Behavior of Eigenvalues & Eigenvectors

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Final Report – Behavior of Eigenvalues and Eigenvectors

Objective

The understanding of eigenvalues and eigenvectors is the purpose of this report. To find the benefits in using these values to solve ordinary differential equation, to find the stability, and also to see the relation of linear transformation.

Introduction

Eigenvalues and eigenvectors are very useful in the modeling of chemical processes, sound vibrations, to see transformation within computer graphics. Linear transformation involves the study matrix theory, and this includes the study of eigenvalues and eigenvectors. This values and vectors will give us a better understanding of these transformations. They are the "axes" (directions) along which a linear transformation acts simply by "stretching/compressing" and/or "flipping"; eigenvalues give you the factors by which this compression occurs. The more directions you have along which you understand the behavior of a linear transformation, the easier it is to understand the linear transformation; so you want to have as many linearly independent eigenvectors as possible associated to a single linear transformation.

When designing the controls for a process it is necessary to create a program to operate these controls. There are programs that use differential equations to operate controls based on certain variables in the system. The equation can either be solved by hand which can take some time or use computer programs. But the similarity to both of these ways is the use of eigenvalues and eigenvectors. When finding the solutions for the differential equations can define the stability of the system. The stability can show different characteristic in a system such as whether it is stable and damped, unstable and undamped, (so that there is constant fluctuation in the system), or as an unstable system in which the amplitude of the fluctuation is always increasing. For stable and damped system, when we see there will be change, the system will modify so it can return to a steady state. As for the remaining cases, I am afraid to say the system will not be able to return to the steady state. For the undamped situation, the system will experience a constant fluctuation and this will be hard on for the system, and eventually lead to failure. In the last situation, with the ever increasing amplitude of the fluctuations this will accomplish even greater failure than the last. In this report on eigenvalues and eigenvectors, we will first show how to use them to solve ordinary differential equations. Next, the eigenvalues will show us the same stability we were discussing about previously. We will show the difference from using eigenvalues, because it can be simpler to use the Routh method.

What are Eigenvalues and Eigenvectors?

Eigenvalues are the special set of scalars associated with the system of linear equations. You would mostly find the use of these is matrices. If you didn't know the word 'Eigen' is actually German, which means 'proper' or 'characteristic'. Hence we can call the eigenvalues a characteristic value or a proper value. So to put it in simple terms, the eigenvalue is a scalar that is used to transform the eigenvector. The basic equation is shown below

$$\mathbf{Ax = \lambda x}$$

The symbol "**λ**" is a number that is representing an eigenvalue of A. As stated in Mathematics, an eigenvector corresponds to the real non zero eigenvalues which point in the direction stretched by the transformation whereas eigenvalue is considered as a factor by which it is stretched. Also, if the eigenvalue appears to be negative then the direction of transformation will also be negative.

Eigenvectors are vectors that are non-zero and they keep their direction when any linear transformation is applied. It changes by the factor a specific scalar. In a brief, we can say, if A is a linear transformation from a vector space V and **x** is a vector in V, which is not a zero vector, then v is an eigenvector of A if A(X) is a scalar multiple of **x**. Now when dealing with eigenspace of a vector x, it would consist of all the eigenvectors with the eigenvalue equivalent collectively with zero vector. Though, the zero vector is not an eigenvector.

Let us say A is an "$n \times n$" matrix and λ is an eigenvalue of matrix A, then **x**, a non-zero vector, is called as eigenvector if it satisfies the given the expression: $A\mathbf{x} = \lambda\mathbf{x}$

x is an eigenvector of A corresponding to eigenvalue, λ. Also we can note, the possibility of being many infinitely eigenvectors, just to only one corresponding eigenvalue. And for distinct eigenvalues, the eigenvectors are linearly dependent.

<u>Brief History</u>

As we learn eigenvalues, we are shown they are loosely related to linear algebra and matrix theory. Historically, however, they actually started with the study of quadratic forms and differential equations. Euler had also studied the rotational motion of a rigid body and discovered the importance of the principal axes. As Lagrange realized, the principal axes are the eigenvectors of the inertia matrix. In the early 19th century, Cauchy saw how their work could be used to classify the quadric surfaces, and generalized it to arbitrary dimensions. Cauchy also coined the characteristic root for what is now called eigenvalue; his term survives in characteristic equation. At the start of the 20th century, Hilbert studied the eigenvalues of integral operators by viewing the operators as infinite matrices. He was the first to use the German word eigen to denote eigenvalues and eigenvectors in 1904, emphasizing how important eigenvalues are to defining the unique nature of a specific transformation. The first numerical algorithm for computing eigenvalues and eigenvectors appeared in 1929, when Von Mises published the power method. One of the most popular methods today, the QR algorithm, was proposed independently by John G.F. Francis and Vera Kublanovskaya in 1961.

Solving Differential Equations

Eigenvalues and eigenvectors can be used as a method for solving linear systems of ordinary differential equations (ODEs). For the smaller systems it is not a tedious by hand work, it is rather a straight forward method. We will see the method of solving systems of ODE's and also for linear algebra/Jacobian matrix. When trying to solve large systems of ODEs however, it is usually best to use some sort program, such python, this is case for myself. Lets see a simple example to solving ODE's. Here we have dx/dt=4x+8y and dy/dt=10x+2y. This will give us,

$$A = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 10 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

We can solve by hand or we can use python coding to solve the differential equations. As you can see we are looking for the eigenvalues and eigenvectors in the first set of equations.

$$det(A - \lambda I) = 0$$
$$det\left(\begin{bmatrix} 4 & 8 \\ 10 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$det \begin{bmatrix} 4-\lambda & 8 \\ 10 & 2-\lambda \end{bmatrix} = 0$$
$$(4-\lambda)(2-\lambda) - 80 = 0$$
$$(\lambda - 12)(\lambda + 6) = 0$$

$$(A - \lambda_1 I)\vec{v} = 0$$
$$\begin{bmatrix} 4-12 & 8-0 \\ 10-0 & 2-12 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$
$$\begin{bmatrix} -8 & 8 \\ 10 & -10 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$
$$y = x$$

$$(A - \lambda_2 I)\vec{v} = 0$$
$$\begin{bmatrix} 4+6 & 8-0 \\ 10-0 & 2+6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$
$$\begin{bmatrix} 10 & 8 \\ 10 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$
$$y = -\frac{5}{4}x$$

These two eigenvalues and associated eigenvectors yield the solution:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{12t} + \ c_2 \begin{bmatrix} 4 \\ -5 \end{bmatrix} e^{-6t}$$

As you can see, it can be simple to perform these equation but they grow to be tedious for more complex systems. Here are examples of finding these values using python. You can see we have about the same values in both examples.

```
A = np.array([[4,8],[10,2]])
eigvals, eigvecs = la.eig(A)
print(eigvals)
print(eigvecs)
```

```
[12.+0.j -6.+0.j]
[[ 0.70710678 -0.62469505]
 [ 0.70710678  0.78086881]]
```

Why are they important?

The eigenvalues and eigenvectors different types of information, for example it provides us with a better understanding of linear transformation. Helps us see if there is stability with the equilibrium point. We can determine if the values are stable, unstable, oscillates, or are complex values. A stable fixed point is such that a system can be initially disturbed around its fixed point yet eventually return to its original location and remain there. An unstable fix point is essentially the opposite to stable. If a point is disturbed in any direction and will cause it to change location, it will be considered a unstable fix point. A fixed point would have eigenvalues of a linearized system around it, this will determine the stability behavior o a system. What can also come into play is the existence of real and imaginary components of the eigenvalues, along with the signs of the real components and the distinctness of their values.

We will start discussing about the imaginary/complex eigenvalues, we will put the eigenvalues in the form of a + bi, where a and b are real scalars and i is the imaginary number, and there will be three cases. The three cases are when the real part is positive, negative, and zero. For real parts being **Positive,** the system becomes unstable and you will see an unstable oscillator. In a graph a vector would be spiraling away from the fix point. This situation is usually undesirable when attempting to control a process or unit. For real parts being **Zero**, now we have a undamped oscillator. To imagine the vector in a graph, the vector acts a circle around the fix point. You would see an undamped oscillation to be common in control schemes arising out of competing controllers. Even though we fond this a common occurrence, its usually undesirable and still to be considered unstable since it will not go back to a steady state. Now we are left with the real parts being **Negative**, the system does become stable and also behaves as a damped oscillator. This can be visualized as a spiral unlike the positive parts, this will spiral inwards to the fix point. The is the desired situation we are looking for. This system is stable since steady state will be reached even after a disturbance to the system. As we see having haib complex eigenvalues with negative parts, is a requirement and sufficient condition of a stable system.

Its time to look at eigenvalues that only deal with the real parts, we will step away from the complex form of things. So now if an eigenvalue has no imaginary part and is equal to zero, it will continue to be unstable. The obvious reason is because the system will not become stable if its eigenvalues have any non-negative real parts. When all eigenvalues are real, positive, and distinct, the system is unstable. This can be said for the same reasoning from the previous case. In a graph, this example would be represented as a typical exponential plot against time. The next case is when the eigenvalues are real, negative and distinct, the system actually still becomes unstable as well. We can see this in a graph, it will output an inverse exponential plot against time. The last case we have to look at is a set of eigenvalues for a system that contains both positive and negative eigenvalues, the fix point will be a unstable saddle point. A saddle point is a series of minimum and maximum points that converges at one area in a gradient field, without hitting the point.

Testing using Python

I have explored the behaviors of the eigenvalues and eigenvectors myself through the use of python programing. First we are going to visually look at the linear transformation that takes place, and the eigenvector helps distinguish the process of transformation. The next example would be exploring the process in solving differential equations and testing the equilibrium point is stable or unstable.

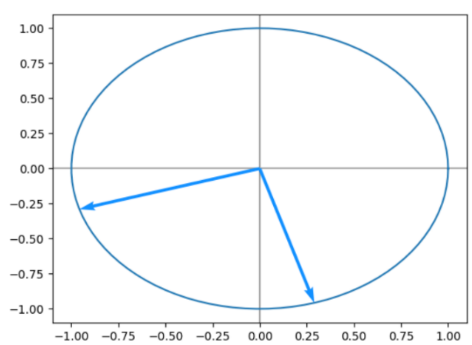
First example is visualizing the eigenvalue acting upon the eigenvector. First we have

Matrix A = $\begin{bmatrix} 5 & 1 \\ 3 & 3 \end{bmatrix}$. We also know one eigenvector of A is v = $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

We can use the equation **Av = λv** as $\begin{bmatrix} 5 & 1 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$. We can also see $6 \times \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$.

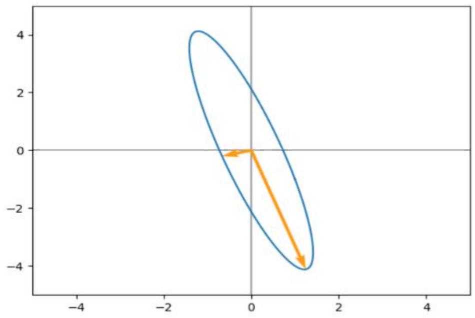
Which means that v is well an eigenvector of A. Also, the corresponding eigenvalue is λ=6. This will be represented through python to see if it holds the same direction. You can see the figure in the appendix section of the report.

The following example will display the linear transformation in action. Linear transformation is a mapping between an input vector and an output vector. Different operations like projection or rotation are linear transformations. Every linear transformations can be though as applying a matrix on the input vector. We will see the meaning of this graphically. For that purpose, let’s start by drawing the set of unit vectors (they are all vectors with a norm of 1).



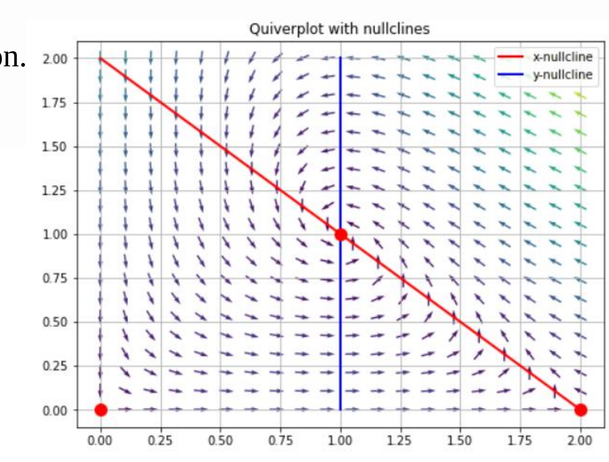
Then, we will transform each of these points by applying a matrix $A = \begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix}$. This is the goal of the function bellow that takes a matrix as input and will draw

- the origin set of unit vectors
- the transformed set of unit vectors
- the eigenvectors
- the eigenvectors scaled by their eigenvalues



You can see the transformation being displayed above. The eigenvectors to this transformation is Eigenvector = $\begin{bmatrix} -0.9570 & 0.2897 \\ -0.2897 & -0.9570 \end{bmatrix}$ and its corresponding eigenvalue are λ = $0.69722, \quad 4.3027$

The last example will be displaying the solution to differential equations. Through finding the solution, we find the real parts of the equations that will lead us to the equilibrium point. Eigenvalue and eigenvectors will provide the information to the stable point or the unstable point. The following example the equations we have used are $2x - x2 - xy$ and $-y + xy$. These equation will act as inputs to the program python, the code will essentially help us find the real parts to the equation. Once the code runs the fix points provided are (0, 0), (1, 1), and (2, 0). The only fix point that hold a real part is (1, 1) because their eigenvalues both came out as negative value, -0.5. Then the use of python, we will see a chart that helps us visualize the fix point. When the real part is negative, then the system is stable and behaves as a damped oscillator. This can be visualized as a vector tracing a spiral toward the fixed point (1, 1) in the graph. This code helped me understand the how to find nullclines in python.



<u>Applications of Eigenvalues and Eigenvectors</u>

Eigenvalues and eigenvectors are not only meant to be used for our understanding of natural occurrences, such as with our mathematical discoveries. It also can help us discover new and more efficient designs for the coming future. A great example would be designing the strongest column for supporting the weight of a roof using only a specified amount of material, what would be the shape of this sustainable column. You would think it would take shape as a ordinary cylinder like any other column you have seen through all the different types of buildings. But Steve Cox of Rice University and Michael Overton of New York University went with a different route in mind, they have discovered that the column would be the strongest if it was larger at the top then the middle and lastly the bottom. At the points of the way from either end, the column could be smaller because the column would not naturally buckle there anyway. The way these scholars achieved this feat was by the study of eigenvalues that involve the column. The calculation that were considered were the weight from above and positioning. They did realize that it is not the most efficient constructed column because it loses its integrity if pressure is applied to the side of it. But the columns main pressure point is the top of the column.

There are many other real life applications of the eigenvalues and eigenvectors of a square matrix. Communication systems is a prime example for the use eigenvalues and eigenvalues. Claude Shannon uses the eigenvalues to determine the theoretical limit to show how much information can be transmitted to a communication system such as telephone lines or even through the air. To achieve this, the eigenvectors and eigenvalues must be calculated of a communication channel, this channel is expressed in matrix form, then waterfilling on the eigenvalues. In the end of all the calculations, the eigenvalues therefor gains the fundamental modes of the channel, meaning the eigenvectors are captured. Another example is when it comes to designing bridges, the natural frequency of a bridge would be the eigenvalue of smallest magnitude of a specific system that models that bridge. Engineers are the ones that exploit this knowledge to ensure the stability of their construction. Designing car stereos systems uses the eigenvalues to analyze in their designs. This would help the car systems to reproduce the vibrations of the car due to the music or any other cause of vibrations from the stereo. Electrical Engineering have a great use of the eigenvalues and eigenvectors, its used for decoupling three phase systems through symmetrical component transformation. Lastly the mechanical engineers uses the values and vectors to "reduce" a linear operation to separate, simpler, problems. For example, if a stress is applied to a "plastic" solid, the deformation can be dissected into "principle directions"- those directions in which the deformation is greatest. Vectors in the principle directions are the eigenvectors and the percentage deformation in each principle direction is the corresponding eigenvalue.

Conclusion

The use of eigenvalues and eigenvectors is tremendous, and the applications continue to surprise me. This report did help me understand the importance of eigenvectors and how exactly it helps through linear transformation, since I am taking a course on computer graphics. It gives a clear understanding what is going on under the hood when coding the changes within objects. Eigenvalues help finding stability of a fix point within a system which proves to a clear help for many applications. Throughout this report I have found the purpose of using eigenvalues and eigenvectors and there clearly benefits of using them. Such as giving a high accuracy for linear systems and a general method that can be applied to a variety of processes. The only disadvantage I have noticed is its mostly applicable to linear systems when finding the stability of a system.

Appendix

Figure 1: Python code

```
A = np.array([[4,8],[10,2]])
eigvals, eigvecs = la.eig(A)
print(eigvals)
print(eigvecs)
```

Figure 2: Python Results

```
[12.+0.j -6.+0.j]
[[ 0.70710678 -0.62469505]
 [ 0.70710678  0.78086881]]
```

Figure 3: Eigenvector

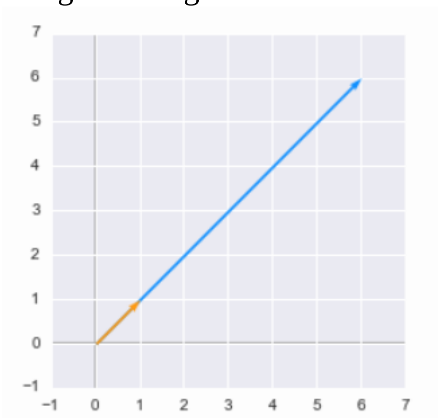


Figure 4: Unit Circle

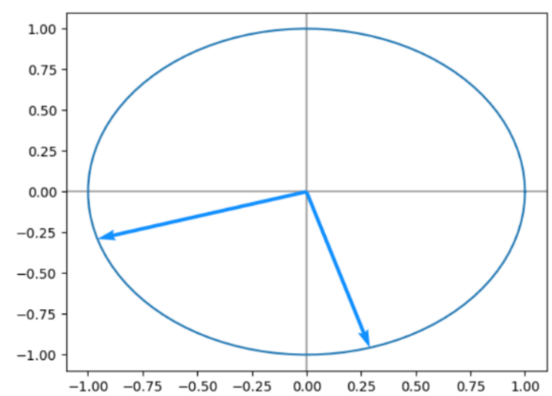


Figure 5: Transformed Unit Circle

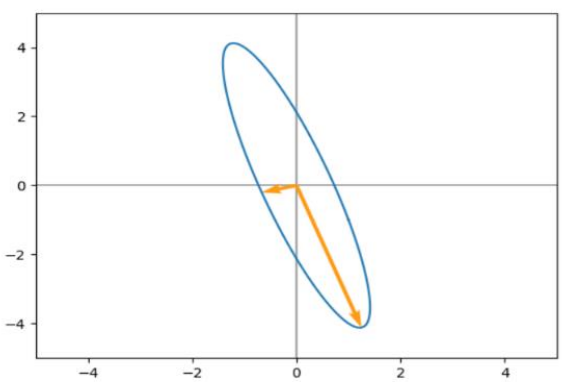
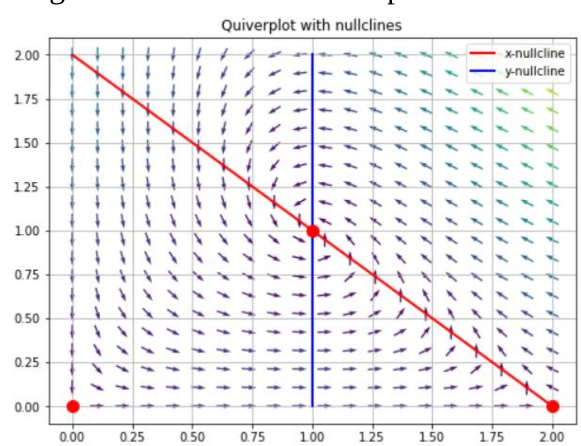


Figure 6: Nullclines of a fix point.



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