Ecuaciones Diferenciales - Variación de Parámetros

1. $y'' + y = \sec x$

Ec. característica: $r^2 + 1 = 0 \Rightarrow r = \pm i$

$y_c = c_1 \cos x + c_2 \sin x$

$y_1 = \cos x,\ y_2 = \sin x$

$y_1' = -\sin x,\ y_2' = \cos x$

$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$

$W_1 = \begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix} = 0 \cdot \cos x - \sec x \cdot \sin x = -\tan x$

$W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix} = \cos x \cdot \sec x - 0 \cdot (-\sin x) = 1$

$u_1 = \int \frac{W_1}{W} dx = \int \frac{-\tan x}{1} dx = \int (-\tan x) dx = \ln|\cos x|$

$u_2 = \int \frac{W_2}{W} dx = \int \frac{1}{1} dx = \int 1\, dx = x$

$y_p = u_1 y_1 + u_2 y_2 = \ln|\cos x| \cdot \cos x + x \cdot \sin x$

$y = c_1 \cos x + c_2 \sin x + \cos x \ln|\cos x| + x \sin x$

2. $y'' + y = \tan x$

Ec. característica: $r^2 + 1 = 0 \Rightarrow r = \pm i$

$y_c = c_1 \cos x + c_2 \sin x$

$y_1 = \cos x,\ y_2 = \sin x$

$y_1' = -\sin x,\ y_2' = \cos x$

$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$

$W_1 = \begin{vmatrix} 0 & \sin x \\ \tan x & \cos x \end{vmatrix} = 0 - \tan x \cdot \sin x = -\sin x \tan x = -\frac{\sin^2 x}{\cos x}$

$W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \tan x \end{vmatrix} = \cos x \tan x - 0 = \sin x$

$u_1 = \int \frac{-\sin^2 x}{\cos x} dx = \int \frac{-(1-\cos^2 x)}{\cos x} dx = \int \left(-\frac{1}{\cos x} + \cos x\right) dx$

$u_1 = \int (-\sec x + \cos x) dx = -\ln|\sec x + \tan x| + \sin x$

$u_2 = \int \sin x\, dx = -\cos x$

$y_p = u_1 y_1 + u_2 y_2 = (-\ln|\sec x + \tan x| + \sin x)\cos x + (-\cos x)\sin x$

$y_p = -\cos x \ln|\sec x + \tan x| + \sin x \cos x - \cos x \sin x = -\cos x \ln|\sec x + \tan x|$

$y = c_1 \cos x + c_2 \sin x - \cos x \ln|\sec x + \tan x|$

3. $y'' + y = \sin x$

Ec. característica: $r^2 + 1 = 0 \Rightarrow r = \pm i$

$y_c = c_1 \cos x + c_2 \sin x$

$y_1 = \cos x,\ y_2 = \sin x$

$y_1' = -\sin x,\ y_2' = \cos x$

$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$

$W_1 = \begin{vmatrix} 0 & \sin x \\ \sin x & \cos x \end{vmatrix} = 0 \cdot \cos x - \sin x \cdot \sin x = -\sin^2 x$

$W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \sin x \end{vmatrix} = \cos x \cdot \sin x - 0 \cdot (-\sin x) = \sin x \cos x$

$u_1 = \int \frac{-\sin^2 x}{1} dx = -\int \sin^2 x\, dx = -\int \frac{1-\cos 2x}{2} dx = -\frac{x}{2} + \frac{\sin 2x}{4}$

$u_2 = \int \frac{\sin x \cos x}{1} dx = \int \sin x \cos x\, dx = \int \frac{\sin 2x}{2} dx = -\frac{\cos 2x}{4}$

$y_p = u_1 y_1 + u_2 y_2 = \left(-\frac{x}{2} + \frac{\sin 2x}{4}\right)\cos x + \left(-\frac{\cos 2x}{4}\right)\sin x$

$= -\frac{x \cos x}{2} + \frac{\sin 2x \cos x}{4} - \frac{\cos 2x \sin x}{4}$

Usando $\sin 2x = 2\sin x \cos x$ y $\cos 2x = \cos^2 x - \sin^2 x$:

$= -\frac{x \cos x}{2} + \frac{2\sin x \cos^2 x}{4} - \frac{(\cos^2 x - \sin^2 x)\sin x}{4}$

$= -\frac{x \cos x}{2} + \frac{\sin x \cos^2 x}{2} - \frac{\sin x \cos^2 x}{4} + \frac{\sin^3 x}{4}$

$= -\frac{x \cos x}{2} + \frac{\sin x \cos^2 x}{4} + \frac{\sin^3 x}{4} = -\frac{x \cos x}{2} + \frac{\sin x(\cos^2 x + \sin^2 x)}{4} = -\frac{x \cos x}{2} + \frac{\sin x}{4}$

Pero como $\frac{\sin x}{4}$ es parte de la solución complementaria, la solución particular es:

$y_p = -\frac{x \cos x}{2}$

$y = c_1 \cos x + c_2 \sin x - \frac{x \cos x}{2}$

4. $y'' + y = \sec\theta \tan\theta$

Ec. característica: $r^2 + 1 = 0 \Rightarrow r = \pm i$

$y_c = c_1 \cos\theta + c_2 \sin\theta$

$y_1 = \cos\theta,\ y_2 = \sin\theta$

$y_1' = -\sin\theta,\ y_2' = \cos\theta$

$W = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = 1$

$W_1 = \begin{vmatrix} 0 & \sin\theta \\ \sec\theta \tan\theta & \cos\theta \end{vmatrix} = -\sec\theta \tan\theta \cdot \sin\theta = -\tan^2\theta$

$W_2 = \begin{vmatrix} \cos\theta & 0 \\ -\sin\theta & \sec\theta \tan\theta \end{vmatrix} = \cos\theta \sec\theta \tan\theta = \tan\theta$

$u_1 = \int (-\tan^2\theta) d\theta = \int (-\sec^2\theta + 1) d\theta = -\tan\theta + \theta$

$u_2 = \int \tan\theta\, d\theta = -\ln|\cos\theta|$

$y_p = u_1 y_1 + u_2 y_2 = (-\tan\theta + \theta)\cos\theta + (-\ln|\cos\theta|)\sin\theta$

$y_p = -\sin\theta + \theta\cos\theta - \sin\theta \ln|\cos\theta|$

$y = c_1 \cos\theta + c_2 \sin\theta - \sin\theta + \theta\cos\theta - \sin\theta \ln|\cos\theta|$

5. $y'' + y = \cos^2 x$

Ec. característica: $r^2 + 1 = 0 \Rightarrow r = \pm i$

$y_c = c_1 \cos x + c_2 \sin x$

$y_1 = \cos x,\ y_2 = \sin x$

$y_1' = -\sin x,\ y_2' = \cos x$

$W = 1$

Identidad: $\cos^2 x = \frac{1+\cos 2x}{2}$

$W_1 = \begin{vmatrix} 0 & \sin x \\ \frac{1+\cos 2x}{2} & \cos x \end{vmatrix} = -\frac{(1+\cos 2x)\sin x}{2}$

$W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \frac{1+\cos 2x}{2} \end{vmatrix} = \frac{(1+\cos 2x)\cos x}{2}$

$u_1 = \int \frac{-(1+\cos 2x)\sin x}{2} dx = \int \left(-\frac{\sin x}{2} - \frac{\sin x \cos 2x}{2}\right) dx$

Para $\int \sin x \cos 2x\, dx$: usar $\cos 2x = 1 - 2\sin^2 x$

$\int \sin x \cos 2x\, dx = \int \sin x(1 - 2\sin^2 x) dx = \int(\sin x - 2\sin^3 x) dx$

$= -\cos x + 2\left(\frac{\cos^3 x}{3} - \cos x\right) = -\cos x + \frac{2\cos^3 x}{3} - 2\cos x = \frac{2\cos^3 x}{3} - 3\cos x$

$u_1 = \frac{\cos x}{2} - \frac{1}{2}\left(\frac{2\cos^3 x}{3} - 3\cos x\right) = \frac{\cos x}{2} - \frac{\cos^3 x}{3} + \frac{3\cos x}{2} = 2\cos x - \frac{\cos^3 x}{3}$

$u_2 = \int \frac{(1+\cos 2x)\cos x}{2} dx = \int \left(\frac{\cos x}{2} + \frac{\cos x \cos 2x}{2}\right) dx$

$= \frac{\sin x}{2} + \frac{1}{2}\int \cos x \cos 2x\, dx$

$\int \cos x \cos 2x\, dx = \int \cos x(2\cos^2 x - 1) dx = \int(2\cos^3 x - \cos x) dx$

$= 2\left(\frac{\sin^3 x}{3} + \sin x\right) - \sin x = \frac{2\sin^3 x}{3} + 2\sin x - \sin x = \frac{2\sin^3 x}{3} + \sin x$

$u_2 = \frac{\sin x}{2} + \frac{1}{2}\left(\frac{2\sin^3 x}{3} + \sin x\right) = \frac{\sin x}{2} + \frac{\sin^3 x}{3} + \frac{\sin x}{2} = \sin x + \frac{\sin^3 x}{3}$

$y_p = \left(2\cos x - \frac{\cos^3 x}{3}\right)\cos x + \left(\sin x + \frac{\sin^3 x}{3}\right)\sin x$

$= 2\cos^2 x - \frac{\cos^4 x}{3} + \sin^2 x + \frac{\sin^4 x}{3}$

$= 2\cos^2 x + \sin^2 x - \frac{\cos^4 x - \sin^4 x}{3} = 2\cos^2 x + \sin^2 x - \frac{(\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x)}{3}$

$= 2\cos^2 x + \sin^2 x - \frac{\cos 2x}{3} = \cos^2 x + 1 - \frac{\cos 2x}{3}$

Simplificando: $y_p = \frac{1}{2} + \frac{\cos 2x}{2}$

$y = c_1 \cos x + c_2 \sin x + \frac{1}{2} + \frac{\cos 2x}{2}$

6. $y'' + y = \sec^2 x$

Ec. característica: $r^2 + 1 = 0 \Rightarrow r = \pm i$

$y_c = c_1 \cos x + c_2 \sin x$

$y_1 = \cos x,\ y_2 = \sin x$

$y_1' = -\sin x,\ y_2' = \cos x$

$W = 1$

$W_1 = \begin{vmatrix} 0 & \sin x \\ \sec^2 x & \cos x \end{vmatrix} = -\sec^2 x \sin x = -\tan x \sec x$

$W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \sec^2 x \end{vmatrix} = \cos x \sec^2 x = \sec x$

$u_1 = \int (-\tan x \sec x) dx = -\sec x$

$u_2 = \int \sec x\, dx = \ln|\sec x + \tan x|$

$y_p = u_1 y_1 + u_2 y_2 = (-\sec x)\cos x + \ln|\sec x + \tan x| \cdot \sin x$

$y_p = -1 + \sin x \ln|\sec x + \tan x|$

$y = c_1 \cos x + c_2 \sin x - 1 + \sin x \ln|\sec x + \tan x|$

7. $y'' - y = \cosh x$

Ec. característica: $r^2 - 1 = 0 \Rightarrow (r-1)(r+1) = 0 \Rightarrow r = 1, -1$

$y_c = c_1 e^x + c_2 e^{-x}$

$y_1 = e^x,\ y_2 = e^{-x}$

$y_1' = e^x,\ y_2' = -e^{-x}$

$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -e^x \cdot e^{-x} - e^x \cdot e^{-x} = -2$

$\cosh x = \frac{e^x + e^{-x}}{2}$

$W_1 = \begin{vmatrix} 0 & e^{-x} \\ \cosh x & -e^{-x} \end{vmatrix} = -e^{-x}\cosh x = -e^{-x} \cdot \frac{e^x + e^{-x}}{2} = -\frac{1 + e^{-2x}}{2}$

$W_2 = \begin{vmatrix} e^x & 0 \\ e^x & \cosh x \end{vmatrix} = e^x \cosh x = e^x \cdot \frac{e^x + e^{-x}}{2} = \frac{e^{2x} + 1}{2}$

$u_1 = \int \frac{-\frac{1+e^{-2x}}{2}}{-2} dx = \int \frac{1+e^{-2x}}{4} dx = \frac{x}{4} + \int \frac{e^{-2x}}{4} dx = \frac{x}{4} - \frac{e^{-2x}}{8}$

$u_2 = \int \frac{\frac{e^{2x}+1}{2}}{-2} dx = -\int \frac{e^{2x}+1}{4} dx = -\frac{e^{2x}}{8} - \frac{x}{4}$

$y_p = u_1 y_1 + u_2 y_2 = \left(\frac{x}{4} - \frac{e^{-2x}}{8} \right) e^x + \left(-\frac{e^{2x}}{8} - \frac{x}{4} \right) e^{-x}$

$= \frac{xe^x}{4} - \frac{e^{-x}}{8} - \frac{e^x}{8} - \frac{xe^{-x}}{4}$

$= \frac{x(e^x - e^{-x})}{4} - \frac{e^x + e^{-x}}{8} = \frac{x \cdot 2\sinh x}{4} - \frac{2\cosh x}{8}$

$= \frac{x\sinh x}{2} - \frac{\cosh x}{4}$

$y = c_1 e^x + c_2 e^{-x} + \frac{x\sinh x}{2} - \frac{\cosh x}{4}$

8. $y'' - y = \sinh 2x$

Ec. característica: $r^2 - 1 = 0 \Rightarrow r = 1, -1$

$y_c = c_1 e^x + c_2 e^{-x}$

$y_1 = e^x,\ y_2 = e^{-x}$

$y_1' = e^x,\ y_2' = -e^{-x}$

$W = -2$

$\sinh 2x = \frac{e^{2x} - e^{-2x}}{2}$

$W_1 = \begin{vmatrix} 0 & e^{-x} \\ \sinh 2x & -e^{-x} \end{vmatrix} = -e^{-x}\sinh 2x = -e^{-x} \cdot \frac{e^{2x}-e^{-2x}}{2} = -\frac{e^x - e^{-3x}}{2}$

$W_2 = \begin{vmatrix} e^x & 0 \\ e^x & \sinh 2x \end{vmatrix} = e^x \sinh 2x = e^x \cdot \frac{e^{2x}-e^{-2x}}{2} = \frac{e^{3x}-e^{-x}}{2}$

$u_1 = \int \frac{-\frac{e^x - e^{-3x}}{2}}{-2} dx = \int \frac{e^x - e^{-3x}}{4} dx = \frac{e^x}{4} - \int \frac{e^{-3x}}{4} dx = \frac{e^x}{4} + \frac{e^{-3x}}{12}$

$u_2 = \int \frac{\frac{e^{3x}-e^{-x}}{2}}{-2} dx = -\int \frac{e^{3x}-e^{-x}}{4} dx = -\frac{e^{3x}}{12} + \int \frac{e^{-x}}{4} dx = -\frac{e^{3x}}{12} - \frac{e^{-x}}{4}$

$y_p = \left(\frac{e^x}{4} + \frac{e^{-3x}}{12} \right) e^x + \left(-\frac{e^{3x}}{12} - \frac{e^{-x}}{4} \right) e^{-x}$

$= \frac{e^{2x}}{4} + \frac{e^{-2x}}{12} - \frac{e^{2x}}{12} - \frac{e^{-2x}}{4}$

$= \frac{3e^{2x}-e^{2x}}{12} + \frac{e^{-2x}-3e^{-2x}}{12} = \frac{2e^{2x}-2e^{-2x}}{12} = \frac{e^{2x}-e^{-2x}}{6}$

$= \frac{2\sinh 2x}{6} = \frac{\sinh 2x}{3}$

$y = c_1 e^x + c_2 e^{-x} + \frac{\sinh 2x}{3}$

9. $y'' - 4y = \frac{e^{2x}}{x}$

Ec. característica: $r^2 - 4 = 0 \Rightarrow (r-2)(r+2) = 0 \Rightarrow r = 2, -2$

$y_c = c_1 e^{2x} + c_2 e^{-2x}$

$y_1 = e^{2x},\ y_2 = e^{-2x}$

$y_1' = 2e^{2x},\ y_2' = -2e^{-2x}$

$W = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -2e^{2x}e^{-2x} - 2e^{2x}e^{-2x} = -4$

$W_1 = \begin{vmatrix} 0 & e^{-2x} \\ \frac{e^{2x}}{x} & -2e^{-2x} \end{vmatrix} = -\frac{e^{2x}e^{-2x}}{x} = -\frac{1}{x}$

$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & \frac{e^{2x}}{x} \end{vmatrix} = \frac{e^{2x} \cdot e^{2x}}{x} = \frac{e^{4x}}{x}$

$u_1 = \int \frac{-\frac{1}{x}}{-4} dx = \int \frac{1}{4x} dx = \frac{\ln|x|}{4}$

$u_2 = \int \frac{\frac{e^{4x}}{x}}{-4} dx = -\frac{1}{4} \int \frac{e^{4x}}{x} dx$ (integral no elemental)

$y_p = \frac{\ln|x|}{4} \cdot e^{2x} - \frac{e^{-2x}}{4} \int \frac{e^{4x}}{x} dx$

$y = c_1 e^{2x} + c_2 e^{-2x} + \frac{e^{2x}\ln|x|}{4} - \frac{e^{-2x}}{4} \int \frac{e^{4x}}{x} dx$

10. $y'' - 9y = \frac{9x}{e^{3x}} = 9xe^{-3x}$

Ec. característica: $r^2 - 9 = 0 \Rightarrow (r-3)(r+3) = 0 \Rightarrow r = 3, -3$

$y_c = c_1 e^{3x} + c_2 e^{-3x}$

$y_1 = e^{3x},\ y_2 = e^{-3x}$

$y_1' = 3e^{3x},\ y_2' = -3e^{-3x}$

$W = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -3e^{3x}e^{-3x} - 3e^{3x}e^{-3x} = -6$

$W_1 = \begin{vmatrix} 0 & e^{-3x} \\ 9xe^{-3x} & -3e^{-3x} \end{vmatrix} = 0 - 9xe^{-3x} \cdot e^{-3x} = -9xe^{-6x}$

$W_2 = \begin{vmatrix} e^{3x} & 0 \\ 3e^{3x} & 9xe^{-3x} \end{vmatrix} = e^{3x} \cdot 9xe^{-3x} - 0 = 9x$

$u_1 = \int \frac{-9xe^{-6x}}{-6} dx = \frac{3}{2} \int xe^{-6x} dx$

Por partes: $\int xe^{-6x} dx = -\frac{xe^{-6x}}{6} - \int -\frac{e^{-6x}}{6} dx = -\frac{xe^{-6x}}{6} - \frac{e^{-6x}}{36}$

$u_1 = \frac{3}{2} \left(-\frac{xe^{-6x}}{6} - \frac{e^{-6x}}{36} \right) = -\frac{3xe^{-6x}}{12} - \frac{3e^{-6x}}{72} = -\frac{xe^{-6x}}{4} - \frac{e^{-6x}}{24}$

$u_2 = \int \frac{9x}{-6} dx = -\frac{3}{2} \int x\, dx = -\frac{3x^2}{4}$

$y_p = \left(-\frac{xe^{-6x}}{4} - \frac{e^{-6x}}{24} \right) e^{3x} + \left(-\frac{3x^2}{4} \right) e^{-3x}$

$= -\frac{xe^{-3x}}{4} - \frac{e^{-3x}}{24} - \frac{3x^2 e^{-3x}}{4}$

$= -e^{-3x} \left(\frac{x}{4} + \frac{1}{24} + \frac{3x^2}{4} \right) = -e^{-3x} \left(\frac{3x^2+x}{4} + \frac{1}{24} \right)$

$= -e^{-3x} \left(\frac{18x^2+6x+1}{24} \right)$

$y = c_1 e^{3x} + c_2 e^{-3x} - \frac{e^{-3x}(18x^2+6x+1)}{24}$

11. $y'' + 3y' + 2y = \frac{1}{1+e^x}$

Ec. característica: $r^2 + 3r + 2 = 0 \Rightarrow (r+1)(r+2) = 0 \Rightarrow r = -1, -2$

$y_c = c_1 e^{-x} + c_2 e^{-2x}$

$y_1 = e^{-x},\ y_2 = e^{-2x}$

$y_1' = -e^{-x},\ y_2' = -2e^{-2x}$

$W = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -2e^{-3x} + e^{-3x} = -e^{-3x}$

$W_1 = \begin{vmatrix} 0 & e^{-2x} \\ \frac{1}{1+e^x} & -2e^{-2x} \end{vmatrix} = -\frac{e^{-2x}}{1+e^x}$

$W_2 = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & \frac{1}{1+e^x} \end{vmatrix} = \frac{e^{-x}}{1+e^x}$

$u_1 = \int \frac{-\frac{e^{-2x}}{1+e^x}}{-e^{-3x}} dx = \int \frac{e^{-2x}}{e^{-3x}(1+e^x)} dx = \int \frac{e^x}{1+e^x} dx$

Sea $u = 1 + e^x,\ du = e^x dx$

$u_1 = \int \frac{du}{u} = \ln|u| = \ln(1+e^x)$

$u_2 = \int \frac{\frac{e^{-x}}{1+e^x}}{-e^{-3x}} dx = -\int \frac{e^{-x}}{e^{-3x}(1+e^x)} dx = -\int \frac{e^{2x}}{1+e^x} dx$

$= -\int \frac{e^x(1+e^x) - e^x}{1+e^x} dx = -\int \left(e^x - \frac{e^x}{1+e^x} \right) dx$

$= -e^x + \ln(1+e^x)$

$y_p = \ln(1+e^x) \cdot e^{-x} + (-e^x + \ln(1+e^x))e^{-2x}$

$= e^{-x}\ln(1+e^x) - e^{-x} + e^{-2x}\ln(1+e^x)$

$y = c_1 e^{-x} + c_2 e^{-2x} + e^{-x}\ln(1+e^x) - e^{-x} + e^{-2x}\ln(1+e^x)$

12. $y'' - 2y' + y = \frac{e^x}{1+x^2}$

Ec. característica: $r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0 \Rightarrow r = 1$ (doble)

$y_c = c_1 e^x + c_2 xe^x$

$y_1 = e^x,\ y_2 = xe^x$

$y_1' = e^x,\ y_2' = e^x + xe^x$

$W = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^x(e^x + xe^x) - xe^x \cdot e^x = e^{2x} + xe^{2x} - xe^{2x} = e^{2x}$

$W_1 = \begin{vmatrix} 0 & xe^x \\ \frac{e^x}{1+x^2} & e^x + xe^x \end{vmatrix} = -\frac{xe^x \cdot e^x}{1+x^2} = -\frac{xe^{2x}}{1+x^2}$

$W_2 = \begin{vmatrix} e^x & 0 \\ e^x & \frac{e^x}{1+x^2} \end{vmatrix} = \frac{e^x \cdot e^x}{1+x^2} = \frac{e^{2x}}{1+x^2}$

$u_1 = \int \frac{-\frac{xe^{2x}}{1+x^2}}{e^{2x}} dx = -\int \frac{x}{1+x^2} dx = -\frac{1}{2}\ln(1+x^2)$

$u_2 = \int \frac{\frac{e^{2x}}{1+x^2}}{e^{2x}} dx = \int \frac{1}{1+x^2} dx = \arctan x$

$y_p = -\frac{\ln(1+x^2)}{2} \cdot e^x + \arctan x \cdot xe^x$

$y = c_1 e^x + c_2 xe^x - \frac{e^x \ln(1+x^2)}{2} + xe^x \arctan x$

13. $y'' + 3y' + 2y = \sin e^x$

Ec. característica: $r^2 + 3r + 2 = 0 \Rightarrow (r+1)(r+2) = 0 \Rightarrow r = -1, -2$

$y_c = c_1 e^{-x} + c_2 e^{-2x}$

$y_1 = e^{-x},\ y_2 = e^{-2x}$

$y_1' = -e^{-x},\ y_2' = -2e^{-2x}$

$W = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -2e^{-3x} + e^{-3x} = -e^{-3x}$

$W_1 = \begin{vmatrix} 0 & e^{-2x} \\ \sin e^x & -2e^{-2x} \end{vmatrix} = 0 - \sin e^x \cdot e^{-2x} = -e^{-2x}\sin e^x$

$W_2 = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & \sin e^x \end{vmatrix} = e^{-x}\sin e^x - 0 = e^{-x}\sin e^x$

$u_1 = \int \frac{-e^{-2x}\sin e^x}{-e^{-3x}} dx = \int e^x \sin e^x\, dx$

Sea $u = e^x,\ du = e^x dx$

$u_1 = \int \sin u\, du = -\cos u = -\cos e^x$

$u_2 = \int \frac{e^{-x}\sin e^x}{-e^{-3x}} dx = -\int e^{2x}\sin e^x\, dx$

Sea $u = e^x,\ du = e^x dx$, entonces $e^{2x} = u^2$ y $dx = \frac{du}{u}$

$u_2 = -\int u^2 \sin u \cdot \frac{du}{u} = -\int u\sin u\, du$

Por partes: $\int u\sin u\, du = -u\cos u + \int \cos u\, du = -u\cos u + \sin u$

$u_2 = -(-e^x\cos e^x + \sin e^x) = e^x\cos e^x - \sin e^x$

$y_p = u_1 y_1 + u_2 y_2 = (-\cos e^x)e^{-x} + (e^x\cos e^x - \sin e^x)e^{-2x}$

$= -e^{-x}\cos e^x + e^{-x}\cos e^x - e^{-2x}\sin e^x$

$= -e^{-2x}\sin e^x$

$y = c_1 e^{-x} + c_2 e^{-2x} - e^{-2x}\sin e^x$

14. $y'' - 2y' + y = e^t \arctan t$

Ec. característica: $r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0 \Rightarrow r = 1$ (doble)

$y_c = c_1 e^t + c_2 te^t$

$y_1 = e^t,\ y_2 = te^t$

$y_1' = e^t,\ y_2' = e^t + te^t$

$W = e^{2t}$

$W_1 = \begin{vmatrix} 0 & te^t \\ e^t \arctan t & e^t + te^t \end{vmatrix} = -te^t \cdot e^t \arctan t = -te^{2t} \arctan t$

$W_2 = \begin{vmatrix} e^t & 0 \\ e^t & e^t \arctan t \end{vmatrix} = e^t \cdot e^t \arctan t = e^{2t} \arctan t$

$u_1 = \int \frac{-te^{2t} \arctan t}{e^{2t}} dt = -\int t \arctan t \, dt$

Por partes: sea $u = \arctan t$, $dv = t \, dt$, $du = \frac{dt}{1+t^2}$, $v = \frac{t^2}{2}$

$\int t \arctan t \, dt = \frac{t^2 \arctan t}{2} - \int \frac{t^2}{2(1+t^2)} dt = \frac{t^2 \arctan t}{2} - \frac{1}{2} \int \frac{t^2}{1+t^2} dt$

$= \frac{t^2 \arctan t}{2} - \frac{1}{2} \int \left(1 - \frac{1}{1+t^2}\right) dt = \frac{t^2 \arctan t}{2} - \frac{1}{2}(t - \arctan t)$

$= \frac{t^2 \arctan t}{2} - \frac{t}{2} + \frac{\arctan t}{2}$

$u_1 = -\frac{t^2 \arctan t}{2} + \frac{t}{2} - \frac{\arctan t}{2}$

$u_2 = \int \frac{e^{2t} \arctan t}{e^{2t}} dt = \int \arctan t \, dt$

Por partes: $u = \arctan t$, $dv = dt$, $du = \frac{dt}{1+t^2}$, $v = t$

$\int \arctan t \, dt = t \arctan t - \int \frac{t}{1+t^2} dt = t \arctan t - \frac{1}{2} \ln(1+t^2)$

$u_2 = t \arctan t - \frac{\ln(1+t^2)}{2}$

$y_p = \left(-\frac{t^2 \arctan t}{2} + \frac{t}{2} - \frac{\arctan t}{2}\right) e^t + \left(t \arctan t - \frac{\ln(1+t^2)}{2}\right) te^t$

$y = c_1 e^t + c_2 te^t + e^t \left(-\frac{t^2 \arctan t}{2} + \frac{t}{2} - \frac{\arctan t}{2}\right) + te^t \left(t \arctan t - \frac{\ln(1+t^2)}{2}\right)$

15. $y'' + 2y' + y = e^{-t} \ln t$

Ec. característica: $r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0 \Rightarrow r = -1$ (doble)

$y_c = c_1 e^{-t} + c_2 te^{-t}$

$y_1 = e^{-t}$, $y_2 = te^{-t}$

$y_1' = -e^{-t}$, $y_2' = e^{-t} - te^{-t}$

$W = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t} - te^{-t} \end{vmatrix} = e^{-t}(e^{-t} - te^{-t}) + te^{-t}e^{-t}$

$= e^{-2t} - te^{-2t} + te^{-2t} = e^{-2t}$

$W_1 = \begin{vmatrix} 0 & te^{-t} \\ e^{-t} \ln t & e^{-t} - te^{-t} \end{vmatrix} = -te^{-t} \cdot e^{-t} \ln t = -te^{-2t} \ln t$

$W_2 = \begin{vmatrix} e^{-t} & 0 \\ -e^{-t} & e^{-t} \ln t \end{vmatrix} = e^{-t} \cdot e^{-t} \ln t = e^{-2t} \ln t$

$u_1 = \int \frac{-te^{-2t} \ln t}{e^{-2t}} dt = -\int t \ln t \, dt$

Por partes: $u = \ln t$, $dv = t \, dt$, $du = \frac{dt}{t}$, $v = \frac{t^2}{2}$

$\int t \ln t \, dt = \frac{t^2 \ln t}{2} - \int \frac{t^2}{2t} dt = \frac{t^2 \ln t}{2} - \int \frac{t}{2} dt = \frac{t^2 \ln t}{2} - \frac{t^2}{4}$

$u_1 = -\frac{t^2 \ln t}{2} + \frac{t^2}{4}$

$u_2 = \int \frac{e^{-2t} \ln t}{e^{-2t}} dt = \int \ln t \, dt$

Por partes: $u = \ln t$, $dv = dt$, $du = \frac{dt}{t}$, $v = t$

$\int \ln t \, dt = t \ln t - \int dt = t \ln t - t$

$u_2 = t \ln t - t$

$y_p = \left(-\frac{t^2 \ln t}{2} + \frac{t^2}{4}\right) e^{-t} + (t \ln t - t) te^{-t}$

$= e^{-t} \left(-\frac{t^2 \ln t}{2} + \frac{t^2}{4} + t^2 \ln t - t^2\right)$

$= e^{-t} \left(\frac{t^2 \ln t}{2} + \frac{t^2}{4} - t^2\right) = e^{-t} \left(\frac{t^2 \ln t}{2} - \frac{3t^2}{4}\right)$

$y = c_1 e^{-t} + c_2 te^{-t} + e^{-t} \left(\frac{t^2 \ln t}{2} - \frac{3t^2}{4}\right)$

16. $2y'' + 2y' + y = \frac{4}{\sqrt{x}}$

Ec. característica: $2r^2 + 2r + 1 = 0$

$r = \frac{-2 \pm \sqrt{4-8}}{4} = \frac{-2 \pm 2i}{4} = \frac{-1 \pm i}{2}$

$\alpha = -\frac{1}{2}$, $\beta = \frac{1}{2}$

$y_c = e^{-x/2} \left(c_1 \cos \frac{x}{2} + c_2 \sin \frac{x}{2}\right)$

$y_1 = e^{-x/2} \cos \frac{x}{2}$, $y_2 = e^{-x/2} \sin \frac{x}{2}$

$y_1' = e^{-x/2} \left(-\frac{1}{2} \cos \frac{x}{2} - \frac{1}{2} \sin \frac{x}{2}\right)$

$y_2' = e^{-x/2} \left(-\frac{1}{2} \sin \frac{x}{2} + \frac{1}{2} \cos \frac{x}{2}\right)$

$W = y_1 y_2' - y_2 y_1'$

$= e^{-x/2} \cos \frac{x}{2} \cdot e^{-x/2} \left(-\frac{1}{2} \sin \frac{x}{2} + \frac{1}{2} \cos \frac{x}{2}\right)$

$- e^{-x/2} \sin \frac{x}{2} \cdot e^{-x/2} \left(-\frac{1}{2} \cos \frac{x}{2} - \frac{1}{2} \sin \frac{x}{2}\right)$

$= e^{-x} \left[-\frac{1}{2} \cos \frac{x}{2} \sin \frac{x}{2} + \frac{1}{2} \cos^2 \frac{x}{2} + \frac{1}{2} \sin \frac{x}{2} \cos \frac{x}{2} + \frac{1}{2} \sin^2 \frac{x}{2}\right]$

$= \frac{e^{-x}}{2} \left(\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2}\right) = \frac{e^{-x}}{2}$

Dividir ecuación por 2: $y'' + y' + \frac{y}{2} = \frac{2}{\sqrt{x}}$

$W_1 = \begin{vmatrix} 0 & e^{-x/2} \sin \frac{x}{2} \\ \frac{2}{\sqrt{x}} & e^{-x/2}(-\frac{1}{2} \sin \frac{x}{2} + \frac{1}{2} \cos \frac{x}{2}) \end{vmatrix}$

$= -\frac{2e^{-x/2} \sin \frac{x}{2}}{\sqrt{x}}$

$W_2 = \begin{vmatrix} e^{-x/2} \cos \frac{x}{2} & 0 \\ e^{-x/2}(-\frac{1}{2} \cos \frac{x}{2} - \frac{1}{2} \sin \frac{x}{2}) & \frac{2}{\sqrt{x}} \end{vmatrix}$

$= \frac{2e^{-x/2} \cos \frac{x}{2}}{\sqrt{x}}$

$u_1 = \int \frac{-\frac{2e^{-x/2} \sin \frac{x}{2}}{\sqrt{x}}}{\frac{e^{-x}}{2}} dx = -\int \frac{4e^{x/2} \sin \frac{x}{2}}{\sqrt{x}} dx$ (integral no elemental)

$u_2 = \int \frac{\frac{2e^{-x/2} \cos \frac{x}{2}}{\sqrt{x}}}{\frac{e^{-x}}{2}} dx = \int \frac{4e^{x/2} \cos \frac{x}{2}}{\sqrt{x}} dx$ (integral no elemental)

$y = c_1 e^{-x/2} \cos \frac{x}{2} + c_2 e^{-x/2} \sin \frac{x}{2} + y_p$

donde y_p requiere integrales no elementales

17. $3y'' - 6y' + 6y = e^x \sec x$

Ec. característica: $3r^2 - 6r + 6 = 0 \Rightarrow r^2 - 2r + 2 = 0$

$r = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$

$y_c = e^x(c_1 \cos x + c_2 \sin x)$

$y_1 = e^x \cos x$, $y_2 = e^x \sin x$

$y_1' = e^x(\cos x - \sin x)$, $y_2' = e^x(\sin x + \cos x)$

$W = e^x \cos x \cdot e^x(\sin x + \cos x) - e^x \sin x \cdot e^x(\cos x - \sin x)$

$= e^{2x}[\cos x \sin x + \cos^2 x - \sin x \cos x + \sin^2 x] = e^{2x}$

Dividir ecuación por 3: $y'' - 2y' + 2y = \frac{e^x \sec x}{3}$

$W_1 = \begin{vmatrix} 0 & e^x \sin x \\ \frac{e^x \sec x}{3} & e^x(\sin x + \cos x) \end{vmatrix} = -\frac{e^x \sin x \cdot e^x \sec x}{3} = -\frac{e^{2x} \tan x}{3}$

$W_2 = \begin{vmatrix} e^x \cos x & 0 \\ e^x(\cos x - \sin x) & \frac{e^x \sec x}{3} \end{vmatrix} = \frac{e^x \cos x \cdot e^x \sec x}{3} = \frac{e^{2x}}{3}$

$u_1 = \int \frac{-\frac{e^{2x} \tan x}{3}}{e^{2x}} dx = -\frac{1}{3} \int \tan x \, dx = -\frac{1}{3}(-\ln|\cos x|) = \frac{\ln|\cos x|}{3}$

$u_2 = \int \frac{\frac{e^{2x}}{3}}{e^{2x}} dx = \int \frac{1}{3} dx = \frac{x}{3}$

$y_p = \frac{\ln|\cos x|}{3} \cdot e^x \cos x + \frac{x}{3} \cdot e^x \sin x$

$y = c_1 e^x \cos x + c_2 e^x \sin x + \frac{e^x \cos x \ln|\cos x|}{3} + \frac{xe^x \sin x}{3}$

18. $4y'' - 4y' + y = \frac{e^{x/2}}{\sqrt{1-x^2}}$

Ec. característica: $4r^2 - 4r + 1 = 0 \Rightarrow (2r-1)^2 = 0 \Rightarrow r = \frac{1}{2}$ (doble)

$y_c = c_1 e^{x/2} + c_2 xe^{x/2}$

$y_1 = e^{x/2}$, $y_2 = xe^{x/2}$

$y_1' = \frac{1}{2} e^{x/2}$, $y_2' = e^{x/2} + \frac{x}{2} e^{x/2} = e^{x/2} \left(1 + \frac{x}{2}\right)$

$W = \begin{vmatrix} e^{x/2} & xe^{x/2} \\ \frac{1}{2} e^{x/2} & e^{x/2}(1 + \frac{x}{2}) \end{vmatrix}$

$= e^{x/2} \cdot e^{x/2} \left(1 + \frac{x}{2}\right) - xe^{x/2} \cdot \frac{1}{2} e^{x/2}$

$= e^x \left(1 + \frac{x}{2} - \frac{x}{2}\right) = e^x$

Dividir ecuación por 4: $y'' - y' + \frac{y}{4} = \frac{e^{x/2}}{4\sqrt{1-x^2}}$

$W_1 = \begin{vmatrix} 0 & xe^{x/2} \\ \frac{e^{x/2}}{4\sqrt{1-x^2}} & e^{x/2}(1 + \frac{x}{2}) \end{vmatrix} = -\frac{xe^{x/2} \cdot e^{x/2}}{4\sqrt{1-x^2}} = -\frac{xe^x}{4\sqrt{1-x^2}}$

$W_2 = \begin{vmatrix} e^{x/2} & 0 \\ \frac{1}{2} e^{x/2} & \frac{e^{x/2}}{4\sqrt{1-x^2}} \end{vmatrix} = \frac{e^{x/2} \cdot e^{x/2}}{4\sqrt{1-x^2}} = \frac{e^x}{4\sqrt{1-x^2}}$

$u_1 = \int \frac{-\frac{xe^x}{4\sqrt{1-x^2}}}{e^x} dx = -\frac{1}{4} \int \frac{x}{\sqrt{1-x^2}} dx$

Sea $u = 1 - x^2$, $du = -2x \, dx$

$u_1 = -\frac{1}{4} \int \frac{-du}{2\sqrt{u}} = \frac{1}{8} \int u^{-1/2} du = \frac{1}{8} \cdot 2\sqrt{u} = \frac{\sqrt{1-x^2}}{4}$

$u_2 = \int \frac{\frac{e^x}{4\sqrt{1-x^2}}}{e^x} dx = \frac{1}{4} \int \frac{1}{\sqrt{1-x^2}} dx = \frac{\arcsin x}{4}$

$y_p = \frac{\sqrt{1-x^2}}{4} \cdot e^{x/2} + \frac{\arcsin x}{4} \cdot xe^{x/2}$

$y = c_1 e^{x/2} + c_2 xe^{x/2} + \frac{e^{x/2}\sqrt{1-x^2}}{4} + \frac{xe^{x/2} \arcsin x}{4}$