Ecuaciones Diferenciales - Variación de Parámetros

1. $y'' + y = \sec x$

$y_c = c_1 \cos x + c_2 \sin x$

$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$

$u_1' = -\frac{\sin x \cdot \sec x}{1} = -\tan x$, $u_1 = \ln|\cos x|$

$u_2' = \frac{\cos x \cdot \sec x}{1} = 1$, $u_2 = x$

$y = c_1 \cos x + c_2 \sin x + \cos x \ln|\cos x| + x \sin x$

2. $y'' + y = \tan x$

$y_c = c_1 \cos x + c_2 \sin x$, $W = 1$

$u_1' = -\frac{\sin x \tan x}{1} = -\frac{\sin^2 x}{\cos x}$

$u_1 = -\int \frac{1-\cos^2 x}{\cos x} dx = -\int (\sec x - \cos x) dx = -\ln|\sec x + \tan x| + \sin x$

$u_2' = \frac{\cos x \tan x}{1} = \sin x$, $u_2 = -\cos x$

$y = c_1 \cos x + c_2 \sin x + \cos x(-\ln|\sec x + \tan x| + \sin x) - \sin x \cos x$

$y = c_1 \cos x + c_2 \sin x - \cos x \ln|\sec x + \tan x|$

3. $y'' + y = \sec x$ (igual que 1)

$y = c_1 \cos x + c_2 \sin x + \cos x \ln|\cos x| + x \sin x$

4. $y'' + y = \sec\theta \tan\theta$

$y_c = c_1 \cos\theta + c_2 \sin\theta$, $W = 1$

$u_1' = -\sin\theta \sec\theta \tan\theta = -\tan^2\theta$

$u_1 = -\int (\sec^2\theta - 1) d\theta = -\tan\theta + \theta$

$u_2' = \cos\theta \sec\theta \tan\theta = \tan\theta$

$u_2 = -\ln|\cos\theta|$

$y = c_1 \cos\theta + c_2 \sin\theta + \cos\theta(-\tan\theta + \theta) - \sin\theta \ln|\cos\theta|$

$y = c_1 \cos\theta + c_2 \sin\theta - \sin\theta + \theta\cos\theta - \sin\theta \ln|\cos\theta|$

5. $y'' + y = \cos^2 x$

$y_c = c_1 \cos x + c_2 \sin x$, $W = 1$

$\cos^2 x = \frac{1+\cos 2x}{2}$

$u_1' = -\sin x \cdot \frac{1+\cos 2x}{2} = -\frac{\sin x}{2} - \frac{\sin x \cos 2x}{2}$

$u_1 = \frac{\cos x}{2} + \frac{1}{6}(2\sin x \sin 2x + \cos 2x)$

$u_2' = \cos x \cdot \frac{1+\cos 2x}{2}$

$u_2 = \frac{\sin x}{2} + \frac{1}{6}(-2\sin 2x \cos x + \sin 2x)$

$y_p = \frac{1}{2}(1+\cos 2x) = \frac{1}{2} + \frac{1}{2}\cos 2x$

$y = c_1 \cos x + c_2 \sin x + \frac{1}{2} + \frac{1}{2}\cos 2x$

6. $y'' + y = \sec^2 x$

$y_c = c_1 \cos x + c_2 \sin x$, $W = 1$

$u_1' = -\sin x \sec^2 x = -\tan x \sec x$

$u_1 = -\sec x$

$u_2' = \cos x \sec^2 x = \sec x$

$u_2 = \ln|\sec x + \tan x|$

$y = c_1 \cos x + c_2 \sin x - \cos x \sec x + \sin x \ln|\sec x + \tan x|$

$y = c_1 \cos x + c_2 \sin x - 1 + \sin x \ln|\sec x + \tan x|$

7. $y'' - y = \cosh x$

$y_c = c_1 e^x + c_2 e^{-x}$

$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2$

$\cosh x = \frac{e^x + e^{-x}}{2}$

$u_1' = -\frac{e^{-x}\cosh x}{-2} = \frac{e^{-x}(e^x + e^{-x})}{4} = \frac{1+e^{-2x}}{4}$

$u_1 = \frac{x}{4} - \frac{e^{-2x}}{8}$

$u_2' = \frac{e^x \cosh x}{-2} = -\frac{e^x(e^x + e^{-x})}{4} = -\frac{e^{2x}+1}{4}$

$u_2 = -\frac{e^{2x}}{8} - \frac{x}{4}$

$y_p = \frac{xe^x}{4} - \frac{e^{-x}}{8} - \frac{e^x}{8} - \frac{xe^{-x}}{4} = \frac{x}{4}(e^x - e^{-x}) - \frac{e^x + e^{-x}}{8}$

$y_p = \frac{x \sinh x}{2} - \frac{\cosh x}{4}$

$y = c_1 e^x + c_2 e^{-x} + \frac{x\sinh x}{2} - \frac{\cosh x}{4}$

8. $y'' - y = \sinh 2x$

$y_c = c_1 e^x + c_2 e^{-x}$, $W = -2$

$\sinh 2x = \frac{e^{2x} - e^{-2x}}{2}$

$u_1' = -\frac{e^{-x}\sinh 2x}{-2} = \frac{e^{-x}(e^{2x} - e^{-2x})}{4} = \frac{e^x - e^{-3x}}{4}$

$u_1 = \frac{e^x}{4} + \frac{e^{-3x}}{12}$

$u_2' = \frac{e^x \sinh 2x}{-2} = -\frac{e^x(e^{2x} - e^{-2x})}{4} = \frac{e^{-x} - e^{3x}}{4}$

$u_2 = -\frac{e^{-x}}{4} - \frac{e^{3x}}{12}$

$y_p = \frac{e^{2x}}{4} + \frac{e^{-2x}}{12} - \frac{e^{-2x}}{4} - \frac{e^{2x}}{12} = \frac{e^{2x}}{6} - \frac{e^{-2x}}{6} = \frac{\sinh 2x}{3}$

$y = c_1 e^x + c_2 e^{-x} + \frac{\sinh 2x}{3}$

9. $y'' - 4y = \frac{e^{2x}}{x}$

$y_c = c_1 e^{2x} + c_2 e^{-2x}$

$W = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -4$

$u_1' = -\frac{e^{-2x} \cdot \frac{e^{2x}}{x}}{-4} = \frac{1}{4x}$, $u_1 = \frac{\ln|x|}{4}$

$u_2' = \frac{e^{2x} \cdot \frac{e^{2x}}{x}}{-4} = -\frac{e^{4x}}{4x}$, $u_2 = -\frac{1}{4}\int \frac{e^{4x}}{x} dx$

$y = c_1 e^{2x} + c_2 e^{-2x} + \frac{e^{2x}\ln|x|}{4} - \frac{e^{-2x}}{4}\int \frac{e^{4x}}{x} dx$

10. $y'' - 9y = \frac{9x}{e^{2x}}$

$y_c = c_1 e^{3x} + c_2 e^{-3x}$

$W = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -6$

$u_1' = -\frac{e^{-3x} \cdot 9xe^{-2x}}{-6} = \frac{3xe^{-5x}}{2}$

$u_1 = \frac{3}{2}\int xe^{-5x} dx = \frac{3}{2}\left(-\frac{xe^{-5x}}{5} + \frac{e^{-5x}}{25}\right) = -\frac{3xe^{-5x}}{10} - \frac{3e^{-5x}}{50}$

$u_2' = \frac{e^{3x} \cdot 9xe^{-2x}}{-6} = -\frac{3xe^x}{2}$

$u_2 = -\frac{3}{2}\int xe^x dx = -\frac{3}{2}(xe^x - e^x) = -\frac{3xe^x}{2} + \frac{3e^x}{2}$

$y_p = e^{3x}\left(-\frac{3xe^{-5x}}{10} - \frac{3e^{-5x}}{50}\right) + e^{-3x}\left(-\frac{3xe^x}{2} + \frac{3e^x}{2}\right)$

$y_p = -\frac{3xe^{-2x}}{10} - \frac{3e^{-2x}}{50} - \frac{3xe^{-2x}}{2} + \frac{3e^{-2x}}{2}$

$y_p = -\frac{8xe^{-2x}}{5} + \frac{37e^{-2x}}{25}$

$y = c_1 e^{3x} + c_2 e^{-3x} - \frac{8xe^{-2x}}{5} + \frac{37e^{-2x}}{25}$

11. $y'' + 3y' + 2y = \frac{1}{1+e^x}$

$r^2 + 3r + 2 = 0 \Rightarrow r = -1, -2$

$y_c = c_1 e^{-x} + c_2 e^{-2x}$

$W = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -e^{-3x}$

$u_1' = -\frac{e^{-2x} \cdot \frac{1}{1+e^x}}{-e^{-3x}} = \frac{e^x}{1+e^x}$

$u_1 = \int \frac{e^x}{1+e^x} dx = \ln(1+e^x)$

$u_2' = \frac{e^{-x}\cdot\frac{1}{1+e^x}}{-e^{-3x}} = -\frac{e^{2x}}{1+e^x}$

$u_2 = -\int \frac{e^{2x}}{1+e^x}dx = -\int \frac{e^x(1+e^x)-e^x}{1+e^x}dx = -e^x + \ln(1+e^x)$

$y = c_1e^{-x}+c_2e^{-2x}+e^{-x}\ln(1+e^x)+e^{-2x}(-e^x+\ln(1+e^x))$

$y = c_1e^{-x}+c_2e^{-2x}+e^{-x}\ln(1+e^x)-e^{-x}+e^{-2x}\ln(1+e^x)$

12. $y''-2y'+y=\frac{e^x}{1+x^2}$

$r^2-2r+1=0 \Rightarrow r=1$ (doble)

$y_c = c_1e^x + c_2xe^x$

$W = \begin{vmatrix} e^x & xe^x \\ e^x & e^x+xe^x \end{vmatrix} = e^{2x}$

$u_1' = -\frac{xe^x\cdot\frac{e^x}{1+x^2}}{e^{2x}} = -\frac{x}{1+x^2}$

$u_1 = -\frac{1}{2}\ln(1+x^2)$

$u_2' = \frac{e^x\cdot\frac{e^x}{1+x^2}}{e^{2x}} = \frac{1}{1+x^2}$

$u_2 = \arctan x$

$y = c_1e^x + c_2xe^x - \frac{e^x}{2}\ln(1+x^2) + xe^x\arctan x$

13. $y''+3y'+2y=\sec e^x$

$y_c = c_1e^{-x}+c_2e^{-2x}$, $W=-e^{-3x}$

$u_1' = -\frac{e^{-2x}\sec e^x}{-e^{-3x}} = e^x\sec e^x$

$u_1 = \int e^x\sec e^x dx = \ln|\sec e^x + \tan e^x|$

$u_2' = \frac{e^{-x}\sec e^x}{-e^{-3x}} = -e^{2x}\sec e^x$

$u_2 = -\int e^{2x}\sec e^x dx$ (integral compleja, no elemental)

$y = c_1e^{-x}+c_2e^{-2x}+e^{-x}\ln|\sec e^x+\tan e^x| + e^{-2x}u_2(x)$

14. $y''-2y'+y=e^t\arctan t$

$y_c = c_1e^t+c_2te^t$, $W=e^{2t}$

$u_1' = -\frac{te^t\cdot e^t\arctan t}{e^{2t}} = -t\arctan t$

$u_1 = -\int t\arctan t\,dt = -\frac{t^2\arctan t}{2}+\frac{t}{2}-\frac{\arctan t}{2}$

$u_2' = \frac{e^t\cdot e^t\arctan t}{e^{2t}} = \arctan t$

$u_2 = t\arctan t - \frac{1}{2}\ln(1+t^2)$

$y = c_1e^t + c_2te^t + e^t\left(-\frac{t^2\arctan t}{2}+\frac{t}{2}-\frac{\arctan t}{2}\right) + te^t\left(t\arctan t - \frac{1}{2}\ln(1+t^2)\right)$

15. $y''+2y'+y=e^{-t}\ln t$

$y_c = c_1e^{-t}+c_2te^{-t}$, $W=e^{-2t}$

$u_1' = -\frac{te^{-t}\cdot e^{-t}\ln t}{e^{-2t}} = -t\ln t$

$u_1 = -\int t\ln t\,dt = -\frac{t^2\ln t}{2}+\frac{t^2}{4}$

$u_2' = \frac{e^{-t}\cdot e^{-t}\ln t}{e^{-2t}} = \ln t$

$u_2 = t\ln t - t$

$y = c_1e^{-t}+c_2te^{-t}+e^{-t}\left(-\frac{t^2\ln t}{2}+\frac{t^2}{4}\right)+te^{-t}(t\ln t - t)$

$y = c_1e^{-t}+c_2te^{-t}+\frac{t^2e^{-t}}{2}\ln t+\frac{t^2e^{-t}}{4}-t^2e^{-t}$

16. $2y''+2y'+y=\frac{4}{\sqrt{x}}$

$2r^2+2r+1=0 \Rightarrow r=\frac{-2\pm\sqrt{4-8}}{4}=\frac{-1\pm i}{2}$

$y_c = e^{-x/2}(c_1\cos\frac{x}{2}+c_2\sin\frac{x}{2})$

$y_1 = e^{-x/2}\cos\frac{x}{2}$, $y_2=e^{-x/2}\sin\frac{x}{2}$

$W = e^{-x/2}\cos\frac{x}{2}\cdot e^{-x/2}(-\frac{1}{2}\sin\frac{x}{2}+\frac{1}{2}\cos\frac{x}{2}) - e^{-x/2}\sin\frac{x}{2}\cdot e^{-x/2}(-\frac{1}{2}\cos\frac{x}{2}-\frac{1}{2}\sin\frac{x}{2}) = \frac{e^{-x}}{2}$

$u_1' = -\frac{e^{-x/2}\sin\frac{x}{2}\cdot\frac{2}{\sqrt{x}}}{\frac{e^{-x}}{2}} = -\frac{4e^{x/2}\sin\frac{x}{2}}{\sqrt{x}}$

$u_2' = \frac{e^{-x/2}\cos\frac{x}{2}\cdot\frac{2}{\sqrt{x}}}{\frac{e^{-x}}{2}} = \frac{4e^{x/2}\cos\frac{x}{2}}{\sqrt{x}}$

$y = c_1e^{-x/2}\cos\frac{x}{2}+c_2e^{-x/2}\sin\frac{x}{2}+y_p$ (integrales complejas)

17. $3y''-6y'+6y=e^x\sec x$

$3r^2-6r+6=0 \Rightarrow r=\frac{6\pm\sqrt{36-72}}{6}=1\pm i$

$y_c = e^x(c_1\cos x + c_2\sin x)$

$y_1 = e^x\cos x$, $y_2 = e^x\sin x$

$W = e^x\cos x\cdot e^x(\sin x+\cos x) - e^x\sin x\cdot e^x(\cos x-\sin x) = e^{2x}$

$u_1' = -\frac{e^x\sin x\cdot\frac{e^x\sec x}{3}}{e^{2x}} = -\frac{\tan x}{3}$

$u_1 = \frac{\ln|\cos x|}{3}$

$u_2' = \frac{e^x\cos x\cdot\frac{e^x\sec x}{3}}{e^{2x}} = \frac{1}{3}$

$u_2 = \frac{x}{3}$

$y = c_1e^x\cos x + c_2e^x\sin x + \frac{e^x\cos x}{3}\ln|\cos x| + \frac{xe^x\sin x}{3}$

18. $4y''-4y'+y=\frac{e^{x/2}}{\sqrt{1-x^2}}$

$4r^2-4r+1=0 \Rightarrow r=\frac{1}{2}$ (doble)

$y_c = c_1e^{x/2}+c_2xe^{x/2}$

$W = \begin{vmatrix} e^{x/2} & xe^{x/2} \\ \frac{1}{2}e^{x/2} & e^{x/2}+\frac{x}{2}e^{x/2} \end{vmatrix} = e^x$

$u_1' = -\frac{xe^{x/2}\cdot\frac{e^{x/2}}{\sqrt{1-x^2}}}{e^x} = -\frac{x}{\sqrt{1-x^2}}$

$u_1 = \sqrt{1-x^2}$

$u_2' = \frac{e^{x/2}\cdot\frac{e^{x/2}}{\sqrt{1-x^2}}}{e^x} = \frac{1}{\sqrt{1-x^2}}$

$u_2 = \arcsin x$

$y = c_1e^{x/2}+c_2xe^{x/2}+e^{x/2}\sqrt{1-x^2}+xe^{x/2}\arcsin x$