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# Galois theory

Notes

Thursday 9<sup>th</sup> February, 2023



# Preface

The notes correspond to the bachelor course *Galois theory* of the Vrije Universiteit Brussel, Faculty of Sciences, Department of Mathematics and Data Sciences. The course is divided into thirteen two-hours lectures.

The material is somewhat standard. Basic texts on fields and Galois theory are for example [2]. . .

As usual, we also mention a set of great expository papers by Keith Conrad available at <https://kconrad.math.uconn.edu/blurbs/>. The notes are extremely well-written and are useful at every stage of a mathematical career.

Thanks go to Alejandro de la Cueva Merino, Wannes Malfait.

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# Lecture 1

## §1. Fields

Recall that a **field** is a commutative ring such that  $1 \neq 0$  and that every non-zero element is invertible. Examples of (infinite) fields are  $\mathbb{Q}$ ,  $\mathbb{R}$  and  $\mathbb{C}$ . If  $p$  is a prime number, then  $\mathbb{Z}/p$  is a field.

**Example 1.1.** The abelian group  $\mathbb{Z}/2 \times \mathbb{Z}/2$  is a field with multiplication

$$(a, b)(c, d) = (ac + bd, ad + bc + bd).$$

**Example 1.2.**  $\mathbb{Q}(i) = \{a + bi : a, b \in \mathbb{Q}\}$  and  $\mathbb{Q}(\sqrt{2})$  are fields.

**Exercise 1.3.** Prove that  $\mathbb{Q}(i)$  and  $\mathbb{Q}(\sqrt{2})$  are not isomorphic as fields.

If  $R$  is a ring, there exists a unique ring homomorphism  $\mathbb{Z} \rightarrow R$ ,  $m \mapsto m1$ . The image  $\{m1 : m \in \mathbb{Z}\}$  of this homomorphism is a subring of  $R$  and it is known as the **ring of integers** of  $R$ . The kernel is a subgroup of  $\mathbb{Z}$  and hence it is generated by some  $t \geq 0$ . The integer  $t$  is the **characteristic** of the ring  $R$ .

**Exercise 1.4.** The characteristic of a field is either zero or a prime number.

Recall that a commutative ring  $R$  is an **integral domain** if  $xy = 0 \implies x = 0$  or  $y = 0$ . Fields are integral domains.

**Exercise 1.5.** Let  $K$  be a field. Prove that the following statements are equivalent:

- 1)  $K$  is of characteristic zero.
- 2) The additive order of 1 is infinite.
- 3) The additive order of each  $x \neq 0$  is infinite.
- 4) The ring of integers of  $K$  is isomorphic to  $\mathbb{Z}$ .

**Exercise 1.6.** Let  $K$  be a field. Prove that the following statements are equivalent:

- 1)  $K$  is of characteristic  $p$ .

- 2) The additive order of 1 is  $p$ .
- 3) The additive order of each  $x \neq 0$  is  $p$ .
- 4) The ring of integers of  $K$  is isomorphic to  $\mathbb{Z}/p$ .

**Definition 1.7.** A **subfield** of a ring  $R$  is a subring of  $R$  that is also a field.

Note that if  $K$  is a subfield of  $E$ , then the characteristic of  $K$  coincides with the characteristic of  $E$ . Moreover, if  $K \rightarrow L$  is a field homomorphism, then  $K$  and  $L$  have the same characteristic.

**Exercise 1.8.** Let  $K$  be a field of characteristic  $p$ . Prove that  $K \rightarrow K, x \mapsto x^{p^n}$ , is a field homomorphism for all  $n \in \mathbb{Z}_{\geq 0}$ .

Note that finite fields are of characteristic  $p$ .

Let  $K$  be a subfield of a field  $E$ . Then  $E$  is a  $K$ -vector space with the usual scalar multiplication  $K \times E \rightarrow E, (\lambda, x) \mapsto \lambda x$ .

**Definition 1.9.** A field  $K$  is **prime** if there are no proper subfields of  $K$ .

Examples of prime fields are  $\mathbb{Q}$  and  $\mathbb{Z}/p$  for a prime number  $p$ .

**Proposition 1.10.** Let  $K$  be a field. The following statements hold:

- 1)  $K$  contains a unique prime field, it is known as the **prime subfield** of  $K$ .
- 2) The prime subfield of  $K$  is either isomorphic to  $\mathbb{Q}$  if the characteristic of  $K$  is zero, or it is isomorphic to  $\mathbb{Z}/p$  for some prime number  $p$  if the characteristic of  $K$  is  $p$ .

*Proof.* To prove the first claim let  $L$  be the intersection of all the subfields of  $K$ . Then  $L$  is a subfield of  $K$ . If  $F$  is a subfield of  $L$ , then  $F$  is a subfield of  $K$ . Thus  $L \subseteq F$  and hence  $F = L$ , which proves that  $L$  is prime. If  $L_1$  is a subfield of  $K$  and  $L_1$  is prime, then  $L \subseteq L_1$  and hence  $L = L_1$ .

Let  $K_0$  be the prime field of  $K$ . Suppose that  $K$  is of characteristic  $p > 0$ . Then the ring  $K_{\mathbb{Z}}$  of integers of  $K$  is a field isomorphic to  $\mathbb{Z}/p$  and hence  $K_0 \simeq K_{\mathbb{Z}}$ . Suppose now that the characteristic of  $K$  is zero. Let  $E = \{m/1/n : m, n \in \mathbb{Z}, n \neq 0\}$ . We claim that  $K_0 = E$ . Since  $K_{\mathbb{Z}} \subseteq K_0$ , it follows that  $E \subseteq K_0$ . Hence  $E = K_0$ , as  $E$  is a subfield of  $K$ .  $\square$

**Definition 1.11.** Let  $E$  be a field and  $K$  be a subfield of  $E$ . Then  $E$  is an **extension** of  $K$ . We will use the notation  $E/K$ .

If  $E$  is an extension of  $K$ , then  $E$  is a  $K$ -vector space.

**Definition 1.12.** The degree of an extension  $E$  of  $K$  is the integer  $\dim_K E$ . It will be denoted by  $[E : K]$ .

We say that  $E$  is a finite extension of  $K$  if  $[E : K]$  is finite.

**Example 1.13.** Let  $K$  be a field. Then  $[K : K] = 1$ . Conversely, if  $E$  is an extension of  $K$  and  $[E : K] = 1$ , then  $K = E$ . If not, let  $x \in E \setminus K$ . We claim that  $\{1, x\}$  is linearly independent over  $K$ . Indeed, if  $a + bx = 0$  for some  $a, b \in K$ , then  $bx = -a$ . If  $b \neq 0$ , then  $x = -a/b \in K$ , a contradiction. If  $b = 0$ , then  $a = 0$ .

We know that  $[\mathbb{C} : \mathbb{R}] = 2$ .

**Example 1.14.** A basis of  $\mathbb{Q}(\sqrt{2})$  over  $\mathbb{Q}$  is given by  $\{1, \sqrt{2}\}$ . Then  $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$ . The calculations can be easily done by computer:

```
julia> E, a = quadratic_field(2)
(Real quadratic field defined by x^2 - 2, sqrt(2))
julia> characteristic(E)
0
julia> K = prime_field(E)
Rational Field
julia> degree(E)
2
julia> basis(E)
2-element Vector{nf_elem}:
 1
 sqrt(2)
julia> one(K) == one(E)
true
julia> zero(K) == zero(E)
true
```

**Example 1.15.** Since  $\mathbb{Q}$  is numerable and  $\mathbb{R}$  is not,  $[\mathbb{R} : \mathbb{Q}] > \aleph_0$ . If  $\{x_i : i \in \mathbb{Z}_{>0}\}$  is a numerable basis of  $\mathbb{R}$  over  $\mathbb{Q}$ , for each  $n$  consider the  $\mathbb{Q}$ -vector space  $V_n$  generated by  $\{x_1, \dots, x_n\}$ . Then

$$\mathbb{R} = \bigcup_{n \geq 1} V_n,$$

is numerable, as each  $V_n$  is numerable, a contradiction.

If  $E$  is an extension of  $K$  and  $E$  is finite, then  $[E : K]$  is finite.

**Proposition 1.16.** Let  $K$  be a finite field. Then  $|K| = p^m$  for some prime number  $p$  and some  $m \geq 1$ .

*Proof.* We know that the prime subfield of  $K$  is isomorphic to  $\mathbb{Z}/p$ . In particular,  $|K_0| = p$ . Since  $K$  is finite,  $[K : K_0] = m$  for some  $m$ . If  $\{x_1, \dots, x_m\}$  is a basis of  $K$  over  $K_0$ , then each element of  $K$  can be written uniquely as  $\sum_{i=1}^m a_i x_i$  for some  $a_1, \dots, a_m \in K_0$ . Then  $K \simeq K_0^m$  and hence  $|K| = |K_0^m| = p^m$ .  $\square$

We now perform some basic calculations with a finite field of eight elements:

```

julia> E, x = FiniteField(2, 3, "x")
(Finite field of degree 3 over F_2, x)
julia> characteristic(E)
2
julia> prime_field(E)
Galois field with characteristic 2
julia> degree(E)
3
julia> size(E)
8
julia> [z for z in E]
8-element Vector{fq_nmod}:
 0
 1
 x
 x + 1
 x^2
 x^2 + 1
 x^2 + x
 x^2 + x + 1

```

**Definition 1.17.** Let  $E$  be an extension of  $K$ . A **subextension**  $F$  of  $K$  is a subfield  $F$  of  $E$  that contains  $K$ , that is  $K \subseteq F \subseteq E$ .

**Definition 1.18.** Let  $E$  and  $E_1$  be extensions over  $K$ . An extension **homomorphism**  $E \rightarrow E_1$  is a field homomorphism  $\sigma: E \rightarrow E_1$  such that  $\sigma(x) = x$  for all  $x \in K$ .

To describe the homomorphism  $\sigma: E \rightarrow E_1$  of the extensions over  $K$  one typically writes the commutative diagram

$$\begin{array}{ccc}
 K & \xlongequal{\quad} & K \\
 \downarrow & & \downarrow \\
 E & \xrightarrow{\sigma} & E_1
 \end{array}$$

We write  $\text{Hom}(E/K, E_1/K)$  to denote the set of homomorphism  $E \rightarrow E_1$  of extensions of  $K$ . Note that if  $\sigma \in \text{Hom}(E/K, E_1/K)$ , then  $\sigma$  is a  $K$ -linear map, as

$$\sigma(\lambda x) = \sigma(\lambda)\sigma(x) = \lambda\sigma(x)$$

for all  $\lambda \in K$  and  $x \in E$ .

**Example 1.19.** The conjugation map  $\mathbb{C} \rightarrow \mathbb{C}, z \mapsto \bar{z}$ , is an endomorphism of  $\mathbb{C}$  as an extension over  $\mathbb{R}$ . Let  $\varphi \in \text{Hom}(\mathbb{C}/\mathbb{R}, \mathbb{C}/\mathbb{R})$ . Then

$$\varphi(x + iy) = \varphi(x) + \varphi(i)\varphi(y) = x + \varphi(i)y$$

§1 Fields

for all  $x, y \in \mathbb{R}$ . Since  $\varphi(i)^2 = \varphi(i^2) = \varphi(-1) = -1$ , it follows that  $\varphi(i) \in \{-i, i\}$ . Thus either  $\varphi(x+iy) = x+iy$  or  $\varphi(x+iy) = x-iy$ .

**Exercise 1.20.** Prove that if  $K$  is a field and  $\sigma: K \rightarrow K$  is a field homomorphism, then  $\sigma \in \text{Hom}(K/K_0, K/K_0)$ .

If  $E/K$  is an extension, then

$$\text{Aut}(E/K) = \{\sigma : \sigma: E \rightarrow E \text{ is a bijective extension homomorphism}\}$$

is a group with composition.

**Definition 1.21.** Let  $E/K$  be an extension. The **Galois group** of  $E/K$  is the group  $\text{Aut}(E/K)$  and it will be denoted by  $\text{Gal}(E/K)$ .

A typical example:  $\text{Gal}(\mathbb{C}/\mathbb{R}) \simeq \mathbb{Z}/2$ .

As an example, we show with the computer that  $\text{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q}) \simeq \mathbb{Z}/2$ :

```
julia> E, x = quadratic_field(2)
(Real quadratic field defined by x^2 - 2, sqrt(2))
julia> characteristic(E)
0
julia> G, C = galois_group(E);
julia> describe(G)
"C2"
julia> order(G)
2
```

**Example 1.22.** Let  $\theta = \sqrt[3]{2}$  and let  $E = \{a + b\theta + c\theta^2 : a, b, c \in \mathbb{Q}\}$ . Note that

$$a + b\theta + c\theta^2 = 0 \iff a = b = c = 0.$$

Then  $E$  is an extension of  $\mathbb{Q}$  such that  $[E : \mathbb{Q}] = 3$ . We claim that  $\text{Gal}(E/\mathbb{Q})$  is trivial. If  $\sigma \in \text{Gal}(E/\mathbb{Q})$  and  $z = a + b\theta + c\theta^2$ , then  $\sigma(z) = a + b\sigma(\theta) + c\sigma^2(\theta)$ . Since  $\sigma(\theta)^3 = \sigma(\theta^3) = \sigma(2) = 2$ , it follows that  $\sigma(\theta) = \theta$  and therefore  $\sigma = \text{id}$ .

**Exercise 1.23.** Prove that the polynomial  $X^3 - 2$  is irreducible in  $\mathbb{Q}[X]$ .

The previous exercise can easily be solved using computers:

```
julia> R, x = PolynomialRing(QQ, "x");
julia> is_irreducible(x^3-2)
true
```

The following exercise is known as the *Eisenstein's irreducibility criterion*:

**Exercise 1.24.** Let  $A$  be a unique factorization domain and  $K$  be its fraction field. Let  $f = \sum_{i=0}^n a_i X^i \in K[X]$  be a polynomial of degree  $n > 0$ . Assume that there exists a prime element  $p \in A$  such that  $p \mid a_i$  for all  $i \in \{0, 1, \dots, n-1\}$ ,  $p \nmid a_n$  and  $p^2 \nmid a_0$ . Then  $f$  is irreducible in  $K[X]$ .

**Exercise 1.25.** Prove that the polynomials  $f = X^{10} + 60X^7 + 82X^6 - 36X^3 + 2$  and  $g = 3X^{10} + 15X^2 - 45$  are irreducible in  $\mathbb{Z}[X]$ .

**Exercise 1.26.** Is the polynomial  $f = 3(X^{10} + 5X^2 - 15)$  irreducible in  $\mathbb{Z}[X]$ ?

## Lecture 2

If  $E/K$  is an extension and  $S$  is a subset of  $E$ , then there exists a unique smallest subextension  $F/K$  of  $E/K$  such that  $S \subseteq F$ . In fact,

$$F = \bigcap \{T : T \text{ is a subfield of } E \text{ that contains } K \cup S\}$$

If  $L/K$  is a subextension of  $E/K$  such that  $S \subseteq L$ , then  $F \subseteq L$  by definition. The extension  $F$  is known as the **subextension generated by  $S$**  and it will be denoted by  $K(S)$ . If  $S = \{x_1, \dots, x_n\}$  is finite, then  $K(S) = K(x_1, \dots, x_n)$  is said to be of **finite type**.

**Example 1.27.** If  $\{e_1, \dots, e_n\}$  is a basis of  $E$  over  $K$ , then  $E = K(e_1, \dots, e_n)$ .

**Example 1.28.** The field  $\mathbb{Q}(\sqrt{2})$  is precisely the extension of  $\mathbb{R}/\mathbb{Q}$  generated by  $\sqrt{2}$ .

Let  $E/K$  be an extension and  $S$  and  $T$  be subsets of  $E$ . Then

$$K(S \cup T) = K(S)(T) = K(T)(S).$$

If, moreover,  $S \subseteq T$ , then  $K(S) \subseteq K(T)$ .

### §2. Algebraic extensions

**Definition 2.1.** Let  $E/K$  be an extension. An element  $x \in E$  is **algebraic** over  $K$  if there exists a non-zero polynomial  $f(X) \in K[X]$  such that  $f(x) = 0$ . If  $x$  is not algebraic over  $K$ , then it is called **transcendent** over  $K$ .

If  $E/K$  is an extension, let

$$\overline{K}_E = \{x \in E : x \text{ is algebraic over } K\}.$$

**Definition 2.2.** An extension  $E/K$  is **algebraic** if every  $x \in E$  is algebraic over  $K$ .

If  $K$  is a field, every  $x \in K$  is algebraic over  $K$ , as  $x$  is a root of  $X - x \in K[X]$ . In particular,  $K/K$  is an algebraic extension.

**Example 2.3.**  $\mathbb{C}/\mathbb{R}$  is an algebraic extension. If  $z \in \mathbb{C} \setminus \mathbb{R}$ , then  $z$  is a root of the polynomial  $X^2 + (z + \bar{z})X + |z|^2 \in \mathbb{R}[X]$ .

If  $F/K$  is an algebraic extension  $x \in E$  is algebraic over  $K$  for some field  $E \supseteq F$ , then  $x$  is algebraic over  $F$ .

**Example 2.4.**  $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$  is algebraic, as the number  $a + b\sqrt{2}$  is a root of the polynomial  $X^2 - 2aX + (a^2 - 2b^2) \in \mathbb{Q}[X]$ .

The extension  $\mathbb{C}/\mathbb{Q}$  is not algebraic.

If  $E/K$  is an extension and  $x \in E$  is algebraic over  $K$ , then the evaluation homomorphism  $K[X] \rightarrow E$ ,  $f \mapsto f(x)$ , is not injective. In particular, its kernel is a non-zero ideal. Hence it is generated by a monic polynomial  $f$ .

**Definition 2.5.** Let  $E/K$  be an extension and  $x \in E$  be an algebraic element. The monic polynomial that generates the kernel of  $K[X] \rightarrow E$ ,  $f \mapsto f(x)$ , is known as the **minimal polynomial** of  $x$  over  $K$  and it will be denoted by  $f(x, K)$ . The **degree** of  $x$  over  $K$  is then  $\deg f(x, K)$ .

Some basic properties of the minimal polynomial of an algebraic element:

**Proposition 2.6.** Let  $E/K$  be an extension and  $x \in E$ .

- 1) If  $g \in K[X] \setminus \{0\}$  is such that  $g(x) = 0$ , then  $f(x, K)$  divides  $g$ . In particular,  $\deg f(x, K) \leq \deg g$ .
- 2)  $f(x, K)$  is irreducible in  $K[X]$ .
- 3) If  $F/K$  is a subextension of  $E/K$ , then  $f(x, F)$  divides  $f(x, K)$ .

*Proof.* Write  $f = f(x, K)$  to denote the minimal polynomial of  $x$ . To prove 1) note that  $g(x) = 0$  implies that  $g$  belongs to the kernel of the evaluation map, so  $g$  is a multiple of  $f$ . To prove 2) note that if  $f = pq$  for some  $p, q \in K[X]$  such that  $0 < \deg p, \deg q < \deg f$ , then  $f(x) = 0$  implies that either  $p(x) = 0$  or  $q(x) = 0$ , a contradiction. Finally we prove 3). Since  $f \in K[X] \subseteq F[X]$  and  $f(x) = 0$ , it follows from 1) that  $f(x, F)$  divides  $f$ .  $\square$

Some easy examples:  $f(i, \mathbb{R}) = f(i, \mathbb{C}) = X^2 + 1$  and  $f(\sqrt[3]{2}, \mathbb{Q}) = X^3 - 2$ :

```
julia> E, x = radical_extension(3, QQ(2), "x");
julia> minpoly(x)
x^3 - 2
julia> F, y = quadratic_field(-1);
julia> minpoly(y)
x^2 + 1
```



**Example 2.7.** Let us compute  $f(\sqrt{2} + \sqrt{3}, \mathbb{Q})$ . Let  $\alpha = \sqrt{2} + \sqrt{3}$ . Then

$$\begin{aligned}\alpha - \sqrt{2} = \sqrt{3} &\implies (\alpha - \sqrt{2})^2 = 3 \implies \alpha^2 - 2\sqrt{2}\alpha + 2 = 3 \\ &\implies \alpha^2 - 1 = 2\sqrt{2}\alpha \implies (\alpha^2 - 1)^2 = 8\alpha^2 \implies \alpha^4 - 10\alpha^2 + 1 = 0.\end{aligned}$$

Thus  $\alpha$  is a root of  $g = X^4 - 10X^2 + 1$ . To prove that  $g = f(\alpha, \mathbb{Q})$  it is enough to prove that  $g$  is irreducible in  $\mathbb{Q}[X]$ . First note that the roots of  $g$  are  $\sqrt{2} + \sqrt{3}$ ,  $\sqrt{2} - \sqrt{3}$ ,  $-\sqrt{2} + \sqrt{3}$  and  $-\sqrt{2} - \sqrt{3}$ . This means that if  $g$  is not irreducible, then  $g = hh_1$  for some polynomials  $h, h_1 \in \mathbb{Q}[X]$  such that  $\deg h = \deg h_1 = 2$ . This is not possible, as  $(\sqrt{2} + \sqrt{3}) + (\sqrt{2} - \sqrt{3}) = 2\sqrt{2} \notin \mathbb{Q}$ ,  $(\sqrt{2} + \sqrt{3}) + (-\sqrt{2} + \sqrt{3}) = 2\sqrt{3} \notin \mathbb{Q}$  and  $(\sqrt{2} + \sqrt{3})(-\sqrt{2} - \sqrt{3}) = -5 - 2\sqrt{6} \notin \mathbb{Q}$ .

**Proposition 2.8.** Let  $F/K$  be a subextension and  $E/K$ . Then

$$[E : K] = [E : F][F : K].$$

*Proof.* Let  $\{e_i : i \in I\}$  be a basis of  $E$  over  $F$  and  $\{f_j : j \in J\}$  be a basis of  $F$  over  $K$ . If  $x \in E$ , then  $x = \sum_i \lambda_i e_i$  (finite sum) for some  $\lambda_i \in F$ . For each  $i$ ,  $\lambda_i = \sum_j a_{ij} f_j$  (finite sum) for some  $a_{ij} \in K$ . Then  $x = \sum_i \sum_j a_{ij} (f_j e_i)$ . This means that  $\{f_j e_i : i \in I, j \in J\}$  generates  $E$  as a  $K$ -vector space. Let us prove that  $\{f_j e_i : i \in I, j \in J\}$  is linearly independent. If  $\sum_i \sum_j a_{ij} (f_j e_i) = 0$  (finite sum) for some  $a_{ij} \in K$ , then

$$\begin{aligned}0 = \sum_i \left( \sum_j a_{ij} f_j \right) e_i &\implies \sum_j a_{ij} f_j = 0 \text{ for all } i \in I \\ &\implies a_{ij} = 0 \text{ for all } i \in I \text{ and } j \in J. \quad \square\end{aligned}$$

We state a lemma:

**Lemma 2.9.** If  $A$  is a finite-dimensional commutative algebra over  $K$  and  $A$  is an integral domain, then  $A$  is a field.

*Proof.* Let  $a \in A \setminus \{0\}$ . We need to prove that there exists  $b \in A$  such that  $ab = 1$ . Let  $\theta : A \rightarrow A$ ,  $x \mapsto ax$ . Clearly  $\theta$  is an algebra homomorphism. It is injective, since  $A$  is an integral domain. Since  $\dim_K A < \infty$ , it follows that  $\theta$  is an isomorphism. In particular,  $\theta(A) = A$ , which means that there exists  $b \in A$  such that  $1 = ab$ .  $\square$

Let  $E/K$  be an extension and  $x \in E \setminus K$ . Then

$$K[x] = \{y = f(x) : \text{for some } f \in K[X]\}$$

is a subring of  $E$  that contains  $K$ . More generally, if  $x_1, \dots, x_n \in E$ , then

$$K[x_1, \dots, x_n] = \{f(x_1, \dots, x_n) : f \in K[X_1, \dots, X_n]\}$$

is a subring of  $E$ . Clearly,  $K[x_1, \dots, x_n]$  is a domain and

$$K(x_1, \dots, x_n) = \left\{ \frac{f(x_1, \dots, x_n)}{g(x_1, \dots, x_n)} : f, g \in K[X_1, \dots, X_m] \text{ with } g(x_1, \dots, x_n) \neq 0 \right\}$$

is the extension of  $K$  generated by  $x_1, \dots, x_n$ . Note that

$$K(x_1, \dots, x_n) = (K(x_1, \dots, x_{n-1}))(x_n).$$

The previous construction can be generalized. Let  $I$  be a non-empty set. For each  $i \in I$  let  $X_i$  be an indeterminate. Consider the polynomial ring  $K[\{X_i : i \in I\}]$  and let  $S = \{x_i : i \in I\}$  be a subset of  $E$ . There exists a unique algebras homomorphism  $K[\{X_i : i \in I\}] \rightarrow E$  such that  $X_i \mapsto x_i$  for all  $i \in I$ . The image is denoted by  $K[S]$ .

**Exercise 2.10.** Prove that  $\mathbb{Q}[\sqrt{2}] = \mathbb{Q}(\sqrt{2})$ .

**Theorem 2.11.** Let  $E/K$  be an extension and  $x \in E \setminus K$ . The following statements are equivalent:

- 1)  $x$  is algebraic over  $K$ .
- 2)  $\dim_K K[x] < \infty$ .
- 3)  $K[x]$  is a field.
- 4)  $K[x] = K(x)$ .

*Proof.* We first prove 1)  $\implies$  2). Let  $z \in K[x]$ , say  $z = h(x)$  for some  $h \in K[X]$ . There exists  $g \in K[X]$  such that  $g \neq 0$  and  $g(x) = 0$ . Divide  $h$  by  $g$  to obtain polynomials  $q, r \in K[X]$  such that  $h = gq + r$ , where  $r = 0$  or  $\deg r < \deg g$ . This implies that

$$z = h(x) = g(x)q(x) + r(x) = r(x).$$

If  $\deg g = m$ , then  $r = \sum_{i=0}^{m-1} a_i X^i$  for some  $a_0, \dots, a_{m-1} \in K$ . Thus  $z = \sum_{i=0}^{m-1} a_i x^i$ , so  $K[x] \subseteq \langle 1, x, \dots, x^{m-1} \rangle$ .

The previous lemma proves that 2)  $\implies$  3).

It is trivial that 3)  $\implies$  4).

It remains to prove that 4)  $\implies$  1). Since  $x \neq 0$ ,  $1/x \in K[x]$ . There exists  $a_0, \dots, a_n \in K$  such that  $1/x = a_0 + a_1 x + \dots + a_n x^n$ . Thus

$$a_n x^{n+1} + \dots + a_1 x^2 + a_0 x - 1 = 0$$

so  $x$  is a root of  $a_n X^{n+1} + \dots + a_0 X - 1 \in K[X] \setminus \{0\}$ . □

Note that if  $x$  is algebraic over  $K$ , then  $K[x] \simeq K[X]/(f(x, K))$ .

**Corollary 2.12.** If  $E/K$  is finite, then  $E/K$  is algebraic.

*Proof.* Let  $n = [E : K]$  and  $x \in E$ . The set  $\{1, x, \dots, x^n\}$  is linearly dependent, so there exist  $a_0, \dots, a_n \in K$  not all zero such that  $a_0 + a_1 x + \dots + a_n x^n = 0$ . Thus  $x$  is a root of the non-zero polynomial  $a_0 + a_1 X + \dots + a_n X^n \in K[X]$ . □

We note that the converse of the previous corollary does not hold.

**Corollary 2.13.** *If  $E/K$  is an extension and  $x_1, \dots, x_n \in E$  are algebraic over  $K$ , then  $K(x_1, \dots, x_n)/K$  is finite and  $K(x_1, \dots, x_n) = K[x_1, \dots, x_n]$ .*

*Proof.* We proceed by induction on  $n$ . The case  $n = 1$  follows immediately from the theorem. So assume the result holds for some  $n \geq 1$ . Since the extensions  $K(x_1, \dots, x_n)/K(x_1, \dots, x_{n-1})$  and  $K(x_1, \dots, x_{n-1})/K$  are both finite, it follows that  $K(x_1, \dots, x_n)/K$  is finite. Moreover,

$$\begin{aligned} K(x_1, \dots, x_n) &= K(x_1, \dots, x_{n-1})(x_n) \\ &= K(x_1, \dots, x_{n-1})[x_n] = K[x_1, \dots, x_{n-1}][x_n] = K[x_1, \dots, x_n]. \quad \square \end{aligned}$$

**Corollary 2.14.** *Let  $E = K(S)$ . Then  $E/K$  is algebraic if and only if  $x$  is algebraic over  $K$  for all  $x \in S$ .*

*Proof.* Let us prove the non-trivial implication. Let  $z \in K(S)$ . In particular, there exists a finite subset  $T \subseteq S$  such that  $z \in K(T)$ . The previous corollary implies that  $K(T)/K$  is algebraic and hence  $z$  is algebraic.  $\square$

**Corollary 2.15.** *If  $E/K$  is an extension, then  $\overline{K}_E$  is a subfield of  $E$  that contains  $K$ . Moreover,  $K(\overline{K}_E)/K$  is algebraic.*

*Proof.* By definition,  $K(\overline{K}_E)/K$  is algebraic. Thus  $K(\overline{K}_E) \subseteq \overline{K}_E$ . From this it follows that  $K(\overline{K}_E) = \overline{K}_E$ .  $\square$

The following exercise is now almost trivial:

**Exercise 2.16.** Let  $E/K$  be an extension of finite type. Prove that  $E/K$  is algebraic if and only if  $E/K$  is finite.

Let  $\overline{\mathbb{Q}} = \{\alpha \in \mathbb{C} : \alpha \text{ is algebraic over } \mathbb{Q}\}$ . Then  $\overline{\mathbb{Q}}$  is the field of algebraic numbers. Can you compute  $[\overline{\mathbb{Q}} : \mathbb{Q}]$ ?

**Exercise 2.17.** Prove that  $[\mathbb{Q}[\sqrt[3]{2}] : \mathbb{Q}] = 3$ .

For the previous exercise, you may use Eisenstein's criterion.

**Exercise 2.18.** Let  $E = \mathbb{Q}[i, \sqrt{2}] = \mathbb{Q}[\sqrt{2}][i]$ . Prove that  $[E : \mathbb{Q}] = 4$ .

**Exercise 2.19.** Let  $E = \mathbb{Q}[\sqrt{2}, \sqrt[3]{5}]$ .

- 1) Compute  $[E : \mathbb{Q}]$ .
- 2) Prove that  $E = \mathbb{Q}[\sqrt{2} + \sqrt[3]{5}]$ .
- 3) Find the minimal polynomial of  $\sqrt{2} + \sqrt[3]{5}$  over  $\mathbb{Q}$ .

**Exercise 2.20.** Find the minimal polynomials of  $\sqrt[4]{3}i$  over  $\mathbb{Q}[i]$  and over  $\mathbb{Q}[\sqrt{3}]$ .

**Exercise 2.21.** Find the minimal polynomial of  $\sqrt{2} + \sqrt[3]{5}i$  over  $\mathbb{Q}[i]$ .



## Lecture 3

Algebraic field extensions form a nice class of extensions. The same happens with finite field extensions.

**Proposition 2.22.** *Let  $F/K$  be a subextension of  $E/K$ . Then  $E/K$  is algebraic if and only if  $E/F$  and  $F/K$  are algebraic.*

*Proof.* We know that if  $E/K$  is algebraic, then  $E/F$  and  $F/K$  are both algebraic. Let us assume that  $E/F$  and  $F/K$  are both algebraic. Let  $x \in E$  and let  $L$  be the subextension over  $K$  generated by the coefficients of  $f(x, F)$ , the minimal polynomial of  $x$  over  $F$ . Then  $L/K$  is finite, since it is generated by finitely many algebraic elements. Moreover,  $x$  is algebraic over  $L$ . Since

$$[L(x) : K] = [L(x) : L][L : K] < \infty,$$

$L(x)/K$  is algebraic. In particular,  $x$  is algebraic over  $K$ . □

**Exercise 2.23.** Let  $F/K$  be a subextension of  $E/K$ . Prove that  $E/K$  is finite if and only if  $E/F$  and  $F/K$  are finite.

Let  $F \subseteq E$  and  $L \subseteq E$ . The composite of  $F$  and  $L$  is defined as

$$FL = K(F \cup L) = F(L) = L(F)$$

and it is equal to the smallest field that contains  $F$  and  $L$ .

**Exercise 2.24.** If  $F = \mathbb{Q}(\sqrt{2})$  and  $L = \mathbb{Q}(\sqrt{3})$ , then  $FL = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ . Compute  $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}]$  and  $\mathbb{Q}(\sqrt{2}) \cap \mathbb{Q}(\sqrt{3})$ .

**Exercise 2.25.** Let  $\xi \in \mathbb{C}$  be a primitive cubic root of one. If  $F = \mathbb{Q}(\sqrt[3]{2})$  and  $L = \mathbb{Q}(\xi)$ , then  $FL = \mathbb{Q}(\sqrt[3]{2}, \xi)$ . Compute  $[\mathbb{Q}(\sqrt[3]{2}, \xi) : \mathbb{Q}]$  and  $\mathbb{Q}(\sqrt[3]{2}) \cap \mathbb{Q}(\xi)$ .

**Exercise 2.26.** Let  $E/K$  and  $F/K$  be extensions, where both  $E$  and  $F$  are subfields of a field  $L$ . If  $F/K$  is algebraic, then  $EF/E$  is algebraic.

**Exercise 2.27.** Let  $E/K$  and  $F/K$  be extensions, where both  $E$  and  $F$  are subfields of a field  $L$ . If  $F/K$  is finite, then  $EF/E$  is finite.

The solution to the previous exercise shows, in particular, that  $[EF : E] \leq [F : K]$ .

**Lemma 2.28.** Let  $\sigma : K \rightarrow L$  be a field homomorphism. Then there exists an extension  $E/K$  and a field isomorphism  $\varphi : E \rightarrow L$  such that  $\varphi|_K = \sigma$ .

*Proof.* Let  $A$  be a set in bijection with  $L \setminus \sigma(K)$  and disjoint with  $K$ . Let  $E = K \cup A$ . If  $\theta : A \rightarrow L \setminus \sigma(K)$  is bijective, then let

$$\varphi : E \rightarrow L, \quad \varphi(x) = \begin{cases} \sigma(x) & \text{if } x \in K, \\ \theta(x) & \text{if } x \in A. \end{cases}$$

Then  $\varphi$  is a bijective map such that  $\varphi|_K = \sigma$ . Transport the operations of  $L$  onto  $E$ , that is to define binary operations on  $E$  as follows:

$$(x, y) \mapsto x \oplus y = \varphi^{-1}(\varphi(x) + \varphi(y)), \quad (x, y) \mapsto x \odot y = \varphi^{-1}(\varphi(x)\varphi(y)).$$

Then, for example,

$$x \oplus y = \varphi^{-1}(\varphi(x) + \varphi(y)) = \varphi^{-1}(\sigma(x) + \sigma(y)) = \varphi^{-1}(\sigma(x+y)) = \varphi^{-1}(\varphi(x+y)) = x+y$$

for all  $x, y \in K$ .  $\square$

If  $\sigma : A \rightarrow B$  is a ring homomorphism, then  $\sigma$  induces a ring homomorphism  $\overline{\sigma} : A[X] \rightarrow B[X]$ ,  $\sum_i a_i X^i \mapsto \sum \sigma(a_i) X^i$ .

**Theorem 2.29.** Let  $K$  be a field and  $f \in K[X]$  be such that  $\deg f > 0$ . Then there exists an extension  $E/K$  such that  $f$  admits a root in  $E$ .

*Proof.* We may assume that  $f$  is irreducible over  $K$ . Let  $L = K[X]/(f)$  and  $\pi : K[X] \rightarrow L$  be the canonical map. Then  $L$  is a field (the reader should explain why). Let  $\sigma : K \rightarrow L$ ,  $a \mapsto \pi(aX^0)$ , and  $g = \overline{\sigma}(f) \in L[X]$ .

We claim that  $\pi(X)$  is a root of  $g$  in  $L$ . Suppose that  $f = \sum_i a_i X^i$ . Then

$$\begin{aligned} g(\pi(X)) &= \overline{\sigma}(f)(\pi(X)) \\ &= \sum_i \sigma(a_i) \pi(X)^i = \sum_i \pi(a_i X^0) \pi(X^i) = \pi\left(\sum_i a_i X^i\right) = \pi(f) = 0. \end{aligned}$$

The previous lemma states that there exists an extension  $E/K$  and an isomorphism  $\varphi : E \rightarrow L$  such that  $\varphi|_K = \sigma$ . Note that  $\varphi(x) = 0$  if and only if  $x = 0$ . If  $u = \pi(X)$ , then  $\varphi^{-1}(u)$  is a root of  $f$  in  $E$ , as

$$\begin{aligned} \varphi(f(\varphi^{-1}(u))) &= \varphi\left(\sum_i a_i \varphi^{-1}(u)^i\right) = \varphi\left(\sum_i a_i \varphi^{-1}(u^i)\right) \\ &= \sum_i \varphi(a_i) u^i = \sum_i \sigma(a_i) u^i = g(u) = 0. \end{aligned} \quad \square$$

### §3 Artin's theorem

As a corollary, if  $K$  is a field and  $f_1, \dots, f_n \in K[X]$  are polynomials of positive degree, then there exists an extension  $E/K$  such that each  $f_i$  admits a root in  $E$ . This is proved by induction on  $n$ .

**Definition 2.30.** A field  $K$  is **algebraically closed** if each  $f \in K[X]$  of positive degree admits a root in  $K$ .

The *fundamental theorem of algebra* states that  $\mathbb{C}$  is algebraically closed. A typical proof uses complex analysis. Later we will give a proof of this result using Galois theory.

**Proposition 2.31.** *The following statements are equivalent:*

- 1)  $K$  is algebraically closed.
- 2) If  $f \in K[X]$  is irreducible, then  $\deg f = 1$ .
- 3) If  $f \in K[X]$  is non-zero, then  $f$  decomposes linearly in  $K[X]$ , that is

$$f = a \prod_{i=1}^n (X - \alpha_i)^{m_i}$$

for some  $a \in K$  and  $\alpha_1, \dots, \alpha_n \in K$ .

- 4) If  $E/K$  is algebraic, then  $E = K$ .

*Proof.* 1)  $\implies$  2  $\implies$  3) are exercises.

Let us prove that 3)  $\implies$  4). Let  $x \in E$ . Decompose  $f(x, K)$  linearly in  $K[X]$  as  $f(x, K) = a \prod_{i=1}^n (X - \alpha_i)^{m_i}$  and evaluate on  $x$  to obtain that  $x = \alpha_j$  for some  $j$ .

To prove that 4)  $\implies$  1) let  $f \in K[X]$  be such that  $\deg f > 0$ . There exists an extension  $E/K$  such that  $f$  has a root  $x$  in  $E$ . The extension  $K(x)/K$  is algebraic and hence  $K(x) = K$ , so  $x \in K$ .  $\square$

### §3. Artin's theorem

**Definition 3.1.** The **algebraic closure** of a field  $K$  is an algebraic extension  $C/K$  such that  $C$  is algebraically closed.

For example,  $\mathbb{C}/\mathbb{R}$  is an algebraic closure but  $\mathbb{C}/\mathbb{Q}$  is not.

**Proposition 3.2.** *Let  $C$  be algebraically closed and  $\sigma: K \rightarrow C$  be a field homomorphism. If  $E/K$  is algebraic, then there exists a field homomorphism  $\varphi: E \rightarrow C$  such that  $\varphi|_K = \sigma$ .*

*Proof.* Suppose first that  $E = K(x)$  and let  $f = f(x, K)$ . Let  $\overline{\sigma}(f) \in C[X]$  and let  $y \in C$  be a root of  $\overline{\sigma}(f)$ . If  $z \in E$ , then  $z = g(x)$  for some  $g \in K[X]$ . Let  $\varphi: E \rightarrow C$ ,  $z \mapsto \overline{\sigma}(g)(y)$ .

The map  $\varphi$  is well-defined. If  $z = h(x)$  for some  $h \in K[X]$ , then

$$0 = g(x) - h(x) = (g - h)(x)$$

and thus  $f$  divides  $g - h$ . In particular,  $\overline{\sigma}(f)$  divides  $\overline{\sigma}(g - h) = \overline{\sigma}(g) - \overline{\sigma}(h)$  and hence  $(\overline{\sigma}(g) - \overline{\sigma}(h))(y) = 0$ .

It is an exercise to show that the map  $\varphi$  is a ring homomorphism.

Let  $a \in K$ . It follows that  $\varphi|_K = \sigma$ , as

$$\varphi(a) = \overline{\sigma}(aX^0)(y) = \sigma(a)$$

Let us now prove the proposition in full generality. Let  $X$  be the set of pairs  $(F, \tau)$ , where  $F$  is a subfield of  $E$  that contains  $K$  and  $\tau: F \rightarrow C$  is a field homomorphism such that  $\tau|_K = \sigma$ . Note that  $(K, \sigma) \in X$ , so  $X$  is non-empty. Moreover,  $X$  is partially ordered by

$$(F, \tau) \leq (F_1, \tau_1) \iff F \subseteq F_1 \text{ and } \tau_1|_F = \tau.$$

If  $\{(F_i, \tau_i) : i \in I\}$  is a chain in  $X$ , then  $F = \cup_{i \in I} F_i$  is a subfield of  $E$  that contains  $K$ . Moreover, if  $z \in F$ , then  $z \in F_i$  for some  $i \in I$  and then one defines  $\tau(z) = \tau_i(z)$ . It is an exercise to prove that  $\tau$  is well-defined. Since  $(F, \tau) \in X$  is an upper bound, Zorn's lemma implies that there exists a maximal element  $(E_1, \theta) \in X$ . We claim that  $E = E_1$ . If not, let  $z \in E \setminus E_1$ . Since we know the proposition is true for the extension  $E_1(z)/E_1$ , let  $\rho: E_1(z) \rightarrow C$  be a field homomorphism such that  $\rho|_{E_1} = \theta$ . Then, in particular,  $\rho|_K = \sigma$ . This implies that  $(E_1(z), \rho) \in X$  and hence  $(E_1, \theta) < (E_1(z), \rho)$ , a contradiction to the maximality of  $(E_1, \theta)$ .  $\square$



## Lecture 4

The previous proposition will be used to prove that the algebraic closure always exists.

**Theorem 3.3 (Artin).** *Let  $K$  be a field. Then  $K$  admits an algebraic closure  $C/K$ . If  $C_1/K$  is an algebraic closure, then the extensions  $C/K$  and  $C_1/K$  are isomorphic.*

*Proof.* Let us first prove the uniqueness. The previous proposition implies the existence of an extensions homomorphism  $\varphi: C \rightarrow C_1$ . Let  $y \in C_1$  and  $f = f(y, K)$  be the minimal polynomial of  $y$  in  $K$ . Since  $f$  admits a factorization

$$f = \lambda \prod (X - \alpha_i)^{m_i}$$

in  $C[X]$ , it follows that

$$f = \overline{\varphi}(f) = \varphi(\lambda) \prod (X - \varphi(\alpha_i))^{m_i}$$

Since  $0 = f(y)$ , we conclude that  $y = \varphi(\alpha_j)$  for some  $j$ . In particular,  $\varphi$  is surjective and hence  $\varphi$  is bijective.

We now prove the existence. Let us assume that  $K$  admits an extension  $E/K$  with  $E$  algebraically closed. We will prove later that this extension indeed exists, at the moment we only want to get an algebraic extension from this setting. Let

$$F = \{x \in E : x \text{ is algebraic over } K\}.$$

Then  $F/K$  is algebraic. Let  $g \in F[X]$  be such that  $\deg g > 0$ . Since  $E$  is algebraically closed,  $g$  admits a root  $\alpha$  in  $E$ . In particular,  $\alpha$  is algebraic over  $F$  and hence  $\alpha$  is algebraic over  $K$ . This implies that  $\alpha \in F$ , thus  $F$  is algebraically closed. This proves that  $F/K$  is an algebraic closure.

Let us prove that there exists an extension  $E_1/K$  such that every polynomial  $f \in K[X]$  with  $\deg f > 0$  has a root in  $E_1$ . Let  $\{f_i : i \in I\}$  be the family of monic irreducible polynomials with coefficients in  $K$ . We may think that  $f_i = f_i(X_i)$ . Let  $R = K[\{X_i : i \in I\}]$  and let  $J$  be the ideal of  $R$  generated by the  $f_i(X_i)$ . We claim that  $J \neq R$ . If not,  $1 \in J$ , so

$$1 = \sum_{j=1}^m g_j f_{i_j}(X_j)$$

for some  $g_1, \dots, g_m \in R$ . There exists an extension  $F/K$  such that  $f_{i_j}$  has a root  $\alpha_j$  in  $F$  for all  $j$ . Let

$$\sigma: R \rightarrow F, \quad \sigma(X_k) = \begin{cases} \alpha_j & \text{if } k = i_j, \\ 0 & \text{if } k \notin \{i_1, \dots, i_m\}. \end{cases}$$

Then  $1 = \sigma(1) = \sum_{j=1}^m \sigma(g_j) f_{i_j}(\alpha_j) = 0$ , a contradiction.

Since  $J$  is a proper ideal, it is contained in a maximal ideal  $M$ . Let  $L = R/M$  and let  $\sigma: K \rightarrow L$  be the composition  $K \hookrightarrow R \rightarrow R/M = L$ , where  $\pi: R \rightarrow R/M$  is the canonical map. As we did before,  $\pi(X_i)$  is a root of  $\overline{\sigma}(f_i)$  for all  $i$ . And there exists an extension  $E_1/K$  such that every  $f_i$  has a root in  $E_1$ . Proceeding in this way, we construct a sequence

$$E_1 \subseteq E_2 \subseteq \dots$$

of fields such that every polynomial of positive degree and coefficients in  $E_k$  admits a root in  $E_{k+1}$ . Let  $E = \cup E_k$ . We claim that  $E$  is algebraically closed. In fact, let  $g \in E[X]$  be such that  $\deg g > 0$ . Then, since  $g \in E_r[X]$  for some  $r$ , it follows that  $g$  has a root in  $E_{r+1} \subseteq E$ .  $\square$

## §4. Decomposition fields

**Definition 4.1.** Let  $K$  be a field and  $f \in K[X]$  be such that  $\deg f > 0$ . A **decomposition field** of  $f$  over  $K$  is field  $E$  that contains  $K$  and that satisfies the following properties:

- 1)  $f$  factorizes linearly in  $E[X]$ .
- 2) if  $F$  is a field such that  $K \subseteq F \subseteq E$  and  $f$  factorizes linearly in  $F[X]$ , then  $F = E$ .

Easy examples:

**Example 4.2.**  $\mathbb{C}$  is a decomposition field of  $X^2 + 1 \in \mathbb{R}[X]$ .

**Example 4.3.**  $\mathbb{Q}[\sqrt{2}]$  is a decomposition field of  $X^2 - 2 \in \mathbb{Q}[X]$ .

**Example 4.4.**  $\mathbb{Q}(\sqrt[3]{2})$  is not a decomposition field of  $X^3 - 2 \in \mathbb{Q}[X]$ . However, if  $\omega$  is a primitive cubic root of one, then  $\mathbb{Q}(\sqrt[3]{2}, \omega)$  is a decomposition field of the polynomial  $X^3 - 2 \in \mathbb{Q}[X]$ .

**Proposition 4.5.**  $E$  is a decomposition field of  $f \in K[X]$  if and only if  $f$  factorizes linearly in  $E[X]$  and  $E = K(x_1, \dots, x_n)$ , where  $x_1, \dots, x_n$  are the roots of  $f$ .

*Proof.* Let  $f = a \prod_{i=1}^r (X - x_i)^{n_i}$  and  $F = K(x_1, \dots, x_r)$  with  $x_1, \dots, x_r \in E$ . Since  $f$  factorizes linearly in  $F[X]$ , it follows that  $F = E$ . Conversely, let  $E = K(x_1, \dots, x_r)$  and assume that  $f$  factorizes linearly in  $F[X]$ . Then, in particular,  $x_1, \dots, x_r \in F$ . Hence  $E \subseteq F$  and  $F = E$ .  $\square$

One immediately obtains the following consequence: If  $E$  is a decomposition field of  $f \in K[X]$ , then  $E/K$  is finite.

**Theorem 4.6.** *Let  $f \in K[X]$  be such that  $\deg f > 0$ . There exists a (unique up to extension isomorphism) decomposition field of  $f$  over  $K$ .*

*Proof.* Let  $C/K$  be an algebraic closure. Write  $f = a \prod_{i=1}^r (X - x_i)^{n_i}$  in  $C[X]$ . Then  $E = K(x_1, \dots, x_r)$  is a decomposition field of  $f$  over  $K$ . Let us prove uniqueness: if  $E_1/K$  is a decomposition field of  $f$  over  $K$ , then  $E_1/K$  is algebraic and thus Proposition 3.2 implies that there exists  $\varphi \in \text{Hom}(E_1/K, C/K)$ , that is  $\varphi: E_1 \rightarrow C$  is a field homomorphism such that  $\varphi|_K$  is the identity. Factorize  $f$  linearly in  $E_1[X]$  and apply  $\bar{\varphi}$ :

$$f = a \prod_{j=1}^s (X - y_j)^{m_j} \implies f = \bar{\varphi}(f) = \varphi(a) \prod_{j=1}^s (X - \varphi(y_j))^{m_j}$$

so  $f$  factorizes linearly in  $\varphi(E_1)$ . Moreover,  $E_1 = K(y_1, \dots, y_s)$  and it follows that  $\varphi(E_1) = K(\varphi(y_1), \dots, \varphi(y_s))$ . Thus  $\varphi(E_1)$  is a decomposition field of  $f$ . Since  $\varphi(E_1) \subseteq C$ , it follows that  $\varphi(E_1) = E$ .  $\square$

**Exercise 4.7.** If  $E/K$  is finite and  $\varphi \in \text{Hom}(E/K, E/K)$ , then  $\varphi$  is an isomorphism.

Let  $C$  be an algebraic closure of  $K$  and  $G = \text{Gal}(C/K)$ . The group  $G$  acts on  $C$

$$\sigma \cdot x = \sigma(x), \quad \sigma \in G, x \in C.$$

The orbits are of the form

$$O_G(x) = \{\sigma(x) : \sigma \in G\} = \{y \in C : y = \sigma(x) \text{ for some } \sigma \in G\}$$

The elements  $x, y \in C$  are **conjugate** if  $y = \sigma(x)$  for some  $\sigma \in G$ .

**Proposition 4.8.** *Let  $C$  be an algebraic closure of  $K$  and  $x, y \in C$ . Then  $x$  and  $y$  are conjugate if and only if  $f(x, K) = f(y, K)$ . In particular,  $O_G(x)$  is finite.*

*Proof.* Let  $G = \text{Gal}(C/K)$ . If  $x$  and  $y$  are conjugate, say  $y = \sigma(x)$  for some  $\sigma \in G$ , let us write  $g = f(x, K)$  as

$$g = X^n + \sum_{i=0}^{n-1} a_i X^i.$$

Then  $0 = g(x) = x^n + \sum_{i=0}^{n-1} a_i x^i$  and hence  $y$  is a root of  $g$ , as

$$\begin{aligned} 0 &= \sigma \left( x^n + \sum_{i=0}^{n-1} a_i x^i \right) = \sigma(x)^n + \sum_{i=0}^{n-1} \sigma(a_i) \sigma(x)^i \\ &= \sigma(x)^n + \sum_{i=0}^{n-1} a_i \sigma(x)^i = y^n + \sum_{i=0}^{n-1} a_i y^i. \end{aligned}$$

Thus  $f(y, K) = g$ .

Conversely, assume that  $f(x, K) = f(y, K)$ . Let  $g = f(x, K) = f(y, K)$  and let

$$\varphi: K[x] \rightarrow K[y], \quad h(x) \mapsto h(y).$$

Let us show that the map  $\varphi$  is well-defined: we need to show that if  $h_1(x) = h_2(x)$ , then  $h_1(y) = \varphi(h_1(x)) = \varphi(h_2(x)) = h_2(y)$ . If  $h_1(x) = h_2(x)$ , then

$$(h_1 - h_2)(x) = h_1(x) - h_2(x) = 0.$$

Thus implies that  $g$  divides  $h_1 - h_2$ . In particular,  $h_1(y) = h_2(y)$ .

A straightforward calculation shows that  $\varphi$  is a field homomorphism such that  $\varphi|_K = \text{id}$ , this means that  $\varphi$  is an extension homomorphism such that  $\varphi(x) = y$ . There exists  $\sigma \in \text{Hom}(C/K, C/K)$  such that  $\sigma|_{K[x]} = \varphi$ . Since  $\sigma$  is bijective (this is left as an exercise, you did something similar before),  $\sigma(x) = \varphi(x) = y$  and hence  $O_G(x) = O_G(y)$ .  $\square$

**Proposition 4.9.** *Let  $C$  be an algebraic closure of  $K$  and  $x$ . Then*

$$f(x, K) = \prod_{y \in O_G(x)} (X - y)^m$$

for some  $m$ .

*Proof.* For each  $y \in O_G(x)$  let  $m_y$  be the multiplicity of  $y$  in  $f(x, K)$ . Then, for example,  $f(x, K) = (X - x)^{m_x} g$  for some  $g$ . If  $y \in O_G(x)$ , then  $y = \sigma(x)$  for some  $\sigma \in \text{Gal}(C/K)$ . Since

$$\overline{\sigma}(f(x, K)) = f(x, K) = (X - y)^{m_x} \overline{\sigma}(g),$$

it follows that  $m_y \geq m_x$ . By symmetry, we conclude that  $m_x = m_y$ .  $\square$

The previous proposition shows, in particular, that all the roots of an irreducible polynomial  $f \in K[X]$  in an algebraic closure  $C$  of  $K$  have the same multiplicity. This is clearly not true if  $f$  is not irreducible. Find an example.

**Definition 4.10.** Let  $K$  be a field and  $\{f_i : i \in I\}$  be a non-empty family of polynomials of positive degree with coefficients in  $K$ . A **decomposition field** of  $\{f_i : i \in I\}$  is an extension  $E/K$  such that every  $f_i$  factorizes linearly in  $E[X]$  and if  $F/K$  is a subextension of  $E/K$  such that every  $f_i$  factorizes linearly in  $F[X]$ , then  $F = E$ .

**Exercise 4.11.** Prove that  $E/K$  is a decomposition field of  $\{f_i : i \in I\}$  if and only if every  $f_i$  factorizes linearly in  $E[X]$  and  $E = K(S)$  where  $S = \{\text{roots of } f_i \text{ for all } i\}$ .

**Exercise 4.12.** Prove that if  $E/K$  is a decomposition field of  $\{f_i : i \in I\}$ , then  $E/K$  is algebraic. If, moreover,  $I$  is finite, then  $E/K$  is a decomposition field of  $\prod_{i \in I} f_i$ .

**Exercise 4.13.** Prove that there exists a decomposition field of  $\{f_i : i \in I\}$  and it is unique up to extension isomorphism.

## §5. Normal extensions

**Proposition 5.1.** *Let  $E/K$  be an algebraic extension and  $\sigma \in \text{Hom}(E/K, E/K)$ . Then  $\sigma$  is bijective.*

*Proof.* Let  $x \in E$  and  $C$  be an algebraic closure of  $K$  that contains  $E$ . There exists  $\varphi: C \rightarrow C$  such that  $\varphi|_E = \sigma$ . Thus  $\varphi|_K = \sigma|_K = \text{id}_K$ . Let  $G = \text{Gal}(C/K)$ . Then  $\varphi \in G$ . If  $z \in O_G(x)$ , then  $z = \tau(x)$  for some  $\tau \in G$  and hence

$$\varphi(z) = \varphi(\tau(x)) = (\varphi\tau)(x).$$

This implies that  $\varphi(z) \in O_G(x)$  and  $\varphi(O_G(x)) = O_G(x)$ . Thus  $\sigma|_{(E \cap O_G(x))}$  is injective and

$$\begin{aligned} \sigma(E \cap O_G(x)) &= \varphi(E \cap O_G(x)) \\ &= \varphi(E) \cap \varphi(O_G(x)) = \sigma(E) \cap O_G(x) \subseteq E \cap O_G(x). \end{aligned}$$

Since  $|E \cap O_G(x)| < \infty$ , it follows that  $E \cap O_G(x) = \sigma(E \cap O_G(x))$  and hence  $x$  belongs to the image of  $\sigma$ .  $\square$



## Lecture 5

**Definition 5.2.** Let  $E/K$  be an algebraic extension and  $C$  be an algebraic closure of  $K$  containing  $E$ . Then  $E/K$  is **normal** if  $\sigma(E) \subseteq E$  for all  $\sigma \in \text{Hom}(E/K, C/K)$ .

Note that  $\sigma(E) \subseteq E$  in the previous definition is equivalent to  $\sigma(E) = E$ .

**Example 5.3.** The extension  $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$  is not normal. Why?

Some trivial examples of normal extensions:  $K/K$  is normal and if  $C$  is an algebraic closure of  $K$ , then  $C/K$  is normal.

**Example 5.4.** The extension  $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$  is normal. In fact, every extension generated by algebraic elements of degree two is normal.

**Exercise 5.5.** Let  $\xi$  be a primitive cubic root of one. Then  $\mathbb{Q}(\sqrt[3]{2}, \xi)/\mathbb{Q}$  is normal.

The following result is useful but technical, that is why we leave the proof as an exercise.

**Exercise 5.6.** Prove that the previous definition depends on  $E$  and not on the algebraic closure  $C$ .

Some properties:

**Proposition 5.7.** Let  $E/K$  be a normal extension and  $f \in K[X]$  be an irreducible polynomial that admits a root  $x$  in  $E$ . Then  $f$  factorizes linearly in  $E$ .

*Proof.* We may assume that  $f$  is monic. Let  $C/K$  be an algebraic closure of  $K$  containing  $E$ . Let  $y$  be a root of  $f$  in  $C$ . Since  $f = f(x, K) = f(y, K)$ , it follows that  $y = \sigma(x)$  for some  $\sigma \in \text{Gal}(C/K)$ . Since  $E/K$  is normal,  $\sigma|_E: E \rightarrow C$  is an automorphism of  $E/K$ , that is  $\sigma(E) \subseteq E$ . In particular,  $y \in E$ .  $\square$

Let  $K \subseteq F \subseteq E$  be a tower of fields. If  $E/K$  is normal, then  $E/F$  is normal. However, Note that  $E/K$  normal does not imply  $F/K$  normal, as this would imply that every extension is normal. Moreover,  $E/F$  normal and  $F/K$  normal do not imply  $E/K$  normal.

**Example 5.8.** The extensions  $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}(\sqrt{2})$  and  $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$  are both normal, but  $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$  is not normal, as the roots of  $X^4 - 2$  are  $\sqrt{2}$ ,  $-\sqrt{2}$ ,  $\sqrt{2}i$  and  $-\sqrt{2}i$ .

Recall that if  $C$  is an algebraic closure of  $K$  and  $x \in C$ , then

$$f(x, K) = \prod (X - y)^m,$$

where the product is taken over all  $y \in O_{\text{Gal}(C/K)}(x)$ . If  $E/K$  is normal and  $x \in E$ , then there exists  $m$  such that

$$f(x, K) = \prod (X - y)^m,$$

where the product is taken over all  $y \in O_{\text{Gal}(E/K)}(x)$ .

**Proposition 5.9.** Let  $E/K$  and  $F/K$  be extensions. If  $F/K$  is normal, then  $EF/E$  is normal.

*Proof.* Let  $C$  be an algebraic closure of  $E$  containing  $EF$ . Let  $\sigma \in \text{Hom}(EF/E, C/E)$ . We claim that  $\sigma(EF) = EF$ . Let

$$\overline{K} = \{x \in C : x \text{ is algebraic over } K\}.$$

Then  $\overline{K}$  is an algebraic closure over  $K$  and  $F \subseteq \overline{K}$ . Since  $F/K$  is normal and  $\sigma|_F \in \text{Hom}(F/K, \overline{K}/K)$ , it follows that  $\sigma(F) = F$ . If  $z \in EF$ , then  $z = \sum_{i=1}^m e_i f_i$  for some  $e_1, \dots, e_m \in E$  and  $f_1, \dots, f_m \in F$ . Since  $\sigma(e_i) = e_i$  for all  $i$ ,

$$\sigma(z) = \sum_{i=1}^m \sigma(e_i) \sigma(f_i) = \sum_{i=1}^m e_i \sigma(f_i) \in EF. \quad \square$$

**Proposition 5.10.** Let  $E/K$  be an algebraic extension. Then  $E/K$  is normal if and only if  $E/K$  is the decomposition field of a family of polynomials of  $K[X]$  of positive degree.

*Proof.* Let  $G = \text{Gal}(E/K)$ . If  $x \in E$  and  $f(x, K) = \prod_{y \in O_G(x)} (X - y)^m$ , then  $f(x, K)$  factorizes linearly in  $E[X]$ . Thus  $E/K$  is a decomposition field of the family  $\{f(x, K) : x \in E\}$ . Conversely, assume that  $E/K$  is a decomposition field of the family  $\{f_i : i \in I\}$ . Then  $E = K(S)$  where  $S$  is the set of roots of the polynomials  $f_i$ . Let  $C/K$  be an algebraic closure of  $K$  that contains  $E$  and let  $\sigma \in \text{Hom}(E/K, C/K)$ . Let  $x \in S$ . Then  $x$  is a root of some  $f_j = \sum a_k X^k$ . Since  $f_j(x) = 0$ , it follows that  $\sigma(x)$  is a root of  $f_j$ , as

$$f_j(\sigma(x)) = \sum a_k \sigma(x)^k = \sum \sigma(a_k) \sigma(x^k) = \sigma\left(\sum a_k x^k\right) = \sigma(0) = 0.$$

Hence  $\sigma(E) \subseteq E$ .  $\square$



## §6. Dedekind's theorem

Note that every extension homomorphism  $E/K \rightarrow F/K$  is, in particular, a  $K$ -linear map  $E \rightarrow F$ , that is

$$\text{Hom}(E/K, F/K) \subseteq \text{Hom}_K(E, F).$$

If  $F/K$  is an extension and  $V$  is a  $K$ -vector space, the set  $\text{Hom}_K(E, F)$  of  $K$ -linear maps is a vector space over  $F$  with  $(a \cdot f)(v) = af(v)$  for  $a \in F$ ,  $f \in \text{Hom}_K(E, F)$  and  $v \in V$ .

**Exercise 6.1.** Prove that  $\dim_F \text{Hom}_K(V, F) \geq \dim_K V$ . Moreover, if  $\dim_K V < \infty$ , then  $\dim_F \text{Hom}_K(V, F) = \dim_K V$ .

If  $V$  is a vector space and  $S$  is a (possibly infinite) subset of  $V$ , then  $S$  is linearly independent if every finite subset of  $S$  is linearly independent.

**Theorem 6.2 (Dedekind).** Let  $E/K$  and  $F/K$  be extensions and let  $\{\varphi_i : i \in I\}$  be a subset of  $\text{Hom}(E/K, F/K)$ , i.e. a family of extension homomorphisms. Assume that  $\varphi_i \neq \varphi_j$  if  $i \neq j$ . Then the subset  $\{\varphi_i : i \in I\} \subseteq \text{Hom}_K(E, F)$  is linearly independent over  $F$ .

*Proof.* Assume it is not. Let  $\{\varphi_1, \dots, \varphi_n\}$  be linearly dependent over  $F$  with  $n$  minimal. Clearly,  $n > 1$ . We may assume that

$$\sum_{i=1}^n a_i \varphi_i = 0 \quad (5.1)$$

for some  $a_1, \dots, a_n \in F$  all different from zero. Let  $z \in E \setminus \{0\}$  be such that  $\varphi_1(z) \neq \varphi_2(z)$ . If  $x \in E$ , then

$$0 = \left( \sum_{i=1}^n a_i \varphi_i \right) (xz) = \sum_{i=1}^n a_i \varphi_i(xz) = \sum_{i=1}^n a_i \varphi_i(x) \varphi_i(z) = \left( \sum_{i=1}^n (a_i \varphi_i(z)) \varphi_i \right) (x).$$

Thus

$$\sum_{i=1}^n (a_i \varphi_i(z)) \varphi_i = 0. \quad (5.2)$$

Since  $\sum_{i=1}^n a_i \varphi_i = 0$  and  $\varphi_1(z) \neq 0$ ,

$$a_1 \varphi_1 + a_2 \frac{\varphi_2(z)}{\varphi_1(z)} \varphi_2 + \dots + a_n \frac{\varphi_n(z)}{\varphi_1(z)} \varphi_n = 0.$$

Thus, subtracting (5.1) and (5.2),

$$\left( a_2 - a_2 \frac{\varphi_2(z)}{\varphi_1(z)} \right) \varphi_2 + \dots + \left( a_n - a_n \frac{\varphi_n(z)}{\varphi_1(z)} \right) \varphi_n = 0.$$

Since  $a_n \neq 0$  and  $\varphi_2(z) \neq \varphi_1(z)$ , the scalar  $a_2 - a_2 \frac{\varphi_2(z)}{\varphi_1(z)} \neq 0$  and hence  $\{\varphi_2, \dots, \varphi_n\}$  is linearly dependent, a contradiction.  $\square$

If  $E/K$  and  $F/K$  are extensions, let  $\gamma(E/K, F/K) = |\text{Hom}(E/K, F/K)|$ .

**Exercise 6.3.** Prove the following statements:

- 1)  $\gamma(E/K, F/K) \leq \dim_F \text{Hom}_K(E, F)$ .
- 2) If  $[E : K] < \infty$ , then  $\gamma(E/K, F/K) \leq [E : K]$ .
- 3) If  $x$  is algebraic over  $K$ , then  $\gamma(K(x)/K, F/K) \leq \deg(x, K)$ .

If  $C$  is an algebraic closure of  $K$ , then we define  $\gamma(E/K) = \gamma(E/K, C/K)$ . This definition does not depend on the algebraic closure.

**Exercise 6.4.** If  $C$  and  $C_1$  are algebraic closures of  $K$ , then

$$|\text{Hom}(E/K, C/K)| = |\text{Hom}(E/K, C_1/K)|.$$

**Proposition 6.5.** Let  $C$  be an algebraic closure of  $K$  and  $G = \text{Gal}(C/K)$ . If  $x \in C$ , then  $\gamma(K(x)/K) = |O_G(x)|$ .

*Proof.* If  $\sigma \in \text{Hom}(K(x)/K, C/K)$ , then there exists  $\phi \in G$  such that  $\phi|_{K(x)} = \sigma$ . Thus  $\sigma(x) = \phi(x) \in O_G(x)$ . Conversely, if  $y \in O_G(x)$ , then there exists  $\tau \in G$  such that  $y = \tau(x)$ . Hence  $\tau|_{K(x)} \in \text{Hom}(K(x)/K, C/K)$  and  $\tau|_{K(x)}(x) = y$ . In particular,  $\gamma(K(x)/K)$  divides  $\deg(x, K)$ .  $\square$

**Exercise 6.6.** If  $E/K$  is finite, then  $|\text{Gal}(E/K)| \leq [E : K]$ . Moreover,  $E/K$  is normal if and only if  $|\text{Gal}(E/K)| = \gamma(E/K)$ .

## Lecture 6

If  $t: A \rightarrow B$  is a surjective map, then  $a \sim a_1 \iff t(a) = t(a_1)$  defines an equivalence relation on  $A$ . The set  $\bar{A}$  of equivalence classes is in bijective correspondence with  $B$ ,  $\bar{A} \rightarrow B, \bar{a} \mapsto t(a)$ . Moreover, if  $|t^{-1}(\{b\})| = m$  for all  $b \in B$ , then  $|A| = m|\bar{A}| = m|B|$ .

**Proposition 6.7.** *Let  $E/K$  be algebraic and  $F/K$  be a subextension such that  $E/F$  is finite. Then  $\gamma(E/K) = \gamma(E/F)\gamma(F/K)$ .*

*Proof.* Assume that  $E = F(x)$ . Let  $f = f(x, F) = \sum b_i X^i$  and let  $G = \text{Gal}(E/F)$ . Let  $C$  be an algebraic closure of  $K$  containing  $E$ . The map

$$\lambda: \text{Hom}(E/K, C/K) \rightarrow \text{Hom}(F/K, C/K), \quad \sigma \mapsto \sigma|_F,$$

is well-defined. It is surjective: if  $\varphi \in \text{Hom}(F/K, C/K)$ , then  $\varphi: F \rightarrow C$  is, in particular, a field homomorphism. Since  $E/F$  is algebraic, by Proposition 3.2 there exists a field homomorphism  $\sigma: E \rightarrow C$  such that  $\sigma|_F = \varphi$ . Since  $\sigma|_K = \varphi|_K = \text{id}$ , in particular  $\sigma \in \text{Hom}(E/K, C/K)$ .

For  $\varphi \in \text{Hom}(F/K, C/K)$ ,

$$\lambda^{-1}(\{\varphi\}) = \{\sigma \in \text{Hom}(E/K, C/K) : \sigma|_F = \varphi\}$$

and let  $R_\varphi$  be the set of roots (in  $C$ ) of the polynomial  $\bar{\varphi}(f) = \sum \varphi(b_i)X^i$ .

*Claim.* The map  $\alpha: \lambda^{-1}(\{\varphi\}) \rightarrow R_\varphi, \sigma \mapsto \sigma(x)$ , is well-defined.

We need to show that  $\sigma(x)$  is a root of  $\bar{\varphi}(f)$ :

$$\begin{aligned} \bar{\varphi}(f)(\sigma(x)) &= \sum \varphi(b_i)\sigma(x)^i = \sum \sigma(b_i)\sigma(x^i) \\ &= \sum \sigma(b_i x^i) = \sigma\left(\sum b_i x^i\right) = \sigma(f(x)) = \sigma(0) = 0. \end{aligned}$$

*Claim.* The map  $\beta: R_\varphi \rightarrow \lambda^{-1}(\{\varphi\}), y \mapsto \sigma_y$ , where  $\sigma_y(z) = \bar{\varphi}(h)(y)$  if  $z = h(x)$ , is well-defined.

We need to show that if  $z = h(x)$  and  $z = h_1(x)$  for some  $h, h_1 \in F[X]$ , then  $\bar{\varphi}(h)(y) = \bar{\varphi}(h_1)(y)$ . The assumptions imply that  $(h - h_1)(x) = 0$  and hence  $f$  divides  $h - h_1$ . Since  $\bar{\varphi}$  is a ring homomorphism,  $\bar{\varphi}(f)$  divides  $\bar{\varphi}(h) - \bar{\varphi}(h_1)$ . This implies  $(\bar{\varphi}(h) - \bar{\varphi}(h_1))(y) = 0$ . We also need to show that  $\sigma_y|_F = \varphi$ : if  $a \in F$ , then write  $a = aX^0 \in F[X]$ . Thus  $\sigma_y(a) = \bar{\varphi}(aX^0)(y) = \varphi(a) \in C$ . It is now an exercise to prove that  $\sigma_y \in \text{Hom}(E/K, C/K)$ .

*Claim.*  $|\lambda^{-1}(\{\varphi\})| = |R_\varphi|$ .

For this we need to show that  $\beta$  is the inverse of  $\alpha$ , that is  $\alpha \circ \beta = \text{id}$  and  $\beta \circ \alpha = \text{id}$ . To prove that  $\beta \circ \alpha = \text{id}$  let  $\sigma$  be such that  $\sigma|_F = \varphi$ . Then  $y = \sigma(x) \in R_\varphi$ . Let  $z = h(x) = \sum a_i x^i \in F[x] = E$ . Then

$$\bar{\varphi}(h)(y) = \sum \varphi(a_i) y^i = \sum \sigma(a_i) y^i = \sigma\left(\sum a_i x^i\right) = \sigma(y).$$

Conversely, if  $y \in R_\varphi$ , then

$$\alpha(\sigma_y) = \sigma_y(x) = y,$$

as  $\sigma_y(x) = \bar{\varphi}(X)(y) = y$ .

*Claim.* If  $\phi \in G$  is such that  $\phi|_F = \varphi$ , then  $O_G(x) = \phi^{-1}(R_\varphi)$ .

Let us first prove  $O_G(x) \supseteq \phi^{-1}(R_\varphi)$ . If  $y \in R_\varphi$ , then

$$\begin{aligned} f(\phi^{-1}(y)) &= \sum b_i \phi^{-1}(y^i) = \phi^{-1}\left(\sum \phi(b_i) y^i\right) \\ &= \phi^{-1}\left(\sum \varphi(b_i) y^i\right) = \phi^{-1} \bar{\varphi}(f)(y) = \phi^{-1}(0) = 0. \end{aligned}$$

Now we prove  $O_G(x) \subseteq \phi^{-1}(R_\varphi)$ . Let  $z \in O_G(x)$  and  $y \in C$  be such that  $\phi^{-1}(y) = z$ . Then  $\bar{\varphi}(f)(y) = 0$ , as

$$\begin{aligned} \bar{\varphi}(f)(y) &= \sum \varphi(b_i) y^i \\ &= \sum \varphi(b_i) \phi(z^i) = \sum \phi(b_i) \phi(z^i) = \phi\left(\sum b_i z^i\right) = \phi(f(z)) = \phi(0) = 0. \end{aligned}$$

It follows that  $|\lambda^{-1}(\varphi)| = |O_G(x)|$  for all  $\varphi$ . By using the argument before the proposition,

$$\begin{aligned} \gamma(E/K) &= |\text{Hom}(E/K, C/K)| \\ &= |O_G(x)| |\text{Hom}(F/K, C/K)| \\ &= |O_G(x)| \gamma(F/K). \end{aligned}$$

Since  $\gamma(K(x)/K) = |O_G(x)|$  by Proposition 6.5, the claim follows.

For the general case we assume that  $E = F(x_1, \dots, x_n)$ . We proceed by induction on  $n$ . If  $n = 0$ , then  $E = F$  and the result is trivial. If  $n > 0$ , let  $L = F[x_1, \dots, x_{n-1}]$

and  $E = L(x_n)$ . The case proved implies that  $\gamma(E/F) = \gamma(E/L)\gamma(L/F)$ . By the inductive hypothesis,  $\gamma(L/K) = \gamma(L/F)\gamma(F/K)$ . Thus

$$\gamma(E/F)\gamma(F/K) = \gamma(E/L)\gamma(L/F)\gamma(F/K) = \gamma(E/L)\gamma(L/K) = \gamma(E/K),$$

again using the previous case.  $\square$

## §7. Separable extensions

**Definition 7.1.** Let  $E/K$  be an algebraic extension and  $x \in E$ . Then  $x$  is **separable** over  $K$  if  $x$  is a simple root of  $f(x, K)$ .

An algebraic extension  $E/K$  is **separable** if every  $x \in E$  is separable over  $K$ . Clearly,  $K/K$  is separable.

**Exercise 7.2.** Prove that an element  $x$  is separable over  $K$  if and only if  $x$  is a simple root of a polynomial with coefficients in  $K$ .

If  $F/K$  is a subextension of  $E/K$  and  $x \in E$  is separable over  $K$ , then  $x$  is separable over  $F$ .

**Exercise 7.3.** If  $C$  is an algebraic closure of  $K$ ,  $x \in C$  and  $G = \text{Gal}(C/K)$ . Prove that the following statements are equivalent:

- 1)  $x$  is separable over  $K$ .
- 2) Every  $y \in O_G(x)$  is separable over  $K$ .
- 3)  $\gamma(K(x)/K) = [K(x) : K] = \deg f(x, K)$ .

Let  $K$  be any field and  $g \in K[X]$ . Let  $z$  be a root of  $g$ . Then  $z$  is a multiple root of  $g$  if and only if  $z$  is a root of  $g'$ .

**Exercise 7.4.** Prove that if  $K$  has characteristic zero or  $K$  is finite, then every algebraic extension of  $K$  is separable.

A consequence: Let  $E/K$  be a finite extension. Then  $E/K$  is separable if and only if  $\gamma(E/K) = [E : K]$ .

**Example 7.5.** Let  $E = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ . Then  $[E : \mathbb{Q}] = 4$  and  $\text{Gal}(E/\mathbb{Q}) \simeq C_2 \times C_2$ . The extension  $E/\mathbb{Q}$  is normal, as it is the decomposition field of  $(X^2 - 2)(X^2 - 3)$  and it is separable as  $\mathbb{Q}$  has characteristic zero.

**Example 7.6.** Let  $E$  be a decomposition field of  $X^4 - 2$  over  $\mathbb{Q}$ . Then  $E/\mathbb{Q}$  is normal and separable. Note that  $E = \mathbb{Q}(\sqrt[4]{2}, i)$ , so  $[E : \mathbb{Q}] = 8 = |\text{Gal}(E/\mathbb{Q})|$ .

Let us compute  $\text{Gal}(E/\mathbb{Q})$ . If  $\sigma \in \text{Gal}(E/\mathbb{Q})$ , then  $\sigma(\sqrt[4]{2}) \in \{\sqrt[4]{2}, -\sqrt[4]{2}, \sqrt[4]{2}i, -\sqrt[4]{2}i\}$  and  $\sigma(i) \in \{-i, i\}$ . Two examples are

$$\alpha: \begin{cases} \sqrt[4]{2} \mapsto \sqrt[4]{2}i, \\ i \mapsto i, \end{cases} \quad \beta: \begin{cases} \sqrt[4]{2} \mapsto \sqrt[4]{2}, \\ i \mapsto -i. \end{cases}$$

It follows that  $\text{Gal}(E/\mathbb{Q})$  is isomorphic to the group  $\langle \alpha, \beta \rangle$ , which turns out to be isomorphic to the dihedral group of eight elements.

Another consequence: If  $E = K(S)$ , then  $E/K$  is separable if and only if every  $x \in S$  is separable over  $K$ . One first does the case  $E = K(x)$  and then proceeds by induction.

**Exercise 7.7.** Let  $K \subseteq F \subseteq E$  be a tower of fields. Prove that if  $E/K$  is separable, then  $F/K$  and  $E/F$  are separable.

**Exercise 7.8.** Let  $E/K$  and  $F/K$  be extensions. Prove that if  $E/K$  is separable, then  $EF/E$  is separable.

## Lecture 7

If  $E/K$  is algebraic, then

$$F = \{x \in E : x \text{ is separable over } K\}$$

is a subfield of  $E$  that contains  $K$ . It is known as the **separable closure** of  $K$  with respect to  $E$ . Note that  $F = K(F)$ , as  $K(F)$  is separable because it is generated by separable elements. Moreover,  $F/K$  is separable and  $E/F$  is a **purely inseparable** extension, meaning that for every  $x \in E \setminus F$ , the polynomial  $f(x, F)$  is not separable.

**Proposition 7.9.** *If  $E/K$  is separable and finite, then  $E = K(x)$  for some  $x \in E$ .*

*Proof.* Let us assume that  $K$  is finite. Then  $E$  is finite and hence the multiplicative group  $E^\times = E \setminus \{0\}$  is cyclic, say  $E^\times = \langle x \rangle$ . It follows that  $E = K(x)$ .

Let us now assume that  $K$  is infinite. We first consider the case  $E = K(x, y)$ . The general case  $E = K(x_1, \dots, x_n)$  is left as an exercise, one needs to proceed by induction. Let  $n = [E : K]$  and  $C$  be an algebraic closure of  $K$  containing  $E$ . Write  $\text{Hom}(E/K, C/K) = \{\sigma_1, \dots, \sigma_n\}$ . Let

$$f = \prod_{1 \leq i < j \leq n} ((\sigma_i(y) - \sigma_j(y)) + X(\sigma_i(x) - \sigma_j(x))) \in C[X].$$

Then  $f \neq 0$ , as  $f$  is a product of non-zero polynomials. Since  $K$  is infinite, there exists  $c \in K$  such that  $f(c) \neq 0$ . For any  $r, s \in \{1, \dots, n\}$  with  $r \neq s$ ,

$$\sigma_r(y) - \sigma_s(y) + c(\sigma_r(x) - \sigma_s(x)) \neq 0,$$

as  $c \in K$ . It follows that  $\sigma_r(y + cx) \neq \sigma_s(y + cx)$ . Thus  $\gamma(K(y + cx)/K) \geq n$ . Now

$$n \geq [K(y + cx) : K] = \gamma(K(y + cx)/K) \geq n,$$

so  $[K(y + cx) : K] = n$  and hence  $K(y + cx) = E$ . □

For example,  $\mathbb{Q}(\sqrt{2}, i) = \mathbb{Q}(\sqrt{2} + i)$ .

**Proposition 7.10.** *Let  $E/K$  be a finite extension. Then  $E = K(x)$  for some  $x \in E$  if and only if  $E/K$  admits finitely many subextensions.*

*Proof.* We first prove  $\implies$ . We may assume that  $K$  is infinite, otherwise the result is trivial. Let us assume that  $E = K(x)$ . We claim that the map

$$\Psi: \{F : K \subseteq F \subseteq E\} \rightarrow \{\text{monic divisors of } f(x, K)\}, \quad F \mapsto f(x, F),$$

is injective. Take  $K \subseteq F_0 \subseteq F \subseteq E$  with  $f(x, F) = f(x, F_0)$ . Then

$$[E : F_0] = [F_0(x) : F_0] = \deg f(x, F_0) = m = [F(x) : F] = [E : F]$$

and hence  $F = F_0$ . It follows that  $\Psi$  is injective and therefore there are finitely many fields between  $K$  and  $E$ .

Let us prove  $\impliedby$ . As before let us assume that  $E = K(x, y)$ . For each  $a \in K$  we consider the extension  $K(ay + x)/K$ . By assumption, there exist  $a, b \in K$  such that  $a \neq b$  and  $K(x + ay) = K(x + by) = L$ . We claim that  $L = E$ . Note that  $x + ay \in L$  and  $x + by \in L$ , so  $(a - b)y \in L$  and hence, since  $K \subseteq L$ , it follows that  $y \in L$ . Thus  $x \in L$  and therefore  $L = E$ .  $\square$

As a consequence, if  $E/K$  is finite and separable, then  $E/K$  admits finitely many subextensions.

## §8. Galois extensions

Let  $E/K$  be an algebraic extension. Assume that  $E = K(S)$  and let  $C$  be an algebraic closure of  $K$  containing  $E$ . Let

$$T = \{y \in C : y \text{ is a root of } f(x, K) \text{ for some } x \in S\}$$

and let  $L = K(T)$ . Then  $E \subseteq L$ , as  $S \subseteq T$ . The extension  $L/K$  is normal, as  $L/K$  is a decomposition field of the family  $\{f(x, K) : x \in S\}$ . Moreover,  $L$  is the smallest normal extension of  $K$  containing  $E$ . The field  $L$  is the **normal closure** of  $E$  (with respect to  $C$ ).

**Exercise 8.1.** If  $E/K$  is finite, then  $L/K$  is finite

**Exercise 8.2.** If  $E/K$  is separable, then  $L/K$  is separable.

Let  $E/K$  be an extension and  $S \subseteq \text{Gal}(E/K)$  be a subset. the set

$${}^S E = \{x \in E : \sigma(x) = x \text{ for all } \sigma \in S\}$$

is a subfield of  $E$  that contains  $K$ . The subfield  ${}^S E$  is known as the **fixed field** of  $S$ .

**Definition 8.3.** Let  $E/K$  be an algebraic extension and  $G = \text{Gal}(E/K)$ . Then  $E/K$  is a **Galois extension** if  ${}^G E = K$ .



Clearly,  $K/K$  is a Galois extension. Note that  $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$  is not a Galois extension. Why?

**Exercise 8.4.** Prove that  $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$  is a Galois extension.

**Exercise 8.5.** If the characteristic of  $K$  is different from two, then every quadratic extension of  $K$  is a Galois extension.

**Exercise 8.6.** Let  $E/K$  be an algebraic extension and  $G = \text{Gal}(E/K)$ . Let  $F = {}^G E$ . Prove that  $\text{Gal}(E/F) = G$  and hence  $E/F$  is a Galois extension.

**Proposition 8.7.** Let  $E/K$  be an algebraic extension. Then  $E/K$  is a Galois extension if and only if  $E/K$  is normal and separable.

*Proof.* Let  $G = \text{Gal}(E/K)$ . Let us first assume that  $E/K$  is Galois. For  $x \in E$  let  $f_x = \prod_{y \in O_G(x)} (X - y) = \sum a_i X^i \in E[X]$ . If  $\varphi \in G$ , then

$$\bar{\varphi}(f_x) = \prod_{y \in O_G(x)} (X - \varphi(y)) = f_x,$$

as if  $O_G(x) = \{\sigma_1(x), \dots, \sigma_r(x)\}$ , then  $\varphi(\sigma_i(x)) = (\varphi\sigma_i)(x) = \sigma_j(x)$  for some  $j$ . Since

$$\sum a_i X^i = f_x = \bar{\varphi}(f_x) = \sum \varphi(a_i) X^i,$$

it follows that  $a_i \in {}^G E = K$  for all  $i$ . Thus  $f_x \in K[X]$  and  $E/K$  is a decomposition field of the family  $\{f_x : x \in E\}$ . In particular,  $E/K$  is normal. Moreover,  $x$  is a simple root of  $f_x \in K[X]$  and hence  $x$  is separable over  $K$ .

Conversely, let  $x \in {}^G E$ . Since  $E/K$  is normal, then  $f(x, K) = \prod_{y \in O_G(x)} (X - y)^m$  for some  $m$ . Since  $E/K$  is separable,  $m = 1$ . Thus  $f(x, K) = \prod_{y \in O_G(x)} (X - y) = X - x$  and  $x \in K$ .  $\square$

**Definition 8.8.** Let  $K$  be a field and  $f \in K[X]$ . Then  $f$  is **separable** if all roots of  $f$  are simple (in some algebraic closure of  $K$ ).

**Proposition 8.9.** Let  $E/K$  be a finite extension. Then  $E/K$  is a Galois extension if and only if  $E$  is a decomposition field over  $K$  of a separable polynomial  $f \in K[X]$ .

*Proof.* Let us assume first that  $E/K$  is a Galois extension. Since  $E/K$  is finite and separable,  $E = K(x)$  by Proposition 7.9. Then  $E/K$  is a decomposition field of  $f(x, K)$  since  $E/K$  is normal. Since  $E/K$  is separable,  $x$  is separable over  $K$ . Thus  $x$  is a simple root of  $f(x, K)$  and hence  $f(x, K)$  is separable.

Conversely, let  $x_1, \dots, x_r$  be the roots of a separable polynomial  $f \in K[X]$ . Then  $E = K(x_1, \dots, x_r)$  is separable and normal.  $\square$

In the previous case,  $\text{Gal}(E/K)$  is known as the **Galois group** of the polynomial  $f$ . The notation is  $\text{Gal}(f, K)$ . If  $n = \deg f$  and  $x_1, \dots, x_n$  are the roots of  $f$ , then any  $\varphi \in \text{Gal}(f, K)$  permutes the roots of  $f$ , that is  $\varphi$  permutes the set  $\{x_1, \dots, x_n\}$ . In particular,  $\text{Gal}(f, K)$  is isomorphic to a subgroup of  $\mathbb{S}_n$  and hence  $|\text{Gal}(f, K)|$  divides  $n!$ .

**Proposition 8.10.** *Let  $E/K$  be a normal extension and  $F$  be the separable closure of  $K$  with respect to  $E$ . Then  $F/K$  is a Galois extension.*

*Proof.* Let  $C/K$  be an algebraic closure such that  $E \subseteq C$ . Let  $\sigma \in \text{Hom}(F/K, C/K)$ . and let  $\varphi \in \text{Hom}(E/K, C/K)$  be such that  $\varphi|_F = \sigma$ . Since  $E/K$  is normal,  $\varphi(E) = E$ . Let  $x \in F$ . Then  $\sigma(x) = \varphi(x) \in E$ . Thus  $f(\sigma(x), K) = f(x, K)$  and  $\sigma(x)$  is separable over  $K$ , which implies that  $\sigma(x) \in F$ . Thus  $F/K$  is normal. Since  $F/K$  is separable, it follows that  $F/K$  is a Galois extension by Proposition 8.7.  $\square$

Some easy facts.

**Exercise 8.11.** Let  $E/K$  be a separable extension and  $L/K$  be the normal closure of  $E$  in some algebraic closure  $C$  that contains  $E$ . Prove that  $L/K$  is a Galois extension.

**Exercise 8.12.** Let  $E/K$  be a finite extension. Prove that  $E/K$  is Galois if and only if  $[E : K] = |\text{Gal}(E/K)|$ .

**Exercise 8.13.** Let  $E/K$  be a Galois extension and  $F/K$  be a subextension of  $E/K$ . Prove that  $E/F$  is a Galois extension.

## Lecture 8

**Theorem 8.14 (Artin).** *Let  $E$  be a field and  $G$  be a finite group of automorphisms of  $E$ . If  $K = {}^G E$ , then  $E/K$  is a Galois extension,  $[E : K] = |G|$  and  $\text{Gal}(E/K) = G$ .*

Before proving the theorem, we need a lemma.

**Lemma 8.15.** *Let  $E/K$  be a separable extension such that  $\deg(x, K) \leq m$  for all  $x \in E$ . Then  $E/K$  is finite and  $[E : K] \leq m$ .*

*Proof.* Let  $z \in E$  be of maximal degree. If  $x \in E$ , then  $K(x, z)/K$  is separable. Then  $K(x, z) = K(y)$  for some  $y$ . It follows that

$$K(z) \subseteq K(x, z) = K(y).$$

Since  $\deg(z, K) \leq \deg(y, K)$ ,  $\deg(z, K) = \deg(y, K)$ . Hence  $K(y) = K(z)$ . In particular,  $x \in K(z)$  and therefore  $E = K(z)$ .  $\square$

Now we are ready to prove Artin's theorem:

*Proof of Theorem 8.14.* Note that  $G \subseteq \text{Gal}(E/K)$ . Let  $x \in E$  and

$$f_x = \prod_{y \in O_G(x)} (X - y).$$

Since  $f_x \in K[X]$ , it follows that the extension  $E/K$  is normal and separable (as it is a decomposition field of a family of separable polynomials), so  $E/K$  is a Galois extension. Moreover,

$$\deg(x, K) \leq \deg f_x = |O_G(x)| \leq |G|.$$

By the previous lemma,  $E/K$  is finite and  $[E : K] \leq |G|$ . This implies that  $|G(E/K)| = [E : K] \leq |G|$  and hence  $|G(E/K)| = |G|$ .  $\square$

**Example 8.16.** Let  $E = K(X, Y)$  and  $\sigma : K[X, Y] \rightarrow E$  be the ring homomorphism given by  $\sigma(X) = Y$  and  $\sigma(Y) = X$ . Note that  $\sigma$  is bijective, as  $\sigma^2 = \text{id}$ . The map

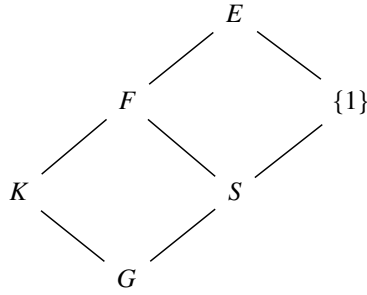
$\sigma$  induces a field homomorphism  $\bar{\sigma}: E \rightarrow E$  such that  $\bar{\sigma}^2 = \text{id}$ . Recall that such a homomorphism is given by  $f/g \mapsto \sigma(f)/\sigma(g)$ . Let  $G = \langle \bar{\sigma} \rangle$ . Then  $|G| = 2$ . We claim that  ${}^G E = K(X+Y, XY)$ . Let  $F = K(X+Y, XY)$ . We only prove that  ${}^G E \subseteq F$ , as the other inclusion is trivial. Artin's theorem implies that  $[E : {}^G E] = 2$  and  $E = F(X)$ , as  $X$  is a root of the polynomial  $Z^2 - (X+Y)Z + XY$ . Then  $[E : F] \leq 2$  and  $[{}^G E : F] = 1$ .

## §9. Galois' correspondence

**Theorem 9.1 (Galois).** *Let  $E/K$  be a finite Galois extension and  $G = \text{Gal}(E/K)$ . There exists a bijective correspondence*

$$\{F : K \subseteq F \subseteq E \text{ subfields}\} \leftrightarrow \{\text{subgroups of } G\}$$

*The correspondence is given by  $F \mapsto G(E/F)$  and  ${}^S E \mapsto S$ . Moreover, normal subextensions of  $E/K$  correspond to normal subgroups of  $G$ .*



*Proof.* Let  $\alpha$  and  $\beta$  be the maps  $\alpha(F) = \text{Gal}(E/F)$  and  $\beta(S) = {}^S E$ . A routine exercise shows that  $\alpha$  and  $\beta$  are well-defined. We first note that

$$\beta(\alpha(F)) = \beta(\text{Gal}(E/F)) = {}^{\text{Gal}(E/F)} E = F$$

since  $E/F$  is a Galois extension. Moreover,

$$\alpha(\beta(S)) = \alpha({}^S E) = \text{Gal}(E/{}^S E) = S$$

by Artin's theorem, as  $S$  is finite.

Let  $F$  be a subfield of  $E$  containing  $K$  and  $S = \alpha(F)$ . Then

$$[F : K] = \frac{[E : K]}{[E : F]} = \frac{|G|}{|S|} = (G : S).$$

Let  $C$  be an algebraic closure of  $K$  that contains  $E$ . If  $S = \text{Gal}(E/F)$ , then  $F = {}^S E$ .

We need to prove that  $F/K$  is normal if and only if  $S$  is normal in  $G$ . Let us first prove  $\implies$ . Let  $\tau \in S$  and  $\sigma \in G$ . Since  $F/K$  is normal,  $\sigma|_F \in \text{Aut}(F)$ . Thus

$\sigma^{-1}(F) = F$ . In particular, if  $x \in F$ , then  $\sigma^{-1}(x) \in F$  and

$$\sigma\tau\sigma^{-1}(x) = \sigma\sigma^{-1}(x) = x.$$

Conversely, let  $\varphi \in \text{Hom}(F/K, C/K)$ . There exists  $\Phi: E \rightarrow C$  such that  $\Phi|_F = \varphi$ . Since  $E/K$  is normal,  $\Phi(E) = E$  and hence  $\Phi \in G$ . We claim that  $\varphi(x) \in F$  for all  $x \in F$ . Note that  $F = {}^S E$ , so

$$\tau\varphi(x) = \tau\Phi(x) = \Phi\Phi^{-1}\tau\Phi(x) = \Phi(x) = \varphi(x)$$

for all  $\tau \in S$ , as  $\Phi^{-1}\tau\Phi \in S$ . This means that  $\varphi(x) \in {}^S E = F$ .

Let us compute  $\text{Gal}(F/K)$ . Since  $F/K$  is normal, the map  $\lambda: G \rightarrow \text{Gal}(F/K)$ ,  $\sigma \mapsto \sigma|_F$ , is a surjective group homomorphism such that  $\ker \lambda = S$ . The first isomorphism theorem implies that  $\text{Gal}(F/K) \simeq G/S$ .  $\square$

Some easy consequences.

**Exercise 9.2.** If  $E/K$  is a Galois extension of degree  $n$  and  $p$  is a prime number dividing  $n$ , then  $E/K$  admits a subextension of degree  $n/p$ .

**Exercise 9.3.** If  $E/K$  is a Galois extension of degree  $p^\alpha m$  with  $p$  a prime number coprime with  $m$ , then  $E/K$  admits a subextension of degree  $m$ .

**Definition 9.4.** An extension  $E/K$  is **abelian** if  $E/K$  is a Galois extension with  $\text{Gal}(E/K)$  abelian.

**Exercise 9.5.** If  $E/K$  is an abelian extension of degree  $n$  and  $d$  divides  $n$ , then  $E/K$  admits a subextension of degree  $d$ .

**Definition 9.6.** An extension  $E/K$  is **cyclic** if  $E/K$  is a Galois extension with  $\text{Gal}(E/K)$  cyclic.

**Example 9.7.** The extension  $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$  admits exactly three non-trivial subextensions:

$$\mathbb{Q}(\sqrt{2})/\mathbb{Q}, \quad \mathbb{Q}(\sqrt{3})/\mathbb{Q}, \quad \mathbb{Q}(\sqrt{6})/\mathbb{Q},$$

as  $\text{Gal}(\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}) \simeq C_2 \times C_2$ .

**Example 9.8.** Let  $\omega \in \mathbb{C} \setminus \{1\}$  be such that  $\omega^5 = 1$ . Then

$$f(\omega, \mathbb{Q}) = 1 + X + X^2 + X^3 + X^4$$

and  $\mathbb{Q}(\omega)/\mathbb{Q}$  has degree four. Moreover,  $\mathbb{Q}(\omega)/\mathbb{Q}$  is a Galois extension and  $\text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) \simeq C_4$ . If  $\sigma \in \text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})$ , then  $\sigma(\omega) = \omega^i$  for some  $i \in \{1, \dots, 4\}$ . Moreover, for every  $i \in \{1, \dots, 4\}$  the map  $\omega \mapsto \omega^i$  induces an automorphism of  $\mathbb{Q}(\omega)/\mathbb{Q}$ . Thus  $|\text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})| = 4$ . Now

$$\sigma_i^k = \text{id} \iff \omega^{i^k} = \sigma_i^k(\omega) = \omega \iff i^k \equiv 1 \pmod{5}.$$

Thus the map  $\sigma_2$  given by  $\omega \mapsto \omega^2$  has order four.

Since  $\text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) = \langle \sigma \rangle$ , where  $\sigma(\omega) = \omega^2$ , is cyclic of order four, the extension  $\mathbb{Q}(\omega)/\mathbb{Q}$  has a unique degree-two subextension  $F/\mathbb{Q}$ . Note that  $|\langle \sigma^2 \rangle| = 2$  and  $\sigma^2(\omega) = \omega^4 = \omega^{-1}$ . Thus  $F = \langle \sigma^2 \rangle \mathbb{Q}(\omega)$ . Let  $\theta = \omega + \omega^{-1}$ . Then

$$\theta^2 = \omega^2 + \omega^3 + 2 = -(1 + \omega + \omega^{-1}) + 2 = 1 - \theta$$

and hence  $\theta$  is a root of  $X^2 + X - 1$ . Since  $\theta \notin \mathbb{Q}$ , it follows that

$$\theta \in \{(-1 + \sqrt{5})/2, (-1 - \sqrt{5})/2\}.$$

Therefore  $F = \mathbb{Q}(\sqrt{5})$ .

Let us mention some other consequences.

**Exercise 9.9.** Let  $E/K$  be a finite Galois extension and  $F_1, \dots, F_n$  fields such that  $K \subseteq F_i \subseteq E$  for all  $i \in \{1, \dots, n\}$ . For every  $i$  let  $S_i = \text{Gal}(E/F_i)$ . Then

$$\text{Gal}\left(E/\bigcap_{i=1}^n F_i\right) = \left\langle \bigcup_{i=1}^n S_i \right\rangle, \quad \text{Gal}\left(E/\prod_{i=1}^n F_i\right) = \bigcap_{i=1}^n S_i.$$

The following statement is a concrete application of the previous exercise.

**Exercise 9.10.** Let  $E/K$  be a finite Galois extension and  $G = \text{Gal}(E/K)$ . Assume that  $G$  is the direct product  $G = S \times T$  of the groups  $S$  and  $T$ . Let  $F = {}^S E$  and  $L = {}^T E$ . Then  $F \cap L = K$  and  $FL = E$ .

**Proposition 9.11.** Let  $E_1/K, \dots, E_r/K$  be Galois extensions. If  $E = \prod_{i=1}^r E_i$ , then  $E/K$  is a Galois extension. If, moreover, each  $E_i/K$  is finite, then

$$\theta: \text{Gal}(E/K) \rightarrow \text{Gal}(E_1/K) \times \cdots \times \text{Gal}(E_r/K), \quad \sigma \mapsto (\sigma|_{E_1}, \dots, \sigma|_{E_r}),$$

is an injective group homomorphism.

*Proof.* We only do the first part in the case  $r = 2$ , the general case is left as an exercise. Since  $E_1/K$  is algebraic, then  $E_1 E_2/E_2$  is algebraic. Since  $E_2/K$  is algebraic,  $E_1 E_2/K$  is algebraic. Similarly,  $E_1 E_2/K$  is separable.

Let  $C/K$  be an algebraic closure such that  $E_1 E_2 \subseteq C$ . If  $\sigma \in \text{Hom}(E_1 E_2/K, C/K)$ , then  $\sigma(E_1 E_2) \subseteq \sigma(E_1)\sigma(E_2) = E_1 E_2$  (do this calculation as an exercise). Thus  $E_1 E_2/K$  is normal.

If both  $E_1/K$  and  $E_2/K$  are finite, then  $E_1 E_2/K$  is finite.

Clearly,  $\theta$  is a group homomorphism. We claim that the map  $\theta$  is injective. Let  $\sigma \in \ker \theta$ . Then  $\sigma|_{E_i} = \text{id}_{E_i}$  for all  $i \in \{1, \dots, r\}$ . Let  $S = \langle \sigma \rangle \subseteq \text{Gal}(E/K)$  and  $F = {}^S E$ . Then  $E_i \subseteq F$  for all  $i \in \{1, \dots, r\}$  and hence  $E \subseteq F$ . It follows that  $F = E = {}^{\{\text{id}\}} E$  and therefore  $S = \{\text{id}\}$ , so  $\sigma = \text{id}$ .  $\square$

**Exercise 9.12.** Let  $E_1/K, \dots, E_r/K$  be finite Galois extensions such that for each  $j$  one has  $E_j \cap (E_1 \cdots E_{j-1} E_{j+1} \cdots E_r) = K$ . Then

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$$\mathrm{Gal}(E/K) \simeq \mathrm{Gal}(E_1/K) \times \cdots \times \mathrm{Gal}(E_r/K).$$

In this case,  $[E : K] = \prod_{i=1}^r [E_i : K]$ .





## Lecture 9

### §10. The fundamental theorem of algebra

We now present an easy proof of the fundamental theorem of algebra based on the ideas of Galois Theory. We need the following well-known facts:

- 1) Every real polynomial of odd degree admits a real root. This means that  $\mathbb{R}$  does not admit extension of odd degree  $> 1$ .
- 2) Every complex number admits a square root in  $\mathbb{C}$ . This means that  $\mathbb{C}$  does not admit degree-two extensions.

**Theorem 10.1.** *The field  $\mathbb{C}$  is algebraically closed.*

*Proof.* Let  $E/\mathbb{C}$  be an algebraic finite extension. Then  $E/\mathbb{R}$  is finite separable extension of even degree. There exists a Galois extension  $L/\mathbb{R}$  such that  $E \subseteq L$ , so  $[L : \mathbb{R}]$  is even. Let  $G = \text{Gal}(L/\mathbb{R})$ . Then  $|G| = 2^m s$  for some odd number  $s$ . If  $T$  is a 2-Sylow subgroup of  $G$ , then there exists a subextension  $F/\mathbb{R}$  of degree  $s$ . Since  $\mathbb{R}$  does not admit extensions of odd degree  $> 1$ ,  $s = 1$  and hence  $G$  is a 2-group. Since  $L/\mathbb{R}$  is a Galois extension,  $L/\mathbb{C}$  is a Galois extension. In particular,  $|\text{Gal}(L/\mathbb{C})| = 2^{m-1}$ . If  $m > 1$ , let  $U$  be a subgroup of  $\text{Gal}(L/\mathbb{C})$  of order  $2^{m-2}$ . Then  $U$  corresponds to a subextension  $L_1/\mathbb{C}$  of degree two, a contradiction. Hence  $m = 1$  and  $[L : \mathbb{C}] = 1$ , so  $L = \mathbb{C}$  and  $E = \mathbb{C}$ .  $\square$

### §11. Purely inseparable extensions

Let  $E/K$  be an algebraic extension. In page 7 we defined the **separable closure** of  $K$  with respect to  $E$  as the field

$$F = \{x \in E : x \text{ is separable over } K\}.$$

Note that  $K \subseteq F \subseteq E$  and  $F = K(F)$ . Moreover,  $F/K$  is separable and  $E/F$  is a **purely inseparable** extension, meaning that for every  $x \in E \setminus F$ , the polynomial  $f(x, F)$  is not separable.

The number  $[E : F]$  is known as the **degree of inseparability** of  $E/K$ . We write  $[E : K]_{\text{ins}} = [E : F]$ . Clearly,  $E/K$  is separable if and only if  $[E : K]_{\text{ins}} = 1$  and  $E/K$  is purely inseparable if and only if  $[E : K]_{\text{ins}} = [E : K]$ .

**Proposition 11.1.** *Let  $K$  be a field of characteristic  $p > 0$  and  $E/K$  be an algebraic extension. The following statements are equivalent:*

- 1)  $E/K$  is purely inseparable.
- 2) If  $x \in E$ , then  $x^{p^m} \in K$  for some  $m \geq 0$ .
- 3) If  $x \in E$ , then  $f(x, K) = X^{p^m} - a$  for some  $a \in K$  and  $m \geq 0$ .
- 4)  $\gamma(E/K) = 1$ .

*Proof.* We first prove 1)  $\implies$  2). Let  $x \in E$  and  $f = f(x, K)$ . Assume  $x$  is not separable. Then  $f(x) = 0$  and  $f'(x) = 0$ , as  $x$  is not a simple root. Since  $\deg f' < \deg f$  and  $f$  is the minimal polynomial of  $x$ , it follows that  $f' = 0$ . The coefficients of  $f'$  are of the form  $ka_k$ . Since  $E$  is a field,  $a_k = 0$  if  $k$  is not divisible by  $p$ . If  $a_k \neq 0$ , then  $k = pm$  for some  $m \geq 0$ . It follows that  $f = g(X^p)$  for some  $g \in K[X]$  with  $\deg g < \deg f$ . We now proceed by induction on the degree of  $x$ . The result is clearly true for elements of degree one. So assume the result holds for element of degree  $\leq n$  for some  $n \geq 1$ . If  $x \in E$  is such that  $\deg(x, K) = n + 1$ , then, since  $f(x, K) = g(X^p)$ , the element  $x^p$  has degree  $\leq n$ . By the inductive hypothesis,  $x^{p^{m+1}} = (x^p)^{p^m} \in K$ . We now prove 2)  $\implies$  3). Let  $x \in E$  and  $m$  be the minimal positive integer such that  $x^{p^m} \in K$ . Then  $x$  is a root of  $X^{p^m} - x^{p^m} \in K[X]$ . Since  $X^{p^m} - x^{p^m} = (X - x)^{p^m}$ , it follows that

$$f(x, K) = (X - x)^r = X^r + \cdots + (-1)^r x^r$$

for some  $r \in \{1, \dots, p^m\}$ . Write  $r = p^s t$  for some integer  $t$  coprime with  $p$  and  $s$  such that  $0 \leq s \leq m$ . Let  $a, b \in \mathbb{Z}$  be such that  $ar + bp^m = p^s$ . Then

$$x^{p^s} = x^{ar+bp^m} = (x^r)^a (x^{p^m})^b \in K.$$

The minimality of  $m$  implies that  $s \geq m$  and hence  $s = m$ . Now  $p^m t = p^s t = r \leq p^m$ , so  $t = 1$ . This means  $f(x, K) = X^{p^m} - x^{p^m}$ .

We now prove 3)  $\implies$  4). Let  $C/K$  be an algebraic closure that contains  $E$  and  $\sigma \in \text{Hom}(E/K, C/K)$ . Let  $x \in E$ . We claim that  $\sigma(x) = x$ . Since  $f(x, K) = X^{p^m} - a$ ,

$$(\sigma(x))^{p^m} = \sigma(x^{p^m}) = \sigma(a) = a = x^{p^m}.$$

It follows that  $\sigma(x)$  is a root of  $X^{p^m} - x^{p^m} = (X - x)^{p^m}$ . Thus  $\sigma(x) = x$ .

Finally, we prove that 4)  $\implies$  1). Let  $C$  be an algebraic closure of  $K$  containing  $E$ . Then  $\text{Gal}(E/K) = \text{Hom}(E/K, C/K) = \{\text{id}\}$ , as  $\gamma(E/K) = 1$ . If  $x \in E$  is separable over  $K$ , then

$$f(x, K) = \prod_{y \in \text{Gal}(E/K)(x)} (X - y) = X - x \in K[X].$$

Thus  $x \in K$  and hence  $E/K$  is purely inseparable.  $\square$

Some consequences:

**Exercise 11.2.** Let  $E/K$  be finite and purely inseparable. Then  $[E : K] = p^s$  for some prime number  $p$  and some  $s$ . Moreover,  $x^{[E:K]} \in K$ .

For the first part of previous exercise write  $E = K(x_1, \dots, x_n)$  and proceed by induction on  $n$ .

**Exercise 11.3.** Let  $K$  be of characteristic  $p > 0$  and  $E/K$  be a finite extension such that  $[E : K]$  is not divisible by  $p$ . Then  $E/K$  is separable.

Let  $E/K$  be finite and  $F$  be the separable closure of  $K$  in  $E$ . Since

$$\gamma(E/K) = \gamma(E/F)\gamma(F/K) = \gamma(F/K),$$

it follows that

$$[E : K] = [E : F]\gamma(E/K) = [E : K]_{\text{ins}}\gamma(E/K).$$

## §12. Norm and trace

**Definition 12.1.** Let  $E/K$  be a finite extension and  $C/K$  be an algebraic closure that contains  $E$ . Let  $A = \text{Hom}(E/K, C/K)$ . For  $x \in E$  we define the **trace** of  $x$  in  $E/K$  as

$$\text{trace}_{E/K}(x) = [E : K]_{\text{ins}} \sum_{\varphi \in A} \varphi(x)$$

and the **norm** of  $x$  in  $E/K$  as

$$\text{norm}_{E/K}(x) = \left( \prod_{\varphi \in A} \varphi(x) \right)^{[E:K]_{\text{ins}}}.$$

As an optional exercise, one can show that these definitions do not depend on the algebraic closure.

We collect some basic properties as an exercise:

**Exercise 12.2.** Let  $E/K$  be a finite extension. The following statements hold:

- 1) If  $E/K$  is not separable, then  $\text{trace}_{E/K}(x) = 0$  for all  $x \in E$ .
- 2) If  $x \in K$ , then  $\text{trace}_{E/K}(x) = [E : K]x$ .
- 3)  $\text{trace}_{E/K}(x) \in K$  for all  $x \in E$ .
- 4)  $\text{norm}_{E/K}(x) = 0$  if and only if  $x = 0$ .
- 5) If  $x \in K$ , then  $\text{norm}_{E/K}(x) = x^{[E:K]}$ .
- 6)  $\text{norm}_{E/K}(x) \in K$  for all  $x \in E$ .

One proves, moreover, that  $\text{trace}_{E/K} : E \rightarrow K$  satisfies

$$\text{trace}_{E/K}(x + \lambda y) = \text{trace}_{E/K}(x) + \lambda \text{trace}_{E/K}(y)$$

for all  $x, y \in E$  and  $\lambda \in K$ , that is to say that  $\text{trace}_{E/K} : E \rightarrow K$  is a linear form in  $E$ . The norm  $\text{norm}_{E/K} : E^\times \rightarrow K^\times$  is a group homomorphism.

**Exercise 12.3.** Let  $E/K$  be a finite extension and  $x \in E$ . If

$$f(x, K) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0,$$

then  $\text{norm}_{E/K}(x) = ((-1)^n a_0)^{[E:K(x)]}$  and  $\text{trace}_{E/K}(x) = -[E : K(x)]a_{n-1}$ .

**Example 12.4.** Let  $E = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ . Then

$$\begin{aligned} \text{trace}_{E/\mathbb{Q}}(\sqrt{2}) &= 0, & \text{norm}_{E/\mathbb{Q}}(\sqrt{2}) &= -4, \\ \text{trace}_{E/\mathbb{Q}(\sqrt{2})}(\sqrt{2}) &= 2\sqrt{2}, & \text{norm}_{E/\mathbb{Q}(\sqrt{2})}(\sqrt{2}) &= 2. \end{aligned}$$

**Example 12.5.** If  $E/K$  is a finite Galois extension, then

$$\text{trace}_{E/K}(x) = \sum_{\sigma \in \text{Gal}(E/K)} \sigma(x) \quad \text{and} \quad \text{trace}_{E/K}(x) = \prod_{\sigma \in \text{Gal}(E/K)} \sigma(x)$$

for all  $x \in E$ . In particular, since  $E = K(y)$  for some  $y$  by Proposition 7.9,

$$\text{trace}_{E/K}(y) = -a_{n-1} \quad \text{and} \quad \text{norm}_{E/K}(y) = (-1)^n a_0,$$

where  $f(y, K) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$ .

### §13. Finite fields

In this section,  $p$  will be a prime number.

**Proposition 13.1.** *Let  $m$  be a positive integer. Up to isomorphism, there exists a unique field  $F_m$  of size  $p^m$ .*

*Proof.* Let  $C$  be an algebraic closure of the field  $\mathbb{Z}/p$  and let  $F_m = \{x \in C : x^{p^m} = x\}$  be the set of roots of  $X^{p^m} - X$ . Since the polynomial  $X^{p^m} - X$  has no multiple roots,  $|F_m| = p^m$ . Moreover,  $F_m$  is the unique subfield of  $C$  of size  $p^m$ .

To prove the uniqueness it is enough to note that if  $K$  is a field of  $p^m$  elements, then  $K$  is the decomposition field of  $X^{p^m} - X$  over  $\mathbb{Z}/p$ .  $\square$

Let  $K = \mathbb{Z}/p$  and  $C$  be an algebraic closure of  $K$ . We claim that  $C = \cup_k F_k$ . If  $x \in C$ , then  $x$  is algebraic over  $K$ . Since  $K(x)/K$  is finite,  $K(x)$  is a finite field, say  $|K| = p^r$  for some  $r$ . Then  $x^{p^r} = x$  and hence  $x \in F_r$ .

**Exercise 13.2.** Prove the following statements:

- 1) If  $x \in F_r$ , then  $x^{p^{rk}} = x$  for all  $k \geq 0$ .
- 2) If  $m \mid n$ , then  $F_m \subseteq F_n$ .
- 3)  $F_m \cap F_n = F_{\gcd(m,n)}$ .
- 4)  $F_m \subseteq F_n$  if and only if  $m \mid n$ .

**Proposition 13.3.** Every finite extension of a finite field is cyclic.

*Proof.* Let  $K = \mathbb{Z}/p$ . It is enough to show that  $F_n/F_m$  is cyclic if  $m$  divides  $n$ .

We first prove that  $F_n/K$  is cyclic. Let  $\sigma \in \text{Gal}(F_n/K)$  be given by  $\sigma(x) = x^p$ . Note that  $|\text{Gal}(F_n/K)| = [F_n : K] = n$ . Since  $\sigma^i(x) = x^{p^i}$  for all  $i \geq 0$ , in particular,  $\sigma^n(x) = x^{p^n} = x$ . Thus  $\sigma^n = \text{id}$  and hence  $|\sigma|$  divides  $n$ . Let  $s = |\sigma|$ . We know that  $F_n^\times = F_n \setminus \{0\}$  is cyclic, say  $F_n^\times = \langle g \rangle$ . Since  $|g| = p^n - 1$ ,

$$g = \sigma^s(g) = g^{p^s}$$

and hence  $p^s \equiv 1 \pmod{p^n - 1}$ . Thus  $p^n - 1$  divides  $p^s - 1$  and hence  $n$  divides  $s$ . Therefore  $n = s$  and  $\text{Gal}(F_n/K) = \langle \sigma \rangle$ .

For the general case note that if  $m$  divides  $n$ , then  $\text{Gal}(F_n/F_m)$  is a subgroup of  $\text{Gal}(F_n/K)$ . Since  $\text{Gal}(F_n/K)$  is cyclic, the claim follows.  $\square$

If  $K = \mathbb{Z}/p$  and  $m$  divides  $n$ , the subextension  $F_m$  corresponds to the unique subgroup of index  $m$  of  $\text{Gal}(F_n/K) = \langle \sigma \rangle$ . This subgroup is  $\langle \sigma^m \rangle$ , where

$$\sigma^m(x) = x^{p^m} = x^{|F_m|}.$$

Note that  $\text{Gal}(F_n/F_m) = \langle \sigma^m \rangle$ . The map  $\sigma^m$  is known as the **Frobenius automorphism**.

**Exercise 13.4.** Let  $E/K$  be an extension of finite fields. Then  $E/K$  is cyclic and  $\text{Gal}(E/K) = \langle \tau \rangle$ , where  $\tau(x) = x^{|K|}$ .



## Lecture 10

### §14. Cyclotomic extensions

For  $n \geq 1$  let  $G_n(K) = \{x \in K : x^n = 1\}$  be the set of  $n$ -roots of one in  $K$ . Note that  $G_n(K)$  is a cyclic subgroup of  $K^\times$  and that  $|G_n(K)|$  divides  $n$ .

**Example 14.1.**  $G_n(\mathbb{R}) = \{-1, 1\}$  if  $n$  is odd and  $G_n = \{1\}$  if  $n$  is even.

**Exercise 14.2.** Let  $K$  be a field of characteristic  $p > 0$ . Let  $n = p^s m$  for some  $m$  not divisible by  $p$ . Then  $G_n(K) = G_m(K)$ .

**Exercise 14.3.** Let  $q$  be a prime number. Then  $G_n(\mathbb{Z}/q) \simeq \mathbb{Z}/\gcd(n, q-1)$ .

Similarly, one can prove that if  $K$  is a finite field, then  $G_n(K)$  is a cyclic group of order  $\gcd(n, |K^\times|)$ .

**Example 14.4.** If  $C$  is algebraically closed of characteristic coprime with  $n$ , then  $G_n(C)$  is cyclic of order  $n$ , as  $X^n - 1$  has all his roots in  $C$  and does not contain multiple roots.

Let  $K$  be an algebraically closed field and  $n$  be such that  $n$  is coprime with the characteristic of  $K$ . The set of **primitive  $n$ -roots** is defined as

$$H_n(K) = \{x \in G_n(K) : |x| = n\}.$$

**Definition 14.5.** Let  $K$  be an algebraically closed field and  $n$  be such that  $n$  is coprime with the characteristic of  $K$ . The  **$n$ -th cyclotomic polynomial** is defined as

$$\Phi_n = \prod_{x \in H_n(K)} (X - x) \in K[X].$$

For  $n \geq 1$  the Euler's function is defined as

$$\varphi(n) = |\{k : 1 \leq k \leq n, \gcd(k, n) = 1\}|.$$

For example,  $\varphi(4) = 2$ ,  $\varphi(8) = \varphi(10) = 4$  and  $\varphi(p) = p - 1$  for every prime  $p$ .

**Proposition 14.6.** *Let  $K$  be an algebraically closed field and  $n$  be such that  $n$  is coprime with the characteristic of  $K$ . Let  $A$  be the ring of integers of  $K$ .*

- 1)  $\deg \Phi_n = \varphi(n)$ .  
 2)  $\Phi_n \in A[X]$ .

*Proof.* The first statement is clear. Let us prove 2) by induction on  $n$ . The case  $n = 1$  is trivial, as  $\Phi_1 = X - 1$ . Assume that  $\Phi_d \in A[X]$  for all  $d$  such that  $d < n$ . In particular,

$$\gamma = \prod_{\substack{d|n \\ d \neq n}} \Phi_d \in A[X].$$

Since  $\gamma$  is monic, it follows that  $\frac{X^n - 1}{\gamma} \in A[X]$ . Now the claim follows from

$$X^n - 1 = \prod_{d|n} \Phi_d = \Phi_n \prod_{\substack{d|n \\ d \neq n}} \Phi_d = \Phi_n \gamma. \quad \square$$

By taking degree in the equality  $X^n - 1 = \prod_{d|n} \Phi_d$  one gets

$$n = \sum_{d|n} \varphi(d).$$

**Definition 14.7.** Let  $n \geq 2$  and  $K$  be a field of characteristic coprime with  $n$ . A **cyclotomic extension** of  $K$  of index  $n$  is a decomposition field of  $X^n - 1$  over  $K$ .

Let  $C$  be an algebraic closure of  $K$  and  $n \geq 2$  be coprime with the characteristic of  $K$ . It follows from Definition 14.7 that a cyclotomic extension of index  $n$  is of the form  $K(\omega)/K$  for some  $\omega \in H_n(C)$ .

**Proposition 14.8.** *A cyclotomic extension of index  $n$  is abelian and of degree a divisor of  $\varphi(n)$ .*

*Proof.* Let  $C$  be an algebraic closure of  $K$  and  $n \geq 2$  be coprime with the characteristic of  $K$ . Let  $\omega \in H_n(C)$  and  $K(\omega)/K$  be a cyclotomic extension. Then  $K(\omega)/K$  is a Galois extension, as it is a decomposition field of a separable polynomial. Let  $U = \mathcal{U}(\mathbb{Z}/n)$  be the group of units of  $\mathbb{Z}/n$  and

$$\lambda: \text{Gal}(K(\omega)/K) \rightarrow U, \quad \sigma \mapsto m_\sigma,$$

where  $m_\sigma$  is such that  $\sigma(\omega) = \omega^{m_\sigma}$ . The map  $\lambda$  is well-defined and it is a group homomorphism, as if  $\sigma, \tau \in \text{Gal}(K(\omega)/K)$ , then, since

$$(\tau\sigma)(\omega) = \tau(\sigma(\omega)) = \tau(\omega^{m_\sigma}) = (\omega^{m_\sigma})^{m_\tau} = \omega^{m_\sigma m_\tau},$$

it follows that  $\lambda(\sigma)\lambda(\tau) = \lambda(\sigma\tau)$ . Since  $\lambda$  is injective,  $\text{Gal}(K(\omega)/K)$  is isomorphic to a subgroup of the abelian group  $U$ . Hence  $\text{Gal}(K(\omega)/K)$  is abelian. Moreover,  $[K(\omega) : K] = |\text{Gal}(K(\omega)/K)|$  is a divisor of  $|U| = \varphi(n)$ .  $\square$



§15 Hilbert's theorem

**Exercise 14.9.** Prove that a cyclotomic extension  $K(\omega)/K$  has degree  $\varphi(n)$  if and only if  $\Phi_n$  is irreducible over  $K$ .

Note that  $\Phi_n$  is irreducible over  $\mathbb{Q}$ . Some concrete examples:

$$\Phi_1 = X - 1, \quad \Phi_2 = X + 1, \quad \Phi_3 = X^2 + X + 1, \quad \Phi_6 = X^2 - X + 1.$$

If  $p$  is a prime number, then  $\Phi_p = X^{p-1} + \cdots + X + 1$ .

**Example 14.10.**  $\Phi_5$  is irreducible over  $\mathbb{Z}/2$ . First note that  $\Phi_5 = X^4 + \cdots + X + 1$  does not have roots in  $\mathbb{Z}/2$ . If  $\Phi_5$  is reducible, then, since  $X^2 + X + 1$  is the unique monic irreducible polynomial over  $\mathbb{Z}/2$ , it follows that

$$\Phi_5 = (X^2 + X + 1)(X^2 + X + 1) = (X^2 + X + 1)^2 = X^4 + X^2 + 1,$$

a contradiction.

**Exercise 14.11.** Prove that  $\Phi_{12} = X^4 - X^2 + 1$  is not irreducible over  $\mathbb{Z}/5$ .

## §15. Hilbert's theorem

Let  $G$  be a group and  $A$  be a (left)  $G$ -module. This means that  $A$  is an abelian group and there exists a map

$$G \times A \rightarrow A, \quad (g, a) \mapsto g \cdot a$$

such that  $1 \cdot a = a$  for all  $a \in A$ ,  $(gh) \cdot a = g \cdot (h \cdot a)$  for all  $g, h \in G$  and  $a \in A$  and  $g \cdot (a + b) = g \cdot a + g \cdot b$  for all  $g \in G$  and  $a, b \in A$ .

**Definition 15.1.** Let  $A$  be a  $G$ -module. A **derivation** of  $A$  is a map  $d: G \rightarrow A$  such that  $d(gh) = g \cdot d(h) + d(g)$  for all  $g, h \in G$ .

Let  $A$  be a  $G$ -module and  $a \in A$ . The map  $d(g) = g \cdot a - a$  is a derivation of  $A$ . Such derivations are known as **inner derivations**.

**Exercise 15.2.** Let  $E/K$  be a finite Galois extension and  $G = \text{Gal}(E/K)$ . Then the (multiplicative) group  $E^\times$  is a  $G$ -module with  $\sigma \cdot x = \sigma(x)$ .

In the context of the previous exercise, let  $d: G \rightarrow E^\times$  be a derivation. For  $\tau \in G$  let  $x_\tau = d(\tau)$ . Then

$$x_{\sigma\tau} = d(\sigma\tau) = (\sigma \cdot d(\tau))d(\sigma) = \sigma(x_\tau)x_\sigma.$$

Is this true that  $x_\sigma = \frac{\sigma(c)}{c} = (\sigma \cdot c)c^{-1}$  for some  $c \in E^\times$ ?

**Proposition 15.3.** Let  $E/K$  be a finite Galois extension and  $G = \text{Gal}(E/K)$ .

1) Let  $\{x_\tau : \tau \in G\} \subseteq E^\times$  be such that  $x_{\sigma\tau} = \sigma(x_\tau)x_\sigma$ . Then there exists  $c \in E^\times$  such that  $x_\sigma = \frac{\sigma(c)}{c}$  for all  $\sigma \in G$ .

2) Let  $\{x_\tau : \tau \in G\} \subseteq E$  be such that  $x_{\sigma\tau} = \sigma(x_\tau) + x_\sigma$ . Then there exists  $c \in E$  such that  $x_\sigma = \sigma(c) - c$  for all  $\sigma \in G$ .

*Proof.* We prove 1). By Dedekind's theorem, the homomorphism  $\sum_{\tau \in G} x_\tau^{-1} \tau$  is non-zero. Thus there exists  $a \in E^\times$  such that  $\sum_{\tau \in G} x_\tau^{-1} \tau(a) = c \in E^\times$ . If  $\sigma \in G$ , then

$$\sigma(c) = \sum_{\tau \in G} \sigma(x_\tau^{-1})(\sigma\tau)(a) = \sum_{\tau \in G} x_{\sigma\tau}^{-1} x_\tau \sigma\tau(a) = x_\sigma \sum_{\tau \in G} x_{\sigma\tau}^{-1} \sigma\tau(a) = x_\sigma c.$$

We now prove 2). By Dedekind's theorem,  $\sum_{\tau \in G} \tau \neq 0$ . Since it is a linear form in  $E$ , it is surjective. In particular, there exists  $a \in E$  such that  $\sum_{\tau \in G} \tau(a) = 1$ . Let  $c = -\sum_{\tau \in G} x_\tau \tau(a)$ . If  $\sigma \in G$ , then

$$\begin{aligned} \sigma(c) &= - \sum_{\tau \in G} \sigma(x_\tau) \sigma\tau(a) \\ &= - \sum_{\tau \in G} (x_{\sigma\tau} - x_\sigma) \sigma\tau(a) \\ &= - \sum_{\tau \in G} x_{\sigma\tau} \sigma\tau(a) + \sum_{\tau \in G} x_\sigma \sigma\tau(a) = c + x_\sigma. \end{aligned}$$

Hence  $x_\sigma = \sigma(c) - c$ . □

The following result is known as Hilbert's 90 theorem.

**Theorem 15.4 (Hilbert).** *Let  $E/K$  be a cyclic extension of degree  $n$  and let  $G = \text{Gal}(E/K) = \langle \sigma \rangle$ .*

- 1) *Let  $x \in E^\times$  be such that  $\text{norm}_{E/K}(x) = 1$ , There exists  $b \in E^\times$  such that  $x = \sigma(b)/b$ .*
- 2) *Let  $x \in E$  be such that  $\text{trace}_{E/K}(x) = 0$ . There exists  $b \in E$  such that  $x = \sigma(b) - b$ .*

*Proof.* Let us prove 1). Note that  $G = \{\sigma^i : 0 \leq i < n\}$ . For  $i \in \{0, \dots, n-1\}$  let  $x_{\sigma^i} = \prod_{k=0}^{i-1} \sigma^k(x)$ . In particular,  $x_\sigma = x$ . We now check that  $\{x_{\sigma^i}\}$  satisfy the assumptions of the previous proposition:

$$\begin{aligned} \sigma^j(x_{\sigma^i}) &= \prod_{k=0}^{i-1} \sigma^{k+j}(x) = \prod_{k=j}^{i+j-1} \sigma^k(x) \\ &= \prod_{k=0}^{i+j-1} \sigma^k(x) \left( \prod_{k=0}^{j-1} \sigma^k(x) \right)^{-1} = \prod_{k=0}^{i+j-1} \sigma^k(x) x_{\sigma^j}^{-1}. \end{aligned}$$

If  $i+j < n$ , then

$$\sigma^j \sigma^i = \sigma^{i+j} \implies \sigma^j(x_{\sigma^i}) = x_{\sigma^i \sigma^j} x_{\sigma^j}^{-1} \implies x_{\sigma^j \sigma^i} = \sigma^j(x_{\sigma^i}) x_{\sigma^j}.$$

If  $i+j = n+r$  for some  $r \in \{0, \dots, n-1\}$ , then, since  $i+j < 2n$  and  $\sigma^i \sigma^j = \sigma^r$ ,

$$\sigma^j(x_{\sigma^i}) = \prod_{k=0}^{n-1} \sigma^k(x) \prod_{k=n}^{n+r-1} \sigma^k(x) x_{\sigma^j}^{-1} = \prod_{k=0}^{r-1} \sigma^k(x) x_{\sigma^j}^{-1} = x_{\sigma^r} x_{\sigma^j}^{-1} = x_{\sigma^j \sigma^i} x_{\sigma^j}^{-1}$$

§15 Hilbert's theorem

and hence  $x_{\sigma^j \sigma^i} = \sigma^j(x_{\sigma^i})x_{\sigma^j}$ . By the previous lemma, there exists  $c \in E^\times$  such that  $x_{\sigma^i} = \frac{\sigma^i(c)}{c}$ . In particular,  $x = x_\sigma = \sigma(c)/c$ .

The second statement is similar and it is left as an exercise.  $\square$

If  $A$ ,  $B$  and  $C$  are groups (written multiplicatively) and  $f: A \rightarrow B$  and  $g: B \rightarrow C$  are group homomorphism, the sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

of groups and homomorphisms is said to be **exact** if  $f(A) = \ker g$ . For example, the sequence

$$1 \longrightarrow B \xrightarrow{g} C$$

is exact if and only if  $g$  is injective, as the first map represents the trivial homomorphism. Similarly, the sequence

$$A \xrightarrow{f} B \longrightarrow 1$$

is exact if and only if  $f$  is surjective.

**Corollary 15.5.** *Let  $E/K$  be finite and cyclic with  $\text{Gal}(E/K) = \langle \sigma \rangle$ .*

1) *The sequence*

$$1 \longrightarrow K^\times \hookrightarrow E^\times \xrightarrow{\rho} E^\times \xrightarrow{\text{norm}_{E/K}} K^\times$$

*is exact, where  $\rho(z) = \sigma(z)/z$ .*

2) *The sequence*

$$0 \longrightarrow K \hookrightarrow E \xrightarrow{\lambda} E \xrightarrow{\text{trace}_{E/K}} K$$

*is exact, where  $\lambda(z) = \sigma(z) - z$ .*

*Proof.* We only prove 1). Note that  $K^\times \hookrightarrow E^\times$  is the inclusion map. If  $z \in \ker \rho$ , then  $\sigma(z) = z$  and hence  $z \in K$ . The sequence

$$E^\times \xrightarrow{\rho} E^\times \xrightarrow{\text{norm}_{E/K}} K^\times$$

is exact by Hilbert's theorem.  $\square$



## Lecture 11

**Proposition 15.6.** *Let  $n \geq 2$  and  $K$  be a field containing a primitive  $n$ -root of one. If  $a \in K^\times$  and  $E/K$  is a decomposition field of  $f = X^n - a$ , then  $E/K$  is cyclic of degree  $d$ , where  $d$  divides  $n$ . Moreover,*

$$d = \min\{k : a^k \in K^\times\},$$

where  $K^n = \{x \in K : x = y^n \text{ for some } y \in K\}$ . Conversely, if  $E/K$  is cyclic of degree  $n$ , then  $E/K$  is a decomposition field of an irreducible polynomial of the form  $X^n - a$  for some  $a \in K^\times$ .

*Proof.* A decomposition field of  $f$  over  $K$  is of the form  $K(\alpha)$ , where  $\alpha^n = a$ . Thus  $K(\alpha)/K$  is a Galois extension. If  $\sigma \in \text{Gal}(K(\alpha)/K)$ , then  $\sigma(\alpha)$  is a root of  $f$ , so  $\sigma(\alpha) = \omega_\sigma \alpha$ , where  $\omega_\sigma \in G_n(K)$ . This means that there exists an injective map

$$\lambda: \text{Gal}(K(\alpha)/K) \rightarrow G_n(K), \quad \sigma \mapsto \omega_\sigma.$$

Moreover,  $\lambda$  is a group homomorphism, as

$$\sigma\tau(\alpha) = \sigma(\tau(\alpha)) = \sigma(\omega_\tau \alpha) = \omega_\tau \sigma(\alpha) = \omega_\tau \omega_\sigma \alpha.$$

Therefore  $\text{Gal}(K(\alpha)/K)$  is isomorphic to a subgroup of  $G_n(K)$ . In particular,  $\text{Gal}(K(\alpha)/K)$  is cyclic and  $|\text{Gal}(K(\alpha)/K)|$  divides  $n$ .

Let  $d = |\text{Gal}(K(\alpha)/K)|$ . Since  $a = \alpha^n$ ,

$$\text{norm}(\alpha)^n = \text{norm}(a) = a^d.$$

Thus  $a^d \in K^n$ , as  $\text{norm}(\alpha) \in K$ . If  $a^k \in K^n$ , say  $a^k = c^n$  for some  $c \in K$ , then

$$a^k = c^n = (\alpha^n)^k = (\alpha^k)^n \implies \alpha^k = c\omega \in K$$

for some  $\omega \in G_n(K)$ . Thus  $\alpha$  is a root of  $X^k - \alpha^k \in K[X]$  and hence  $k \geq d$ .

Note that  $f(\alpha, K) = X^d - \alpha^d$ .

Let  $E/K$  be cyclic of degree  $n$ . Assume that  $\text{Gal}(E/K) = \langle \sigma \rangle$ . If  $\omega$  is a primitive  $n$ -root of one,

$$\text{norm}_{E/K}(\omega) = 1 = \omega^n.$$

By Hilbert's theorem, there exists  $b \in E^\times$  such that  $\omega = \sigma(b)/b$ . Thus  $\sigma(b) = \omega b$  and hence  $\sigma^i(b) = \omega^i b$  for all  $i \geq 0$ . Since  $|\{b, \sigma(b), \dots, \sigma^{n-1}(b)\}| = n$ , it follows that  $E = K(b)$ . Moreover,

$$\sigma(b^n) = \sigma(b)^n = (\omega b)^n = b^n$$

and hence  $b^n \in K$ . This means that  $E/K$  is a decomposition field of  $X^n - b^n$ . Note that  $X^n - b^n$  is irreducible, as  $[E : K] = [K(b) : K] = n$ .  $\square$

**Proposition 15.7.** *Let  $K$  be a field of characteristic  $p > 0$ .*

- 1) *Let  $a \in K$  and  $f = X^p - X - a$ . Then  $f$  is irreducible over  $K$  or all the roots of  $f$  belong to  $K$ . In the first case, if  $b$  is a root of  $f$ , then  $K(b)/K$  is a cyclic extension of degree  $p$ .*
- 2) *Every cyclic extension of degree  $p$  is a decomposition field of an irreducible polynomial of the form  $X^p - X - a$ .*

*Proof.* We first prove 1). Let  $K_0$  be the prime field of  $K$ . Note that  $K_0 \simeq \mathbb{Z}/p$ . Let  $b$  be a root of  $f$  and let  $x \in K_0$ . Then

$$f(b+x) = (b+x)^p - (b+x) - a = (b^p - b - a) + (x^p - x) = 0$$

and thus  $\{b+x : x \in K_0\}$  is the set of roots of  $f$ . Note that  $f' = -1$ , so  $f$  has no multiple roots.

We claim that if  $b \notin K$ , then  $f$  is irreducible. If  $f$  is not irreducible, then  $f = gh$  for some  $g, h \in K[X]$  such that  $0 < \deg g < p$ . There exists a subset  $S$  of  $K_0$  such that  $g = \prod_{x \in S} (X - (b+x))$  and hence

$$|S|b + \sum_{x \in S} x = \sum_{x \in S} (b+x) \in K.$$

This implies that  $|S|b \in K$  and hence, since  $|S| \in K^\times$ , it follows that  $b \in K$ .

Since  $K(b)/K$  is a decomposition field of a separable polynomial,  $K(b)/K$  is a Galois extension. Moreover,  $|\text{Gal}(K(b)/K)| = [K(b) : K] = p$  and hence  $\text{Gal}(K(b)/K)$  is cyclic.

We now prove 2). Let  $E/K$  be cyclic of degree  $p$ . Assume that  $\text{Gal}(E/K) = \langle \sigma \rangle$ . Since  $\text{trace}_{E/K}(1) = p = 0$ , Hilbert's theorem implies that there exists  $b \in E$  such that  $\sigma(b) = b+1$ . In particular,  $b \notin K$  and thus  $E = K(b)$ . Moreover, since

$$\sigma(b^p - b) = \sigma(b)^p - \sigma(b) = (b+1)^p - (b+1) = b^p - b,$$

it follows that  $b^p - b \in K$ . Thus  $f(b, K) = X^p - X - (b^p - b) \in K[X]$ .  $\square$

## §16. Symmetric polynomials

Let  $K$  be a field and  $\{t_1, \dots, t_n\}$  be an algebraic independent set over  $K$ . Let  $E = K(t_1, \dots, t_n)$  and  $f = \prod_{i=1}^n (X - t_i) \in E[X]$ . Then

$$f = X^n + \sum_{i=1}^n (-1)^i s_i X^{n-i},$$

where

$$\begin{aligned} s_1 &= t_1 + t_2 + \cdots + t_n, \\ s_2 &= \sum_{1 \leq i < j \leq n} t_i t_j, \\ &\vdots \\ s_n &= t_1 t_2 \cdots t_n. \end{aligned}$$

For example,

$$(X - t_1)(X - t_2)(X - t_3) = X^3 - (t_1 + t_2 + t_3)X^2 + (t_1 t_2 + t_2 t_3 + t_1 t_3)X - t_1 t_2 t_3.$$

The polynomials  $s_1, s_2, \dots, s_n$  are known as the **elementary symmetric polynomials** in the variables  $t_1, \dots, t_n$ . Note that  $\deg s_i = i$ .

Let  $\sigma \in \mathbb{S}_n$  and

$$\alpha_\sigma: K[t_1, \dots, t_n] \rightarrow K[t_1, \dots, t_n], \quad t_i \mapsto t_{\sigma(i)} \quad \text{for all } i.$$

Then  $\alpha_\sigma$  is a bijective homomorphism of  $K$ -algebras. In fact,  $\alpha_\sigma^{-1} = \alpha_{\sigma^{-1}}$ . Note that

$$\alpha_\sigma(h(t_1, \dots, t_n)) = h(t_{\sigma(1)}, \dots, t_{\sigma(n)}).$$

Since  $\alpha_\sigma$  is injective, it induces an element  $\widehat{\sigma} \in \text{Gal}(E/K)$  given by

$$\widehat{\sigma} \left( \frac{h}{g} \right) = \frac{\alpha_\sigma(h)}{\alpha_\sigma(g)}.$$

The map  $\mathbb{S}_n \rightarrow \text{Gal}(E/K)$ ,  $\sigma \mapsto \widehat{\sigma}$ , is an injective group homomorphism. Thus  $\{\widehat{\sigma} : \sigma \in \mathbb{S}_n\} \simeq \mathbb{S}_n$ .

**Definition 16.1.** Let  $g \in K[t_1, \dots, t_n]$ . Then  $g$  is **symmetric** if  $\widehat{\sigma}(g) = g$  for all  $\sigma \in \mathbb{S}_n$ .

We write  $P$  to denote the set of symmetric polynomials in  $K[t_1, \dots, t_n]$ . Clearly,  $P$  is a subalgebra of  $K[t_1, \dots, t_n]$ . The following statements hold:

- 1)  $K \subseteq P$ .
- 2)  $\sum_{i=1}^n t_i^r \in P$  for all  $r \geq 1$ .

- 3)  $s_i \in P$  for all  $i$ .  
 4)  $K(P) \subseteq {}^G E$ , where  $G = \{\widehat{\sigma} : \sigma \in \mathbb{S}_n\}$ .

Let  $F = K(s_1, s_2, \dots, s_n)$ . Then  $E/F$  is a Galois extension, as it is a decomposition field of  $f$ .

**Proposition 16.2.**  $[E : F] \leq n!$ .

*Proof.* We proceed by induction on  $n$ . The case  $n = 1$  is clear, as  $E = F$ . Assume that  $n > 1$ . Let  $u_1, \dots, u_{n-1}$  be the elementary symmetric polynomials in  $t_1, \dots, t_{n-1}$ . Then

$$s_i = u_i + t_n u_{i-1}$$

for all  $i \in \{1, \dots, n\}$ , where  $u_0 = 1$  and  $u_n = 0$ . Note that  $u_1 = s_1 - t_n$  and  $u_i = s_i - t_n u_{i-1}$  for all  $i$ . Since  $K(s_1, \dots, s_n, t_n) = K(u_1, \dots, u_{n-1}, t_n)$ ,

$$F(t_n) = K(u_1, \dots, u_{n-1}, t_n) = K(t_n)(u_1, \dots, u_{n-1})$$

and

$$[E : F] = [E : F(t_n)][F(t_n) : F] \leq n[E : F(t_n)].$$

Note that  $E = K(t_1, \dots, t_n) = K(t_n)(t_1, \dots, t_{n-1})$ . By the inductive hypothesis,  $[E : F(t_n)] \leq (n-1)!$  and hence  $[E : F] \leq n!$ , as desired.  $\square$

**Theorem 16.3.**  ${}^G E = F$ .

*Proof.* By Artin's theorem,

$$[{}^G E : F] = \frac{[E : F]}{[E : {}^G E]} \leq \frac{n!}{[E : {}^G E]} = 1$$

and hence  ${}^G E = F$ .  $\square$

**Exercise 16.4.** Prove that  $\text{Gal}(E/F) \simeq \mathbb{S}_n$ .

**Exercise 16.5.** Prove that  $\{s_1, \dots, s_n\}$  is algebraically independent over  $K$ .

**Exercise 16.6.** Prove that every symmetric polynomial in  $t_1, \dots, t_n$  can be written as a rational fraction in  $s_1, \dots, s_n$ .

## §17. Solvable groups

Let  $G$  be a group. If  $x, y \in G$  we define the **commutator** of  $x$  and  $y$  as

$$[x, y] = xyx^{-1}y^{-1}.$$

Note that  $[x, y] = 1$  if and only if  $xy = yx$ . Moreover,  $[x, y]^{-1} = [y, x]$ . The **commutator subgroup**  $[G, G]$  of  $G$  is defined as the subgroup of  $G$  generated by all commutators, i.e.



$$[G, G] = \langle [x, y] : x, y \in G \rangle.$$

This means that every element of  $[G, G]$  is a finite product of commutators, so every element of  $[G, G]$  is of the form  $\prod_{i=1}^m [x_i, y_i]$ . In general, the commutator subgroup is not equal to the set of commutators!

**Exercise 17.1.** Let  $G$  be a group. Prove the following facts:

- 1)  $G$  is abelian if and only if  $[G, G] = \{1\}$ .
- 2)  $[G, G]$  is a normal subgroup of  $G$ .
- 3)  $G/[G, G]$  is abelian.
- 4) If  $H$  is a subgroup of  $G$  and  $[G, G] \subseteq H$ , then  $H$  is normal in  $G$ .
- 5) If  $H$  is a normal subgroup of  $G$ , then  $G/H$  is abelian if and only if  $[G, G] \subseteq H$ .

**Definition 17.2.** Let  $G$  be a group. The **derived series** of  $G$  is defined as  $G^{(0)} = G$  and  $G^{(k+1)} = [G^{(k)}, G^{(k)}]$  for  $k \geq 0$ .

**Exercise 17.3.** Prove that  $G^{(k)}$  is normal in  $G$  for all  $k$ .

Why derived series? We cannot explain this here, but let us use the following notation. We write  $G' = [G, G]$ ,  $G'' = [G', G']$ ... Note that

$$G \supseteq G' \supseteq G'' \supseteq \dots$$

**Exercise 17.4.** Let  $n \geq 3$ . Prove that  $[\mathbb{S}_n, \mathbb{S}_n] = \mathbb{A}_n$ .

**Example 17.5.** Let  $K = \{\text{id}, (12)(34), (13)(24), (14)(23)\}$ . Then  $K$  is a normal subgroup of  $\mathbb{A}_4$ . One proves that  $[\mathbb{A}_4, \mathbb{A}_4] = K$ .

**Example 17.6.** Let  $n \geq 5$ . Since  $\mathbb{A}_n$  is a non-abelian simple group, it follows that  $[\mathbb{A}_n, \mathbb{A}_n] = \mathbb{A}_n$ .

Let us show that  $\mathbb{A}_5$  is a non-abelian simple group. Hence it is not solvable:

```
julia> A5 = alternating_group(5)
Alt( [ 1 .. 5 ] )
julia> is_abelian(A5)
false
julia> is_simple(A5)
true
julia> is_solvable(A5)
false
```

**Definition 17.7.** A group  $G$  is **solvable** if and only if  $G^{(m)} = \{1\}$  for some  $m$ .

Clearly, every abelian group is solvable.

**Exercise 17.8.** Prove that  $\mathbb{S}_n$  is solvable if and only if  $n \leq 4$ .

Let us compute (with the computer software Oscar) the derived series of the symmetric group  $\mathbb{S}_4$ . The calculation shows that  $\mathbb{S}_4$  is solvable:

```
julia> G = symmetric_group(4);
julia> derived_series(G)
4-element Vector{PermGroup}:
 Sym( [ 1 .. 4 ] )
 Alt( [ 1 .. 4 ] )
 Group([ (1,4) (2,3), (1,2) (3,4) ])
 Group(())
julia> [order(x) for x in derived_series(G)]
4-element Vector{fmpz}:
 24
 12
 4
 1
julia> is_solvable(G)
true
```

**Proposition 17.9.** *Let  $G$  be a group and  $H$  be a subgroup of  $G$ . The following statements hold:*

- 1) *If  $G$  is solvable, then  $H$  is solvable.*
- 2) *If  $H$  is normal in  $G$  and  $G$  is solvable, then  $G/H$  is solvable.*
- 3) *If  $H$  is normal in  $G$  and  $H$  and  $G/H$  are solvable, then  $G$  is solvable.*

*Proof.* The first statement follows from the fact that  $H^{(i)} \subseteq G^{(i)}$  holds for all  $i$ .

Assume now that  $H$  is normal in  $G$ . Let  $Q = G/H$  and  $\pi: G \rightarrow Q$  be the canonical map. By induction one proves that  $\pi(G^{(i)}) = Q^{(i)}$  for all  $i \geq 0$ . The case where  $i = 0$  is trivial, as  $\pi$  is surjective. If the result holds for some  $i \geq 0$ , then

$$\pi(G^{(i+1)}) = \pi([G^{(i)}, G^{(i)}]) = [\pi(G^{(i)}), \pi(G^{(i)})] = [Q^{(i)}, Q^{(i)}] = Q^{(i+1)}.$$

We now prove 2). Since  $G$  is solvable,  $G^{(n)} = \{1\}$  for some  $n$ . Thus  $Q$  is solvable, as  $Q^n = \pi(G^{(n)}) = \pi(\{1\}) = \{1\}$ .

We finally prove 3). Since  $Q$  is solvable,  $Q^{(n)} = \{1\}$  for some  $n$ . Since  $\pi(G^{(n)}) = Q^{(n)} = \{1\}$ , it follows that  $G^{(n)} \subseteq H$ . Since  $H$  is solvable,

$$G^{(n+m)} \subseteq (G^{(n)})^{(m)} \subseteq H^{(m)} = \{1\}$$

for some  $m$ . Thus  $G$  is solvable. □

An application:

**Proposition 17.10.** *Let  $G$  be a finite  $p$ -group. Then  $G$  is solvable.*

*Proof.* Assume the result is not true. Let  $G$  be a finite  $p$ -group of minimal order that is not solvable. Since  $G$  is a  $p$ -group,  $Z(G) \neq \{1\}$ . Since  $|G|$  is minimal,  $G/Z(G)$  is

a solvable  $p$ -group. Since  $Z(G)$  is abelian,  $Z(G)$  is solvable. Now  $G$  is solvable by Proposition 17.9.  $\square$

We finish this discussion with two important theorems (without proof) about finite solvable groups.

**Theorem 17.11 (Burnside).** *Let  $p$  and  $q$  be prime numbers. If  $G$  is a group of order  $p^a q^b$ , then  $G$  is solvable.*

The proof appears in courses on representation theory of finite groups.

**Theorem 17.12 (Feit–Thompson).** *Every finite group of odd order is solvable.*

The proof of the theorem is extremely hard. It occupies a full volume of *Pacific Journal of Mathematics*, see [1].



## Lecture 12

### §18. Radical extensions

**Definition 18.1.** An extension  $E/K$  is **radical** if  $E = K(x_1, \dots, x_m)$  such that for each  $i \in \{1, \dots, m\}$  there exists  $a_i \in \mathbb{Z}$  such that  $x_i^{a_i} \in K(x_1, \dots, x_{i-1})$ .

Note that radical extensions are finite.

**Example 18.2.** Let  $E$  be a decomposition field of  $X^4 - 2$  over  $\mathbb{Q}$ . Then  $E/\mathbb{Q}$  is radical, as  $E = \mathbb{Q}(\sqrt[4]{2}, i)$ .

**Example 18.3.** Let  $\alpha, \beta \in \mathbb{C}$  be such that  $\alpha^2 = 2$  and  $\beta^5 = 1 + \alpha$ . The number  $\sqrt[5]{1 + \sqrt{2}}$  belongs to the radical extension  $\mathbb{Q}(\alpha, \beta)/\mathbb{Q}$ .

**Theorem 18.4.** Let  $K$  be of characteristic zero and  $R/K$  be a radical extension. If  $E/K$  is a subextension of  $R/K$ , then  $\text{Gal}(E/K)$  is solvable.

*Proof.* Without loss of generality, we may assume that  $E/K$  is a Galois extension. In fact, let  $G = \text{Gal}(E/K)$  and  $K_0 = {}^G E$ . Then  $E/K_0$  is a Galois extension and  $\text{Gal}(E/K_0) = G$  by Artin's theorem. Thus, replacing  $K$  by  $K_0$  if needed, we may assume that  $E/K$  is Galois.

Let  $L$  be the normal closure of  $R$  in some algebraic closure  $C$  that contains  $R$ . Note that if  $R = K(x_1, \dots, x_m)$ , then

$$L = K(\{\sigma_i(x_j) : 1 \leq i \leq s, 1 \leq j \leq m\}),$$

where  $\text{Hom}(R/K, C/K) = \{\sigma_1, \dots, \sigma_s\}$ .

*Claim.*  $L/K$  is radical.

Since  $x_j^{a_j} \in K(x_1, \dots, x_{j-1})$  for some integer  $a_j$ ,

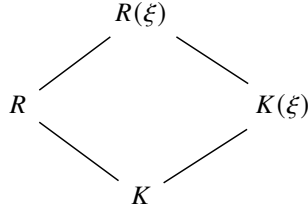
$$\sigma_i(x_j)^{a_j} = \sigma_i(x_j^{a_j}) \in \sigma_i(K(x_1, \dots, x_{j-1})) = K(\sigma_i(x_1), \dots, \sigma_i(x_{j-1}))$$

Thus  $L/K$  is radical and Galois.

We may assume then that  $E/K$  and  $R/K$  are both Galois.

Since  $\text{Gal}(E/K) \simeq \text{Gal}(R/K)/\text{Gal}(R/E)$ , we only need to prove that  $\text{Gal}(R/K)$  is solvable.

Let  $\xi$  be a primitive  $n$ -th root of one (in some algebraic closure of  $K$  that contains  $R$ ). Consider the diagram



Then

- 1)  $K(\xi)/K$  and  $R(\xi)/R$  are abelian.
- 2)  $R(\xi)/K$  is Galois.
- 3)  $\text{Gal}(R/K) \simeq \text{Gal}(R(\xi)/K)/\text{Gal}(R(\xi)/R)$ .
- 4)  $\text{Gal}(K(\xi)/K) \simeq \text{Gal}(R(\xi)/K)/\text{Gal}(R(\xi)/K(\xi))$ .

The third item implies that we need to show that  $\text{Gal}(R(\xi)/K)$  is solvable. By the fourth item it suffices to show that  $\text{Gal}(R(\xi)/K(\xi))$  is solvable.

Since  $R = K(x_1, \dots, x_m)$ ,

$$R(\xi) = K(x_1, \dots, x_m, \xi) = K(\xi)(x_1, \dots, x_m)$$

and hence  $R(\xi)/K(\xi)$  is radical. This means that without loss of generality we may assume that  $K$  contains primitive  $n$ -roots of one. For example, if  $R = K(x_1, \dots, x_m)$  and  $x_i^{a_i} \in K(x_1, \dots, x_{i-1})$ , then we may assume that  $K$  contains a primitive  $a_i$ -root of one. We proceed by induction on  $m$ . The case  $m = 0$  is trivial. Assume that the claim holds for some  $m \geq 0$ . Let  $L = K(x_1)$ . Then  $L/K$  is a decomposition field of  $X^{a_1} - x_1^{a_1}$ , and hence  $L/K$  is a cyclic extension. Thus  $\text{Gal}(L/K)$  is cyclic (and hence, in particular, solvable). Let  $H$  be the subgroup that corresponds to  $L$ , that is  $H = \text{Gal}(R/L)$  (here we use Galois' correspondence). Then  $H$  is normal in  $\text{Gal}(R/K)$ . Since  $R = K(x_1, \dots, x_m) = L(x_2, \dots, x_m)$ ,  $R/L$  is radical and Galois. By the inductive hypothesis,  $\text{Gal}(R/L)$  is solvable. Since

$$\text{Gal}(L/K) \simeq \text{Gal}(R/K)/\text{Gal}(R/L),$$

it follows that  $\text{Gal}(R/K)$  is solvable. □

**Definition 18.5.** Let  $f \in K[X]$  and  $E$  be a decomposition field of  $f$  over  $K$ . We say that  $f$  is **solvable by radicals** if there is a radical extension  $R/K$  such that  $E \subseteq R$ .

The general polynomial of degree two is solvable by radicals, as its Galois group is solvable (in fact, it is isomorphic to  $\mathbb{S}_2$ ).

**Exercise 18.6.** Prove that  $f = X^2 - s_1X + s_2 \in \mathbb{Q}[X]$  is solvable by radicals.

Theorem 18.4 translates into the following result:

**Exercise 18.7.** If  $f \in K[X]$  is solvable by radicals, then  $\text{Gal}(f, K)$  is solvable.

As a consequence, the general polynomial of degree  $n \geq 5$  is not solvable by radicals, as its Galois group is isomorphic to  $\mathbb{S}_5$ .

**Example 18.8.** Let  $p$  be a prime number and  $f = X^5 - 2pX + p \in \mathbb{Q}[X]$ . We claim that  $f$  is not solvable by radicals.

By Gauss' theorem one proves that  $f$  has no rational roots.

Since  $f(-1)f(1) < 0$  and  $\deg f$  is odd, one proves that  $f$  has at least three real roots. Moreover,  $f$  has exactly three real roots, as  $f' = 5X^4 - 2p$ . Let  $x_1, x_2 \in \mathbb{C} \setminus \mathbb{R}$  and  $x_3, x_4, x_5 \in \mathbb{R}$  be the roots of  $f$ .

By Eisenstein's theorem,  $f$  is irreducible.

Let  $E/\mathbb{Q}$  be a decomposition field of  $f$ . Then  $\text{Gal}(f, \mathbb{Q}) = \text{Gal}(E/\mathbb{Q})$  is isomorphic to a subgroup  $G$  of  $\mathbb{S}_5$ . Since  $f$  is irreducible, 5 divides  $[E : \mathbb{Q}] = |G|$ . In particular, by Cauchy's theorem,  $G$  contains an element  $\sigma$  of order five. This element is a 5-cycle, so without loss of generality we may assume that  $\sigma = (x_1x_2x_3x_4x_5)$ . Note that  $(x_1x_2) \in G$ . Thus  $G \simeq \mathbb{S}_5$  and hence  $G$  is not solvable.

**Exercise 18.9.** Let  $f = X^6 + 2X^5 - 5X^4 + 9X^3 - 5X^2 + 2X + 1 \in \mathbb{Q}[X]$  is solvable by radicals.





## Some solutions

**6.1** Let  $\{v_i : i \in I\}$  be a basis of  $V$  over  $K$ . For each  $i \in I$  let  $f_i : V \rightarrow F$ ,  $f_i(v_j) = \delta_{ij}$ . Then  $\{f_i : i \in I\}$  is linearly independent over  $F$ . In fact, let  $\sum a_i f_i = 0$ , where each  $a_i \in F$ . Then  $a_i = 0$  for almost all  $i$ . If  $j \in I$ , then

$$0 = \left( \sum a_i f_i \right) (v_j) = \sum a_i f_i(v_j) = a_j.$$

Now assume that  $\dim_K V = n$ . Let  $\{v_1, \dots, v_n\}$  be a basis of  $V$  over  $K$ . We claim that  $\{f_1, \dots, f_n\}$  is a basis of  $\text{Hom}_K(V, F)$  over  $F$ . If  $g \in \text{Hom}_K(V, F)$ , then  $g = \sum g(v_i) f_i$ . If  $1 \leq k \leq n$ , then

$$\left( \sum g(v_i) f_i \right) (v_k) = \sum g(v_i) f_i(v_k) = g(v_k).$$

**6.4** We need to find a bijective map

$$\text{Hom}(E/K, C/K) \rightarrow \text{Hom}(E/K, C_1/K).$$

If  $\sigma \in \text{Hom}(E/K, C/K)$ , then  $\theta^{-1}\sigma \in \text{Hom}(E/K, C_1/K)$ . If  $\varphi \in \text{Hom}(E/K, C_1/K)$ , then  $\theta\varphi \in \text{Hom}(E/K, C/K)$ . The maps  $\sigma \mapsto \theta^{-1}\sigma$  and  $\varphi \mapsto \theta\varphi$  are inverse to each other.



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