Convex functions

A function $f(x)$ is *convex* if for any real numbers $a < b$, each point (c, d) on the line segment joining $(a, f(a))$ and $(b, f(b))$ lies above or at the point $(c, f(c))$ on the graph of f with the same x -coordinate.

Algebraically, this condition says that

$$f((1-t)a+tb) \leq (1-t)f(a)+tf(b). \tag{1}$$

whenever $a < b$ and for all $t \in [0,1]$. (The left hand side represents the height of the graph of the function above the x-value $x = (1-t)a + tb$ which is a fraction t of the way from a to b, and the right hand side represents the height of the line segment above the same x-value.)

Those who know what a convex set in geometry is can interpret the condition as saying that the set $S = \{(x,y) : y \geq f(x)\}$ of points above the graph of f is a convex set. Loosely speaking, this will hold if the graph of f curves in the shape of a smile instead of a frown. For example, the function $f(x) = x^2$ is convex, as is $f(x) = x^n$ for any positive *even* integer.

One can also speak of a function $f(x)$ being convex *on an interval* I. This means that the condition (1) above holds at least when $a, b \in I$ (and $a < b$ and $t \in [0,1]$). For example, one can show that $f(x) = x^3$ is convex on $[0, \infty)$, and that $f(x) = \sin x$ is convex on $(-\pi, 0)$.

Finally one says that a function $f(x)$ on an interval I is *strictly convex*, if

$$f((1-t)a+tb) < (1-t)f(a)+tf(b)$$

whenever $a, b \in I$ and $a < b$ and $t \in (0,1)$. In other words, the line segment connecting two points on the graph of f should lie entirely above the graph of f, except where it touches at its endpoints.

For convenience, here is a brief list of some convex functions. In these, k represents a positive integer, r, s represent real constants, and x is the variable. In fact, all of these are strictly convex on the interval given, except for x^r and $-x^r$ when r is 0 or 1.

$$x^{2k}\text{, on all of } \mathbb{R}$$

$$x^r\text{, on } [0,\infty)\text{, if } r \geq 1$$

$$-x^r\text{, on } [0,\infty)\text{, if } r \in [0,1]$$

$$x^r\text{, on } (0,\infty)\text{, if } r \leq 0$$

$$-\log x\text{, on } (0,\infty)$$

$$-\sin x\text{, on } [0,\pi]$$

$$-\cos x\text{, on } [-\pi/2,\pi/2]$$

$$\tan x\text{, on } [0,\pi/2)$$

$$e^x\text{, on all of } \mathbb{R}$$

$$r/(s+x)\text{ on } (-s,\infty)\text{, if } r > 0$$

A sum of convex functions is convex. Adding a constant or linear function to a function does not affect convexity.

Remarks (for those who know about continuity and derivatives):
If one wants to prove rigorously that a function is convex, instead of just guessing it from the graph, it is often easier to use one of the criteria below instead of the definition of convexity.

1. Let $f(x)$ be a continuous function on an interval I. Then $f(x)$ is convex if and only if $(f(a)+f(b))/2 \geq f((a+b)/2)$ holds for all $a, b \in I$. Also, $f(x)$ is strictly convex if and only if $(f(a)+f(b))/2 > f((a+b)/2)$ whenever $a, b \in I$ and $a < b$.

2. Let $f(x)$ be a differentiable function on an interval I. Then $f(x)$ is convex if and only if $f'(x)$ is increasing on I. Also, $f(x)$ is strictly convex if and only if $f'(x)$ is strictly increasing on the interior of I.

3. Let $f(x)$ be a twice differentiable function on an interval I. Then $f(x)$ is convex if and only if $f''(x) \geq 0$ for all $x \in I$. Also, $f(x)$ is strictly convex if and only if $f''(x) > 0$ for all x in the interior of I.

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