

Compendium of Solved Quantum Field Theory Problems

A work under construction

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Abstract

This compendium of solved problems is intended to include the problems of David Tong Notes, Quantum Field Theory in a Nutshell by Zee, and A First Book of Quantum Field Theory.

The reasons for constructing this stationary are manifold. On a personal level, we're engaged in an independent study of the subject, and having this resource at hand accelerates our ability to revisit various topics swiftly, maintaining a steady pace of progress.

Beyond personal use, the creation of this compendium also serves a broader purpose: to give back to the scientific community. This field of study has become more accessible to researchers over the past years, thanks to the collective contributions of countless scientists. We hope to extend this collaborative spirit by providing our own problem solutions. While we don't claim these solutions are perfect, we do aspire to present them with a level of detail that can assist anyone who may be struggling to understand the topic or offer a fresh perspective to those who are on a smoother path. We have no pretense of expertise on the subject, and the true value of this compendium is something we leave to the community to assess. Based on our experience, we believe that having detailed solutions can be a significant tool for deliberate learning. We welcome all contributions that help to correct any errors or fill in any gaps in the document. We view this compendium as an ongoing collaborative effort, where each correction or suggestion further refines this learning resource.

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Contents

Abstract	i
Acknowledgements	iii
Contents	v
List of Figures	vii
List of Tables	ix
1 Introduction	1
2 David Tong Solution Manual	3
2.1 Sheet 1: Classical Field Theory	3
Appendices	7

List of Figures

List of Tables

CHAPTER 1

Introduction

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No intro is still given

CHAPTER 2

David Tong Solution Manual

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2.1 Sheet 1: Classical Field Theory

A Vibrating String

A string of length a , mass per unit length σ and under tension T is fixed at each end. The Lagrangian governing the time evolution of the transverse displacement $y(x, t)$ is:

$$L = \int_0^a dx \left[\frac{\sigma}{2} \left(\frac{\partial y}{\partial t} \right)^2 - \frac{T}{2} \left(\frac{\partial y}{\partial x} \right)^2 \right] \quad (2.1) \quad \{\text{eq:L_y}\}$$

where x identifies position along the string from one end point. By expressing the displacement as a sine series Fourier expansion in the form:

$$y(x, t) = \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right) q_n(t) \quad (2.2) \quad \{\text{eq:y_sine_four}\}$$

show that the Lagrangian becomes

$$L = \sum_{n=1}^{\infty} \frac{\sigma}{2} \dot{q}_n^2 - \frac{T}{2} \left(\frac{n\pi}{a} q_n \right)^2 \quad (2.3) \quad \{\text{eq:res_L}\}$$

Derive the equations of motion. Hence show that the string is equivalent to an infinite set of decoupled harmonic oscillators with frequencies:

$$\omega_n = \sqrt{\frac{T}{\sigma}} \left(\frac{n\pi}{a} \right) \quad (2.4) \quad \{\text{eq1:ex1_freq}\}$$

Solution

Replacing expression 2.2 in 2.1 we readily obtain:

$$L = \sum_{n,m}^{\infty,\infty} \left[\frac{\sigma}{a} \dot{q}_n \dot{q}_m \int_0^a dx \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right) - \frac{T\pi^2}{a^3} \int_0^a dx nm \cos\left(\frac{n\pi x}{a}\right) \cos\left(\frac{m\pi x}{a}\right) \right] \quad (2.5) \quad \{\text{eq:step1}\}$$

Integrating:

2. David Tong Solution Manual

$$L = \sum_{n,m}^{\infty,\infty} \left[\frac{\sigma}{a} \dot{q}_n \dot{q}_m \frac{a}{2} \delta_{mn} - \frac{T\pi^2}{a^3} \frac{a}{2} nm \delta_{mn} \right] = \sum_{n=1}^{\infty} \frac{\sigma}{2} \dot{q}_n^2 - \frac{T}{2} \left(\frac{n\pi}{a} q_n \right)^2 \quad (2.6)$$

{eq:step2}

From the above equation it is seen that the generalized coordinates are given by q_n . It is important to note that by integrating in the last step we do not longer need to study the Lagrange equations for the Lagrange density in the integrand 2.1 if we want to solve for y . That since we can simply apply the Lagrange equations for 2.6 and replace the resultant q_n in 2.2. So, we have:

$$0 = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_m} \right) - \frac{\partial L}{\partial q_m} = \sigma \ddot{q}_m - T \left(\frac{n\pi}{a} \right)^2 q_m \quad (2.7)$$

Or more simply:

$$\ddot{q}_m = -\frac{T}{\sigma} \left(\frac{n\pi}{a} \right)^2 q_m = -w_m^2 q_m \quad (2.8)$$

as a result it is clear that each q_m is an oscillator with the frequency w_n given by 2.4. The fact that they are decouple is evident from the fact the the solution of one of them does not affect the other. The last was evident even without finding the equations of motion since L can be written as $L_{w_1}(q_1) + L_{w_2}(q_2) + \dots + L_{w_n}(q_n) + \dots$, where $L_{w_n}(q_n)$ corresponds to the Lagrangian of the oscillator with frequency w_n .

Klein-Gordon Invariance

Show that the Klein-Gordon equations is covariant under Lorentz transformations. In other words: show that if $\phi(x)$ satisfies the Klein-Gordon equation, then $\phi(\Lambda^{-1}x)$ also satisfies this equation for any Lorentz transformation given by Λ .

Solution

In an operational sense we want to probe that in a new reference frame $x' = \Lambda x$, then:

$$\eta_{\nu\mu} \partial'^\nu \partial'^\mu \phi'(x') - m^2 \phi'(x') = 0 \quad (2.9)$$

{eq1:
ex2_kg_step1}

where $\phi'(x')$ is the field as observed by the prime observer. Furthermore, it can be seen that $\phi'(x') = \phi(\Lambda^{-1}x')$ since the primed observed will see the same behaviour of the field $\phi(x')$ at the point $\Lambda^{-1}x'$. As an example, consider $x' = x + a$, clearly $\phi'(x') = \phi(x' - a)$.

By supposition:

$$\eta_{\nu\mu} \partial^\nu \partial^\mu \phi(x) - m^2 \phi(x) = 0 \quad (2.10)$$

{eq1:
ex2_kg_step2}

Applying $x' = \Lambda x$:

$$\eta_{\nu\mu} \partial^\nu \partial^\mu \phi(\Lambda^{-1}x') - m^2 \phi(\Lambda^{-1}x') = 0 \quad (2.11)$$

{eq1:
ex2_kg_step3}

rewriting the derivatives to operate over the prime system we have:

$$\eta_{\nu\mu} \Lambda^{-1\nu}{}_\alpha \partial'^\alpha \Lambda^{-1\mu}{}_\sigma \partial'^\sigma \phi(\Lambda^{-1}x') - m^2 \phi(\Lambda^{-1}x') = 0 \quad (2.12)$$

{eq1:
ex2_kg_step4}

using the symmetry of $\eta_{\mu\nu}$, that is, $\eta = \Lambda^T \eta \Lambda$, where T is the transpose:

$$\Lambda^\kappa{}_\nu \eta_{\kappa\beta} \Lambda^\beta{}_\mu \Lambda^{-1}{}^\nu{}_\alpha \Lambda^{-1}{}^\mu{}_\sigma \partial'^\alpha \partial'^\sigma \phi(\Lambda^{-1}x') - m^2 \phi(\Lambda^{-1}x') = 0 \quad (2.13)$$

 {eq1:
ex2_kg_step5}

identifying the Kronecker's delta functions:

$$\delta^\kappa{}_\alpha \delta^\beta{}_\sigma \eta_{\kappa\beta} \partial'^\alpha \partial'^\sigma \phi(\Lambda^{-1}x') - m^2 \phi(\Lambda^{-1}x') = 0 \quad (2.14)$$

 {eq1:
ex2_kg_step6}

identifying $\phi'(x')$ and applying the sum we finally obtain 2.9 as desired. It is to be noted that the identification of $\phi'(x')$ is done for the purpose of clarity and to make more explicit that the equation had the same form. However, this identification is somewhat irrelevant since x is always a function of the new coordinates x' and $\phi(x)$ is always the same object. This is why when proving invariant form of an scalar equation we can disregard this fact.

Invariance under a phase

The motion of a complex field $\psi(x)$ is governed by the Lagrangian:

$$\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi - m^2 \psi^* \psi - \frac{\lambda}{2} (\psi^* \psi)^2 \quad (2.15)$$

{eq1:ex3_step1}

Write down the Euler-Lagrange field equations for this system. Verify that the Lagrangian is invariant under the infinitesimal transformation:

$$\delta\psi = i\alpha\psi \quad \delta\psi^* = -i\alpha\psi^* \quad (2.16)$$

{eq1:ex3_step2}

Derive the Noether's current associated with this transformation and verify explicitly that it is conserved using the field equations satisfied by ψ .

Solution

We can see that in general any multiplication of two linear function $f_1(\psi)f_2(\psi^*)$ which are linear in ψ and ψ^* respectively are invariant under the transformation 2.16:

$$\begin{aligned} f_1(\psi + i\alpha\psi)f_2(\psi^* - i\alpha\psi^*) &= [f_1(\psi) + i\alpha f_1(\psi)][f_2(\psi^*) - i\alpha f_2(\psi^*)] \\ &= f_1(\psi)f_2(\psi^*) + O(\alpha^2) = f_1(\psi)f_2(\psi^*) \end{aligned} \quad (2.17)$$

{eq1:ex3_step3}

It is readily seen that if a term $f_1(\psi)f_2(\psi^*)$ is invariant, so it is $[f_1(\psi)f_2(\psi^*)]^n$ for $n \geq 0$.

As each individual term in the Lagrangian density satisfy having the form $[f_1(\psi)f_2(\psi^*)]^n$ it is seen that \mathcal{L} is invariant under the proposed transformation. As $\delta\mathcal{L} = 0$ it is clear the current is given by:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_a)} \psi_a \right) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \psi \right) + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^*)} \psi^* \right) \quad (2.18)$$

{eq1:ex3_step4}

From where we see that the current is:

$$j_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^*)} \psi^* \quad (2.19)$$

{eq1:ex3_step4}

On the other hand, the field equation satisfy by ψ is:

Appendices
