

Ecuaciones Diferenciales - Variación de Parámetros

1. $y'' + y = \sec x$

Ec. característica: $r^2 + 1 = 0 \Rightarrow r = \pm i$

$y_c = c_1 \cos x + c_2 \sin x$

$y_1 = \cos x, y_2 = \sin x$

$y'_1 = -\sin x, y'_2 = \cos x$

$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$

$W_1 = \begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix} = 0 \cdot \cos x - \sec x \cdot \sin x = -\tan x$

$W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix} = \cos x \cdot \sec x - 0 \cdot (-\sin x) = 1$

$u_1 = \int \frac{W_1}{W} dx = \int \frac{-\tan x}{1} dx = \int (-\tan x) dx = \ln |\cos x|$

$u_2 = \int \frac{W_2}{W} dx = \int \frac{1}{1} dx = \int 1 dx = x$

$y_p = u_1 y_1 + u_2 y_2 = \ln |\cos x| \cdot \cos x + x \cdot \sin x$

$y = c_1 \cos x + c_2 \sin x + \cos x \ln |\cos x| + x \sin x$

2. $y'' + y = \tan x$

Ec. característica: $r^2 + 1 = 0 \Rightarrow r = \pm i$

$y_c = c_1 \cos x + c_2 \sin x$

$y_1 = \cos x, y_2 = \sin x$

$y'_1 = -\sin x, y'_2 = \cos x$

$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$

$W_1 = \begin{vmatrix} 0 & \sin x \\ \tan x & \cos x \end{vmatrix} = 0 - \tan x \cdot \sin x = -\sin x \tan x = -\frac{\sin^2 x}{\cos x}$

$W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \tan x \end{vmatrix} = \cos x \tan x - 0 = \sin x$

$u_1 = \int \frac{-\sin^2 x}{\cos x} dx = \int \frac{-(1-\cos^2 x)}{\cos x} dx = \int \left(-\frac{1}{\cos x} + \cos x\right) dx$

$u_1 = \int (-\sec x + \cos x) dx = -\ln |\sec x + \tan x| + \sin x$

$u_2 = \int \sin x dx = -\cos x$

$y_p = u_1 y_1 + u_2 y_2 = (-\ln |\sec x + \tan x| + \sin x) \cos x + (-\cos x) \sin x$

$y_p = -\cos x \ln |\sec x + \tan x| + \sin x \cos x - \cos x \sin x = -\cos x \ln |\sec x + \tan x|$

$y = c_1 \cos x + c_2 \sin x - \cos x \ln |\sec x + \tan x|$

3. $y'' + y = \sec x$

Ec. característica: $r^2 + 1 = 0 \Rightarrow r = \pm i$

$y_c = c_1 \cos x + c_2 \sin x$

$y_1 = \cos x, y_2 = \sin x$

$y'_1 = -\sin x, y'_2 = \cos x$

$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$

$W_1 = \begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix} = 0 \cdot \cos x - \sec x \cdot \sin x = -\tan x$

$W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix} = \cos x \cdot \sec x - 0 = 1$

$u_1 = \int \frac{-\tan x}{1} dx = \int (-\tan x) dx = \ln |\cos x|$

$u_2 = \int \frac{1}{1} dx = x$

$y_p = u_1 y_1 + u_2 y_2 = \ln |\cos x| \cdot \cos x + x \cdot \sin x$

$y = c_1 \cos x + c_2 \sin x + \cos x \ln |\cos x| + x \sin x$

4. $y'' + y = \sec \theta \tan \theta$

Ec. característica: $r^2 + 1 = 0 \Rightarrow r = \pm i$

$y_c = c_1 \cos \theta + c_2 \sin \theta$

$y_1 = \cos \theta, y_2 = \sin \theta$

$y'_1 = -\sin \theta, y'_2 = \cos \theta$

$W = \begin{vmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{vmatrix} = 1$

$W_1 = \begin{vmatrix} 0 & \sin \theta \\ \sec \theta \tan \theta & \cos \theta \end{vmatrix} = -\sec \theta \tan \theta \cdot \sin \theta = -\tan^2 \theta$

$W_2 = \begin{vmatrix} \cos \theta & 0 \\ -\sin \theta & \sec \theta \tan \theta \end{vmatrix} = \cos \theta \sec \theta \tan \theta = \tan \theta$

$u_1 = \int (-\tan^2 \theta) d\theta = \int (-\sec^2 \theta + 1) d\theta = -\tan \theta + \theta$

$u_2 = \int \tan \theta d\theta = -\ln |\cos \theta|$

$y_p = u_1 y_1 + u_2 y_2 = (-\tan \theta + \theta) \cos \theta + (-\ln |\cos \theta|) \sin \theta$

$y_p = -\sin \theta + \theta \cos \theta - \sin \theta \ln |\cos \theta|$

$y = c_1 \cos \theta + c_2 \sin \theta - \sin \theta + \theta \cos \theta - \sin \theta \ln |\cos \theta|$

5. $y'' + y = \cos^2 x$

Ec. característica: $r^2 + 1 = 0 \Rightarrow r = \pm i$

$y_c = c_1 \cos x + c_2 \sin x$

$y_1 = \cos x, y_2 = \sin x$

$y'_1 = -\sin x, y'_2 = \cos x$

$W = 1$

Identidad: $\cos^2 x = \frac{1+\cos 2x}{2}$

$W_1 = \begin{vmatrix} 0 & \sin x \\ \frac{1+\cos 2x}{2} & \cos x \end{vmatrix} = -\frac{(1+\cos 2x) \sin x}{2}$

$W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \frac{1+\cos 2x}{2} \end{vmatrix} = \frac{(1+\cos 2x) \cos x}{2}$

$u_1 = \int \frac{-(1+\cos 2x) \sin x}{2} dx = \int \left(-\frac{\sin x}{2} - \frac{\sin x \cos 2x}{2}\right) dx$

Para $\int \sin x \cos 2x dx$: usar $\cos 2x = 1 - 2 \sin^2 x$

$\int \sin x \cos 2x dx = \int \sin x (1 - 2 \sin^2 x) dx = \int (\sin x - 2 \sin^3 x) dx$

$= -\cos x + 2 \left(\frac{\cos^3 x}{3} - \cos x\right) = -\cos x + \frac{2 \cos^3 x}{3} - 2 \cos x = \frac{2 \cos^3 x}{3} - 3 \cos x$

$u_1 = \frac{\cos x}{2} - \frac{1}{2} \left(\frac{2 \cos^3 x}{3} - 3 \cos x\right) = \frac{\cos x}{2} - \frac{\cos^3 x}{3} + \frac{3 \cos x}{2} = 2 \cos x - \frac{\cos^3 x}{3}$

$u_2 = \int \frac{(1+\cos 2x) \cos x}{2} dx = \int \left(\frac{\cos x}{2} + \frac{\cos x \cos 2x}{2}\right) dx$

$= \frac{\sin x}{2} + \frac{1}{2} \int \cos x \cos 2x dx$

$\int \cos x \cos 2x dx = \int \cos x (2 \cos^2 x - 1) dx = \int (2 \cos^3 x - \cos x) dx$

$= 2 \left(\frac{\sin^3 x}{3} + \sin x\right) - \sin x = \frac{2 \sin^3 x}{3} + 2 \sin x - \sin x = \frac{2 \sin^3 x}{3} + \sin x$

$u_2 = \frac{\sin x}{2} + \frac{1}{2} \left(\frac{2 \sin^3 x}{3} + \sin x\right) = \frac{\sin x}{2} + \frac{\sin^3 x}{3} + \frac{\sin x}{2} = \sin x + \frac{\sin^3 x}{3}$

$y_p = \left(2 \cos x - \frac{\cos^3 x}{3}\right) \cos x + \left(\sin x + \frac{\sin^3 x}{3}\right) \sin x$

$= 2 \cos^2 x - \frac{\cos^4 x}{3} + \sin^2 x + \frac{\sin^4 x}{3}$

$= 2 \cos^2 x + \sin^2 x - \frac{\cos^4 x - \sin^4 x}{3} = 2 \cos^2 x + \sin^2 x - \frac{(\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x)}{3}$

$= 2 \cos^2 x + \sin^2 x - \frac{\cos 2x}{3} = \cos^2 x + 1 - \frac{\cos 2x}{3}$

Simplificando: $y_p = \frac{1}{2} + \frac{\cos 2x}{2}$

$y = c_1 \cos x + c_2 \sin x + \frac{1}{2} + \frac{\cos 2x}{2}$

6. $y'' + y = \sec^2 x$

Ec. característica: $r^2 + 1 = 0 \Rightarrow r = \pm i$

$y_c = c_1 \cos x + c_2 \sin x$

$y_1 = \cos x, y_2 = \sin x$

$y'_1 = -\sin x, y'_2 = \cos x$

$W = 1$

$W_1 = \begin{vmatrix} 0 & \sin x \\ \sec^2 x & \cos x \end{vmatrix} = -\sec^2 x \sin x = -\tan x \sec x$

$W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \sec^2 x \end{vmatrix} = \cos x \sec^2 x = \sec x$

$u_1 = \int (-\tan x \sec x) dx = -\sec x$

$u_2 = \int \sec x dx = \ln |\sec x + \tan x|$

$y_p = u_1 y_1 + u_2 y_2 = (-\sec x) \cos x + \ln |\sec x + \tan x| \cdot \sin x$

$y_p = -1 + \sin x \ln |\sec x + \tan x|$

$y = c_1 \cos x + c_2 \sin x - 1 + \sin x \ln |\sec x + \tan x|$

7. $y'' - y = \cosh x$

Ec. característica: $r^2 - 1 = 0 \Rightarrow (r-1)(r+1) = 0 \Rightarrow r = 1, -1$

$y_c = c_1 e^x + c_2 e^{-x}$

$y_1 = e^x, y_2 = e^{-x}$

$y'_1 = e^x, y'_2 = -e^{-x}$

$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -e^x \cdot e^{-x} - e^x \cdot e^{-x} = -2$

$\cosh x = \frac{e^x + e^{-x}}{2}$

$W_1 = \begin{vmatrix} 0 & e^{-x} \\ \cosh x & -e^{-x} \end{vmatrix} = -e^{-x} \cosh x = -e^{-x} \cdot \frac{e^x + e^{-x}}{2} = -\frac{1+e^{-2x}}{2}$

$W_2 = \begin{vmatrix} e^x & 0 \\ e^x & \cosh x \end{vmatrix} = e^x \cosh x = e^x \cdot \frac{e^x + e^{-x}}{2} = \frac{e^{2x} + 1}{2}$

$u_1 = \int \frac{-\frac{1+e^{-2x}}{2}}{-2} dx = \int \frac{1+e^{-2x}}{4} dx = \frac{x}{4} + \int \frac{e^{-2x}}{4} dx = \frac{x}{4} - \frac{e^{-2x}}{8}$

$u_2 = \int \frac{\frac{e^{2x} + 1}{2}}{-2} dx = -\int \frac{e^{2x} + 1}{4} dx = -\frac{e^{2x}}{8} - \frac{x}{4}$

$y_p = u_1 y_1 + u_2 y_2 = \left(\frac{x}{4} - \frac{e^{-2x}}{8}\right) e^x + \left(-\frac{e^{2x}}{8} - \frac{x}{4}\right) e^{-x}$

$= \frac{x e^x}{4} - \frac{e^{-x}}{8} - \frac{e^x}{8} - \frac{x e^{-x}}{4}$

$= \frac{x(e^x - e^{-x})}{4} - \frac{e^x + e^{-x}}{8} = \frac{x \cdot 2 \sinh x}{4} - \frac{2 \cosh x}{8}$

$= \frac{x \sinh x}{2} - \frac{\cosh x}{4}$

$$y = c_1 e^x + c_2 e^{-x} + \frac{x \sinh x}{2} - \frac{\cosh x}{4}$$

$$\mathbf{8.} \quad y'' - y = \sinh 2x$$

$$\text{Ec. característica: } r^2 - 1 = 0 \Rightarrow r = 1, -1$$

$$y_c = c_1 e^x + c_2 e^{-x}$$

$$y_1 = e^x, y_2 = e^{-x}$$

$$y'_1 = e^x, y'_2 = -e^{-x}$$

$$W = -2$$

$$\sinh 2x = \frac{e^{2x} - e^{-2x}}{2}$$

$$W_1 = \begin{vmatrix} 0 & e^{-x} \\ \sinh 2x & -e^{-x} \end{vmatrix} = -e^{-x} \sinh 2x = -e^{-x} \cdot \frac{e^{2x} - e^{-2x}}{2} = -\frac{e^x - e^{-3x}}{2}$$

$$W_2 = \begin{vmatrix} e^x & 0 \\ e^x & \sinh 2x \end{vmatrix} = e^x \sinh 2x = e^x \cdot \frac{e^{2x} - e^{-2x}}{2} = \frac{e^{3x} - e^{-x}}{2}$$

$$u_1 = \int \frac{-\frac{e^x - e^{-3x}}{2}}{-2} dx = \int \frac{e^x - e^{-3x}}{4} dx = \frac{e^x}{4} - \int \frac{e^{-3x}}{4} dx = \frac{e^x}{4} + \frac{e^{-3x}}{12}$$

$$u_2 = \int \frac{\frac{e^{3x} - e^{-x}}{2}}{-2} dx = -\int \frac{e^{3x} - e^{-x}}{4} dx = -\frac{e^{3x}}{12} + \int \frac{e^{-x}}{4} dx = -\frac{e^{3x}}{12} - \frac{e^{-x}}{4}$$

$$y_p = \left(\frac{e^x}{4} + \frac{e^{-3x}}{12} \right) e^x + \left(-\frac{e^{3x}}{12} - \frac{e^{-x}}{4} \right) e^{-x}$$

$$= \frac{e^{2x}}{4} + \frac{e^{-2x}}{12} - \frac{e^{2x}}{12} - \frac{e^{-2x}}{4}$$

$$= \frac{3e^{2x} - e^{-2x}}{12} + \frac{e^{-2x} - 3e^{-2x}}{12} = \frac{2e^{2x} - 2e^{-2x}}{12} = \frac{e^{2x} - e^{-2x}}{6}$$

$$= \frac{2 \sinh 2x}{6} = \frac{\sinh 2x}{3}$$

$$y = c_1 e^x + c_2 e^{-x} + \frac{\sinh 2x}{3}$$

$$\mathbf{9.} \quad y'' - 4y = \frac{e^{2x}}{x}$$

$$\text{Ec. característica: } r^2 - 4 = 0 \Rightarrow (r - 2)(r + 2) = 0 \Rightarrow r = 2, -2$$

$$y_c = c_1 e^{2x} + c_2 e^{-2x}$$

$$y_1 = e^{2x}, y_2 = e^{-2x}$$

$$y'_1 = 2e^{2x}, y'_2 = -2e^{-2x}$$

$$W = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -2e^{2x}e^{-2x} - 2e^{2x}e^{-2x} = -4$$

$$W_1 = \begin{vmatrix} 0 & e^{-2x} \\ \frac{e^{2x}}{x} & -2e^{-2x} \end{vmatrix} = -\frac{e^{2x}e^{-2x}}{x} = -\frac{1}{x}$$

$$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & \frac{e^{2x}}{x} \end{vmatrix} = \frac{e^{2x} \cdot e^{2x}}{x} = \frac{e^{4x}}{x}$$

$$u_1 = \int \frac{-\frac{1}{x}}{-4} dx = \int \frac{1}{4x} dx = \frac{\ln|x|}{4}$$

$$u_2 = \int \frac{\frac{e^{4x}}{x}}{-4} dx = -\frac{1}{4} \int \frac{e^{4x}}{x} dx \text{ (integral no elemental)}$$

$$y_p = \frac{\ln|x|}{4} \cdot e^{2x} - \frac{e^{-2x}}{4} \int \frac{e^{4x}}{x} dx$$

$$y = c_1 e^{2x} + c_2 e^{-2x} + \frac{e^{2x} \ln|x|}{4} - \frac{e^{-2x}}{4} \int \frac{e^{4x}}{x} dx$$

$$\mathbf{10.} \quad y'' - 9y = \frac{9x}{e^{2x}} = 9xe^{-2x}$$

$$\text{Ec. característica: } r^2 - 9 = 0 \Rightarrow (r - 3)(r + 3) = 0 \Rightarrow r = 3, -3$$

$$y_c = c_1 e^{3x} + c_2 e^{-3x}$$

$$y_1 = e^{3x}, y_2 = e^{-3x}$$

$$y'_1 = 3e^{3x}, y'_2 = -3e^{-3x}$$

$$W = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -3e^{3x}e^{-3x} - 3e^{3x}e^{-3x} = -6$$

$$W_1 = \begin{vmatrix} 0 & e^{-3x} \\ 9xe^{-2x} & -3e^{-3x} \end{vmatrix} = -9xe^{-2x}e^{-3x} = -9xe^{-5x}$$

$$W_2 = \begin{vmatrix} e^{3x} & 0 \\ 3e^{3x} & 9xe^{-2x} \end{vmatrix} = 9xe^{-2x}e^{3x} = 9xe^x$$

$$u_1 = \int \frac{-9xe^{-5x}}{-6} dx = \frac{3}{2} \int xe^{-5x} dx$$

$$\text{Por partes: } \int xe^{-5x} dx = -\frac{xe^{-5x}}{5} - \int -\frac{e^{-5x}}{5} dx = -\frac{xe^{-5x}}{5} - \frac{e^{-5x}}{25}$$

$$u_1 = \frac{3}{2} \left(-\frac{xe^{-5x}}{5} - \frac{e^{-5x}}{25} \right) = -\frac{3xe^{-5x}}{10} - \frac{3e^{-5x}}{50}$$

$$u_2 = \int \frac{9xe^x}{-6} dx = -\frac{3}{2} \int xe^x dx$$

$$\text{Por partes: } \int xe^x dx = xe^x - e^x$$

$$u_2 = -\frac{3}{2}(xe^x - e^x) = -\frac{3xe^x}{2} + \frac{3e^x}{2}$$

$$y_p = \left(-\frac{3xe^{-5x}}{10} - \frac{3e^{-5x}}{50} \right) e^{3x} + \left(-\frac{3xe^x}{2} + \frac{3e^x}{2} \right) e^{-3x}$$

$$= -\frac{3xe^{-2x}}{10} - \frac{3e^{-5x}}{50} - \frac{3xe^{-2x}}{2} + \frac{3e^{-2x}}{2}$$

$$= xe^{-2x} \left(-\frac{3}{10} - \frac{3}{2} \right) + e^{-2x} \left(-\frac{3}{50} + \frac{3}{2} \right)$$

$$= xe^{-2x} \left(-\frac{3}{10} - \frac{15}{10} \right) + e^{-2x} \left(-\frac{3}{50} + \frac{75}{50} \right)$$

$$= -\frac{18xe^{-2x}}{10} + \frac{72e^{-2x}}{50} = -\frac{9xe^{-2x}}{5} + \frac{36e^{-2x}}{25}$$

$$y = c_1 e^{3x} + c_2 e^{-3x} - \frac{9xe^{-2x}}{5} + \frac{36e^{-2x}}{25}$$

$$\mathbf{11.} \quad y'' + 3y' + 2y = \frac{1}{1+e^x}$$

$$\text{Ec. característica: } r^2 + 3r + 2 = 0 \Rightarrow (r + 1)(r + 2) = 0 \Rightarrow r = -1, -2$$

$$y_c = c_1 e^{-x} + c_2 e^{-2x}$$

$$y_1 = e^{-x}, y_2 = e^{-2x}$$

$$y'_1 = -e^{-x}, y'_2 = -2e^{-2x}$$

$$W = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -2e^{-3x} + e^{-3x} = -e^{-3x}$$

$$W_1 = \begin{vmatrix} 0 & e^{-2x} \\ \frac{1}{1+e^x} & -2e^{-2x} \end{vmatrix} = -\frac{e^{-2x}}{1+e^x}$$

$$W_2 = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & \frac{1}{1+e^x} \end{vmatrix} = \frac{e^{-x}}{1+e^x}$$

$$u_1 = \int \frac{-\frac{e^{-2x}}{1+e^x}}{-e^{-3x}} dx = \int \frac{e^{-2x}}{e^{-3x}(1+e^x)} dx = \int \frac{e^x}{1+e^x} dx$$

$$\text{Sea } u = 1 + e^x, du = e^x dx$$

$$u_1 = \int \frac{du}{u} = \ln|u| = \ln(1 + e^x)$$

$$u_2 = \int \frac{\frac{e^{-x}}{1+e^x}}{-e^{-3x}} dx = -\int \frac{e^{-x}}{e^{-3x}(1+e^x)} dx = -\int \frac{e^{2x}}{1+e^x} dx$$

$$= -\int \frac{e^x(1+e^x) - e^x}{1+e^x} dx = -\int \left(e^x - \frac{e^x}{1+e^x} \right) dx$$

$$= -e^x + \ln(1 + e^x)$$

$$y_p = \ln(1 + e^x) \cdot e^{-x} + (-e^x + \ln(1 + e^x))e^{-2x}$$

$$= e^{-x} \ln(1 + e^x) - e^{-x} + e^{-2x} \ln(1 + e^x)$$

$$y = c_1 e^{-x} + c_2 e^{-2x} + e^{-x} \ln(1 + e^x) - e^{-x} + e^{-2x} \ln(1 + e^x)$$

$$\mathbf{12.} \quad y'' - 2y' + y = \frac{e^x}{1+x^2}$$

$$\text{Ec. característica: } r^2 - 2r + 1 = 0 \Rightarrow (r - 1)^2 = 0 \Rightarrow r = 1 \text{ (doble)}$$

$$y_c = c_1 e^x + c_2 x e^x$$

$$y_1 = e^x, y_2 = x e^x$$

$$y'_1 = e^x, y'_2 = e^x + x e^x$$

$$W = \begin{vmatrix} e^x & x e^x \\ e^x & e^x + x e^x \end{vmatrix} = e^x(e^x + x e^x) - x e^x \cdot e^x = e^{2x} + x e^{2x} - x e^{2x} = e^{2x}$$

$$W_1 = \begin{vmatrix} 0 & x e^x \\ \frac{e^x}{1+x^2} & e^x + x e^x \end{vmatrix} = -\frac{x e^x \cdot e^x}{1+x^2} = -\frac{x e^{2x}}{1+x^2}$$

$$W_2 = \begin{vmatrix} e^x & 0 \\ e^x & \frac{e^x}{1+x^2} \end{vmatrix} = \frac{e^x \cdot e^x}{1+x^2} = \frac{e^{2x}}{1+x^2}$$

$$u_1 = \int \frac{-\frac{x e^{2x}}{1+x^2}}{e^{2x}} dx = -\int \frac{x}{1+x^2} dx = -\frac{1}{2} \ln(1 + x^2)$$

$$u_2 = \int \frac{\frac{e^{2x}}{1+x^2}}{e^{2x}} dx = \int \frac{1}{1+x^2} dx = \arctan x$$

$$y_p = -\frac{\ln(1+x^2)}{2} \cdot e^x + \arctan x \cdot x e^x$$

$$y = c_1 e^x + c_2 x e^x - \frac{e^x \ln(1+x^2)}{2} + x e^x \arctan x$$

$$\mathbf{13.} \quad y'' + 3y' + 2y = \sec e^x$$

$$\text{Ec. característica: } r^2 + 3r + 2 = 0 \Rightarrow (r + 1)(r + 2) = 0 \Rightarrow r = -1, -2$$

$$y_c = c_1 e^{-x} + c_2 e^{-2x}$$

$$y_1 = e^{-x}, y_2 = e^{-2x}$$

$$y'_1 = -e^{-x}, y'_2 = -2e^{-2x}$$

$$W = -e^{-3x}$$

$$W_1 = \begin{vmatrix} 0 & e^{-2x} \\ \sec e^x & -2e^{-2x} \end{vmatrix} = -e^{-2x} \sec e^x$$

$$W_2 = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & \sec e^x \end{vmatrix} = e^{-x} \sec e^x$$

$$u_1 = \int \frac{-e^{-2x} \sec e^x}{-e^{-3x}} dx = \int e^x \sec e^x dx$$

$$\text{Sea } u = e^x, du = e^x dx$$

$$u_1 = \int \sec u du = \ln|\sec e^x + \tan e^x|$$

$$u_2 = \int \frac{e^{-x} \sec e^x}{-e^{-3x}} dx = -\int e^{2x} \sec e^x dx \text{ (integral no elemental)}$$

$$y_p = \ln|\sec e^x + \tan e^x| \cdot e^{-x} - e^{-2x} \int e^{2x} \sec e^x dx$$

$$y = c_1 e^{-x} + c_2 e^{-2x} + e^{-x} \ln|\sec e^x + \tan e^x| - e^{-2x} \int e^{2x} \sec e^x dx$$

$$\mathbf{14.} \quad y'' - 2y' + y = e^t \arctan t$$

$$\text{Ec. característica: } r^2 - 2r + 1 = 0 \Rightarrow (r - 1)^2 = 0 \Rightarrow r = 1 \text{ (doble)}$$

$$y_c = c_1 e^t + c_2 t e^t$$

$$y_1 = e^t, y_2 = t e^t$$

$$y'_1 = e^t, y'_2 = e^t + t e^t$$

$$W = e^{2t}$$

$$W_1 = \begin{vmatrix} 0 & t e^t \\ e^t \arctan t & e^t + t e^t \end{vmatrix} = -t e^t \cdot e^t \arctan t = -t e^{2t} \arctan t$$

$$W_2 = \begin{vmatrix} e^t & 0 \\ e^t & e^t \arctan t \end{vmatrix} = e^t \cdot e^t \arctan t = e^{2t} \arctan t$$

$$u_1 = \int \frac{-t e^{2t} \arctan t}{e^{2t}} dt = -\int t \arctan t dt$$

$$\text{Por partes: sea } u = \arctan t, dv = t dt, du = \frac{dt}{1+t^2}, v = \frac{t^2}{2}$$

$$\int t \arctan t dt = \frac{t^2 \arctan t}{2} - \int \frac{t^2}{2(1+t^2)} dt = \frac{t^2 \arctan t}{2} - \frac{1}{2} \int \frac{t^2}{1+t^2} dt$$

$$= \frac{t^2 \arctan t}{2} - \frac{1}{2} \int \left(1 - \frac{1}{1+t^2} \right) dt = \frac{t^2 \arctan t}{2} - \frac{1}{2} (t - \arctan t)$$

$$= \frac{t^2 \arctan t}{2} - \frac{t}{2} + \frac{\arctan t}{2}$$

$$u_1 = -\frac{t^2 \arctan t}{2} + \frac{t}{2} - \frac{\arctan t}{2}$$

$$u_2 = \int \frac{e^{2t} \arctan t}{e^{2t}} dt = \int \arctan t dt$$

Por partes: $u = \arctan t$, $dv = dt$, $du = \frac{dt}{1+t^2}$, $v = t$

$$\int \arctan t \, dt = t \arctan t - \int \frac{t}{1+t^2} dt = t \arctan t - \frac{1}{2} \ln(1+t^2)$$

$$u_2 = t \arctan t - \frac{\ln(1+t^2)}{2}$$

$$y_p = \left(-\frac{t^2 \arctan t}{2} + \frac{t}{2} - \frac{\arctan t}{2} \right) e^t + \left(t \arctan t - \frac{\ln(1+t^2)}{2} \right) t e^t$$

$$y = c_1 e^t + c_2 t e^t + e^t \left(-\frac{t^2 \arctan t}{2} + \frac{t}{2} - \frac{\arctan t}{2} \right) + t e^t \left(t \arctan t - \frac{\ln(1+t^2)}{2} \right)$$

$$\mathbf{15.} \quad y'' + 2y' + y = e^{-t} \ln t$$

Ec. característica: $r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0 \Rightarrow r = -1$ (doble)

$$y_c = c_1 e^{-t} + c_2 t e^{-t}$$

$$y_1 = e^{-t}, y_2 = t e^{-t}$$

$$y'_1 = -e^{-t}, y'_2 = e^{-t} - t e^{-t}$$

$$W = \begin{vmatrix} e^{-t} & t e^{-t} \\ -e^{-t} & e^{-t} - t e^{-t} \end{vmatrix} = e^{-t}(e^{-t} - t e^{-t}) + t e^{-t} e^{-t} = e^{-2t} - t e^{-2t} + t e^{-2t} = e^{-2t}$$

$$W_1 = \begin{vmatrix} 0 & t e^{-t} \\ e^{-t} \ln t & e^{-t} - t e^{-t} \end{vmatrix} = -t e^{-t} \cdot e^{-t} \ln t = -t e^{-2t} \ln t$$

$$W_2 = \begin{vmatrix} e^{-t} & 0 \\ -e^{-t} & e^{-t} \ln t \end{vmatrix} = e^{-t} \cdot e^{-t} \ln t = e^{-2t} \ln t$$

$$u_1 = \int \frac{-t e^{-2t} \ln t}{e^{-2t}} dt = -\int t \ln t \, dt$$

Por partes: $u = \ln t$, $dv = t \, dt$, $du = \frac{dt}{t}$, $v = \frac{t^2}{2}$

$$\int t \ln t \, dt = \frac{t^2 \ln t}{2} - \int \frac{t^2}{2t} dt = \frac{t^2 \ln t}{2} - \int \frac{t}{2} dt = \frac{t^2 \ln t}{2} - \frac{t^2}{4}$$

$$u_1 = -\frac{t^2 \ln t}{2} + \frac{t^2}{4}$$

$$u_2 = \int \frac{e^{-2t} \ln t}{e^{-2t}} dt = \int \ln t \, dt$$

Por partes: $u = \ln t$, $dv = dt$, $du = \frac{dt}{t}$, $v = t$

$$\int \ln t \, dt = t \ln t - \int dt = t \ln t - t$$

$$u_2 = t \ln t - t$$

$$y_p = \left(-\frac{t^2 \ln t}{2} + \frac{t^2}{4} \right) e^{-t} + (t \ln t - t) t e^{-t}$$

$$= e^{-t} \left(-\frac{t^2 \ln t}{2} + \frac{t^2}{4} + t^2 \ln t - t^2 \right)$$

$$= e^{-t} \left(\frac{t^2 \ln t}{2} + \frac{t^2}{4} - t^2 \right) = e^{-t} \left(\frac{t^2 \ln t}{2} - \frac{3t^2}{4} \right)$$

$$y = c_1 e^{-t} + c_2 t e^{-t} + e^{-t} \left(\frac{t^2 \ln t}{2} - \frac{3t^2}{4} \right)$$

$$\mathbf{16.} \quad 2y'' + 2y' + y = \frac{4}{\sqrt{x}}$$

Ec. característica: $2r^2 + 2r + 1 = 0$

$$r = \frac{-2 \pm \sqrt{4-8}}{4} = \frac{-2 \pm 2i}{4} = \frac{-1 \pm i}{2}$$

$$\alpha = -\frac{1}{2}, \beta = \frac{1}{2}$$

$$y_c = e^{-x/2} \left(c_1 \cos \frac{x}{2} + c_2 \sin \frac{x}{2} \right)$$

$$y_1 = e^{-x/2} \cos \frac{x}{2}, y_2 = e^{-x/2} \sin \frac{x}{2}$$

$$y'_1 = e^{-x/2} \left(-\frac{1}{2} \cos \frac{x}{2} - \frac{1}{2} \sin \frac{x}{2} \right)$$

$$y'_2 = e^{-x/2} \left(-\frac{1}{2} \sin \frac{x}{2} + \frac{1}{2} \cos \frac{x}{2} \right)$$

$$W = y_1 y'_2 - y_2 y'_1$$

$$= e^{-x/2} \cos \frac{x}{2} \cdot e^{-x/2} \left(-\frac{1}{2} \sin \frac{x}{2} + \frac{1}{2} \cos \frac{x}{2} \right)$$

$$- e^{-x/2} \sin \frac{x}{2} \cdot e^{-x/2} \left(-\frac{1}{2} \cos \frac{x}{2} - \frac{1}{2} \sin \frac{x}{2} \right)$$

$$= e^{-x} \left[-\frac{1}{2} \cos \frac{x}{2} \sin \frac{x}{2} + \frac{1}{2} \cos^2 \frac{x}{2} + \frac{1}{2} \sin \frac{x}{2} \cos \frac{x}{2} + \frac{1}{2} \sin^2 \frac{x}{2} \right]$$

$$= \frac{e^{-x}}{2} (\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2}) = \frac{e^{-x}}{2}$$

Dividir ecuación por 2: $y'' + y' + \frac{y}{2} = \frac{2}{\sqrt{x}}$

$$W_1 = \begin{vmatrix} 0 & e^{-x/2} \sin \frac{x}{2} \\ \frac{2}{\sqrt{x}} & e^{-x/2} \left(-\frac{1}{2} \sin \frac{x}{2} + \frac{1}{2} \cos \frac{x}{2} \right) \end{vmatrix}$$

$$= -\frac{2e^{-x/2} \sin \frac{x}{2}}{\sqrt{x}}$$

$$W_2 = \begin{vmatrix} e^{-x/2} \cos \frac{x}{2} & 0 \\ e^{-x/2} \left(-\frac{1}{2} \cos \frac{x}{2} - \frac{1}{2} \sin \frac{x}{2} \right) & \frac{2}{\sqrt{x}} \end{vmatrix}$$

$$= \frac{2e^{-x/2} \cos \frac{x}{2}}{\sqrt{x}}$$

$$u_1 = \int \frac{-\frac{2e^{-x/2} \sin \frac{x}{2}}{\sqrt{x}}}{\frac{e^{-x}}{2}} dx = -\int \frac{4e^{x/2} \sin \frac{x}{2}}{\sqrt{x}} dx \text{ (integral no elemental)}$$

$$u_2 = \int \frac{\frac{2e^{-x/2} \cos \frac{x}{2}}{\sqrt{x}}}{\frac{e^{-x}}{2}} dx = \int \frac{4e^{x/2} \cos \frac{x}{2}}{\sqrt{x}} dx \text{ (integral no elemental)}$$

$$y = c_1 e^{-x/2} \cos \frac{x}{2} + c_2 e^{-x/2} \sin \frac{x}{2} + y_p$$

donde y_p requiere integrales no elementales

$$\mathbf{17.} \quad 3y'' - 6y' + 6y = e^x \sec x$$

Ec. característica: $3r^2 - 6r + 6 = 0 \Rightarrow r^2 - 2r + 2 = 0$

$$r = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

$$y_c = e^x (c_1 \cos x + c_2 \sin x)$$

$$y_1 = e^x \cos x, y_2 = e^x \sin x$$

$$y'_1 = e^x (\cos x - \sin x), y'_2 = e^x (\sin x + \cos x)$$

$$W = e^x \cos x \cdot e^x (\sin x + \cos x) - e^x \sin x \cdot e^x (\cos x - \sin x)$$

$$= e^{2x} [\cos x \sin x + \cos^2 x - \sin x \cos x + \sin^2 x] = e^{2x}$$

Dividir ecuación por 3: $y'' - 2y' + 2y = \frac{e^x \sec x}{3}$

$$W_1 = \begin{vmatrix} 0 & e^x \sin x \\ \frac{e^x \sec x}{3} & e^x (\sin x + \cos x) \end{vmatrix} = -\frac{e^x \sin x \cdot e^x \sec x}{3} = -\frac{e^{2x} \tan x}{3}$$

$$W_2 = \begin{vmatrix} e^x \cos x & 0 \\ e^x (\cos x - \sin x) & \frac{e^x \sec x}{3} \end{vmatrix} = \frac{e^x \cos x \cdot e^x \sec x}{3} = \frac{e^{2x}}{3}$$

$$u_1 = \int \frac{-\frac{e^{2x} \tan x}{3}}{e^{2x}} dx = -\frac{1}{3} \int \tan x \, dx = -\frac{1}{3} (-\ln |\cos x|) = \frac{\ln |\cos x|}{3}$$

$$u_2 = \int \frac{\frac{e^{2x}}{3}}{e^{2x}} dx = \int \frac{1}{3} dx = \frac{x}{3}$$

$$y_p = \frac{\ln |\cos x|}{3} \cdot e^x \cos x + \frac{x}{3} \cdot e^x \sin x$$

$$y = c_1 e^x \cos x + c_2 e^x \sin x + \frac{e^x \cos x \ln |\cos x|}{3} + \frac{x e^x \sin x}{3}$$

$$\mathbf{18.} \quad 4y'' - 4y' + y = \frac{e^{x/2}}{\sqrt{1-x^2}}$$

Ec. característica: $4r^2 - 4r + 1 = 0 \Rightarrow (2r-1)^2 = 0 \Rightarrow r = \frac{1}{2}$ (doble)

$$y_c = c_1 e^{x/2} + c_2 x e^{x/2}$$

$$y_1 = e^{x/2}, y_2 = x e^{x/2}$$

$$y'_1 = \frac{1}{2} e^{x/2}, y'_2 = e^{x/2} + \frac{x}{2} e^{x/2} = e^{x/2} \left(1 + \frac{x}{2} \right)$$

$$W = \begin{vmatrix} e^{x/2} & x e^{x/2} \\ \frac{1}{2} e^{x/2} & e^{x/2} \left(1 + \frac{x}{2} \right) \end{vmatrix}$$

$$= e^{x/2} \cdot e^{x/2} \left(1 + \frac{x}{2} \right) - x e^{x/2} \cdot \frac{1}{2} e^{x/2}$$

$$= e^x \left(1 + \frac{x}{2} - \frac{x}{2} \right) = e^x$$

Dividir ecuación por 4: $y'' - y' + \frac{y}{4} = \frac{e^{x/2}}{4\sqrt{1-x^2}}$

$$W_1 = \begin{vmatrix} 0 & x e^{x/2} \\ \frac{e^{x/2}}{4\sqrt{1-x^2}} & e^{x/2} \left(1 + \frac{x}{2} \right) \end{vmatrix} = -\frac{x e^{x/2} \cdot e^{x/2}}{4\sqrt{1-x^2}} = -\frac{x e^x}{4\sqrt{1-x^2}}$$

$$W_2 = \begin{vmatrix} e^{x/2} & 0 \\ \frac{1}{2} e^{x/2} & \frac{e^{x/2}}{4\sqrt{1-x^2}} \end{vmatrix} = \frac{e^{x/2} \cdot e^{x/2}}{4\sqrt{1-x^2}} = \frac{e^x}{4\sqrt{1-x^2}}$$

$$u_1 = \int \frac{-\frac{x e^x}{4\sqrt{1-x^2}}}{e^x} dx = -\frac{1}{4} \int \frac{x}{\sqrt{1-x^2}} dx$$

Sea $u = 1 - x^2$, $du = -2x \, dx$

$$u_1 = -\frac{1}{4} \int \frac{-du}{2\sqrt{u}} = \frac{1}{8} \int u^{-1/2} du = \frac{1}{8} \cdot 2\sqrt{u} = \frac{\sqrt{1-x^2}}{4}$$

$$u_2 = \int \frac{\frac{e^x}{4\sqrt{1-x^2}}}{e^x} dx = \frac{1}{4} \int \frac{1}{\sqrt{1-x^2}} dx = \frac{\arcsin x}{4}$$

$$y_p = \frac{\sqrt{1-x^2}}{4} \cdot e^{x/2} + \frac{\arcsin x}{4} \cdot x e^{x/2}$$

$$y = c_1 e^{x/2} + c_2 x e^{x/2} + \frac{e^{x/2} \sqrt{1-x^2}}{4} + \frac{x e^{x/2} \arcsin x}{4}$$