamma}{k}\right). \tag{13}$$

Proof Sketch: A detailed proof of (13) is provided in the Appendix of [4]. Here we highlight the key ideas:

- **Separation of Variables:** The use of a CARA utility function allows one to factor the value function into a product of an exponential term (capturing the mark-to-market component $x + qs$) and a function $w_q(t)$ that encapsulates the dynamic aspects of the liquidation problem. This step is crucial in reducing the original HJB equation to a more tractable system of ODEs.

- **Reduction to a System of ODEs:** By substituting the ansatz (9) into the HJB equation, one obtains a system for $w_q(t)$ as shown in (10). The boundary and terminal conditions are set by the requirement that the utility function matches the prescribed behavior at $t = T$ and when $q = 0$.

- **First-Order Optimality Condition:** The supremum over δ^a in the HJB equation leads to a first-order condition. Solving this condition yields the candidate optimal control in closed form. A verification theorem then confirms that this candidate indeed maximizes the expected utility.

The framework described above unifies the trade scheduling and limit order placement problems into a single model, enabling a compact treatment of both price risk and non-execution risk. In the following sections, we detail the numerical methods employed to solve (10) and simulate the trading process, and we compare the optimal strategy against various benchmark strategies.

3 Methodology

In this section, we detail the computational procedures used to both replicate and extend the results of [4]. Our approach consists of three main parts: (i) the numerical solution of the ODE system derived from the HJB equation, (ii) a simulation framework to study the trading dynamics and sensitivity with respect to key parameters, and (iii) an extended analysis comparing the optimal strategy with several benchmark strategies.

3.1 Numerical Resolution of the ODE System

The core of our approach lies in solving the following system of ordinary differential equations (ODEs) for the functions $w_q(t)$ contained in (10) with terminal conditions of (11).

For this matter, we discretize the time interval $[0, T]$ into N uniform steps, with step size $\Delta t = T/N$. To ensure numerical stability, we integrate the ODE system backward in time from $t = T$ to $t = 0$. The backward Euler scheme is applied as follows:

$$w_q(t_i) = w_q(t_{i+1}) - \Delta t \left[\left(\alpha q^2 - \beta q\right) w_q(t_{i+1}) - \eta \, w_{q-1}(t_{i+1}) \right], \tag{14}$$

where t_i represents the i-th time node in the discretized grid (with $t_N = T$ and $t_0 = 0$). This implicit scheme is chosen because it offers favorable stability properties, particularly important for stiff ODE systems that can arise in control problems.

The algorithm is implemented by initializing $w_0(t) = 1$ for all time steps and setting $w_q(T) = \exp(-kq\, b)$ for $q \geq 1$. Then, for each time step $i = N-1, N-2, \ldots, 0$ and for each inventory level $q = 1, 2, \ldots, q_{\max}$, we update $w_q(t_i)$ using (14). The computed values $w_q(0)$ at $t = 0$ are later used to derive the optimal quote.

Once the functions $w_q(t)$ are obtained, the optimal ask quote at time t for inventory level q is computed via (13). Our numerical implementation computes $\delta^{a*}(t, q)$ at each time step and for every inventory level using the pre-computed $w_q(t)$ values.

3.2 Monte Carlo Simulation of Trading Paths

To analyze the trading dynamics under the computed optimal strategy, we simulate the evolution of three key processes:

1. **Inventory Process q_t:** The trader's inventory decreases by one unit upon each execution.
2. **Price Process S_t:** Modeled using the Euler–Maruyama discretization of the SDE in (1).
3. **Cash Process X_t:** Updated whenever an execution occurs, where the cash increment is $S_t + \delta^{a*}(t, q)$.

At each discrete time step, we determine whether an order is executed based on the probability $p_{\text{fill}} = \lambda^a(\delta^{a*}(t, q))\Delta t$, where the intensity is given by $\lambda^a(\delta) = A \exp(-k\delta)$. Uniform random numbers are generated to decide if a fill occurs. For each fill, the inventory decreases by one, and the cash account is increased by the execution price.

Repeating this process over a large number of paths enables us to compute ensemble averages, such as the average trading curve (i.e., the expected remaining inventory as a function of time) and the distribution of final cash outcomes.

3.3 Limit Cases

We also examine several limit cases discussed in [4] to further understand the model's behavior under extreme conditions. Our numerical implementation is compared against the analytical formulas provided in the paper.

Asymptotic Regime ($T \to \infty$). In the limit of an infinite time horizon, the optimal ask quote at time $t = 0$ is given by (see Proposition 2 in [4]):

$$\lim_{T \to \infty} \delta^{a*}(0, q) \;=\; \frac{1}{k} \ln\!\left(\frac{A}{1 + \frac{\gamma}{k}} \cdot \frac{1}{\alpha q^2 - \beta q} \right),$$

where $\alpha = \frac{1}{2} k\, \gamma\, \sigma^2$ and $\beta = k\, \mu$. This expression indicates that the asymptotic quote decreases with increasing inventory q, capturing the trade-off between execution probability and price risk.

No Drift/No Volatility ($\mu = \sigma = 0$). When the price dynamics are deterministic, the optimal quote simplifies considerably. In this case (see Proposition 3 in [4]), the closed-form expression becomes:

$$\delta^{a*}(t,q) \;=\; -b + \frac{1}{k}\ln\left(1 + \frac{\eta^q (T-t)^q}{q! \displaystyle\sum_{j=0}^{q-1} \frac{\eta^j (T-t)^j}{j!}\, e^{-k\, b\, (q-j)}}\right) + \frac{1}{\gamma}\ln\Big(1+\frac{\gamma}{k}\Big),$$

with $\eta = A\left(1+\frac{\gamma}{k}\right)^{-\left(1+\frac{k}{\gamma}\right)}$. Here, the absence of price risk shifts the balance towards minimizing the penalty for unsold inventory.

High Liquidation Cost ($b \to +\infty$). In the limit of an extremely high liquidation cost, the trader is strongly incentivized to liquidate completely before the terminal time. The analytical expression for the optimal quote in this regime is given by (see Proposition 5 in [4]):

$$\lim_{b\to+\infty} \delta^{a*}(t,q) \;=\; \frac{1}{k}\ln\left(\frac{A}{1+\frac{\gamma}{k}} \cdot \frac{1}{q} \cdot \frac{e^{\beta(T-t)}-1}{\beta}\right),$$

(with the appropriate limit taken when $\beta = 0$). This formula confirms that as b increases, the optimal quotes become lower (more aggressive), thereby driving the final inventory towards zero.

3.4 Sensitivity Analysis

To further understand the impact of key parameters (μ, σ, A, k, γ, and b) on the optimal strategy, we conduct a detailed sensitivity analysis. In this analysis, we systematically vary one parameter at a time while keeping the others fixed, and we use plots to visually compare how the optimal quote $\delta^{a*}(0,q)$ (computed at $t = 0$ for simplicity) changes with the inventory q under different settings. This visual approach replaces the original numerical tables and allows for a clearer interpretation of the results.

We also expand the analysis by studying the influence of b in the average final inventory, where we expect an inverse relationship.

3.5 Extended Analysis: Comparison with Benchmark Strategies

Beyond replicating the original results, we extend the analysis by comparing the optimal strategy with several benchmark strategies. These alternative strategies are defined as follows:

- **Safe Strategy:** The trader posts conservative quotes that are negatively correlated with the current inventory, ensuring rapid execution at the cost of lower execution prices. The safe quotes follow:

$$\delta^{\text{safe}}(q) = -mq \tag{15}$$

  Where m is a parameter to choose that quantifies how conservative the trader is.

- **Greedy Strategy:** A fixed, high quote is always posted, favoring better prices when executed, but at the risk of slower execution and increased price risk. The greedy quotes follow:

$$\delta^{\text{greedy}}(q) = \delta \tag{16}$$

  Where δ is a fixed quote, multiple of the number of ticks.

- **Random Strategy:** The quote is drawn uniformly at random from a predefined interval, representing a heuristic, non-adaptive approach. The random quotes are given by:

$$\delta^{\text{random}}(q,t) \sim \mathcal{U}\left[\delta_{\min}, \delta_{\max}\right], \tag{17}$$

  where $\delta_{\min}$ and $\delta_{\max}$ denote the lower and upper bounds of the interval, respectively.

Using the same Monte Carlo simulation framework, we evaluate each strategy in terms of:

1. Trading Curve: The evolution of the average remaining inventory over time.

2. Final Cash Distribution: The statistical distribution of the final cash balance after liquidation.

These performance metrics allow us to assess the trade-offs inherent in each strategy. The optimal strategy, derived from solving the HJB equation, is expected to balance execution speed and price risk more effectively than the benchmark strategies.

The following section presents and discusses the results obtained from these numerical experiments.

4 Results

In this section, we present the numerical and simulation results obtained from our implementation of the optimal liquidation model. Our analysis is organized into several parts. First, we show the replication of the different figures appearing in [4], except for the Historical Backtest section since we do not have access to the high-frequency prices of AXA. We also show the results of our extension by analysing the effect of b in the average terminal inventory, as well as comparing the optimal strategy with several benchmark strategies (safe, greedy, and random approaches) in terms of trading speed and final cash outcomes. Note that unless otherwise stated, the parameters specified for the simulations are the same as in Section 4 in [4] for the replication of figures.

4.1 Optimal Quotes

Figures 1 and 2 replicate the short- and long-horizon scenarios from [4] using our backward Euler solution to the ODE system and the optimal quote expression provided in Equation (13). In both cases, we rely on the routines outlined in Section 3, which compute the functions $w_q(t)$ and then apply the fill-intensity logic to ensure consistency with the original framework.

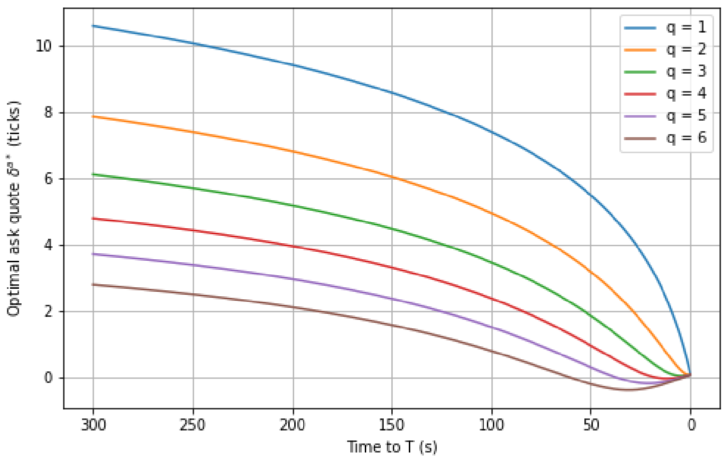

Figure 1: Optimal ask quotes over a 5-minute horizon.

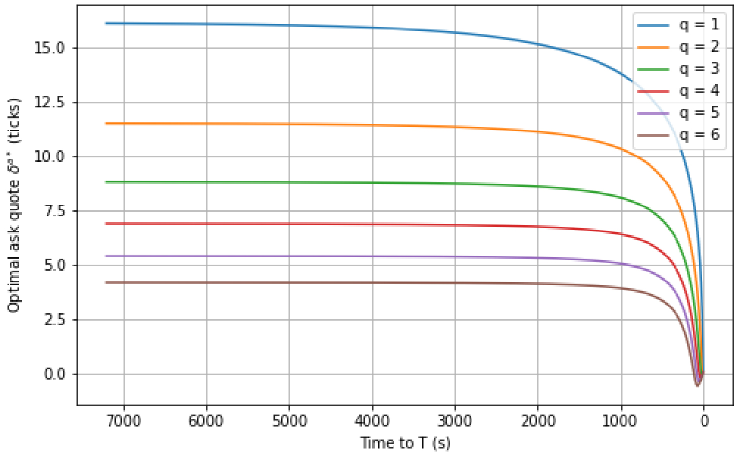

Figure 2: Optimal ask quotes over a 2-hour horizon.

Figure 1 (5 minutes) shows that when q is large, the trader posts more aggressive quotes to accelerate execution within the limited time, reaching even negative quotes, whereas for smaller q, the quotes remain higher. In contrast, Figure 2 (2 hours) displays a more gradual decline, allowing the trader to start with relatively higher quotes and only become increasingly aggressive as T approaches. These behaviors are consistent with the derivations in [4], particularly the asymptotic regime for large T and the time-varying impact of inventory on $\delta^{a*}(t, q)$ as set out in Equation (13).

4.2 Trading Curve

Figure 3 replicates the trading curve result in [4], depicting the expected inventory level over time when the initial inventory is set to $q = 6$ and the liquidation window spans 5 minutes. The simulation proceeds by applying the optimal quotes derived from Equation (13) alongside the execution intensity in Equation (3), thereby determining the probability of each unit being sold at each discrete time step.

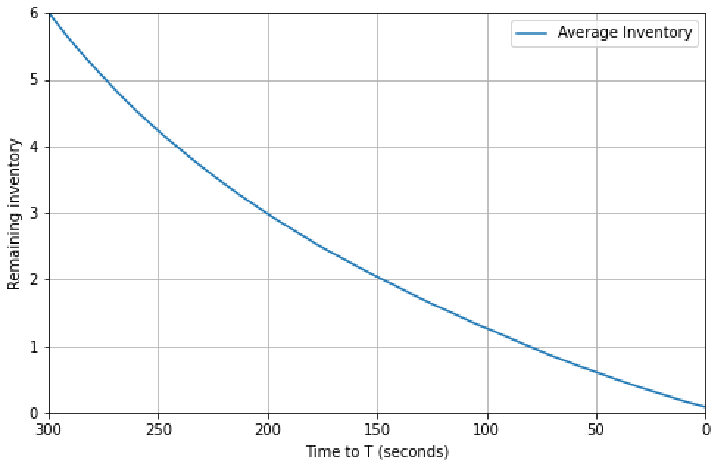

Figure 3: Trading curve (average remaining inventory) for $q = 6$ over 5 minutes.

As time advances, the inventory steadily declines, reflecting the increased likelihood of execution under the optimal quote schedule. Because the penalty cost b is finite, the model does not strictly force complete liquidation before T, so a small residual position can remain if quoting aggressively enough is not warranted by the trade-off between potential price improvement and non-execution risk. We will later see how the limit of b high affect the optimal quotes and, therefore, the trading curve.

In summary, the trading curve is decreasing and convex, two behaviors that match our expecations.

4.3 Limit Cases: $T \to \infty$, $\mu = \sigma = 0$, and $b \to +\infty$

In [4], Section 3 analyzes three special cases of the model: (1) an asymptotic horizon $T \to \infty$ (Case 3.1), (2) no drift/no volatility $\mu = \sigma = 0$ (Case 3.2), and (3) a high liquidation cost $b \to +\infty$ (Case 3.3). Each scenario admits an exact formula that can be compared with the numerical solution obtained by our backward Euler approach by taking limit values. Figures 4, 5, and 6 show how the "General Solver" (i.e. the full ODE solution from Section 2) closely matches the "Analytical" expressions derived for these limit regimes in [4].

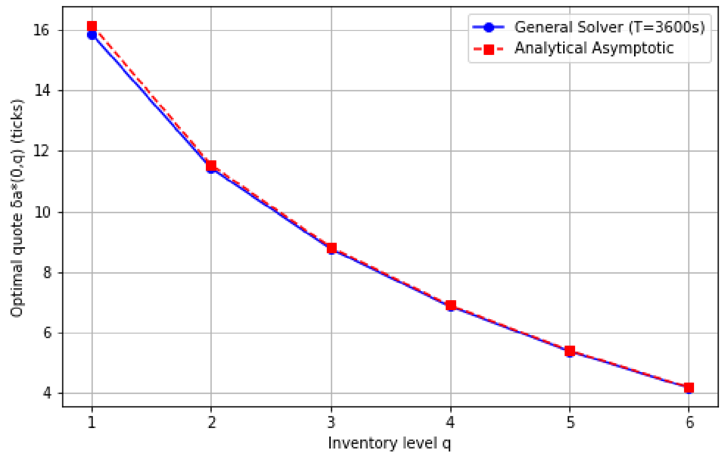

Figure 4: $T \to \infty$. The numerical solution versus the closed-form expression.

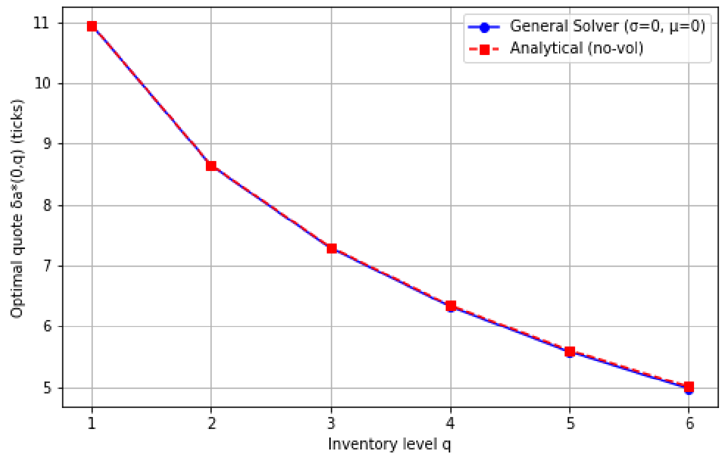

Figure 5: $\mu = 0$, $\sigma = 0$. The numerical solution versus the closed-form expression.

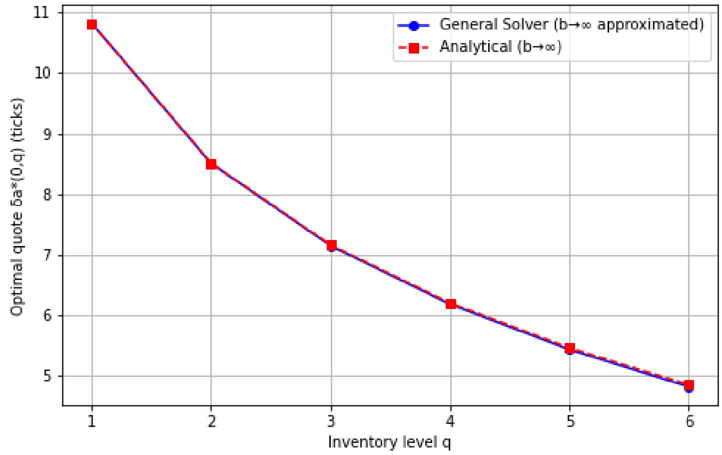

Figure 6: $b \to +\infty$ (No-vol). The numerical solution versus the closed-form expression.

In Figure 4, we fix $T = 3600$ seconds as an approximation to $T \to \infty$, and observe that the optimal quotes $\delta^{a*}(0, q)$ from the full solver nearly coincide with the asymptotic formula in [4]. Figure 5 sets $\mu = \sigma = 0$, yielding a purely deterministic price path; the general solver again aligns with the analytical solution for $\delta^{a*}(0, q)$. Finally, Figure 6 shows the effect of letting $b \to +\infty$, where the incentive to avoid any leftover inventory dominates the strategy. These plots confirm that our numerical implementation accurately captures the model's behavior in these limit scenarios, matching the special-case closed formulas.

4.4 Sensitivity Analysis

In this subsection, we present the sensitivity of the optimal ask quote $\delta^{a*}(0, q)$ with respect to various model parameters. For each parameter, the following figures illustrate how changes affect the quotes.

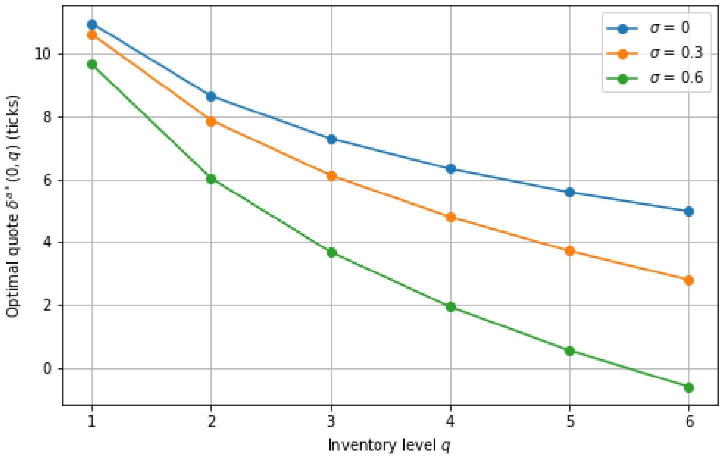

Figure 7: Sensitivity of the optimal quote with respect to σ.

Increasing σ amplifies the price risk, which according to the model (via the parameter $\alpha = \frac{1}{2}k\,\gamma\,\sigma^2$ in Equation (12)), forces the trader to lower the optimal quote $\delta^{a*}(0, q)$ to have a faster liquidation.

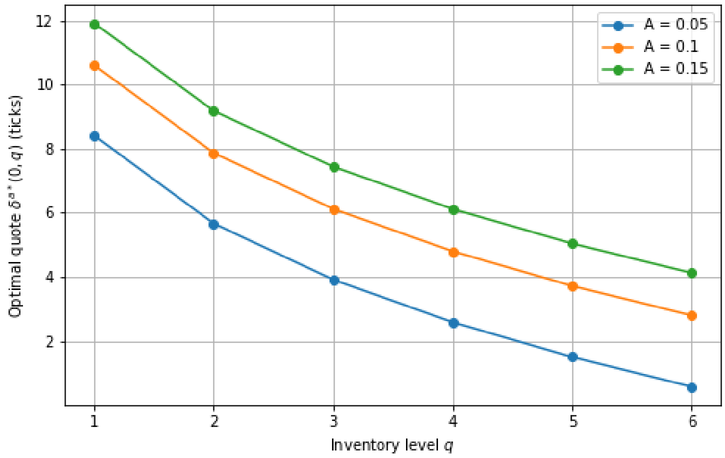

Figure 8: Sensitivity of the optimal quote with respect to A.

An increase in the intensity scale A raises the execution likelihood at a given quote, making the trader post higher quotes. This effect is clearly visible in Figure 8, in line with the formulation of the execution intensity $\lambda^a(\delta) = A \exp(-k\,\delta)$.

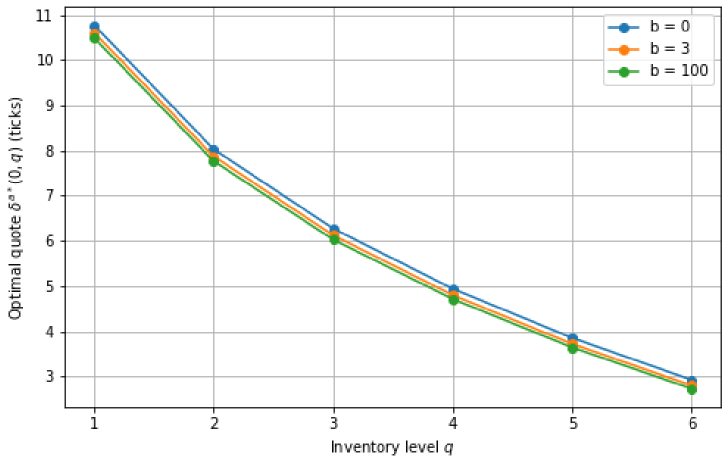

Figure 9: Sensitivity of the optimal quote with respect to b.

Figure 9 shows that as the liquidation cost b increases, the trader is incentivized to lower the quote to reduce the penalty for unsold inventory. This behavior, however, is not as clear as in other cases (the differences in quotes are not as high as in other cases), and therefore it will require some further analysis that we will carry out in Section 4.5.

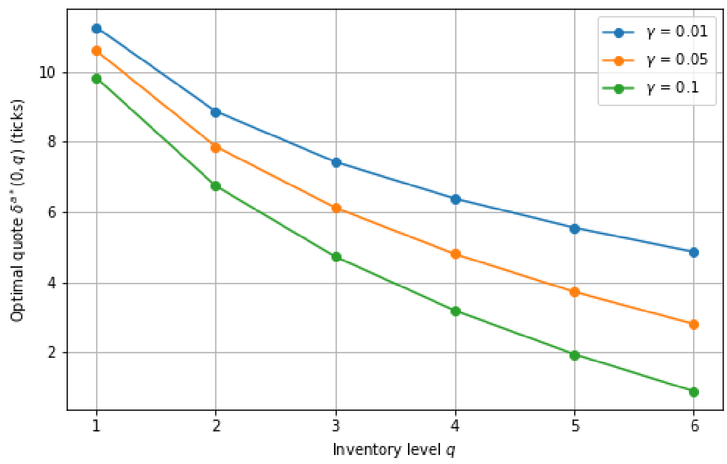

Figure 10: Sensitivity of the optimal quote with respect to γ.

As depicted in Figure 10, a higher risk aversion γ leads to a lower optimal quote. This result follows from the fact that increased risk aversion intensifies both the price risk and non-execution risk, as reflected in the optimal quote formula (see Equation (13)) and the parameter dependencies outlined in [4].

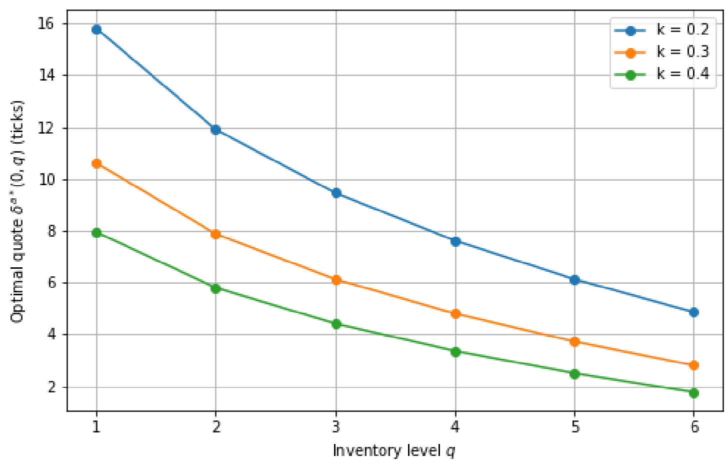

Figure 11: Sensitivity of the optimal quote with respect to k.

Figure 11 illustrates that an increase in the parameter k (which governs the exponential decay of the execution intensity) generally reduces the optimal quote. This is expected because a higher k means that the probability of execution drops faster with an increase in δ, thereby necessitating a lower quote to maintain a sufficient execution probability.

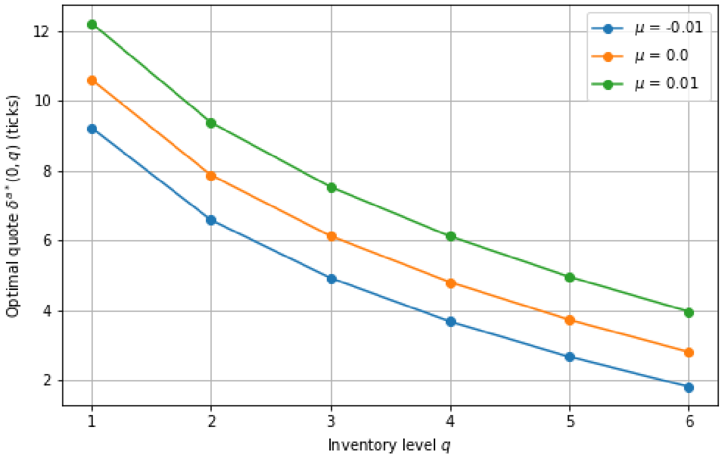

Figure 12: Sensitivity of the optimal quote with respect to μ.

Finally, Figure 12 demonstrates that an increase in the drift μ leads the trader to post higher quotes. A positive drift implies an expected upward movement in the reference price, so the trader may afford to be less aggressive in liquidating, as captured by the term $\beta = k\,\mu$ in Equation (12).

4.5 Impact of the Liquidation Cost b on Final Inventory

We include an additional experiment not covered in [4] to illustrate how the liquidation cost b influences the final average inventory. In particular, we run Monte Carlo simulations of the optimal strategy for various values of b, and we record the average inventory remaining at the terminal time T. As we saw in Section 4.4, since b appears in the terminal condition $w_q(T) = e^{-kqb}$, an increase in b imposes a higher penalty for leftover shares, which motivates the trader to liquidate more aggressively. Conversely, when b is small, the penalty for holding inventory is less severe, and the final average inventory is higher (with an apparent limit close to $\mathbb{E}[q_T]_{b=0} \approx 0.28$).

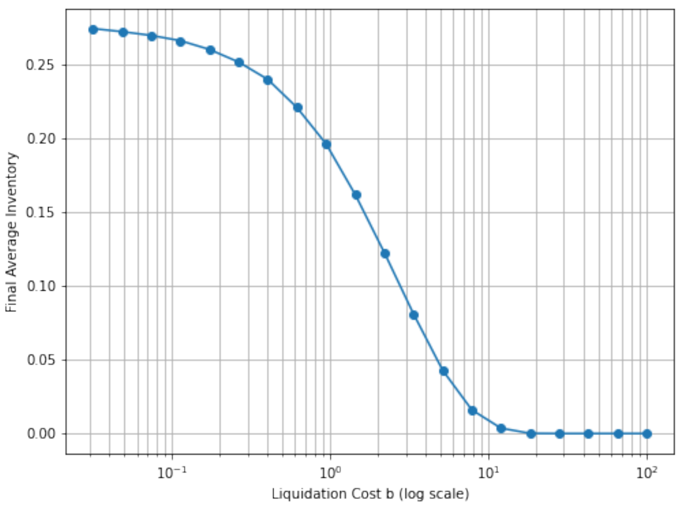

Figure 13: Final average inventory as a function of the liquidation cost b.

In Figure 13 we observe how, for larger values of b, the terminal inventory tends to 0, a behavior that is expected. With these results, we conclude that even though in Section 4.4 we observed that the differences in quotes for varying values of b were not high, this small difference leads to a final inventory that converges to 0.

4.6 Comparison with Benchmark Strategies

Finally, and in addition to the main results of [4], we consider three alternative strategies to benchmark the "optimal" approach derived from Equation (13): a *safe* strategy that posts lower quotes tied to the current inventory, a *greedy* strategy that fixes a constant (higher) quote regardless of inventory, and a *random* strategy that samples quotes uniformly from a predefined range. We then simulate the trading curve (average inventory over time) and the distribution of final cash for each strategy, holding all other model parameters the same.

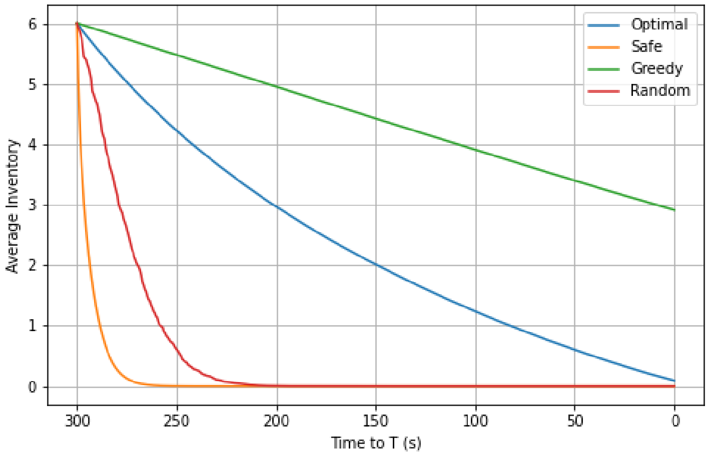

Figure 14: Trading curve for the four strategies (optimal, safe, greedy, and random).

Figure 14 shows the average remaining inventory over time for an initial inventory of six units, under a 5-minute liquidation horizon. The *safe* strategy, by posting the most aggressive (lowest) quotes, achieves the quickest liquidation. Conversely, the *greedy* strategy, which insists on high quotes, gets rid of its inventory at a much slower rate. The *random* approach displays an intermediate path, neither as quick as *safe* nor as slow as *greedy*. Meanwhile, the *optimal* strategy adaptively accelerates or moderates liquidation according to inventory and time, achieving a balanced trade-off between speed and price.

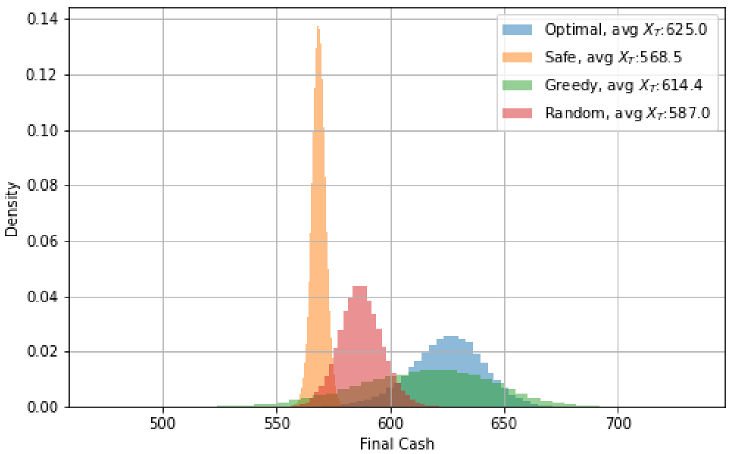

Figure 15: Distribution of final cash for the four strategies (optimal, safe, greedy, and random).

In terms of generated cash, Figure 15 presents the final cash distribution for each strategy. The *optimal* method achieves a higher average final cash, while the *safe* and *random* approaches exhibit lower distributions and average final cash, according to what is expected. Surprinsingly, the *greedy* appraoch does not achieve a higher final average cash, which is something that could be expected. This can be reasoned since, as seen in Figure 14, the average final inventory of this strategy is significantly larger than 0. This makes the trader to not be able to sell all of its inventory, and therefore preventing him from obtaining the corresponding cash.

In Figure 15 we also observe that some strategies have wider cash distributions than others. Specifically, we see that the *faster* strategies have thinner distributions than the *slower* ones. This comes from the fact that *faster* strategies get rid of the inventory more quickly, therefore limiting their exposure to price fluctuations and leading to a narrower range of possible final cash outcomes.

5 Conclusions

In this project, we have successfully replicated the key theoretical and numerical results of [4] using our backward Euler scheme to solve the ODE system arising from the HJB formulation. Our numerical experiments confirm that the optimal ask quotes, trading curves, and limit-case behaviors (such as the asymptotic regime $T \to \infty$, the no-drift/no-volatility case, and the high liquidation cost scenario) align closely with the analytical expressions provided in the original paper. In particular, the behavior of $\delta^{a*}(t, q)$ as detailed in Equation (13) is consistently observed across various time horizons and parameter regimes.

Furthermore, our extended analysis—including a detailed study of the impact of the liquidation cost b in the average final inventory, and a comparison with benchmark strategies (safe, greedy, and random)—provides additional insight into the model's practical performance. The results indicate that strategies which liquidate inventory more quickly tend to exhibit narrower cash distributions, due to their reduced exposure to market fluctuations, whereas slower strategies show a broader range of outcomes. Overall, the optimal strategy adaptively balances execution speed and price risk, achieving superior final cash outcomes compared to simpler heuristics.

There remain several promising directions for further research. One natural extension is to incorporate market orders into the framework, allowing the trader to more efficiently manage inventory when immediate execution is required. Such an approach could enable a hybrid strategy that combines the benefits of both limit and market orders. Additionally, the current model assumes that only one share is traded per execution; relaxing this assumption to allow for multi-share trades at each decision epoch would more accurately reflect the realities of high-volume trading.

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