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Introduction to Python and Numerical Methods

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Stellar Evolution

Protostar formation

- **Molecular cloud**: If it is above the Jeans mass it will undergo gravitational collapse.
- Core of the cloud becomes opaque. Radiation associated with hydrogen dissociation **balances** gravity.
- **Protostar** formed. It acreates the rest of the cloud.

Early stages of a star

- If the temperature and the pressure of the protostar rise enough **thermonuclear fusion** starts and we say that a star has borned.
- The **mass of the newborn star** will determine its evolution and its end.

Stellar Evolution

Massive stars

- Above $8M_\odot$
- **Mechanisms of nuclear burning**: proton-proton chain, CNO cycle and triple-alpha process.
- Formation of an **iron-rich core**.

Neutron star formation

- When the nuclear fuel at the core is exhausted **degeneracy pressure of electrons** holds together the core.
- Growth of the core. **Neutrons form** via electron capture and a flood of neutrinos is released.
- **Supernova**. Neutrino flood eject outwards the in-falling outer layers.
- Remnant: **Neutron star**.

Neutron Stars

NS models

- Hydrostatic equilibrium
- Spherical symmetry:

$$ds^2 = -e^{2\phi(r)}dt^2 + \left(1 - \frac{2Gm\left(r\right)}{rc^2}\right)^{-1}dr^2 + r^2 d\Omega^2$$

- Ideal fluid:

$$T^{\mu\nu} = \left(\mu + \frac{P}{c^2}\right)u^\mu u^\nu + \frac{P}{c^2}g^{\mu\nu}$$

TOV equations

System of Coupled Ordinary Differential Equations

$$
\begin{aligned}
\frac{dP}{dr} &= -G(\mu + P/c^2)\frac{m + 4\pi r^3 P/c^2}{r(r - 2Gm/c^2)}, \\
\frac{dm}{dr} &= 4\pi r^2 \mu, \\
\frac{d\phi}{dr} &= \frac{m + 4\pi r^3 P/c^2}{r(r - 2Gm/c^2)}.
\end{aligned}
$$

Polytropic equations of state

$$
\begin{aligned}
\mu &= \rho\,(1 + \epsilon)\,, \\
P &= (\Gamma - 1)\,\rho\epsilon, \\
P &= K\rho^{\Gamma}.
\end{aligned}
$$

Boundary conditions

$$
\begin{aligned}
&P(r = 0) = K\rho_c^{\Gamma}, \quad && P(r = R) = 0, \\
&m(r = 0) = 0, && m(r = R) = M, \\
&\phi(r = 0) = 0, && \\
&\phi(r = R) = \frac{1}{2}\log\left(1 - 2GM/rc^2\right). &&
\end{aligned}
$$

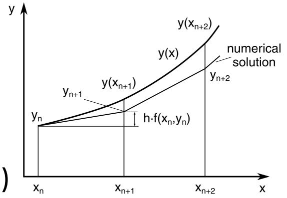
Numerical methods

Given an ODE:

$$\frac{dy}{dx} = f(x, y)$$

The solution would be a curve that for each point (x, y) its slope is $f(x, y)$. We can approximate the solution curve $y(x)$ by the following algorithm:

- Calculate $y'_0 = f(x_0, y_0)$
- Move to $x_1 = x_0 + h$, $y_1 = y_0 + hy'_0$
- Calculate $y'_1 = f(x_1, y_1)$
- Move to $x_2 = x_1 + h$, $y_2 = y_1 + hy'_1$
- And so on ... (where $h = \Delta x = x_{i+1} - x_i$)



Euler method

www.freecodecamp.org

$$y_{n+1} = y_n + hy'_n \quad , \quad x_{n+1} = x_n + h$$

Numerical methods

Runge-Kutta 2nd order method (Modified Euler method)

$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2} k_1)$$

$$y_{n+1} = y_n + hk_2 \quad , \quad x_{n+1} = x_i + h$$

Heun method

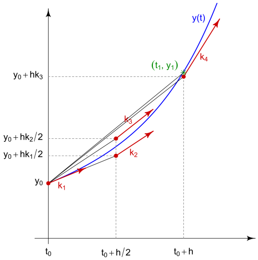
$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + h, y_n + hk_1)$$

$$y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2) \quad , \quad x_{n+1} = x_i + h$$

Numerical methods

$$\frac{dy}{dx} = f(x, y)$$



www.wikipedia.org

Runge-Kutta 4th order method

$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1)$$

$$k_3 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_2)$$

$$k_4 = f(x_n + h, y_n + hk_3)$$

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad , \quad x_{n+1} = x_i + h$$

Convergence test

$$
\begin{aligned}
\frac{y^{(1)} - y^{(2)}}{y^{(2)} - y^{(4)}} &= \frac{h^2 - \frac{1}{4}h^2 + \mathcal{O}(h^3)}{\frac{1}{4}h^2 - \frac{1}{16}h^2 + \mathcal{O}(h^3)} \\
&= \frac{h^2 + \mathcal{O}(h^3)}{\frac{1}{4}h^2 + \mathcal{O}(h^3)} \\
&= 4 + \mathcal{O}(h^3)
\end{aligned}
$$

Where $y^{(1)}$ is a solution calculated with an h resolution. $y^{(2)}$ has $h/2$ resolution and $y^{(4)}$ has $h/4$ resolution.