Galois theory

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Abstract. The notes correspond to the bachelor course **Galois Theory** of the Vrije Universiteit Brussel, Faculty of Sciences, Department of Mathematics and Data Sciences.

Contents

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Introduction

The notes correspond to the bachelor course **Galois theory** of the Vrije Universiteit Brussel, Faculty of Sciences, Department of Mathematics and Data Sciences. The course is divided into twelve two-hour lectures.

The material is somewhat standard. Basic texts on fields and Galois theory are for example [**3**] and [**4**].

As usual, we also mention a set of great expository papers by Keith Conrad, the notes are extremely well-written and useful at every stage of a mathematical career.

The notes include Magma code, which we use to verify examples and offer alternative solutions to certain exercises. Magma is a powerful software tool designed for working with algebraic structures. There is a free online version of Magma available.

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1. Lecture – Week 1

§ 1.1. Fields. Recall that a **field** is a commutative ring such that $1 \neq 0$ and every non-zero element is invertible. Examples of (infinite) fields are $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$. If p is a prime number, then $\mathbb{Z}/p$ is a field.

1.1. Example. The abelian group $\mathbb{Z}/2 \times \mathbb{Z}/2$ is a field with multiplication

$$(a,b)(c,d) = (ac + bd, ad + bc + bd).$$

1.2. Example. $\mathbb{Q}[i] = \{a + bi : a, b \in \mathbb{Q}\}$ and $\mathbb{Q}[\sqrt{2}]$ are fields.

$$\begin{tikzcd}
\mathbb{C} \arrow[drr, dash] \arrow[dd, dash] & & \\
 & & \mathbb{R} \arrow[d, dash] \\
\mathbb{Q}[i] \arrow[dr, dash] & & \mathbb{Q}[\sqrt{2}] \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}$$

1.3. Exercise. Prove that $\mathbb{Q}[i]$ and $\mathbb{Q}[\sqrt{2}]$ are not isomorphic as fields.

If R is a ring, there exists a unique ring homomorphism $\mathbb{Z} \to R$, $m \mapsto m1$. The image

$$\{m1 : m \in \mathbb{Z}\}$$

of this homomorphism is a subring of R and it is known as the **ring of integers** of R. The kernel is a subgroup of $\mathbb{Z}$ generated by some $t \geq 0$. The integer t is the **characteristic** of the ring R.

1.4. Exercise. The characteristic of a field is either zero or a prime number.

1.5. Example. The characteristic of the field of Example 1.1 is two. Why?

Recall that a commutative ring R is an **integral domain** if $xy = 0 \implies x = 0$ or $y = 0$. Fields are integral domains.

1.6. Exercise. Let K be a field. Prove that the following statements are equivalent:

1) K is of characteristic zero.
2) The additive order of 1 is infinite.
3) The additive order of each $x \neq 0$ is infinite.
4) The ring of integers of K is isomorphic to $\mathbb{Z}$.

1.7. Exercise. Let K be a field. Prove that the following statements are equivalent:

1) K is of characteristic p.
2) The additive order of 1 is p.
3) The additive order of each $x \neq 0$ is p.
4) The ring of integers of K is isomorphic to $\mathbb{Z}/p$.

1.8. Proposition. *Let K be a field. Then any finite subgroup of $K^\times = K \setminus \{0\}$ is cyclic.*

Proof. Let G be a finite subgroup of $K^\times$. Let $x \in G$ of maximal order N. We will show that any element of G is a power of x. Let $y \in G$ and $n = |y|$. Note that $X^n - 1 \in K[X]$ has at most n roots in K.

We claim that n divides N. If not, there exists a prime number p and a power $q = p^\beta$ of p such that $q \mid n$ and $q \nmid N$. Let $z = xy^{n/q}$. Since G is abelian,

$$|z| = \operatorname{lcm}\{N, q\} = Nq > N,$$

a contradiction.

The polynomial $X^n - 1 \in K[X]$ has n distinct roots in K, these are the elements $x^{kN/n}$ for $k \in \{0, \dots, n-1\}$. Since y has order n, it has to be one of these roots. Thus $y = x^{kN/n}$ for some $k \in \{0, \dots, n-1\}$. $\square$

1.9. Definition. A **subfield** of a ring R is a subring of R that is also a field.

Note that if K is a subfield of E, then the characteristic of K coincides with the characteristic of E. Moreover, if $K \to L$ is a field homomorphism, then K and L have the same characteristic.

1.10. Exercise. Let K be a field of characteristic p. Prove that $K \to K$, $x \mapsto x^{p^n}$, is a field homomorphism for all $n \in \mathbb{Z}_{\geq 0}$.

Note that finite fields are of prime characteristic.

Let K be a subfield of a field E. Then E is a K-vector space with the usual scalar multiplication $K \times E \to E$, $(\lambda, x) \mapsto \lambda x$.

1.11. Definition. A field K is **prime** if there are no proper subfields of K.

Examples of prime fields are $\mathbb{Q}$ and $\mathbb{Z}/p$ for a prime number p.

1.12. Proposition. *Let K be a field. The following statements hold:*

1) *K contains a unique prime field, it is known as the **prime subfield** of K.*
2) *The prime subfield of K is either isomorphic to $\mathbb{Q}$ if the characteristic of K is zero, or it is isomorphic to $\mathbb{Z}/p$ for some prime number p if the characteristic of K is p.*

Proof. To prove the first claim let L be the intersection of all the subfields of K. Then L is a subfield of K. If F is a subfield of L, then F is a subfield of K. Thus $L \subseteq F$ and hence $F = L$, which proves that L is prime. If L_1 is a subfield of K and L_1 is prime, then $L \subseteq L_1$ and hence $L = L_1$.

Let K_0 be the prime field of K. Suppose that K is of characteristic $p > 0$. Then the ring $K_{\mathbb{Z}}$ of integers of K is a field isomorphic to $\mathbb{Z}/p$ and hence $K_0 \simeq K_{\mathbb{Z}}$. Suppose now that the characteristic of K is zero. Let $E = \{m1/n1 : m, n \in \mathbb{Z}, n \neq 0\}$. We claim that $K_0 = E$. Since $K_{\mathbb{Z}} \subseteq K_0$, it follows that $E \subseteq K_0$. Hence $E = K_0$, as E is a subfield of K. $\square$

1.13. Definition. Let E be a field and K be a subfield of E. Then E is a **field extension** of K. We will use the notation E/K.

If E is an extension of K, then E is a K-vector space.

1.14. Definition. The **degree** of an extension E of K is the integer $\dim_K E$. It will be denoted by $[E : K]$.

We say that E is a **finite extension** of K if $[E : K]$ is finite.

1.15. Example. Let K be a field. Then $[K : K] = 1$. Conversely, if E is an extension of K and $[E : K] = 1$, then $K = E$. If not, let $x \in E \setminus K$. We claim that $\{1, x\}$ is linearly independent over K. Indeed, if $a1 + bx = 0$ for some $a, b \in K$, then $bx = -a$. If $b \neq 0$, then $x = -a/b \in K$, a contradiction. If $b = 0$, then $a = 0$.

We know that $[\mathbb{C} : \mathbb{R}] = 2$.

1.16. Example. A basis of $\mathbb{Q}[\sqrt{2}]$ over $\mathbb{Q}$ is given by $\{1, \sqrt{2}\}$. Then $[\mathbb{Q}[\sqrt{2}] : \mathbb{Q}] = 2$. The calculations can be easily done by computer:

```
> Q<z> := QuadraticField(2);
> Characteristic(Q);
0
> K := PrimeField(Q);
Rational Field
> Degree(Q);
2
> Basis(Q);
[ 1, z ]
> z^2
2
> One(K) = One(Q);
1 = 1
> G := GaloisGroup(Q);
> G;
Symmetric group G acting on a set of cardinality 2
Order = 2
```

1.17. Example. Since $\mathbb{Q}$ is numerable and $\mathbb{R}$ is not, $[\mathbb{R} : \mathbb{Q}] > \aleph_0$. If $\{x_i : i \in \mathbb{Z}_{>0}\}$ is a numerable basis of $\mathbb{R}$ over $\mathbb{Q}$, for each n consider the $\mathbb{Q}$-vector space V_n generated by $\{x_1, \dots, x_n\}$. Then

$$\mathbb{R} = \bigcup_{n \geq 1} V_n,$$

is numerable, as each V_n is numerable, a contradiction.

If E is an extension of K and E is finite, then $[E : K]$ is finite.

1.18. Proposition. *Let K be a finite field. Then $|K| = p^m$ for some prime number p and some $m \geq 1$.*

Proof. We know the prime subfield K_0 of K is isomorphic to $\mathbb{Z}/p$. In particular, $|K_0| = p$. Since K is finite, $[K : K_0] = m$ for some m. If $\{x_1, \dots, x_m\}$ is a basis of K over K_0, then each element of K can be written uniquely as $\sum_{i=1}^{m} a_i x_i$ for some $a_1, \dots, a_m \in K_0$. Then there is a bijection between K and K_0^m and hence $|K| = |K_0^m| = p^m$. $\square$

We now perform some basic calculations with a finite field of eight elements:

```
> E<x> := FiniteField(8);
> PrimeField(E);
Finite field of size 2
> Degree(E);
3
> Characteristic(E);
2
> #E;
8
> { x : x in E };
{ 1, x, x^2, x^3, x^4, x^5, x^6, 0 }
```

Here is an alternative construction if a field of eight elements as a quotient of a polynomial ring:

```
> R<x> := PolynomialRing(PrimeField(E));
> IsIrreducible(x^3+x+1);
true
> I := ideal< R | x^3+x+1 >;
> IsMaximal(I);
true
> Q<y> := quo< R | I >;
> IsField(Q);
true
> { y : y in Q };
{
    y^2 + 1,
    0,
    y^2,
    y + 1,
    1,
    y,
    y^2 + y + 1,
    y^2 + y
}
> #Q;
8
```

1.19. Definition. Let E be an extension of K. A **subextension** F/K of E/K is a subfield F of E that contains K, that is $K \subseteq F \subseteq E$.

1.20. Definition. Let E and E_1 be extensions over K. An **extension homomorphism**

$$E/K \to E_1/K$$

is a field homomorphism $\sigma\colon E \to E_1$ such that $\sigma(x) = x$ for all $x \in K$.

To describe the homomorphism $\sigma\colon E/K \to E_1/K$ of the extensions over K one typically writes the commutative diagram

$$\begin{tikzcd} K \arrow[r, equal] \arrow[d, hook] & K \arrow[d, hook] \\ E \arrow[r, "\sigma"] & E_1 \end{tikzcd}$$

We write $\mathrm{Hom}(E/K, E_1/K)$ to denote the set of homomorphism $E/K \to E_1/K$ of extensions of K. Note that if $\sigma \in \mathrm{Hom}(E/K, E_1/K)$, then σ is a K-linear map, as

$$\sigma(\lambda x) = \sigma(\lambda)\sigma(x) = \lambda\sigma(x)$$

for all $\lambda \in K$ and $x \in E$.

1.21. Example. The conjugation map $\mathbb{C} \to \mathbb{C}$, $z \mapsto \overline{z}$, is an endomorphism of $\mathbb{C}$ as an extension over $\mathbb{R}$. Let $\varphi \in \mathrm{Hom}(\mathbb{C}/\mathbb{R}, \mathbb{C}/\mathbb{R})$. Then

$$\varphi(x + iy) = \varphi(x) + \varphi(i)\varphi(y) = x + \varphi(i)y$$

for all $x, y \in \mathbb{R}$. Since $\varphi(i)^2 = \varphi(i^2) = \varphi(-1) = -1$, it follows that $\varphi(i) \in \{-i, i\}$. Thus either $\varphi(x + iy) = x + iy$ or $\varphi(x + iy) = x - iy$.

1.22. Exercise. Let K be a field, K_0 be its prime field and $\sigma\colon K \to K$ be a field homomorphism. Prove that $\sigma \in \mathrm{Hom}(K/K_0, K/K_0)$.

If E/K is an extension, then

$$\begin{aligned} \mathrm{Aut}(E/K) &= \{\sigma\colon E/K \to E/K \text{ is a bijective extension homomorphism}\} \\ &= \{\sigma\colon E \to E : \sigma \text{ is a bijective field homomorphism with } \sigma|_K = \mathrm{id}\} \end{aligned}$$

is a group with composition.

1.23. Definition. Let E/K be an extension. The **Galois group** of E/K is the group $\mathrm{Aut}(E/K)$ and it will be denoted by $\mathrm{Gal}(E/K)$.

A typical example: $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \simeq \mathbb{Z}/2$.

As an example, we show with the computer that $\mathrm{Gal}(\mathbb{Q}[\sqrt{2}]/\mathbb{Q}) \simeq \mathbb{Z}/2$:

```
> Q<z> := QuadraticField(2);
> G := GaloisGroup(Q);
> G;
Symmetric group G acting on a set of cardinality 2
Order = 2
> GroupName(G);
C2
```

1.24. Example. Let $\theta = \sqrt[3]{2}$ and let $E = \{a + b\theta + c\theta^2 : a, b, c \in \mathbb{Q}\}$. Note that

$$a + b\theta + c\theta^2 = 0 \iff a = b = c = 0.$$

Then E is an extension of $\mathbb{Q}$ such that $[E : \mathbb{Q}] = 3$. We claim that $\mathrm{Gal}(E/\mathbb{Q})$ is trivial. If $\sigma \in \mathrm{Gal}(E/\mathbb{Q})$ and $z = a + b\theta + c\theta^2$, then $\sigma(z) = a + b\sigma(\theta) + c\sigma^2(\theta)$. Since

$$\sigma(\theta)^3 = \sigma(\theta^3) = \sigma(2) = 2,$$

it follows that $\sigma(\theta) = \theta$ and therefore $\sigma = \mathrm{id}$.

1.25. Exercise. Prove that the polynomial $X^3 - 2$ is irreducible in $\mathbb{Q}[X]$.

The previous exercise can easily be solved using Magma:

```
> Q<x> := PolynomialRing(Rationals());
> f := x^3-2;
> IsIrreducible(f);
true
```

The following exercise is known as the **Eisenstein's irreducibility criterion**:

1.26. Exercise. Let A be a unique factorization domain and K be its fraction field. Let $f = \sum_{i=0}^{n} a_i X^i \in A[X]$ be a polynomial of degree $n > 0$. Assume that there exists a prime element $p \in A$ such that $p \mid a_i$ for all $i \in \{0, 1, \dots, n-1\}$, $p \nmid a_n$ and $p^2 \nmid a_0$. Then f is irreducible in $K[X]$.

1.27. Exercise. Prove that the polynomials

$$f = X^{10} + 60X^7 + 82X^6 - 36X^3 + 2,$$
$$g = 3X^{10} + 15X^2 - 45,$$

are irreducible in $\mathbb{Q}[X]$.

The previous exercise is easy for Magma:

```
> R<x> := PolynomialRing(Rationals());
> g := 3*x^10+15*x^2-45;
> IsIrreducible(g);
true
> f := x^10+60*x^7+82*x^6-36*x^3+2;
> IsIrreducible(f);
true
```

1.28. Exercise. Is the polynomial $f = 3(X^{10} + 5X^2 - 15)$ irreducible in $\mathbb{Z}[X]$?

Let us see what Magma can do here:

```
> R<x> := PolynomialRing(Integers());
> f := 3*x^10+15*x^2-45;
> IsIrreducible(f);
false
> Factorization(f);
[
    <3, 1>,
    <x^10 + 5*x^2 - 15, 1>
]
```

If E/K is an extension and S is a subset of E, then there exists a unique smallest subextension F/K of E/K such that $S \subseteq F$. In fact,

$$F = \bigcap\{T : T \text{ is a subfield of } E \text{ that contains } K \cup S\}$$

If L/K is a subextension of E/K such that $S \subseteq L$, then $F \subseteq L$ by definition. The extension F is known as the **subextension generated by** S and it will be denoted by $K(S)$. If $S = \{x_1, \dots, x_n\}$ is finite, then $K(S) = K(x_1, \dots, x_n)$ is said to be of **finite type**.

1.29. Example. If $\{e_1, \dots, e_n\}$ is a basis of E over K, then $E = K(e_1, \dots, e_n)$.

1.30. Example. The field $\mathbb{Q}(\sqrt{2})$ is precisely the extension of $\mathbb{R}/\mathbb{Q}$ generated by $\sqrt{2}$.

Let E/K be an extension and S and T be subsets of E. Then

$$K(S \cup T) = K(S)(T) = K(T)(S).$$

If, moreover, $S \subseteq T$, then $K(S) \subseteq K(T)$.

§ 1.2. Algebraic extensions.

1.31. Definition. Let E/K be an extension. An element $x \in E$ is **algebraic** over K if there exists a non-zero polynomial $f(X) \in K[X]$ such that $f(x) = 0$. If x is not algebraic over K, then it is called **transcendental** over K.

1.32. Definition. An extension E/K is **algebraic** if every $x \in E$ is algebraic over K.

If K is a field, every $x \in K$ is algebraic over K, as x is a root of $X - x \in K[X]$. In particular, K/K is an algebraic extension.

1.33. Example. $\mathbb{C}/\mathbb{R}$ is an algebraic extension. If $z \in \mathbb{C} \setminus \mathbb{R}$, then z is a root of the polynomial $X^2 - (z + \overline{z})X + |z|^2 \in \mathbb{R}[X]$.

If F/K is an extension $x \in E$ is algebraic over K for some field $E \supseteq F$, then x is algebraic over F.

1.34. Example. $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is algebraic, as the number $a+b\sqrt{2}$ is a root of the polynomial $X^2 - 2aX + (a^2 - 2b^2) \in \mathbb{Q}[X]$.

The extension $\mathbb{C}/\mathbb{Q}$ is not algebraic. For example, Hermite proved that e is transcendental over $\mathbb{Q}$; see [**4**, Theorem 24.4]. Lindemann's theorem states that π is not algebraic over $\mathbb{Q}$; see [**4**, Theorem 24.5].

1.35. Example. Let $a = \sqrt{2}$ and $b = \sqrt[3]{3}$. Both a and b are algebraic numbers over $\mathbb{Q}$. Let us show that $a + b$ is also algebraic. Let $f(X) = X^3 - 3 \in \mathbb{Q}[X]$. Then $f(b) = 0$. Note that the polynomial

$$g(X) = f(X - a) = X^3 - 3aX^2 + 3aX - a^3 - 3 \in \mathbb{Q}(a)[X]$$

is such that $g(a + b) = 0$. How can we find a polynomial with coefficients in $\mathbb{Q}$ that vanishes on $a + b$? We do the "conjugation" trick:

$$h(X) = f(X - a)f(X + a) = X^6 - 6X^4 - 6X^3 + 12X^2 - 36X + 1 \in \mathbb{Q}[X].$$

Note that $h(a + b) = 0$. How can you prove that ab is also algebraic over $\mathbb{Q}$?

2. Lecture – Week 2

If E/K is an extension and $x \in E$ is algebraic over K, then the evaluation homomorphism $K[X] \to E$, $p \mapsto p(x)$, is not injective. In particular, its kernel is a non-zero ideal. Hence it is generated by a monic polynomial f.

2.1. Definition. Let E/K be an extension and $x \in E$ be an algebraic element. The monic polynomial that generates the kernel of $K[X] \to E$, $f \mapsto f(x)$, is known as the **minimal polynomial** of x over K and it will be denoted by $f(x, K)$. The **degree** of x over K is then $\deg f(x, K)$.

Some basic properties of the minimal polynomial of an algebraic element:

2.2. Proposition. *Let E/K be an extension and $x \in E$. Assume that x is algebraic over K.*

1) *If $g \in K[X] \setminus \{0\}$ is such that $g(x) = 0$, then $f(x, K)$ divides g and $\deg f(x, K) \leq \deg g$.*
2) *$f(x, K)$ is irreducible in $K[X]$.*
3) *If F/K is a subextension of E/K, then $f(x, F)$ divides $f(x, K)$.*

Proof. Write $f = f(x, K)$ to denote the minimal polynomial of x. To prove 1) note that $g(x) = 0$ implies that g belongs to the kernel of the evaluation map, so g is a multiple of f. To prove 2) note that if $f = pq$ for some $p, q \in K[X]$ such that $0 < \deg p, \deg q < \deg f$, then $f(x) = 0$ implies that either $p(x) = 0$ or $q(x) = 0$, a contradiction. Finally, we prove 3). Since $f \in K[X] \subseteq F[X]$ and $f(x) = 0$, it follows from 1) that $f(x, F)$ divides f. $\square$

Some easy examples: $f(i, \mathbb{Q}) = X^2 + 1$, $f(i, \mathbb{C}) = X - i$ and $f(\sqrt[3]{2}, \mathbb{Q}) = X^3 - 2$. Here is the code:

```
> R<x> := PolynomialRing(Rationals());
> Q<i> := QuadraticField(-1);
> MinimalPolynomial(i);
x^2 + 1
```

A bit harder: $f(\sqrt[3]{2} + 1, \mathbb{Q}) = X^3 - 3X^2 + 3X - 3$. Here is the code:

```
> R<x> := PolynomialRing(Rationals());
> E<z> := NumberField(x^3-2);
> MinimalPolynomial(z+1);
x^3 - 3*x^2 + 3*x - 3
```

2.3. Example. Let us compute $f(\sqrt{2} + \sqrt{3}, \mathbb{Q})$. Let $\alpha = \sqrt{2} + \sqrt{3}$. Then

$$\begin{aligned} \alpha - \sqrt{2} = \sqrt{3} &\implies (\alpha - \sqrt{2})^2 = 3 \implies \alpha^2 - 2\sqrt{2}\alpha + 2 = 3 \\ &\implies \alpha^2 - 1 = 2\sqrt{2}\alpha \implies (\alpha^2 - 1)^2 = 8\alpha^2 \implies \alpha^4 - 10\alpha^2 + 1 = 0. \end{aligned}$$

Thus α is a root of $g = X^4 - 10X^2 + 1$. To prove that $g = f(\alpha, \mathbb{Q})$ it is enough to prove that g is irreducible in $\mathbb{Q}[X]$. First note that the roots of g are $\sqrt{2} + \sqrt{3}$, $\sqrt{2} - \sqrt{3}$, $-\sqrt{2} + \sqrt{3}$ and $-\sqrt{2} - \sqrt{3}$. This means that if g is not irreducible, then $g = hh_1$ for some polynomials

$h, h_1 \in \mathbb{Q}[X]$ such that $\deg h = \deg h_1 = 2$. This is not possible, as

$$(\sqrt{2}+\sqrt{3}) + (\sqrt{2}-\sqrt{3}) = 2\sqrt{2} \notin \mathbb{Q},$$
$$(\sqrt{2}+\sqrt{3}) + (-\sqrt{2}+\sqrt{3}) = 2\sqrt{3} \notin \mathbb{Q},$$
$$(\sqrt{2}+\sqrt{3})(-\sqrt{2}-\sqrt{3}) = -5-2\sqrt{6} \notin \mathbb{Q}.$$

2.4. Proposition. *Let F/K be a subextension of E/K. Then*

$$[E:K] = [E:F][F:K].$$

Proof. Let $\{e_i : i \in I\}$ be a basis of E over F and $\{f_j : j \in J\}$ be a basis of F over K. If $x \in E$, then $x = \sum_i \lambda_i e_i$ (finite sum) for some $\lambda_i \in F$. For each i, $\lambda_i = \sum_j a_{ij} f_j$ (finite sum) for some $a_{ij} \in K$. Then $x = \sum_i \sum_j a_{ij}(f_j e_i)$. This means that $\{f_j e_i : i \in I, j \in J\}$ generates E as a K-vector space. Let us prove that $\{f_j e_i : i \in I, j \in J\}$ is linearly independent. If $\sum_i \sum_j a_{ij}(f_j e_i) = 0$ (finite sum) for some $a_{ij} \in K$, then

$$\begin{aligned} 0 = \sum_i \left(\sum_j a_{ij} f_j \right) e_i &\implies \sum_j a_{ij} f_j = 0 \text{ for all } i \in I \\ &\implies a_{ij} = 0 \text{ for all } i \in I \text{ and } j \in J. \end{aligned}$$ □

We state a lemma:

2.5. Lemma. *If A is a finite-dimensional commutative algebra over K and A is an integral domain, then A is a field.*

Proof. Let $a \in A \setminus \{0\}$. We need to prove that there exists $b \in A$ such that $ab = 1$. Let $\theta\colon A \to A$, $x \mapsto ax$. Note that θ is K-linear transformation, as

$$\theta(x+y) = a(x+y) = ax + ay = \theta(x) + \theta(y), \quad \theta(\lambda x) = a(\lambda x) = \lambda(ax) = \lambda\theta(x),$$

for all $x, y \in A$ and $\lambda \in K$. It is injective since A is an integral domain. Since $\dim_K A < \infty$, it follows that θ is an isomorphism. In particular, $\theta(A) = A$, which implies that there exists $b \in A$ such that $1 = ab$. □

Let E/K be an extension and $x \in E$. Then

$$K[x] = \{f(x) : f \in K[X]\}$$

is a subring of E that contains K. Note that $K[x]$ is a K-vector space.

More generally, if $x_1, \dots, x_n \in E$, then

$$K[x_1, \dots, x_n] = \{f(x_1, \dots, x_n) : f \in K[X_1, \dots, X_n]\}$$

is a subring of E. Note that $K[x_1, \dots, x_n]$ is a K-vector space. Clearly, $K[x_1, \dots, x_n]$ is a domain and

$$K(x_1, \dots, x_n) = \left\{ \frac{f(x_1, \dots, x_n)}{g(x_1, \dots, x_n)} : f, g \in K[X_1, \dots, X_n] \text{ with } g(x_1, \dots, x_n) \neq 0 \right\}$$

is the extension of K generated by $x_1, \dots, x_n$. Note that

$$K(x_1, \dots, x_n) = (K(x_1, \dots, x_{n-1}))(x_n).$$

The previous construction can be generalized. Let I be a non-empty set. For each $i \in I$, let X_i be a variable. Consider the polynomial ring $K[\{X_i : i \in I\}]$ and let $S = \{x_i : i \in I\}$ be a subset of E. There exists a unique algebra homomorphism

$$K[\{X_i : i \in I\}] \to E$$

such that $X_i \mapsto x_i$ for all $i \in I$. The image is denoted by $K[S]$. In particular, an element $z \in K[S]$ is of the form

$$z = h(x_1, \dots, x_n)$$

for a polynomial $h \in K[X_1, \dots, X_n]$ in finitely many variables $X_1, \dots, X_n$ and $x_1, \dots, x_n \in S$.

2.6. Exercise. Prove that $\mathbb{Q}[\sqrt{2}] = \mathbb{Q}(\sqrt{2})$.

The exercise is not an accident.

2.7. Theorem. *Let E/K be an extension and $x \in E \setminus K$. The following statements are equivalent:*

1) *x is algebraic over K.*
2) $\dim_K K[x] < \infty$.
3) *$K[x]$ is a field.*
4) $K[x] = K(x)$.

Proof. We first prove 1) $\implies$ 2). Let $z \in K[x]$, say $z = h(x)$ for some $h \in K[X]$. There exists $g \in K[X]$ such that $g \neq 0$ and $g(x) = 0$. Divide h by g to obtain polynomials $q, r \in K[X]$ such that $h = gq + r$, where $r = 0$ or $\deg r < \deg g$. This implies that

$$z = h(x) = g(x)q(x) + r(x) = r(x).$$

If $\deg g = m$, then $r = \sum_{i=0}^{m-1} a_i X^i$ for some $a_0, \dots, a_{m-1} \in K$. Thus

$$z = \sum_{i=0}^{m-1} a_i x^i$$

and hence $K[x] \subseteq \langle 1, x, \dots, x^{m-1} \rangle$.

The previous lemma proves that 2) $\implies$ 3).

It is trivial that 3) $\implies$ 4).

It remains to prove that 4) $\implies$ 1). Since $x \neq 0$, $1/x \in K(x) = K[x]$. There exists $a_0, \dots, a_n \in K$ such that $1/x = a_0 + a_1 x + \cdots + a_n x^n$. Thus

$$a_n x^{n+1} + \cdots + a_1 x^2 + a_0 x - 1 = 0,$$

and hence x is a root of $a_n X^{n+1} + \cdots + a_0 X - 1 \in K[X] \setminus \{0\}$. $\square$

Note that if x is algebraic over K, then $K[x] \simeq K[X]/(f(x, K))$.

2.8. Exercise. Let E/K be an extension and $x \in E$ be an algebraic element over K. Prove that the degree of x over K is equal to $[K(x) : K]$.

2.9. Corollary. *If E/K is finite, then E/K is algebraic.*

Proof. Let $n=[E:K]$ and $x\in E\setminus K$. The set $\{1,x,\dots,x^n\}$ has $n+1$ elements, so it is linearly dependent. There exist $a_0,\dots,a_n\in K$, not all zero, such that

$$a_0+a_1x+\cdots+a_nx^n=0.$$

Thus x is a root of the non-zero polynomial $a_0+a_1X+\cdots+a_nX^n\in K[X]$. $\square$

In Example 1.35 we proved that $\sqrt{2}+\sqrt[3]{3}$ and $\sqrt{2}\sqrt[3]{3}$ are algebraic over $\mathbb{Q}$. This can be easily proved now with Corollary 2.9.

2.10. Exercise. Let E/K be an extension and a and b be algebraic over K. Prove that $a+b$ and ab are algebraic over K.

We note that the converse of Corollary 2.9 result does not hold.

2.11. Corollary. *If E/K is an extension and $x_1,\dots,x_n\in E$ are algebraic over K, then $K(x_1,\dots,x_n)/K$ is finite and $K(x_1,\dots,x_n)=K[x_1,\dots,x_n]$.*

Proof. We proceed by induction on n. The case $n=1$ follows immediately from the theorem. So assume the result holds for some $n\geq 1$. Since the extensions

$$K(x_1,\dots,x_n)/K(x_1,\dots,x_{n-1})\quad\text{and}\quad K(x_1,\dots,x_{n-1})/K$$

are both finite, it follows that $K(x_1,\dots,x_n)/K$ is finite. Moreover,

$$\begin{aligned}K(x_1,\dots,x_n)&=K(x_1,\dots,x_{n-1})(x_n)\\&=K(x_1,\dots,x_{n-1})[x_n]=K[x_1,\dots,x_{n-1}][x_n]=K[x_1,\dots,x_n].\end{aligned}$$

$\square$

2.12. Corollary. *Let $E=K(S)$ for some set S. Then E/K is algebraic if and only if x is algebraic over K for all $x\in S$.*

Proof. Let us prove the non-trivial implication. Let $z\in K(S)$. In particular, there exists a finite subset $T\subseteq S$ such that $z\in K(T)$. The previous result implies that $K(T)/K$ is algebraic, and hence z is algebraic. $\square$

If E/K is an extension, let

$$\overline{K}_E=\{x\in E: x \text{ is algebraic over } K\}.$$

2.13. Corollary. *If E/K is an extension, then $\overline{K}_E$ is a subfield of E that contains K. Moreover, $K(\overline{K}_E)=\overline{K}_E$ and $K(\overline{K}_E)/K$ is algebraic.*

Proof. By definition, $K(\overline{K}_E)/K$ is algebraic. Thus $K(\overline{K}_E)\subseteq\overline{K}_E$. From this, it follows that $K(\overline{K}_E)=\overline{K}_E$. $\square$

The following exercise is now almost trivial:

2.14. Exercise. Let E/K be an extension of finite type; this means that $E=K(S)$ for some finite set S. Prove that E/K is algebraic if and only if E/K is finite.

Let $\overline{\mathbb{Q}}=\{\alpha\in\mathbb{C}:\alpha\text{ is algebraic over }\mathbb{Q}\}$. Then $\overline{\mathbb{Q}}$ is the field of algebraic numbers. Can you compute $[\overline{\mathbb{Q}}:\mathbb{Q}]$?

2.15. Exercise. Prove that $[\mathbb{Q}[\sqrt[3]{2}]:\mathbb{Q}]=3$.

For the previous exercise, you may use Eisenstein's criterion.

2.16. Exercise. Let $E = \mathbb{Q}[i, \sqrt{2}] = \mathbb{Q}[\sqrt{2}][i]$. Prove that $[E : \mathbb{Q}] = 4$.

2.17. Exercise. Let $E = \mathbb{Q}[\sqrt{2}, \sqrt[3]{5}]$.

1) Compute $[E : \mathbb{Q}]$.
2) Prove that $E = \mathbb{Q}[\sqrt{2} + \sqrt[3]{5}]$.
3) Find the minimal polynomial of $\sqrt{2} + \sqrt[3]{5}$ over $\mathbb{Q}$.

2.18. Exercise. Find the minimal polynomials of $\sqrt[4]{3}i$ over $\mathbb{Q}[i]$ and over $\mathbb{Q}[\sqrt{3}]$.

2.19. Exercise. Find the minimal polynomial of $\sqrt{2} + \sqrt[3]{5}i$ over $\mathbb{Q}[i]$.

Algebraic field extensions form a nice class of extensions. The same happens with finite field extensions.

2.20. Proposition. *Let F/K be a subextension of E/K. Then E/K is algebraic if and only if E/F and F/K are algebraic.*

Proof. If E/K is algebraic, then E/F and F/K are both algebraic, as $K \subseteq F \subseteq E$. Let us assume that E/F and F/K are both algebraic. Let $x \in E$ and let L be the subextension over K generated by the coefficients of $f(x, F)$, the minimal polynomial of x over F. Then L/K is finite, since it is generated by finitely many algebraic elements. Moreover, x is algebraic over L. Since

$$[L(x) : K] = [L(x) : L][L : K] < \infty,$$

$L(x)/K$ is algebraic. In particular, x is algebraic over K. $\square$

2.21. Exercise. Let F/K be a subextension of E/K. Prove that E/K is finite if and only if E/F and F/K are finite.

Let K be a field and $K \subseteq F \subseteq L$ and $K \subseteq F \subseteq L$ be fields. The **composite** of E and F is defined as

$$EF = K(E \cup F) = F(E) = E(F)$$

and it is equal to the smallest field that contains E and F. Here is the picture:

$$\begin{tikzcd} & L \arrow[d, dash] & \\ & EF & \\ E \arrow[ur, dash] & & F \arrow[ul, dash] \\ & K \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

2.22. Exercise. Let E/K and F/K be algebraic field extensions. Prove that

$$EF = \left\{ \sum_{i=1}^{m} e_i f_i : m \in \mathbb{Z}_{>0}, e_i \in E, f_i \in F \text{ for all } i \in \{1, \dots, m\} \right\}.$$

2.23. Exercise. If $E = \mathbb{Q}(\sqrt{2})$ and $F = \mathbb{Q}(\sqrt{3})$, then $EF = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Compute $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}]$ and $\mathbb{Q}(\sqrt{2}) \cap \mathbb{Q}(\sqrt{3})$.

2.24. Exercise. Let $\xi \in \mathbb{C}$ be a primitive cubic root of one. If $E = \mathbb{Q}(\sqrt[3]{2})$ and $F = \mathbb{Q}(\xi)$, then $EF = \mathbb{Q}(\sqrt[3]{2}, \xi)$. Compute $[\mathbb{Q}(\sqrt[3]{2}, \xi) : \mathbb{Q}]$ and $\mathbb{Q}(\sqrt[3]{2}) \cap \mathbb{Q}(\xi)$.

2.25. Exercise. Let E/K and F/K be extensions, where both E and F are subfields of a field L. If F/K is algebraic, then EF/E is algebraic.

2.26. Exercise. Let E/K and F/K be extensions, where both E and F are subfields of a field L. If F/K is finite, then EF/E is finite.

The solution to the previous exercise shows, in particular, that $[EF : E] \leq [F : K]$.

3. Lecture – Week 3

3.1. Lemma. *Let $\sigma\colon K\to L$ be a field homomorphism. Then there exists an extension E/K and a field isomorphism $\varphi\colon E\to L$ such that $\varphi|_K=\sigma$.*

Proof. Note that $\sigma\colon K\to\sigma(K)$ is bijective. Let A be a set in bijection with $L\setminus\sigma(K)$ and disjoint with K. Let $E=K\cup A$. If $\theta\colon A\to L\setminus\sigma(K)$ is bijective, then let

$$\varphi\colon E\to L,\quad \varphi(x)=\begin{cases}\sigma(x) & \text{if } x\in K,\\ \theta(x) & \text{if } x\in A.\end{cases}$$

Then φ is a bijective map such that $\varphi|_K=\sigma$. Transport the operations of L onto E, that is to define binary operations on E as follows:

$$(x,y)\mapsto x\oplus y=\varphi^{-1}(\varphi(x)+\varphi(y)),\qquad (x,y)\mapsto x\odot y=\varphi^{-1}(\varphi(x)\varphi(y)).$$

Then, for example,

$$x\oplus y=\varphi^{-1}(\varphi(x)+\varphi(y))=\varphi^{-1}(\sigma(x)+\sigma(y))=\varphi^{-1}(\sigma(x+y))=\varphi^{-1}(\varphi(x+y))=x+y$$

for all $x,y\in K$. □

If $\sigma\colon A\to B$ is a ring homomorphism, then σ induces a ring homomorphism

$$\overline{\sigma}\colon A[X]\to B[X],\quad \sum_i a_iX^i\mapsto\sum_i\sigma(a_i)X^i.$$

3.2. Theorem. *Let K be a field and $f\in K[X]$ be such that* $\deg f>0$. *Then there exists an extension E/K such that f admits a root in E.*

Proof. We may assume that f is irreducible over K. Let $L=K[X]/(f)$ and

$$\pi\colon K[X]\to L$$

be the canonical map. Then L is a field (the reader should explain why). Let $\sigma\colon K\to L$, $a\mapsto\pi(aX^0)$, and $g=\overline{\sigma}(f)\in L[X]$.

We claim that $\pi(X)$ is a root of g in L. Suppose that $f=\sum_i a_iX^i$. Then

$$\begin{aligned}g(\pi(X))&=\overline{\sigma}(f)(\pi(X))\\&=\sum_i\sigma(a_i)\pi(X)^i=\sum_i\pi(a_iX^0)\pi(X^i)=\pi(\sum a_iX^i)=\pi(f)=0.\end{aligned}$$

By Lemma 3.1, there exists an extension E/K and an isomorphism $\varphi\colon E\to L$ such that $\varphi|_K=\sigma$. Note that $\varphi(x)=0$ if and only if $x=0$. If $u=\pi(X)$, then $\varphi^{-1}(u)$ is a root of f in E, as

$$\begin{aligned}\varphi(f(\varphi^{-1}(u)))&=\varphi\left(\sum_i a_i\varphi^{-1}(u)^i\right)=\varphi\left(\sum_i a_i\varphi^{-1}(u^i)\right)\\&=\sum_i\varphi(a_i)u^i=\sum_i\sigma(a_i)u^i=g(u)=0.\end{aligned}$$

□

As a corollary, if K is a field and $f_1,\dots,f_n\in K[X]$ are polynomials of positive degree, then there exists an extension E/K such that each f_i admits a root in E. This is proved by induction on n.

3.3. Definition. A field K is **algebraically closed** if each $f \in K[X]$ of positive degree admits a root in K.

The **fundamental theorem of algebra** states that $\mathbb{C}$ is algebraically closed. A typical proof uses complex analysis. Later we will give a proof of this result using Galois theory.

3.4. Proposition. *The following statements are equivalent:*

1) *K is algebraically closed.*
2) *If $f \in K[X]$ is irreducible, then $\deg f = 1$.*
3) *If $f \in K[X]$ is non-zero, then f decomposes linearly in $K[X]$, that is*

$$f = a \prod_{i=1}^{n} (X - \alpha_i)^{m_i}$$

for some $a \in K$ and $\alpha_1, \dots, \alpha_n \in K$.
4) *If E/K is algebraic, then $E = K$.*

Proof. 1) $\implies$ 2 $\implies$ 3) are exercises.

Let us prove that 3) $\implies$ 4). Let $x \in E$. Decompose $f(x, K)$ linearly in $K[X]$ as

$$f(x, K) = a \prod_{i=1}^{n} (X - \alpha_i)^{m_i}$$

and evaluate on x to obtain that $x = \alpha_j$ for some j.

To prove that 4) $\implies$ 1) let $f \in K[X]$ be such that $\deg f > 0$. There exists an extension E/K such that f has a root x in E. The extension $K(x)/K$ is algebraic and hence $K(x) = K$, so $x \in K$. $\square$

§ 3.1. Artin's theorem.

3.5. Definition. An **algebraic closure** of a field K is an algebraic extension C/K such that C is algebraically closed.

For example, $\mathbb{C}/\mathbb{R}$ is an algebraic closure but $\mathbb{C}/\mathbb{Q}$ is not.

3.6. Proposition. *Let C be algebraically closed and $\sigma\colon K \to C$ be a field homomorphism. If E/K is algebraic, then there exists a field homomorphism $\varphi\colon E \to C$ such that $\varphi|_K = \sigma$.*

Proof. Suppose first that $E = K(x)$ and let $f = f(x, K)$. Let $\overline{\sigma}(f) \in C[X]$ and let $y \in C$ be a root of $\overline{\sigma}(f)$. If $z \in E$, then $z = g(x)$ for some $g \in K[X]$. Let $\varphi\colon E \to C$, $z \mapsto \overline{\sigma}(g)(y)$.

The map φ is well-defined. If $z = h(x)$ for some $h \in K[X]$, then

$$0 = g(x) - h(x) = (g - h)(x)$$

and thus f divides $g - h$. In particular, $\overline{\sigma}(f)$ divides $\overline{\sigma}(g - h) = \overline{\sigma}(g) - \overline{\sigma}(h)$ and hence

$$(\overline{\sigma}(g) - \overline{\sigma}(h))(y) = 0.$$

It is an exercise to show that the map φ is a ring homomorphism.

Let $a \in K$. It follows that $\varphi|_K = \sigma$, as

$$\varphi(a) = \overline{\sigma}(aX^0)(y) = \sigma(a)$$

Let us now prove the proposition in full generality. Let X be the set of pairs (F, τ), where F is a subfield of E that contains K and $\tau\colon F \to C$ is a field homomorphism such that $\tau|_K = \sigma$. Note that $(K, \sigma) \in X$, so X is non-empty. Moreover, X is partially ordered by

$$(F, \tau) \leq (F_1, \tau_1) \iff F \subseteq F_1 \text{ and } \tau_1|_F = \tau.$$

If $\{(F_i, \tau_i) : i \in I\}$ is a chain in X, then $F = \cup_{i \in I} F_i$ is a subfield of E that contains K. Moreover, if $z \in F$, then $z \in F_i$ for some $i \in I$ and then one defines $\tau(z) = \tau_i(z)$. It is an exercise to prove that τ is well-defined. Since $(F, \tau) \in X$ is an upper bound, Zorn's lemma implies that there exists a maximal element $(E_1, \theta) \in X$. We claim that $E = E_1$. If not, let $z \in E \setminus E_1$. Since we know the proposition is true for the extension $E_1(z)/E_1$, let $\rho\colon E_1(z) \to C$ be a field homomorphism such that $\rho|_{E_1} = \theta$. Then, in particular, $\rho|_K = \sigma$. This implies that $(E_1(z), \rho) \in X$ and hence $(E_1, \theta) < (E_1(z), \rho)$, a contradiction to the maximality of (E_1, θ). $\square$

4. Lecture – Week 4

The previous proposition will be used to prove that the algebraic closure always exists.

4.1. Theorem (Artin). *Let K be a field. Then K admits an algebraic closure C/K. If C_1/K is an algebraic closure, then the extensions C/K and C_1/K are isomorphic.*

Proof. Let us first prove the uniqueness. The previous proposition implies the existence of an extension homomorphism $\varphi\colon C \to C_1$. Let $y \in C_1$ and $f = f(y, K)$ be the minimal polynomial of y in K. Since f admits a factorization

$$f = \lambda \prod (X - \alpha_i)^{m_i}$$

in $C[X]$, it follows that

$$f = \overline{\varphi}(f) = \varphi(\lambda) \prod (X - \varphi(\alpha_i))^{m_i}$$

Since $0 = f(y)$, we conclude that $y = \varphi(\alpha_j)$ for some j. In particular, φ is surjective and hence φ is bijective.

We now prove the existence. Let us assume that K admits an extension E/K with E algebraically closed. We will prove later that this extension indeed exists; at the moment, we only want to get an algebraic extension from this setting. Let

$$F = \{x \in E : x \text{ is algebraic over } K\}.$$

Then F/K is algebraic. Let $g \in F[X]$ be such that $\deg g > 0$. Since E is algebraically closed, g admits a root α in E. In particular, α is algebraic over F and hence α is algebraic over K. This implies that $\alpha \in F$, thus F is algebraically closed. This proves that F/K is an algebraic closure.

Let us prove that there exists an extension E_1/K such that every polynomial $f \in K[X]$ with $\deg f > 0$ has a root in E_1. Let $\{f_i : i \in I\}$ be the family of monic irreducible polynomials with coefficients in K. We may think that $f_i = f_i(X_i)$. Let $R = K[\{X_i : i \in I\}]$ and let J be the ideal of R generated by the $f_i(X_i)$. We claim that $J \neq R$. If not, $1 \in J$, so

$$1 = \sum_{j=1}^{m} g_j f_{i_j}(X_{i_j})$$

for some $g_1, \dots, g_m \in R$. There exists an extension F/K such that f_{i_j} has a root α_j in F for all j. Let

$$\tau\colon R \to F, \quad \tau(X_k) = \begin{cases} \alpha_j & \text{if } k = i_j, \\ 0 & \text{if } k \notin \{i_1, \dots, i_m\}. \end{cases}$$

Then τ is a ring homomorphism and

$$1 = \tau(1) = \sum_{j=1}^{m} \tau(g_j) f_{i_j}(\alpha_j) = 0,$$

a contradiction.

Since J is a proper ideal, it is contained in a maximal ideal M. Let $L = R/M$ and let $\sigma\colon K \to L$ be the composition $K \hookrightarrow R \to R/M = L$, where $\pi\colon R \to R/M$ is the canonical map. As we did before, $\pi(X_i)$ is a root of $\overline{\sigma}(f_i)$ for all i. And there exists an extension E_1/K such that every f_i has a root in E_1. Proceeding in this way, we construct a sequence

$$E_1 \subseteq E_2 \subseteq \cdots$$

of fields such that every polynomial of positive degree and coefficients in E_k admits a root in E_{k+1}. Let $E = \cup E_k$. We claim that E is algebraically closed. In fact, let $g \in E[X]$ be such that $\deg g > 0$. Then, since $g \in E_r[X]$ for some r, it follows that g has a root in $E_{r+1} \subseteq E$. □

§ 4.1. Decomposition fields.

4.2. Definition. Let K be a field and $f \in K[X]$ be such that $\deg f > 0$. A **decomposition field** (or **splitting field**) of f over K is a field E that contains K and that satisfies the following properties:

1) f factorizes linearly in $E[X]$.

2) If F is a field such that $K \subseteq F \subseteq E$ and f factorizes linearly in $F[X]$, then $F = E$.

Easy examples:

4.3. Example. $\mathbb{C}$ is a decomposition field of $X^2 + 1 \in \mathbb{R}[X]$.

4.4. Example. $\mathbb{Q}[\sqrt{2}]$ is a decomposition field of $X^2 - 2 \in \mathbb{Q}[X]$.

4.5. Example. The decomposition field of $f = X^2 - 2$ over $\mathbb{Z}/7$ is precisely $\mathbb{Z}/7$, as 3 and 4 are the roots of f in $\mathbb{Z}/7$.

4.6. Example. $\mathbb{Q}(\sqrt[3]{2})$ is not a decomposition field of $X^3 - 2 \in \mathbb{Q}[X]$. However, if ω is a primitive cubic root of one, then $\mathbb{Q}(\sqrt[3]{2}, \omega)$ is a decomposition field of the polynomial $X^3 - 2 \in \mathbb{Q}[X]$.

4.7. Proposition. *E is a decomposition field of $f \in K[X]$ if and only if f factorizes linearly in $E[X]$ and $E = K(x_1, \dots, x_n)$, where $x_1, \dots, x_n$ are the roots of f.*

Proof. Let $f = a\prod_{i=1}^{r}(X - x_i)^{n_i}$ and $F = K(x_1, \dots, x_r)$ with $x_1, \dots, x_r \in E$. Since f factorizes linearly in $F[X]$, it follows that $F = E$. Conversely, let $E = K(x_1, \dots, x_r)$ and assume that f factorizes linearly in $F[X]$. Then, in particular, $x_1, \dots, x_r \in F$. Hence $E \subseteq F$ and $F = E$. □

One immediately obtains the following consequence: If E is a decomposition field of $f \in K[X]$, then E/K is finite.

4.8. Theorem. *Let $f \in K[X]$ be such that $\deg f > 0$. There exists a (unique up to extension isomorphism) decomposition field of f over K.*

Proof. Let C/K be an algebraic closure of K. Write

$$f = a\prod_{i=1}^{r}(X - x_i)^{n_i}$$

in $C[X]$. Then $E = K(x_1, \dots, x_r)$ is a decomposition field of f over K.

Let us prove the uniqueness: if E_1/K is a decomposition field of f over K, then E_1/K is algebraic and thus Proposition 3.6 implies that there exists $\varphi \in \operatorname{Hom}(E_1/K, C/K)$, that is $\varphi\colon E_1 \to C$ is a field homomorphism such that $\varphi|_K$ is the identity. Factorize f linearly in $E_1[X]$ and apply $\overline{\varphi}$:

$$f = a\prod_{j=1}^{s}(X - y_j)^{m_j} \implies f = \overline{\varphi}(f) = \varphi(a)\prod_{j=1}^{s}(X - \varphi(y_j))^{m_j}$$

so f factorizes linearly in $\varphi(E_1)[X]$. Moreover, $E_1 = K(y_1, \dots, y_s)$ and

$$\varphi(E_1) = K(\varphi(y_1), \dots, \varphi(y_s)).$$

Thus $\varphi(E_1)$ is a decomposition field of f. Since $\varphi(E_1) \subseteq C$, it follows that $\varphi(E_1) = E$. $\square$

4.9. Exercise. If C is an algebraic closure of K and $\varphi \in \operatorname{Hom}(C/K, C/K)$, then φ is an isomorphism.

Let C be an algebraic closure of K and $G = \operatorname{Gal}(C/K)$. The group G acts on C

$$\sigma \cdot x = \sigma(x), \quad \sigma \in G,\, x \in C.$$

The orbits are of the form

$$O_G(x) = \{\sigma(x) : \sigma \in G\} = \{y \in C : y = \sigma(x) \text{ for some } \sigma \in G\}$$

The elements $x, y \in C$ are **conjugate** if $y = \sigma(x)$ for some $\sigma \in G$.

4.10. Proposition. *Let C be an algebraic closure of K and $x, y \in C$. Then x and y are conjugate if and only if $f(x, K) = f(y, K)$. In particular, $O_G(x)$ is finite.*

Proof. Let $G = \operatorname{Gal}(C/K)$. If x and y are conjugate, say $y = \sigma(x)$ for some $\sigma \in G$, let us write $g = f(x, K)$ as

$$g = X^n + \sum_{i=0}^{n-1} a_i X^i$$

for some $n \geq 1$ and $a_0, \dots, a_{n-1} \in K$. Then $0 = g(x) = x^n + \sum_{i=0}^{n-1} a_i x^i$ and hence y is a root of g, as

$$\begin{aligned} 0 = \sigma\left(x^n + \sum_{i=0}^{n-1} a_i x^i\right) &= \sigma(x)^n + \sum_{i=0}^{n-1} \sigma(a_i)\sigma(x)^i \\ &= \sigma(x)^n + \sum_{i=0}^{n-1} a_i \sigma(x)^i = y^n + \sum_{i=0}^{n-1} a_i y^i. \end{aligned}$$

Thus $f(y, K) = g$.

Conversely, assume that $f(x, K) = f(y, K)$. Let $g = f(x, K) = f(y, K)$ and let

$$\varphi\colon K[x] \to K[y], \quad h(x) \mapsto h(y).$$

Let us show that the map φ is well-defined: we need to show that if $h_1(x) = h_2(x)$, then

$$h_1(y) = \varphi(h_1(x)) = \varphi(h_2(x)) = h_2(y).$$

If $h_1(x) = h_2(x)$, then

$$(h_1 - h_2)(x) = h_1(x) - h_2(x) = 0.$$

This implies that g divides $h_1 - h_2$. In particular, $h_1(y) = h_2(y)$.

A straightforward calculation shows that φ is a field homomorphism such that $\varphi|_K = \mathrm{id}$, this means that φ is an extension homomorphism such that $\varphi(x) = y$. There exists $\sigma \in \operatorname{Hom}(C/K, C/K)$ such that $\sigma|_{K[x]} = \varphi$. Since σ is bijective (this is left as an exercise, you did something similar before), $\sigma(x) = \varphi(x) = y$ and hence $O_G(x) = O_G(y)$. $\square$

4.11. Proposition. *Let C be an algebraic closure of K and $x \in C$. Then*

$$f(x,K) = \prod_{y \in O_G(x)} (X-y)^m$$

for some m.

Proof. For each $y \in O_G(x)$ let m_y be the multiplicity of y in $f(x,K)$. Then, for example, $f(x,K) = (X-x)^{m_x} g$ for some g. If $y \in O_G(x)$, then $y = \sigma(x)$ for some $\sigma \in \mathrm{Gal}(C/K)$. Since

$$\overline{\sigma}(f(x,K)) = f(x,K) = (X-y)^{m_x} \overline{\sigma}(g),$$

it follows that $m_y \geq m_x$. By symmetry, we conclude that $m_x = m_y$. $\square$

The previous proposition shows, in particular, that all the roots of an irreducible polynomial $f \in K[X]$ in an algebraic closure C of K have the same multiplicity. This is not true if f is not irreducible. Find an example.

4.12. Definition. Let K be a field and $\{f_i : i \in I\}$ be a non-empty family of polynomials of positive degree with coefficients in K. A **decomposition field** of $\{f_i : i \in I\}$ is an extension E/K such that every f_i factorizes linearly in $E[X]$ and if F/K is a sub extension of E/K such that every f_i factorizes linearly in $F[X]$, then $F = E$.

4.13. Exercise. Prove that E/K is a decomposition field of $\{f_i : i \in I\}$ if and only if every f_i factorizes linearly in $E[X]$ and $E = K(S)$ where $S = \{\text{roots of } f_i \text{ for all } i\}$.

4.14. Exercise. Prove that if E/K is a decomposition field of $\{f_i : i \in I\}$, then E/K is algebraic. If, moreover, I is finite, then E/K is a decomposition field of $\prod_{i \in I} f_i$.

4.15. Exercise. Prove that there exists a decomposition field of $\{f_i : i \in I\}$ and it is unique up to extension isomorphism.

4.16. Exercise. Let $f = X^3 - X - 1 \in (\mathbb{Z}/3)[X]$ and E be a decomposition field of f. Compute $[E : \mathbb{Z}/3]$.

What about the decomposition field of $f = X^3 - X - 1 \in \mathbb{Q}[X]$?

4.17. Exercise. Let $f = X^4 - 5x^2 + 5 \in \mathbb{Q}[X]$ and E be a decomposition field of f. Compute $[E : \mathbb{Q}]$ and $\mathrm{Gal}(E/\mathbb{Q})$.

5. Lecture – Week 5

§ 5.1. Normal extensions.

5.1. Proposition. *Let E/K be an algebraic extension and $\sigma \in \mathrm{Hom}(E/K, E/K)$. Then σ is bijective.*

Proof. It is enough to prove that σ is surjective. Why? Let $x \in E$ and C be an algebraic closure of K that contains E. By Proposition 3.6, there exists a field homomorphism $\varphi\colon C \to C$ such that $\varphi|_E = \sigma$. Thus $\varphi|_K = \sigma|_K = \mathrm{id}_K$. Let $G = \mathrm{Gal}(C/K)$. Then $\varphi \in G$. If $z \in O_G(x)$, then $z = \tau(x)$ for some $\tau \in G$ and hence

$$\varphi(z) = \varphi(\tau(x)) = (\varphi\tau)(x).$$

This implies that $\varphi(z) \in O_G(x)$ and $\varphi(O_G(x)) = O_G(x)$. The restriction $\sigma|_{E \cap O_G(x)}$ is injective. Then

$$\begin{aligned}\sigma(E \cap O_G(x)) &= \varphi(E \cap O_G(x)) \\ &= \varphi(E) \cap \varphi(O_G(x)) = \sigma(E) \cap O_G(x) \subseteq E \cap O_G(x).\end{aligned}$$

Since $|E \cap O_G(x)| < \infty$, it follows that $E \cap O_G(x) = \sigma(E \cap O_G(x))$ and hence x belongs to the image of σ. □

5.2. Definition. Let E/K be an algebraic extension and C be an algebraic closure of K containing E. Then E/K is **normal** if $\sigma(E) \subseteq E$ for all $\sigma \in \mathrm{Hom}(E/K, C/K)$.

Note that $\sigma(E) \subseteq E$ in the previous definition is equivalent to $\sigma(E) = E$.

5.3. Example. The extension $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is not normal. Why?

Some trivial examples of normal extensions: K/K is normal and if C is an algebraic closure of K, then C/K is normal.

5.4. Example. The extension $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is normal. Every extension generated by algebraic elements of degree two is normal.

5.5. Exercise. Let ξ be a primitive cubic root of one. Then $\mathbb{Q}(\sqrt[3]{2}, \xi)/\mathbb{Q}$ is normal.

The following result is practical but technical. That is why we leave the proof as an exercise.

5.6. Exercise. Prove that the previous definition depends only on E (and not on the algebraic closure C).

Some properties:

5.7. Proposition. *Let E/K be a normal extension and $f \in K[X]$ be an irreducible polynomial that admits a root x in E. Then f factorizes linearly in E.*

Proof. We may assume that f is monic. Let C/K be an algebraic closure of K containing E. Let y be a root of f in C. Since $f = f(x, K) = f(y, K)$, it follows that $y = \sigma(x)$ for some $\sigma \in \mathrm{Gal}(C/K)$. Since E/K is normal, $\sigma|_E\colon E \to C$ is an automorphism of E/K, that is $\sigma(E) \subseteq E$. In particular, $y \in E$. □

Let $K \subseteq F \subseteq E$ be a tower of fields. If E/K is normal, then E/F is normal. However, Note that E/K normal does not imply F/K normal, as this would imply that every extension is normal. Moreover, E/F normal and F/K normal do not imply E/K normal.

5.8. Example. The extensions $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ are both normal, but the extension $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$ is not, as the roots of X^4-2 are $\sqrt[4]{2}$, $-\sqrt[4]{2}$, $\sqrt[4]{2}i$ and $-\sqrt[4]{2}i$.

Recall that if C is an algebraic closure of K and $x \in C$, then

$$f(x,K)=\prod(X-y)^m,$$

where the product is taken over all $y \in O_{\mathrm{Gal}(C/K)}(x)$. If E/K is normal and $x \in E$, then there exists m such that

$$f(x,K)=\prod(X-y)^m,$$

where the product is taken over all $y \in O_{\mathrm{Gal}(E/K)}(x)$.

5.9. Proposition. *Let E/K and F/K be extensions. If F/K is normal, then EF/E is normal.*

Proof. Let C be an algebraic closure of E containing EF (this exists because EF/E is algebraic). Let $\sigma \in \mathrm{Hom}(EF/E, C/E)$. We claim that $\sigma(EF)=EF$. Let

$$\overline{K}=\{x \in C : x \text{ is algebraic over } K\}.$$

Then $\overline{K}$ is an algebraic closure over K and $F \subseteq \overline{K}$. Since F/K is normal and $\sigma|_F \in \mathrm{Hom}(F/K, \overline{K}/K)$, it follows that $\sigma(F)=F$. If $z \in EF$, then $z=\sum_{i=1}^m e_i f_i$ for some $e_1,\dots,e_m \in E$ and $f_1,\dots,f_m \in F$. Since $\sigma(e_i)=e_i$ for all i,

$$\sigma(z)=\sum_{i=1}^m \sigma(e_i)\sigma(f_i)=\sum_{i=1}^m e_i\sigma(f_i) \in EF. \qquad \square$$

What is the relation between normal extensions and decomposition fields? The notions look deeply related. The following proposition serves as an explanation:

5.10. Proposition. *Let E/K be an algebraic extension. Then E/K is normal if and only if E/K is the decomposition field of a family of polynomials of $K[X]$ of positive degree.*

Proof. Assume first that E/K is a normal extension. Let $G=\mathrm{Gal}(E/K)$. If $x \in E$ and $f(x,K)=\prod_{y\in O_G(x)}(X-y)^m$, then $f(x,K)$ factorizes linearly in $E[X]$. Thus E/K is a decomposition field of the family $\{f(x,K) : x \in E\}$.

Conversely, assume that E/K is a decomposition field of the family $\{f_i : i \in I\}$. Then $E=K(S)$ where S is the set of roots of the polynomials f_i. Let C/K be an algebraic closure of K that contains E and let $\sigma \in \mathrm{Hom}(E/K, C/K)$. Let $x \in S$. Then x is a root of some $f_j=\sum a_k X^k$. Since $f_j(x)=0$, it follows that $\sigma(x)$ is a root of f_j, as

$$f_j(\sigma(x))=\sum a_k\sigma(x)^k=\sum \sigma(a_k)\sigma(x^k)=\sigma\left(\sum a_k x^k\right)=\sigma(0)=0.$$

Hence $\sigma(E)\subseteq E$. $\square$

5.11. Exercise. Let $E=\mathbb{Q}[\sqrt[4]{7}+\sqrt{2}]$.

1) Prove that $E/\mathbb{Q}$ is not normal.
2) Compute $[E:\mathbb{Q}]$.
3) Compute $\mathrm{Gal}(E/\mathbb{Q})$.

§ 5.2. Dedekind's theorem. Note that every extension homomorphism $E/K \to F/K$ is, in particular, a K-linear map $E \to F$, that is

$$\mathrm{Hom}(E/K, F/K) \subseteq \mathrm{Hom}_K(E, F).$$

If F/K is an extension and V is a K-vector space, the set $\mathrm{Hom}_K(V, F)$ of K-linear maps is a vector space over F with $(a \cdot f)(v) = af(v)$ for $a \in F$, $f \in \mathrm{Hom}_K(V, F)$ and $v \in V$.

5.12. Exercise. Let V be a K-vector space. Prove that $\dim_F \mathrm{Hom}_K(V, F) \geq \dim_K V$. Moreover, if $\dim_K V < \infty$, then $\dim_F \mathrm{Hom}_K(V, F) = \dim_K V$.

If V is a vector space and S is a (possibly infinite) subset of V, then S is linearly independent if every finite subset of S is linearly independent.

5.13. Theorem (Dedekind). *Let E/K and F/K be extensions and let $\{\varphi_i : i \in I\}$ be a subset of* $\mathrm{Hom}(E/K, F/K)$*, i.e. a family of extension homomorphisms. Assume that $\varphi_i \neq \varphi_j$ if $i \neq j$. Then the subset $\{\varphi_i : i \in I\} \subseteq \mathrm{Hom}_K(E, F)$ is linearly independent over F.*

Proof. Assume it is not. Let $\{\varphi_1, \dots, \varphi_n\}$ be linearly dependent over F with n minimal. Clearly, $n > 1$. Without loss of generality, we may assume that

$$\sum_{i=1}^{n} a_i \varphi_i = 0 \tag{5.1}$$

for some $a_1, \dots, a_n \in F$ all different from zero. Let $z \in E \setminus \{0\}$ be such that $\varphi_1(z) \neq \varphi_2(z)$. If $x \in E$, then

$$0 = \left(\sum_{i=1}^{n} a_i \varphi_i\right)(xz) = \sum_{i=1}^{n} a_i \varphi_i(xz) = \sum_{i=1}^{n} a_i \varphi_i(x)\varphi_i(z) = \left(\sum_{i=1}^{n} (a_i \varphi_i(z))\varphi_i\right)(x).$$

Thus

$$\sum_{i=1}^{n} (a_i \varphi_i(z))\varphi_i = 0.$$

Since $\varphi_1(z) \neq 0$,

$$a_1\varphi_1 + a_2\frac{\varphi_2(z)}{\varphi_1(z)}\varphi_2 + \cdots + a_n\frac{\varphi_n(z)}{\varphi_1(z)}\varphi_n = 0. \tag{5.2}$$

Thus, subtracting (5.1) and (5.2),

$$\left(a_2 - a_2\frac{\varphi_2(z)}{\varphi_1(z)}\right)\varphi_2 + \cdots + \left(a_n - a_n\frac{\varphi_n(z)}{\varphi_1(z)}\right)\varphi_n = 0.$$

Since $a_n \neq 0$ and $\varphi_2(z) \neq \varphi_1(z)$, the scalar $a_2 - a_2\frac{\varphi_2(z)}{\varphi_1(z)} \neq 0$ and hence $\{\varphi_2, \dots, \varphi_n\}$ is linearly dependent, a contradiction. $\square$

If E/K and F/K are extensions, let $\gamma(E/K, F/K) = |\mathrm{Hom}(E/K, F/K)|$.

5.14. Exercise. Prove the following statements:

1) $\gamma(E/K, F/K) \leq \dim_F \mathrm{Hom}_K(E, F)$.
2) If $[E : K] < \infty$, then $\gamma(E/K, F/K) \leq [E : K]$.
3) If x is algebraic over K, then $\gamma(K(x)/K, F/K) \leq \deg f(x, K)$.

If C is an algebraic closure of K, then we define $\gamma(E/K) = \gamma(E/K, C/K)$. This definition does not depend on the algebraic closure.

5.15. Exercise. If C and C_1 are algebraic closures of K, then

$$|\operatorname{Hom}(E/K, C/K)| = |\operatorname{Hom}(E/K, C_1/K)|.$$

5.16. Proposition. *Let C be an algebraic closure of K and $G = \operatorname{Gal}(C/K)$. If $x \in C$, then $\gamma(K(x)/K) = |O_G(x)|$.*

Proof. If $\sigma \in \operatorname{Hom}(K(x)/K, C/K)$, then there exists $\phi \in G$ such that $\phi|_{K(x)} = \sigma$. Thus

$$\sigma(x) = \phi(x) \in O_G(x).$$

Conversely, if $y \in O_G(x)$, then there exists $\tau \in G$ such that $y = \tau(x)$. Hence

$$\tau|_{K(x)} \in \operatorname{Hom}(K(x)/K, C/K)$$

and $\tau|_{K(x)}(x) = y$. Since our sets are then in bijective correspondence, the claim follows. □

5.17. Exercise. If E/K is finite, then $|\operatorname{Gal}(E/K)| \leq [E : K]$. Moreover, E/K is normal if and only if $|\operatorname{Gal}(E/K)| = \gamma(E/K)$.

6. Lecture – Week 6

If $t\colon A \to B$ is a surjective map, then $a \sim a_1 \iff t(a) = t(a_1)$ defines an equivalence relation on A. The set $\overline{A}$ of equivalence classes is in bijective correspondence with B, $\overline{A} \to B$, $\overline{a} \mapsto t(a)$. Moreover, if $|t^{-1}(\{b\})| = m$ for all $b \in B$, then $|A| = m|\overline{A}| = m|B|$.

6.1. Proposition. *Let E/K be algebraic and F/K be a subextension such that E/F is finite. Then $\gamma(E/K) = \gamma(E/F)\gamma(F/K)$.*

Proof. Assume first that $E = F(x)$. Let C be an algebraic closure of K containing E and $G = \mathrm{Gal}(C/F)$. Let $f = f(x, F) = \sum b_i X^i$.

The map

$$\lambda\colon \mathrm{Hom}(E/K, C/K) \to \mathrm{Hom}(F/K, C/K), \quad \sigma \mapsto \sigma|_F,$$

is well-defined. It is surjective: if $\varphi \in \mathrm{Hom}(F/K, C/K)$, then $\varphi\colon F \to C$ is, in particular, a field homomorphism. Since E/F is algebraic, by Proposition 3.6 there exists a field homomorphism $\sigma\colon E \to C$ such that $\sigma|_F = \varphi$. Since $\sigma|_K = \varphi|_K = \mathrm{id}$, in particular $\sigma \in \mathrm{Hom}(E/K, C/K)$.

For $\varphi \in \mathrm{Hom}(F/K, C/K)$,

$$\lambda^{-1}(\{\varphi\}) = \{\sigma \in \mathrm{Hom}(E/K, C/K) : \sigma|_F = \varphi\}$$

and let R_φ be the set of roots (in C) of the polynomial $\overline{\varphi}(f) = \sum \varphi(b_i) X^i$.

Claim. The map $\alpha\colon \lambda^{-1}(\{\varphi\}) \to R_\varphi$, $\sigma \mapsto \sigma(x)$, is well-defined.

We need to show that $\sigma(x)$ is a root of $\overline{\varphi}(f)$:

$$\begin{aligned}\overline{\varphi}(f)(\sigma(x)) &= \sum \varphi(b_i)\sigma(x)^i = \sum \sigma(b_i)\sigma(x^i) \\ &= \sum \sigma(b_i x^i) = \sigma\left(\sum b_i x^i\right) = \sigma(f(x)) = \sigma(0) = 0.\end{aligned}$$

Claim. The map $\beta\colon R_\varphi \to \lambda^{-1}(\{\varphi\})$, $y \mapsto \sigma_y$, where $\sigma_y(z) = \overline{\varphi}(h)(y)$ if $z = h(x)$, is well-defined.

We need to show that if $z = h(x)$ and $z = h_1(x)$ for some $h, h_1 \in F[X]$, then $\overline{\varphi}(h)(y) = \overline{\varphi}(h_1)(y)$. The assumptions imply that $(h - h_1)(x) = 0$ and hence f divides $h - h_1$. Since $\overline{\varphi}$ is a ring homomorphism, $\overline{\varphi}(f)$ divides $\overline{\varphi}(h) - \overline{\varphi}(h_1)$. This implies $(\overline{\varphi}(h) - \overline{\varphi}(h_1))(y) = 0$. We also need to show that $\sigma_y|_F = \varphi$: if $a \in F$, then write $a = aX^0 \in F[X]$. Thus $\sigma_y(a) = \overline{\varphi}(aX^0)(y) = \varphi(a) \in C$. It is now an exercise to prove that $\sigma_y \in \mathrm{Hom}(E/K, C/K)$.

Claim. $|\lambda^{-1}(\{\varphi\})| = |R_\varphi|$.

For this we need to show that β is the inverse of α, that is $\alpha \circ \beta = \mathrm{id}$ and $\beta \circ \alpha = \mathrm{id}$. To prove that $\beta \circ \alpha = \mathrm{id}$ let σ be such that $\sigma|_F = \varphi$. Then $y = \sigma(x) \in R_\varphi$. Let $z = h(x) = \sum a_i x^i \in F[x] = E$. Then

$$\overline{\varphi}(h)(y) = \sum \varphi(a_i) y^i = \sum \sigma(a_i) y^i = \sigma\left(\sum a_i x^i\right) = \sigma(z).$$

Conversely, if $y \in R_\varphi$, then

$$\alpha(\sigma_y) = \sigma_y(x) = y,$$

as $\sigma_y(x) = \overline{\varphi}(X)(y) = y$.

Claim. If $\phi \in \mathrm{Gal}(C/K)$ is such that $\phi|_F = \varphi$, then $|\phi^{-1}(R_\varphi)| = |R_\varphi|$ and

$$O_G(x) = \phi^{-1}(R_\varphi).$$

Let us first prove $O_G(x) \supseteq \phi^{-1}(R_\varphi)$. If $y \in R_\varphi$, then

$$\begin{aligned} f(\phi^{-1}(y)) &= \sum b_i \phi^{-1}(y^i) = \phi^{-1}\left(\sum \phi(b_i) y^i\right) \\ &= \phi^{-1}(\overline{\varphi}(f)(y)) = \phi^{-1}(0) = 0. \end{aligned}$$

Then $f(x, F) = f(\phi^{-1}(y), F)$. By Proposition 4.10, $\phi^{-1}(y) \in O_G(x)$.

Now we prove $O_G(x) \subseteq \phi^{-1}(R_\varphi)$. Let $z \in O_G(x)$. Then $\overline{\varphi}(f)(\phi(z)) = 0$, as

$$\begin{aligned} \overline{\varphi}(f)(\phi(z)) &= \sum \varphi(b_i)\phi(z^i) \\ &= \sum \phi(b_i)\phi(z^i) = \phi\left(\sum b_i z^i\right) = \phi(f(z)) = \phi(0) = 0. \end{aligned}$$

Thus $\phi(z) \in R_\varphi$ and hence $z \in \phi^{-1}(R_\varphi)$. It follows that $|\lambda^{-1}(\{\varphi\})| = |O_G(x)|$ for all φ. By using the argument before the proposition,

$$\begin{aligned} \gamma(E/K) &= |\operatorname{Hom}(E/K, C/K)| \\ &= |O_G(x)||\operatorname{Hom}(F/K, C/K)| \\ &= |O_G(x)|\gamma(F/K). \end{aligned}$$

Since $\gamma(E/F) = \gamma(F(x)/F) = |O_G(x)|$ by Proposition 5.16, the claim follows.

For the general case, we assume that $E = F(x_1, \dots, x_n)$. We proceed by induction on n. If $n = 0$, then $E = F$ and the result is trivial. If $n > 0$, let $L = F[x_1, \dots, x_{n-1}]$ and $E = L(x_n)$. The case proved implies that $\gamma(E/F) = \gamma(E/L)\gamma(L/F)$. By the inductive hypothesis, $\gamma(L/K) = \gamma(L/F)\gamma(F/K)$. Thus

$$\gamma(E/F)\gamma(F/K) = \gamma(E/L)\gamma(L/F)\gamma(F/K) = \gamma(E/L)\gamma(L/K) = \gamma(E/K),$$

again using the previous case. □

§ 6.1. Separable extensions.

6.2. Definition. Let E/K be an extension and $x \in E$ an algebraic element over K. Then x is **separable** over K if x is a simple root of $f(x, K)$.

An algebraic extension E/K is **separable** if every $x \in E$ is separable over K. Then K/K is separable.

6.3. Exercise. Prove that an element x is separable over K if and only if x is a simple root of a polynomial with coefficients in K.

If F/K is a subextension of E/K and $x \in E$ is separable over K, then x is separable over F.

6.4. Exercise. If C is an algebraic closure of K, $x \in C$ and $G = \operatorname{Gal}(C/K)$. Prove that the following statements are equivalent:

1) x is separable over K.
2) Every $y \in O_G(x)$ is separable over K.
3) $\gamma(K(x)/K) = [K(x) : K] = \deg f(x, K)$.

Let K be any field and $g \in K[X]$. Let z be a root of g. Then z is a multiple root of g if and only if z is a root of g'.

6.5. Exercise. Prove that if K has characteristic zero or K is finite, then every algebraic extension of K is separable.

6.6. Example. Let $E = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Then $[E : \mathbb{Q}] = 4$ and $\mathrm{Gal}(E/Q) \simeq C_2 \times C_2$. The extension E/Q is normal, as it is the decomposition field of $(X^2 - 2)(X^2 - 3)$ and it is separable as $\mathbb{Q}$ has characteristic zero.

6.7. Example. Let E be a decomposition field of $X^4 - 2$ over $\mathbb{Q}$. Then $E/\mathbb{Q}$ is normal and separable. Note that $E = \mathbb{Q}(\sqrt[4]{2}, i)$, so

$$[E : \mathbb{Q}] = 8 = |\,\mathrm{Gal}(E/\mathbb{Q})|.$$

Let us compute $\mathrm{Gal}(E/\mathbb{Q})$. If $\sigma \in \mathrm{Gal}(E/\mathbb{Q})$, then $\sigma(\sqrt[4]{2}) \in \{\sqrt[4]{2}, -\sqrt[4]{2}, \sqrt[4]{2}i, -\sqrt[4]{2}i\}$ and $\sigma(i) \in \{-i, i\}$. Two examples are

$$\alpha\colon \begin{cases} \sqrt[4]{2} \mapsto \sqrt[4]{2}i, \\ i \mapsto i, \end{cases} \qquad \beta\colon \begin{cases} \sqrt[4]{2} \mapsto \sqrt[4]{2}, \\ i \mapsto -i. \end{cases}$$

It follows that $\mathrm{Gal}(E/\mathbb{Q})$ is isomorphic to the group $\langle \alpha, \beta \rangle$, which turns out to be isomorphic to the dihedral group of eight elements.

Another consequence: If $E = K(S)$, then E/K is separable if and only if every $x \in S$ is separable over K. One first does the case $E = K(x)$ and then proceeds by induction.

6.8. Exercise. Let $K \subseteq F \subseteq E$ be a tower of fields. Prove that E/K is separable if and only if F/K and E/F are separable.

6.9. Exercise. Let E/K and F/K be extensions. Prove that if F/K is separable, then EF/E is separable.

If E/K is algebraic, then

$$F = \{x \in E : x \text{ is separable over } K\}$$

is a subfield of E that contains K. It is known as the **separable closure** of K with respect to E. Note that $F = K(F)$, as $K(F)$ is separable because it is generated by separable elements. Moreover, F/K is separable and E/F is a **purely inseparable** extension, meaning that for every $x \in E \setminus F$, the polynomial $f(x, F)$ is not separable.

6.10. Proposition. *If E/K is separable and finite, then $E = K(x)$ for some $x \in E$.*

Proof. Let us assume that K is finite. Then E is finite and hence the multiplicative group $E^\times = E \setminus \{0\}$ is cyclic, say $E^\times = \langle x \rangle$. It follows that $E = K(x)$.

Let us now assume that K is infinite. We first consider the case $E = K(x, y)$. The general case $E = K(x_1, \dots, x_n)$ is left as an exercise, one needs to proceed by induction. Let $n = [E : K]$ and C be an algebraic closure of K containing E. Write $\mathrm{Hom}(E/K, C/K) = \{\sigma_1, \dots, \sigma_n\}$. Let

$$f = \prod_{1 \leq i < j \leq n} ((\sigma_i(y) - \sigma_j(y)) + X(\sigma_i(x) - \sigma_j(x))) \in C[X].$$

Then $f \neq 0$, as f is a product of non-zero polynomials. Since K is infinite, there exists a non-zero $c \in K$ such that $f(c) \neq 0$. For any $r, s \in \{1, \dots, n\}$ with $r \neq s$,

$$\sigma_r(y) - \sigma_s(y) + c(\sigma_r(x) - \sigma_s(x)) \neq 0,$$

as $f(c) \neq 0$. It follows that $\sigma_r(y + cx) \neq \sigma_s(y + cx)$. Thus $\gamma(K(y + cx)/K) \geq n$. Now

$$n \geq [K(y + cx) : K] = \gamma(K(y + cx)/K) \geq n,$$

so $[K(y + cx) : K] = n$ and hence $K(y + cx) = E$. $\square$

For example, $\mathbb{Q}(\sqrt{2}, i) = \mathbb{Q}(\sqrt{2} + i)$.

7. Lecture – Week 7

7.1. Theorem (Steinitz). *Let E/K be a finite extension. Then $E = K(x)$ for some $x \in E$ if and only if E/K admits finitely many subextensions.*

Proof. We may assume that K is infinite; otherwise, the result is trivial. We first prove $\implies$. Let us assume that $E = K(x)$ for some x. We claim that the map

$$\Psi\colon \{F : K \subseteq F \subseteq E\} \to \{g \in K[X] : g \text{ is a monic divisor of } f(x,K)\},$$
$$F \mapsto f(x,F),$$

is injective. Take F_0 such that $K \subseteq F_0 \subseteq F \subseteq E$ and $f(x,F) = f(x,F_0)$. Then

$$[E : F_0] = [F_0(x) : F_0] = \deg f(x,F_0) = m = [F(x) : F] = [E : F]$$

and hence $F = F_0$.

In general, let F_1 and F_2 be such that $K \subseteq F_1, F_2 \subseteq E$ and $f(x,F_1) = f(x,F_2)$. Let $F_0 = F_1 \cap F_2$. Then $f = f(x,F_1) = f(x,F_2) \in F_0[X]$ and hence $f(x,F_0) = f$. Hence we can apply what we proved before to $F_0 \subseteq F_1$ and $F_0 \subseteq F_2$, to obtain that $F_1 = F_0 = F_2$. It follows that Ψ is injective and hence there are finitely many fields between K and E.

Let us prove $\impliedby$. Let us assume that $E = K(x,y)$. For each $a \in K$, we consider the extension $K(ay+x)/K$. By assumption, there exist $a, b \in K$ such that $a \neq b$ and

$$K(x+ay) = K(x+by) = L.$$

We claim that $L = E$. Note that $x + ay \in L$ and $x + by \in L$, so $(a-b)y \in L$ and hence, since $K \subseteq L$, it follows that $y \in L$. Thus $x \in L$ and therefore $L = E$. □

As a consequence, if E/K is finite and separable, then E/K admits finitely many subextensions.

§ 7.1. Galois extensions.

Let E/K be an algebraic extension. Assume that $E = K(S)$ and let C be an algebraic closure of K containing E. Let

$$T = \{y \in C : y \text{ is a root of } f(x,K) \text{ for } x \in S\}$$

and let $L = K(T)$. Then $E \subseteq L$, as $S \subseteq T$. The extension L/K is normal, as L/K is a decomposition field of the family $\{f(x,K) : x \in S\}$. Moreover, L is the smallest normal extension of K containing E. The field L is the **normal closure** of E (with respect to C).

7.2. Exercise. If E/K is finite, then L/K is finite

7.3. Exercise. If E/K is separable, then L/K is separable.

Let E/K be an extension and $S \subseteq \mathrm{Gal}(E/K)$ be a subset. the set

$${}^{S}E = \{x \in E : \sigma(x) = x \text{ for all } \sigma \in S\}$$

is a subfield of E that contains K. The subfield ${}^{S}E$ is known as the **fixed field** of S.

7.4. Definition. Let E/K be an algebraic extension and $G = \mathrm{Gal}(E/K)$. Then E/K is a **Galois extension** if ${}^{G}E = K$.

Clearly, K/K is a Galois extension. Note that $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is not a Galois extension. Why?

7.5. Exercise. Prove that $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$ is a Galois extension.

7.6. Exercise. If the characteristic of K is different from two, then every quadratic extension of K is a Galois extension.

7.7. Exercise. Let E/K be an algebraic extension and $G = \mathrm{Gal}(E/K)$. Let $F = {}^{G}E$. Prove that $\mathrm{Gal}(E/F) = G$ and hence E/F is a Galois extension.

7.8. Proposition. *Let E/K be an algebraic extension. Then E/K is a Galois extension if and only if E/K is normal and separable.*

Proof. Let $G = \mathrm{Gal}(E/K)$. Let us first assume that E/K is Galois. For $x \in E$ let

$$f_x = \prod_{y \in O_G(x)} (X - y) = \sum a_i X^i \in E[X].$$

If $\varphi \in G$, then

$$\overline{\varphi}(f_x) = \prod_{y \in O_G(x)} (X - \varphi(y)) = f_x,$$

as if $O_G(x) = \{\sigma_1(x), \dots, \sigma_r(x)\}$, then $\varphi(\sigma_i(x)) = (\varphi\sigma_i)(x) = \sigma_j(x)$ for some j. Since

$$\sum a_i X^i = f_x = \overline{\varphi}(f_x) = \sum \varphi(a_i) X^i,$$

it follows that $a_i \in {}^{G}E = K$ for all i. Thus $f_x \in K[X]$ and E/K is a decomposition field of the family $\{f_x : x \in E\}$. In particular, E/K is normal. Moreover, x is a simple root of $f_x \in K[X]$ and hence x is separable over K.

Conversely, let $x \in {}^{G}E$. Since E/K is normal, then $f(x, K) = \prod_{y \in O_G(x)} (X - y)^m$ for some m. Since E/K is separable, $m = 1$. Moreover $x \in {}^{G}E$, so $O_G(x) = \{x\}$. Thus $f(x, K) = \prod_{y \in O_G(x)} (X - y) = X - x$ and $x \in K$. $\square$

7.9. Definition. Let K be a field and $f \in K[X]$. Then f is **separable** if all roots of f are simple (in some algebraic closure of K).

7.10. Proposition. *Let E/K be a finite extension. Then E/K is a Galois extension if and only if E is a decomposition field over K of a separable polynomial $f \in K[X]$.*

Proof. Let us assume first that E/K is a Galois extension. Since E/K is finite and separable, $E = K(x)$ by Proposition 6.10. Then E/K is a decomposition field of $f(x, K)$ since E/K is normal. Since E/K is separable, x is separable over K. Thus x is a simple root of $f(x, K)$ and hence $f(x, K)$ is separable. Conversely, let $x_1, \dots, x_r$ be the roots of a separable polynomial $f \in K[X]$. Then $E = K(x_1, \dots, x_r)$ is separable and normal. $\square$

In the previous case, $\mathrm{Gal}(E/K)$ is known as the **Galois group** of the polynomial f. The notation is $\mathrm{Gal}(f, K)$. If $n = \deg f$ and $x_1, \dots, x_n$ are the roots of f, then any $\varphi \in \mathrm{Gal}(f, K)$ permutes the roots of f, that is φ permutes the set $\{x_1, \dots, x_n\}$. In particular, $\mathrm{Gal}(f, K)$ is isomorphic to a subgroup of $\mathbb{S}_n$ and hence $|\mathrm{Gal}(f, K)|$ divides $n!$.

7.11. Proposition. *Let E/K be a normal extension and F be the separable closure of K with respect to E. Then F/K is a Galois extension.*

Proof. Let C/K be an algebraic closure such that $E \subseteq C$. Let $\sigma \in \operatorname{Hom}(F/K, C/K)$. and let $\varphi \in \operatorname{Hom}(E/K, C/K)$ be such that $\varphi|_F = \sigma$. Since E/K is normal, $\varphi(E) = E$. Let $x \in F$. Then $\sigma(x) = \varphi(x) \in E$. Thus $f(\sigma(x), K) = f(x, K)$ and $\sigma(x)$ is separable over K, which implies that $\sigma(x) \in F$. Thus F/K is normal. Since F/K is separable, it follows that F/K is a Galois extension by Proposition 7.8. □

Some easy facts.

7.12. Exercise. Let E/K be a separable extension and L/K be the normal closure of E in some algebraic closure C that contains E. Prove that L/K is a Galois extension.

7.13. Exercise. Let E/K be a finite extension. Prove that E/K is Galois if and only if $[E : K] = |\operatorname{Gal}(E/K)|$.

For the previous exercise, note that if E/K is a finite extension, then

$$|\operatorname{Gal}(E/K)| \leq \gamma(E/K) \leq [E : K].$$

The first inequality is equality if and only if E/K is normal. The second inequality is equality if and only if E/K is separable.

7.14. Exercise. Let E/K be a Galois extension and F/K be a subextension of E/K. Prove that E/F is a Galois extension.

7.15. Theorem (Artin). *Let E be a field and G be a finite group of automorphisms of E. If $K = {}^G E$, then E/K is a Galois extension, $[E : K] = |G|$ and $\operatorname{Gal}(E/K) = G$.*

Before proving the theorem, we need a lemma.

7.16. Lemma. *Let E/K be a separable extension such that $\deg f(x, K) \leq m$ for all $x \in E$. Then E/K is finite and $[E : K] \leq m$.*

Proof. Let $z \in E$ be of maximal degree. If $x \in E$, then $K(x, z)/K$ is separable. There exists y such that $K(x, z) = K(y)$. Then

$$K(z) \subseteq K(x, z) = K(y).$$

Since $\deg f(z, K) \leq \deg f(y, K)$, $\deg f(z, K) = \deg f(y, K)$. Hence $K(y) = K(z)$. In particular, $x \in K(z)$ and therefore $E = K(z)$. □

Now we are ready to prove Artin's theorem:

Proof of Theorem 7.15. Note that $G \subseteq \operatorname{Gal}(E/K)$. Let $x \in E$ and

$$f_x = \prod_{y \in O_G(x)} (X - y).$$

Since $f_x \in K[X]$, the extension E/K is normal and separable (as it is a decomposition field of a family of separable polynomials), so E/K is a Galois extension. Moreover,

$$\deg f(x, K) \leq \deg f_x = |O_G(x)| \leq |G|.$$

By the previous lemma, E/K is finite and $[E : K] \leq |G|$. This implies that $|\operatorname{Gal}(E/K)| = [E : K] \leq |G|$ and hence $|\operatorname{Gal}(E/K)| = |G|$. □

7.17. Example. Let $E = K(X,Y)$ and $\sigma\colon K[X,Y] \to E$ be the ring homomorphism given by $\sigma(X) = Y$ and $\sigma(Y) = X$. Note that σ is bijective, as $\sigma^2 = \mathrm{id}$. The map σ induces a field homomorphism $\overline{\sigma}\colon E \to E$ such that $\overline{\sigma}^2 = \mathrm{id}$. Recall that such a homomorphism is given by $f/g \mapsto \sigma(f)/\sigma(g)$. Let $G = \langle \overline{\sigma} \rangle$. Then $|G| = 2$. We claim that ${}^GE = K(X+Y, XY)$. Let $F = K(X+Y, XY)$. We only prove that ${}^GE \subseteq F$, as the other inclusion is trivial. Artin's theorem implies that $[E : {}^GE] = 2$ and $E = F(X)$, as X is a root of the polynomial $Z^2 - (X+Y)Z + XY$. Then $[E : F] \leq 2$ and $[{}^GE : F] = 1$.

8. Lecture – Week 8

§ 8.1. Galois' correspondence. A **partially order set** (or **poset**) is a pair $(X, \leq)$, where X is a non-empty set and $\leq$ is a reflexive, antisymmetric and transitive relation on X. This means:

1) $x \leq x$ for all $x \in X$,
2) $x \leq y$ and $y \leq x$ imply $x = y$, and
3) $x \leq y$ and $y \leq z$ imply $x \leq z$.

Let $(X, \leq)$ be a partially ordered set and $x, y \in X$. An element z of a poset $(X, \leq)$ is an **upper bound** of x and y if $x \leq z$ and $y \leq z$. And ξ is a **least upper bound** of x and y if it is an upper bound with $\xi \leq z$ for every upper bound z of x and y. Similarly, one defines lower bounds and greatest lower bounds.

8.1. Definition. A **lattice** if a partially ordered set L in which each pair of elements $x, y \in L$ has a least upper bound $x \vee y$ and a greatest lower bound $x \wedge y$.

The basic example is the following. Let X be a set and $\mathcal{P}(X)$ be the collection of all subsets of X. The relation $A \leq B \iff A \subseteq B$ turns $\mathcal{P}(X)$ into a lattice with $A \vee B = A \cup B$ and $A \wedge B = A \cap B$.

8.2. Example. Let G be a group and $L(G)$ be the collection of subgroups of G. The relation

$$H \leq K \iff H \subseteq K$$

turns $L(G)$ into a lattice with $H \vee K = \langle H, K \rangle$ and $H \wedge K = H \cap K$.

8.3. Example. Let E/K be a field extension and $L(E/K)$ be the collection of intermediate fields. The relation

$$F \leq L \iff F \subseteq L$$

turns $L(E/K)$ into a lattice with $F \vee L = FL$ and $F \wedge L = F \cap L$.

A map $f \colon L \to L_1$ between two lattices is said to be **order-reversing** if $x \leq y$ implies $f(y) \leq f(x)$.

We shall need an exercise.

8.4. Exercise. Let L_1 and L_2 be lattices and $f \colon L_1 \to L_2$ be a bijection such that f and its inverse are both order reversing. Then

$$f(x \vee y) = f(x) \wedge f(y), \quad f(x \wedge y) = f(x) \vee f(y)$$

for all $x, y \in L_1$.

8.5. Theorem (Galois). *Let E/K be a finite Galois extension and $G = \mathrm{Gal}(E/K)$. There exists a bijective correspondence*

$$L(E/K) = \{F : K \subseteq F \subseteq E \text{ subfields}\} \leftrightarrow \{S : S \text{ is a subgroup of } G\} = L(G).$$

The correspondence is given by $\alpha \colon F \mapsto \mathrm{Gal}(E/F)$ and $\beta \colon {}^{S}E \leftarrow S$. Moreover, the following conditions hold:

1) *α and β are order-reversing bijections.*
2) *$[F : K] = (G : \mathrm{Gal}(E/F))$ and $(G : S) = [{}^{S}E : K]$.*
3) *F/K is a Galois extension if and only if $\mathrm{Gal}(E/F)$ is a normal subgroup of G.*

$$\begin{tikzcd}
 & & E \arrow[dl, dash] \arrow[dr, dash] & \\
 & F \arrow[dl, dash] \arrow[dr, dash] & & \{1\} \arrow[dl, dash] \\
K \arrow[dr, dash] & & S \arrow[dl, dash] & \\
 & G & &
\end{tikzcd}$$

Proof. Let $\alpha\colon L(E/K)\to L(G)$, $\alpha(F)=\mathrm{Gal}(E/F)$, and $\beta\colon L(G)\to L(E/K)$, $\beta(S)={}^{S}E$. A routine exercise shows that α and β are well-defined. We first note that

$$\beta(\alpha(F))=\beta(\mathrm{Gal}(E/F))={}^{\mathrm{Gal}(E/F)}E=F$$

since E/F is a Galois extension. Moreover,

$$\alpha(\beta(S))=\alpha({}^{S}E)=\mathrm{Gal}(E/{}^{S}E)=S$$

by Artin's theorem, as S is finite.

It is straightforward to check that α and β are order-reversing bijections.

Let F be a subfield of E containing K and $S=\alpha(F)$. Then

$$[F:K]=\frac{[E:K]}{[E:F]}=\frac{|G|}{|S|}=(G:S).$$

Let C be an algebraic closure of K that contains E. If $S=\mathrm{Gal}(E/F)$, then $F={}^{S}E$.

We need to prove that F/K is normal if and only if S is normal in G. Let us first prove $\Longrightarrow$. Let $\tau\in S$ and $\sigma\in G$. Since F/K is normal, $\sigma|_F\in\mathrm{Aut}(F)$. Thus $\sigma^{-1}(F)=F$. In particular, if $x\in F$, then $\sigma^{-1}(x)\in F$ and

$$\sigma\tau\sigma^{-1}(x)=\sigma\sigma^{-1}(x)=x.$$

Conversely, let $\varphi\in\mathrm{Hom}(F/K,C/K)$. There exists $\Phi\colon E\to C$ such that $\Phi|_F=\varphi$. Since E/K is normal, $\Phi(E)=E$ and hence $\Phi\in G$. We claim that $\varphi(x)\in F$ for all $x\in F$. Note that $F={}^{S}E$, so

$$\tau\varphi(x)=\tau\Phi(x)=\Phi\Phi^{-1}\tau\Phi(x)=\Phi(x)=\varphi(x)$$

for all $\tau\in S$, as $\Phi^{-1}\tau\Phi\in S$. This means that $\varphi(x)\in{}^{S}E=F$.

Let us compute $\mathrm{Gal}(F/K)$. Since F/K is normal, the map $\lambda\colon G\to\mathrm{Gal}(F/K)$, $\sigma\mapsto\sigma|_F$, is a surjective group homomorphism such that $\ker\lambda=S$. The first isomorphism theorem implies that $\mathrm{Gal}(F/K)\simeq G/S$. □

Some easy consequences.

8.6. Exercise. If E/K is a Galois extension of degree n and p is a prime number dividing n, then E/K admits a subextension of degree n/p.

8.7. Exercise. If E/K is a Galois extension of degree $p^{\alpha}m$ with p a prime number coprime with m, then E/K admits a subextension of degree m.

8.8. Definition. An extension E/K is **abelian** if E/K is a Galois extension with $\operatorname{Gal}(E/K)$ abelian.

8.9. Exercise. If E/K is an abelian extension of degree n and d divides n, then E/K admits a subextension of degree d.

8.10. Definition. An extension E/K is **cyclic** if E/K is a Galois extension with $\operatorname{Gal}(E/K)$ cyclic.

8.11. Example. The extension $\mathbb{Q}(\sqrt{2},\sqrt{3})/\mathbb{Q}$ admits exactly three non-trivial subextensions:

$$\mathbb{Q}(\sqrt{2})/\mathbb{Q},\quad \mathbb{Q}(\sqrt{3})/\mathbb{Q},\quad \mathbb{Q}(\sqrt{6})/\mathbb{Q},$$

as $\operatorname{Gal}(\mathbb{Q}(\sqrt{2},\sqrt{3})/Q)\simeq C_2\times C_2$.

8.12. Example. Let $\omega\in\mathbb{C}\setminus\{1\}$ be such that $\omega^5=1$. Then

$$f(\omega,\mathbb{Q})=1+X+X^2+X^3+X^4$$

and $\mathbb{Q}(\omega)/\mathbb{Q}$ has degree four. Moreover, $\mathbb{Q}(\omega)/\mathbb{Q}$ is a Galois extension and

$$\operatorname{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})\simeq C_4.$$

If $\sigma\in\operatorname{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})$, then $\sigma(\omega)=\omega^i$ for some $i\in\{1,\dots,4\}$. Moreover, each map $\omega\mapsto\omega^i$, for $i\in\{1,\dots,4\}$, induces an automorphism of $\mathbb{Q}(\omega)/\mathbb{Q}$. Thus $|\operatorname{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})|=4$. Now

$$\sigma_i^k=\mathrm{id}\iff\omega^{i^k}=\sigma_i^k(\omega)=\omega\iff i^k\equiv 1 \bmod 5.$$

Thus the map σ_2 given by $\omega\mapsto\omega^2$ has order four.

Since $\operatorname{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})=\langle\sigma\rangle$, where $\sigma(\omega)=\omega^2$, is cyclic of order four, the extension $\mathbb{Q}(\omega)/\mathbb{Q}$ has a unique degree-two subtextension $F/\mathbb{Q}$. Note that $|\langle\sigma^2\rangle|=2$ and $\sigma^2(\omega)=\omega^4=\omega^{-1}$. Thus $F={}^{\langle\sigma^2\rangle}\mathbb{Q}(\omega)$. Let $\theta=\omega+\omega^{-1}$. Then

$$\theta^2=\omega^2+\omega^3+2=-(1+\omega+\omega^{-1})+2=1-\theta$$

and hence θ is a root of X^2+X-1. It follows that

$$\theta\in\{(-1+\sqrt{5})/2,(-1-\sqrt{5})/2\}.$$

Therefore $F=\mathbb{Q}(\sqrt{5})$.

Let us mention some other consequences (of the fact that the correspondence depends on order-reversing bijections).

8.13. Exercise. Let E/K be a finite Galois extension and $G=\operatorname{Gal}(E/K)$. If S and T are subgroups of G, then ${}^{\langle S,T\rangle}E={}^{S}E\cap{}^{T}E$ and ${}^{S\cap T}E={}^{S}E\,{}^{T}E$.

8.14. Exercise. Let E/K be a finite Galois extension and $F,L\in L(E/K)$. Prove that $\operatorname{Gal}(E/FL)=\operatorname{Gal}(E/F)\cap\operatorname{Gal}(E/L)$ and $\operatorname{Gal}(E/F\cap L)=\langle\operatorname{Gal}(E/F),\operatorname{Gal}(E/L)\rangle$.

8.15. Exercise. Let E/K be a finite Galois extension and $G=\operatorname{Gal}(E/K)$. Assume that G is the direct product $G=S\times T$ of the groups S and T. Let $F={}^{S}E$ and $L={}^{T}E$. Then $F\cap L=K$ and $FL=E$.

8.16. PROPOSITION. *Let $E_1/K, E_2/K$ be Galois extensions. If $E = E_1E_2$, then E/K is a Galois extension. If, moreover, E_1/K and E_2/K are finite, then*

$$\theta\colon \operatorname{Gal}(E/K) \to \operatorname{Gal}(E_1/K) \times \operatorname{Gal}(E_2/K), \quad \sigma \mapsto (\sigma|_{E_1}, \sigma|_{E_2}),$$

is an injective group homomorphism.

PROOF. Since E_1/K is algebraic, then E_1E_2/E_2 is algebraic. Since E_2/K is algebraic, E_1E_2/K is algebraic. Similarly, E_1E_2/K is separable.

Let C/K be an algebraic closure such that $E_1E_2 \subseteq C$. If $\sigma \in \operatorname{Hom}(E_1E_2/K, C/K)$, then $\sigma(E_1E_2) \subseteq \sigma(E_1)\sigma(E_2) = E_1E_2$ (do this calculation as an exercise using the fact that E_1/K and E_2/K are normal extensions). Thus E_1E_2/K is normal.

If both E_1/K and E_2/K are finite, then E_1E_2/K is finite.

Then θ is a group homomorphism. We claim that the map θ is injective. Let $\sigma \in \ker\theta$. Then $\sigma|_{E_i} = \mathrm{id}_{E_i}$ for all $i \in \{1, 2\}$. Let $S = \langle\sigma\rangle \subseteq \operatorname{Gal}(E/K)$ and $F = {}^{S}E$. Then $E_i \subseteq F$ for all $i \in \{1, 2\}$ and hence $E \subseteq F$. It follows that $F = E = {}^{\{\mathrm{id}\}}E$ and therefore $S = \{\mathrm{id}\}$, so $\sigma = \mathrm{id}$. □

8.17. EXERCISE. Let $E_1/K, \dots, E_r/K$ be finite Galois extensions such that for each j one has $E_j \cap (E_1 \cdots E_{j-1}E_{j+1} \cdots E_r) = K$. Then

$$\operatorname{Gal}(E/K) \simeq \operatorname{Gal}(E_1/K) \times \cdots \times \operatorname{Gal}(E_r/K).$$

In this case, $[E : K] = \prod_{i=1}^{r}[E_i : K]$.

§ 8.2. The fundamental theorem of algebra. We now present an easy proof of the fundamental theorem of algebra based on the ideas of Galois Theory. We need the following well-known facts:

1) Every real polynomial of odd degree admits a real root. This means that $\mathbb{R}$ does not admit extension of odd degree > 1.

2) Every complex number admits a square root in $\mathbb{C}$. This means that $\mathbb{C}$ does not admit degree-two extensions.

8.18. THEOREM. *The field $\mathbb{C}$ is algebraically closed.*

PROOF. Let $E/\mathbb{C}$ be an algebraic finite extension. Then $E/\mathbb{R}$ is finite separable extension of even degree. There exists a Galois extension $L/\mathbb{R}$ such that $E \subseteq L$, so $[L : \mathbb{R}]$ is even. Let $G = \operatorname{Gal}(L/\mathbb{R})$. Then $|G| = 2^m s$ for some odd number s. If T is a 2-Sylow subgroup of G, then there exists a subextension $F/\mathbb{R}$ of degree s. Since $\mathbb{R}$ does not admit extensions of odd degree > 1, $s = 1$ and hence G is a 2-group. Since $L/\mathbb{R}$ is a Galois extension, $L/\mathbb{C}$ is a Galois extension. In particular, $|\operatorname{Gal}(L/\mathbb{C})| = 2^{m-1}$. If $m > 1$, let U be a subgroup of $\operatorname{Gal}(L/\mathbb{C})$ of order 2^{m-2}. Then U corresponds to a subextension $L_1/\mathbb{C}$ of degree two, a contradiction. Hence $m = 1$ and $[L : \mathbb{C}] = 1$, so $L = \mathbb{C}$ and $E = \mathbb{C}$. □

§ 8.3. Purely inseparable extensions. Let E/K be an algebraic extension. In page 6.1 we defined the **separable closure** of K with respect to E as the field

$$F = \{x \in E : x \text{ is separable over } K\}.$$

Note that $K \subseteq F \subseteq E$ and $F = K(F)$. Moreover, F/K is separable and E/F is a **purely inseparable** extension, meaning that for every $x \in E \setminus F$, the polynomial $f(x, F)$ is not separable.

The number $[E:K]_{\text{ins}} = [E:F]$ is known as the **degree of inseparability** of E/K. Clearly, E/K is separable if and only if $[E:K]_{\text{ins}} = 1$ and E/K is purely inseparable if and only if $[E:K]_{\text{ins}} = [E:K]$.

8.19. Exercise. Let K be a field of characteristic $p > 0$ and $f \in K[X]$ be irreducible. If f is not separable, then $f = g(X^p)$ for some $g \in K[X]$.

8.20. Proposition. *Let K be a field of characteristic $p > 0$ and E/K be an algebraic extension. The following statements are equivalent:*

1) *E/K is purely inseparable.*
2) *If $x \in E$, then $x^{p^m} \in K$ for some $m \geq 0$.*
3) *If $x \in E$, then $f(x, K) = X^{p^m} - a$ for some $a \in K$ and $m \geq 0$.*
4) *$\gamma(E/K) = 1$.*

Proof. We first prove 1) $\implies$ 2). Let $x \in E$ and $f = f(x, K)$. Assume x is not separable. Then $f(x) = 0$ and $f'(x) = 0$, as x is not a simple root. By Exercise 8.19, $f = g(X^p)$ for some $g \in K[X]$. We now proceed by induction on the degree of x. The result is true for elements of degree one. So assume the result holds for the element of degree $\leq n$ for some $n \geq 1$. If $x \in E$ is such that $\deg f(x, K) = n + 1$, then, since $f(x, K) = g(X^p)$, the element x^p has degree $\leq n$. By the inductive hypothesis, $x^{p^{m+1}} = (x^p)^{p^m} \in K$.

We now prove 2) $\implies$ 3). Let $x \in E$ and m be the minimal positive integer such that $x^{p^m} \in K$. Then x is a root of $X^{p^m} - x^{p^m} \in K[X]$. Since $X^{p^m} - x^{p^m} = (X - x)^{p^m}$, it follows that

$$f(x, K) = (X - x)^r = X^r + \cdots + (-1)^r x^r$$

for some $r \in \{1, \ldots, p^m\}$. Write $r = p^s t$ for some integer t coprime with p and s such that $0 \leq s \leq m$. Let $a, b \in \mathbb{Z}$ be such that $ar + bp^m = p^s$. Then

$$x^{p^s} = x^{ar+bp^m} = (x^r)^a \left(x^{p^m}\right)^b \in K.$$

The minimality of m implies that $s \geq m$ and hence $s = m$. Now $p^m t = p^s t = r \leq p^m$, so $t = 1$. This means $f(x, K) = X^{p^m} - x^{p^m}$.

We now prove 3) $\implies$ 4). Let C/K be an algebraic closure of K containing E and $x \in E$. Let $\sigma \in \operatorname{Hom}(E/K, C/K)$. We claim that $\sigma(x) = x$. Since $f(x, K) = X^{p^m} - a$,

$$(\sigma(x))^{p^m} = \sigma\left(x^{p^m}\right) = \sigma(a) = a = x^{p^m}.$$

It follows that $\sigma(x)$ is a root of $X^{p^m} - x^{p^m} = (X - x)^{p^m}$. Thus $\sigma(x) = x$.

Finally, we prove that 4) $\implies$ 1). Let C be an algebraic closure of K containing E. Then $\operatorname{Gal}(E/K) = \operatorname{Hom}(E/K, C/K) = \{\mathrm{id}\}$, as $\gamma(E/K) = 1$. If $x \in E$ is separable over K, then

$$f(x, K) = \prod_{y \in O_{\operatorname{Gal}(E/K)}(x)} (X - y) = X - x \in K[X].$$

Thus $x \in K$ and hence E/K is purely inseparable. □

Some consequences:

8.21. Exercise. Let K be a field of characteristic $p > 0$ and E/K be finite and purely inseparable. Then $[E:K] = p^s$ for some prime number p and some s. Moreover, $x^{[E:K]} \in K$.

For the first part of the previous exercise, write $E = K(x_1, \dots, x_n)$ and proceed by induction on n.

8.22. Exercise. Let K be of characteristic $p > 0$ and E/K be a finite extension such that $[E : K]$ is not divisible by p. Then E/K is separable.

Let K be of characteristic $p > 0$, E/K be finite and F be the separable closure of K in E. Since

$$\gamma(E/K) = \gamma(E/F)\gamma(F/K) = \gamma(F/K),$$

it follows that

$$[E : K] = [E : F]\gamma(E/K) = [E : K]_{\text{ins}}\gamma(E/K).$$

9. Lecture – Week 9

§ 9.1. Norm and trace.

9.1. DEFINITION. Let E/K be a finite extension and C/K be an algebraic closure that contains E. Let $A = \operatorname{Hom}(E/K, C/K)$. For $x \in E$ we define the **trace** of x in E/K as

$$\operatorname{trace}_{E/K}(x) = [E:K]_{\mathrm{ins}} \sum_{\varphi \in A} \varphi(x)$$

and the **norm** of x in E/K as

$$\operatorname{norm}_{E/K}(x) = \left(\prod_{\varphi \in A} \varphi(x) \right)^{[E:K]_{\mathrm{ins}}}.$$

As an optional exercise, one can show that these definitions do not depend on the algebraic closure.

We collect some basic properties as an exercise:

9.2. EXERCISE. Let E/K be a finite extension. The following statements hold:

1) If E/K is not separable, then $\operatorname{trace}_{E/K}(x) = 0$ for all $x \in E$.
2) If $x \in K$, then $\operatorname{trace}_{E/K}(x) = [E:K]x$.
3) $\operatorname{trace}_{E/K}(x) \in K$ for all $x \in E$.
4) $\operatorname{norm}_{E/K}(x) = 0$ if and only if $x = 0$.
5) If $x \in K$, then $\operatorname{norm}_{E/K}(x) = x^{[E:K]}$.
6) $\operatorname{norm}_{E/K}(x) \in K$ for all $x \in E$.

One proves, moreover, that $\operatorname{trace}_{E/K} \colon E \to K$ satisfies

$$\operatorname{trace}_{E/K}(x + \lambda y) = \operatorname{trace}_{E/K}(x) + \lambda \operatorname{trace}_{E/K}(y)$$

for all $x, y \in E$ and $\lambda \in K$, that is to say that $\operatorname{trace}_{E/K} \colon E \to K$ is a linear form in E The norm $\operatorname{norm}_{E/K} \colon E^\times \to K^\times$ is a group homomorphism.

9.3. EXERCISE. Let E/K be a finite extension and $x \in E$ be algebraic. If

$$f(x, K) = X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0,$$

then $\operatorname{norm}_{E/K}(x) = ((-1)^n a_0)^{[E:K(x)]}$ and $\operatorname{trace}_{E/K}(x) = -[E:K(x)]a_{n-1}$.

9.4. EXAMPLE. Let $E = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Then

$$\operatorname{trace}_{E/\mathbb{Q}}(\sqrt{2}) = 0, \qquad \operatorname{norm}_{E/\mathbb{Q}}(\sqrt{2}) = 4,$$
$$\operatorname{trace}_{E/\mathbb{Q}(\sqrt{2})}(\sqrt{2}) = 2\sqrt{2}, \qquad \operatorname{norm}_{E/\mathbb{Q}(\sqrt{2})}(\sqrt{2}) = 2.$$

9.5. EXAMPLE. If E/K is a finite Galois extension, then

$$\operatorname{trace}_{E/K}(x) = \sum_{\sigma \in \operatorname{Gal}(E/K)} \sigma(x) \quad \text{and} \quad \operatorname{norm}_{E/K}(x) = \prod_{\sigma \in \operatorname{Gal}(E/K)} \sigma(x)$$

for all $x \in E$. In particular, since $E = K(y)$ for some y by Proposition 6.10,

$$\operatorname{trace}_{E/K}(y) = -a_{n-1} \quad \text{and} \quad \operatorname{norm}_{E/K}(y) = (-1)^n a_0,$$

where $f(y, K) = X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0$.

§ 9.2. Finite fields. In this section, p will be a prime number.

9.6. Proposition. *Let m be a positive integer. Up to isomorphism, there exists a unique field F_m of size p^m.*

Proof. Let C be an algebraic closure of the field $\mathbb{Z}/p$ and let $F_m = \{x \in C : x^{p^m} = x\}$ be the set of roots of $X^{p^m} - X$. Since the polynomial $X^{p^m} - X$ has no multiple roots, $|F_m| = p^m$. Moreover, F_m is the unique subfield of C of size p^m.

To prove the uniqueness, it is enough to note that if K is a field of p^m elements, then K is the decomposition field of $X^{p^m} - X$ over $\mathbb{Z}/p$. □

Let $K = \mathbb{Z}/p$ and C be an algebraic closure of K. We claim that $C = \cup_k F_k$. If $x \in C$, then x is algebraic over K. Since $K(x)/K$ is finite, $K(x)$ is a finite field, say $|K| = p^r$ for some r. Then $x^{p^r} = x$ and hence $x \in F_r$.

9.7. Exercise. Prove the following statements:

1) If $x \in F_r$, then $x^{p^{rk}} = x$ for all $k \geq 0$.
2) $F_m \subseteq F_n$ if and only if $m \mid n$.
3) $F_m \cap F_n = F_{\gcd(m,n)}$.

9.8. Proposition. *Every finite extension of a finite field is cyclic.*

Proof. Let $K = \mathbb{Z}/p$. It is enough to show that F_n/F_m is cyclic if m divides n.

We first prove that F_n/K is cyclic. Let

$$\sigma \colon F_n \to F_n, \quad x \mapsto x^p.$$

Then $\sigma \in \operatorname{Gal}(F_n/K)$ (it is bijective because all field homomorphisms are injective and F_n is finite).

Note that F_n/K is a Galois extension, as F_n is the splitting field over K of the separable polynomial $X^{p^n} - X \in K[X]$. Thus $|\operatorname{Gal}(F_n/K)| = [F_n : K] = n$.

We claim that σ generated $\operatorname{Gal}(F_n/K)$. Since $\sigma^i(x) = x^{p^i}$ for all $i \geq 0$, in particular,

$$\sigma^n(x) = x^{p^n} = x.$$

Thus $\sigma^n = \mathrm{id}$ and hence $|\sigma|$ divides n. Let $s = |\sigma|$. We know that $F_n^\times = F_n \setminus \{0\}$ is cyclic, say $F_n^\times = \langle g \rangle$. Since $|g| = p^n - 1$,

$$g = \sigma^s(g) = g^{p^s}$$

and hence $p^s \equiv 1 \bmod (p^n - 1)$. Thus $p^n - 1$ divides $p^s - 1$ and hence n divides s. Therefore $n = s$ and $\operatorname{Gal}(F_n/K) = \langle \sigma \rangle$.

For the general case note that if m divides n, then $\operatorname{Gal}(F_n/F_m)$ is a subgroup of $\operatorname{Gal}(F_n/K)$. Since $\operatorname{Gal}(F_n/K)$ is cyclic, the claim follows. □

If $K = \mathbb{Z}/p$ and m divides n, the subextension F_m corresponds to the unique subgroup of index m of $\operatorname{Gal}(F_n/K) = \langle \sigma \rangle$. This subgroup is $\langle \sigma^m \rangle$, where

$$\sigma^m(x) = x^{p^m} = x^{|F_m|}.$$

Note that $\operatorname{Gal}(F_n/F_m) = \langle \sigma^m \rangle$. The map σ^m is known as the **Frobenius automorphism**.

9.9. Exercise. Let E/K be an extension of finite fields. Then E/K is cyclic. Moreover, $\operatorname{Gal}(E/K) = \langle \tau \rangle$, where $\tau(x) = x^{|K|}$.

§ 9.3. Cyclotomic extensions. For $n \geq 1$ let $G_n(K) = \{x \in K : x^n = 1\}$ be the set of n-roots of one in K. Note that $G_n(K)$ is a cyclic subgroup of $K^\times$ and that $|G_n(K)|$ divides n.

9.10. Example. $G_n(\mathbb{R}) = \{-1, 1\}$ if n is odd and $G_n(\mathbb{R}) = \{1\}$ if n is even.

9.11. Exercise. Let K be a field of characteristic $p > 0$. Let $n = p^s m$ for some m not divisible by p. Then $G_n(K) = G_m(K)$.

9.12. Exercise. Let q be a prime number. Then $G_n(\mathbb{Z}/q) \simeq \mathbb{Z}/\gcd(n, q-1)$.

Similarly, one can prove that if K is a finite field, then $G_n(K)$ is a cyclic group of order $\gcd(n, |K^\times|)$.

9.13. Example. If C is algebraically closed of characteristic coprime with n, then $G_n(C)$ is cyclic of order n, as $X^n - 1$ has all its roots in C and does not contain multiple roots.

Let K be an algebraically closed field and n be such that n is coprime with the characteristic of K. The set of **primitive n-roots** is defined as

$$H_n(K) = \{x \in G_n(K) : |x| = n\}.$$

9.14. Definition. Let K be an algebraically closed field and n be such that n is coprime with the characteristic of K. The **n-th cyclotomic polynomial** is defined as

$$\Phi_n = \prod_{x \in H_n(K)} (X - x) \in K[X].$$

For $n \geq 1$ the Euler's function is defined as

$$\varphi(n) = |\{k : 1 \leq k \leq n,\ \gcd(k, n) = 1\}|.$$

For example, $\varphi(4) = 2$, $\varphi(8) = \varphi(10) = 4$ and $\varphi(p) = p - 1$ for every prime p.

9.15. Proposition. *Let K be an algebraically closed field and n be such that n is coprime with the characteristic of K. Let A be the ring of integers of K.*

1) $\deg \Phi_n = \varphi(n)$.
2) $\Phi_n \in A[X]$.

Proof. The first statement is clear. Let us prove 2) by induction on n. The case $n = 1$ is trivial, as $\Phi_1 = X - 1$. Assume that $\Phi_d \in A[X]$ for all d such that $d < n$. In particular,

$$\gamma = \prod_{\substack{d \mid n \\ d \neq n}} \Phi_d \in A[X].$$

Since γ is monic, it follows that $\frac{X^n - 1}{\gamma} \in A[X]$. Now the claim follows from

$$X^n - 1 = \prod_{d \mid n} \Phi_d = \Phi_n \prod_{\substack{d \mid n \\ d \neq n}} \Phi_d = \Phi_n \gamma. \qquad \square$$

By taking degree in the equality $X^n - 1 = \prod_{d \mid n} \Phi_d$ one gets

$$n = \sum_{d \mid n} \varphi(d).$$

9.16. Definition. Let $n \geq 2$ and K be a field of characteristic coprime with n. A **cyclotomic extension** of K of index n is a decomposition field of $X^n - 1$ over K.

Let C be an algebraic closure of K and $n \geq 2$ be coprime with the characteristic of K. If follows from Definition 9.16 that a cyclotomic extension of index n is of the form $K(\omega)/K$ for some $\omega \in H_n(K)$.

9.17. Proposition. *A cyclotomic extension of index* n *is abelian and of degree a divisor of* $\varphi(n)$.

Proof. Let C be an algebraic closure of K and $n \geq 2$ be coprime with the characteristic of K. Let $\omega \in H_n(C)$ and $K(\omega)/K$ be a cyclotomic extension. Then $K(\omega)/K$ is a Galois extension, as it is a decomposition field of a separable polynomial. Let $U = \mathcal{U}(\mathbb{Z}/n)$ be the group of units of $\mathbb{Z}/n$ and

$$\lambda\colon \operatorname{Gal}(K(\omega)/K) \to U, \quad \sigma \mapsto m_\sigma,$$

where m_σ is such that $\sigma(\omega) = \omega^{m_\sigma}$. The map λ is well-defined and it is a group homomorphism, as if $\sigma, \tau \in \operatorname{Gal}(K(\omega)/K)$, then, since

$$(\tau\sigma)(\omega) = \tau(\sigma(\omega)) = \tau(\omega^{m_\sigma}) = (\omega^{m_\sigma})^{m_\tau} = \omega^{m_\sigma m_\tau},$$

it follows that $\lambda(\sigma)\lambda(\tau) = \lambda(\sigma\tau)$. Since λ is injective, $\operatorname{Gal}(K(\omega)/K)$ is isomorphic to a subgroup of the abelian group U. Hence $\operatorname{Gal}(K(\omega)/K)$ is abelian. Moreover, $[K(\omega) : K] = |\operatorname{Gal}(K(\omega)/K)|$ is a divisor of $|U| = \varphi(n)$. □

9.18. Exercise. Prove that a cyclotomic extension $K(\omega)/K$ has degree $\varphi(n)$ if and only if Φ_n is irreducible over K.

Note that Φ_n is irreducible over $\mathbb{Q}$. Some concrete examples:

$$\Phi_1 = X - 1, \quad \Phi_2 = X + 1, \quad \Phi_3 = X^2 + X + 1, \quad \Phi_6 = X^2 - X + 1.$$

If p is a prime number, then $\Phi_p = X^{p-1} + \cdots + X + 1$.

9.19. Example. Φ_5 is irreducible over $\mathbb{Z}/2$. First note that $\Phi_5 = X^4 + \cdots + X + 1$ does not have roots in $\mathbb{Z}/2$. If Φ_5 is reducible, then, since $X^2 + X + 1$ is the unique degree-two monic irreducible polynomial over $\mathbb{Z}/2$, it follows that

$$\Phi_5 = (X^2 + X + 1)(X^2 + X + 1) = (X^2 + X + 1)^2 = X^4 + X^2 + 1,$$

a contradiction.

9.20. Exercise. Prove that $\Phi_{12} = X^4 - X^2 + 1$ is not irreducible over $\mathbb{Z}/5$.

§ 9.4. Hilbert's theorem 90.

9.21. Theorem (Hilbert). *Let* E/K *be a cyclic extension. Assume that* $\operatorname{Gal}(E/K)$ *is generated by* τ. *For* $a \in E$, $\operatorname{norm}_{E/K}(a) = 1$ *if and only if* $a = b/\tau(b)$ *for some* $b \in E \setminus \{0\}$.

Proof. Let $n = |G|$. We first prove $\Longleftarrow$. If $a = b/\tau(b)$ and $b \neq 0$, then

$$\operatorname{norm}_{E/K}(a) = a\tau(a)\tau^2(a)\cdots\tau^{n-1}(a) = \frac{b}{\tau(b)}\frac{\tau(b)}{\tau^2(b)}\cdots\frac{\tau^{n-1}(b)}{\tau^n(b)} = 1.$$

Now we prove $\implies$. Let $a \in E$ be such that $\operatorname{norm}_{E/K}(a) = 1$. For $c \in E$ let

$$\begin{aligned} d_0 &= ac, \\ d_1 &= a\tau(a)\tau(c), \\ d_2 &= a\tau(a)\tau^2(a)\tau^2(c), \\ &\vdots \\ d_{n-1} &= \underbrace{a\tau(a)\cdots\tau^{n-1}(a)}_{=\operatorname{norm}_{E/K}(a)}\tau^{n-1}(c) = \tau^{n-1}(c). \end{aligned}$$

Then

$$a\tau(d_j) = a\tau(a)\cdots\tau^{j+1}(a)\tau^{j+1}(c) = d_{j+1}$$

for all $j \in \{0, \dots, n-2\}$. Let $b = d_0 + \cdots + d_{n-1}$. We claim that $b \neq 0$ for some c. Suppose this is not true, say $b = 0$ for all c. Then

$$0 = ac + (a\tau(a))\tau(c) + \cdots + (a\tau(a)\cdots\tau^{n-1}(a))\tau^{n-1}(c)$$

for every $c \in E$. This implies that $a = 0$ by Dedekind's theorem, a contradiction.

So let $c \in E$ be such that $b \neq 0$. Then

$$\begin{aligned} \tau(b) &= \tau(d_0) + \cdots + \tau(d_{n-1}) \\ &= \tau(ac) + \tau(a\tau(c)) + \cdots + \tau(\tau^{n-1}(c)) \\ &= \frac{1}{a}(d_1 + \cdots + d_{n-1}) + \tau^n(c) \\ &= \frac{1}{a}(d_0 + \cdots + d_{n-1}) \\ &= b/a. \end{aligned}$$

□

9.22. Exercise. Let E/K be a cyclic extension. Assume that $\operatorname{Gal}(E/K)$ is generated by τ. Prove that for $a \in E$, $\operatorname{trace}_{E/K}(a) = 0$ if an only if $a = b - \tau(b)$ for some $b \in L \setminus \{0\}$.

10. Lecture – Week 10

10.1. Corollary. *Let $a, b, c \in \mathbb{Z}$ be such that $a^2 + b^2 = c^2$. Then*

$$(a, b, c) = \lambda(r^2 - s^2, -2rs, r^2 + s^2)$$

for some $r, s \in \mathbb{Z}$ and some $\lambda \in \mathbb{Z}$.

Proof. We work with the extension $\mathbb{Q}(i)/\mathbb{Q}$. Note that $\mathrm{Gal}(\mathbb{Q}(i), \mathbb{Q}) = \{\mathrm{id}, \gamma\}$ is cyclic, where $\gamma\colon \mathbb{Q}(i) \to \mathbb{Q}(i)$, $z \mapsto \overline{z}$, is the complex conjugation. We may assume that $c \neq 0$, otherwise $a = b = 0$ and the result is trivial. Write $(a/c)^2 + (b/c)^2 = 1$ and let $\alpha = (a/c) + (b/c)i \in \mathbb{Q}(i)$. Then $\mathrm{norm}_{\mathbb{Q}(i)/\mathbb{Q}}(\alpha) = 1$. By Hilbert's theorem, there exists $\beta \in \mathbb{Q}(i) \setminus \{0\}$ such that

$$\alpha = a + bi = \frac{\gamma(\beta)}{\beta}.$$

Note that if $m \in \mathbb{Z} \setminus \{0\}$, then $\frac{\gamma(m\beta)}{m\beta} = \frac{\gamma(\beta)}{\beta}$. There exists $m \in \mathbb{Z} \setminus \{0\}$ such that $m\beta \in \mathbb{Z}[i]$, say $m\beta = r + is$ with $r, s \in \mathbb{Z}$. Then

$$\alpha = \frac{\gamma(\beta)}{\beta} = \frac{\gamma(m\beta)}{m\beta} = \frac{r - is}{r + is} = \frac{r^2 - s^2 - 2rsi}{r^2 + s^2}.$$

From this the claim follows. □

10.2. Exercise. Let $A, B \in \mathbb{Z}$ be such that $A^2 - 4B$ is not a square. Prove that a solution $(x, y, z) \in \mathbb{Z}^3$ of $x^2 + Axy + By^2 = z^2$ is proportional to

$$(r^2 - Bs^2, 2rs + As^2, r^2 + Ars + Bs^2).$$

10.3. Proposition. *Let $n \geq 2$ and K be a field containing a primitive n-root of one. If $a \in K^\times$ and E/K is a decomposition field of $f = X^n - a$, then E/K is cyclic of degree d, where d divides n. Moreover,*

$$d = \min\{k : a^k \in K^n\},$$

where $K^n = \{x \in K : x = y^n \text{ for some } y \in K\}$. Conversely, if E/K is cyclic of degree n, then E/K is a decomposition field of an irreducible polynomial of the form $X^n - a$ for some $a \in K^\times$.

Proof. A decomposition field of f over K is of the form $K(\alpha)$, where $\alpha^n = a$. Thus $K(\alpha)/K$ is a Galois extension. If $\sigma \in \mathrm{Gal}(K(\alpha)/K)$, then $\sigma(\alpha)$ is a root of f, so $\sigma(\alpha) = \omega_\sigma \alpha$, where $\omega_\sigma \in G_n(K)$. This means that there exists an injective map

$$\lambda\colon \mathrm{Gal}(K(\alpha)/K) \to G_n(K), \quad \sigma \mapsto \omega_\sigma.$$

Moreover, λ is a group homomorphism, as

$$\sigma\tau(\alpha) = \sigma(\tau(\alpha)) = \sigma(\omega_\tau \alpha) = \omega_\tau \sigma(\alpha) = \omega_\tau \omega_\sigma \alpha.$$

Therefore $\mathrm{Gal}(K(\alpha)/K)$ is isomorphic to a subgroup of $G_n(K)$. In particular, $\mathrm{Gal}(K(\alpha)/K)$ is cyclic and $|\mathrm{Gal}(K(\alpha)/K)|$ divides n.

Let $d = |\mathrm{Gal}(K(\alpha)/K)|$. Since $a = \alpha^n$,

$$\mathrm{norm}_{K(\alpha)/K}(\alpha)^n = \mathrm{norm}_{K(\alpha)/K}(a) = a^d.$$

Thus $a^d \in K^n$, as $\mathrm{norm}_{K(\alpha)/K}(\alpha) \in K$. If $a^k \in K^n$, say $a^k = c^n$ for some $c \in K$, then

$$c^n = a^k = (\alpha^n)^k = (\alpha^k)^n \implies \alpha^k = c\omega \in K$$

for some $\omega \in G_n(K)$. Thus α is a root of $X^k - \alpha^k \in K[X]$ and hence $k \geq d$.

Note that $f(\alpha, K) = X^d - \alpha^d$.

Let E/K be cyclic of degree n. Assume that $\mathrm{Gal}(E/K) = \langle \sigma \rangle$. If ω is a primitive n-root of one,

$$\mathrm{norm}_{E/K}(\omega) = \omega^n = 1.$$

By Hilbert's theorem 90, there exists $b \in E^\times$ such that $\omega = \sigma(b)/b$. Thus $\sigma(b) = \omega b$ and hence $\sigma^i(b) = \omega^i b$ for all $i \geq 0$. Since $|\{b, \sigma(b), \dots, \sigma^{n-1}(b)\}| = n$, it follows that $E = K(b)$. Moreover,

$$\sigma(b^n) = \sigma(b)^n = (\omega b)^n = b^n$$

and hence $b^n \in K$. This means that E/K is a decomposition field of $X^n - b^n$. Note that $X^n - b^n$ is irreducible, as $[E : K] = [K(b) : K] = n$. □

10.4. Proposition. *Let K be a field of characteristic $p > 0$.*

1) *Let $a \in K$ and $f = X^p - X - a$. Then f is irreducible over K or all the roots of f belong to K. In the first case, if b is a root of f, then $K(b)/K$ is a cyclic extension of degree p.*
2) *Every cyclic extension of degree p is a decomposition field of an irreducible polynomial of the form $X^p - X - a$.*

Proof. We first prove 1). Let K_0 be the prime field of K. Note that $K_0 \simeq \mathbb{Z}/p$. Let b be a root of f and let $x \in K_0$. Then

$$f(b+x) = (b+x)^p - (b+x) - a = (b^p - b - a) + (x^p - x) = 0$$

and thus $\{b + x : x \in K_0\}$ is the set of roots of f. Note that $f' = -1$, so f has no multiple roots.

We claim that if $b \notin K$, then f is irreducible. If f is not irreducible, then $f = gh$ for some $g, h \in K[X]$ such that $0 < \deg g < p$. There exists a subset S of K_0 such that $g = \prod_{x \in S}(X - (b + x))$ and hence

$$|S|b + \sum_{x \in S} x = \sum_{x \in S}(b + x) \in K.$$

This implies that $|S|b \in K$ and hence, since $|S| \in K^\times$, it follows that $b \in K$.

Since $K(b)/K$ is a decomposition field of a separable polynomial, $K(b)/K$ is a Galois extension. Moreover, $|\mathrm{Gal}(K(b)/K)| = [K(b) : K] = p$ and hence $\mathrm{Gal}(K(b)/K)$ is cyclic.

We now prove 2). Let E/K be cyclic of degree p. Assume that $\mathrm{Gal}(E/K) = \langle \sigma \rangle$. Since $\mathrm{trace}_{E/K}(1) = p = 0$, Hilbert's theorem implies that there exists $b \in E$ such that $\sigma(b) = b+1$. In particular, $b \notin K$ and thus $E = K(b)$. Moreover, since

$$\sigma(b^p - b) = \sigma(b)^p - \sigma(b) = (b+1)^p - (b+1) = b^p - b,$$

it follows that $b^p - b \in K$. Thus $f(b, K) = X^p - X - (b^p - b) \in K[X]$. □

§ 10.1. Symmetric polynomials. Let K be a field and $\{t_1, \dots, t_n\}$ be a commuting set of independent variables. Let $E = K(t_1, \dots, t_n)$ and $f = \prod_{i=1}^n (X - t_i) \in E[X]$. Then

$$f = X^n + \sum_{i=1}^n (-1)^i s_i X^{n-i},$$

where

$$\begin{aligned} s_1 &= t_1 + t_2 + \cdots + t_n, \\ s_2 &= \sum_{1 \le i < j \le n} t_i t_j, \\ &\vdots \\ s_n &= t_1 t_2 \cdots t_n. \end{aligned}$$

For example,

$$(X - t_1)(X - t_2)(X - t_3) = X^3 - (t_1 + t_2 + t_3)X^2 + (t_1t_2 + t_2t_3 + t_1t_3)X - t_1t_2t_3.$$

The polynomials $s_1, s_2, \dots, s_n$ are known as the **elementary symmetric polynomials** in the variables $t_1, \dots, t_n$. Note that $\deg s_i = i$.

Let $\sigma \in \mathbb{S}_n$ and

$$\alpha_\sigma \colon K[t_1, \dots, t_n] \to K[t_1, \dots, t_n], \quad t_i \mapsto t_{\sigma(i)} \quad \text{for all } i.$$

Then α_σ is a bijective homomorphism of K-algebras. In fact, $\alpha_\sigma^{-1} = \alpha_{\sigma^{-1}}$. Note that

$$\alpha_\sigma(h(t_1, \dots, t_n)) = h(t_{\sigma(1)}, \dots, t_{\sigma(n)}).$$

Since α_σ is injective, it induces an element $\widehat{\sigma} \in \mathrm{Gal}(E/K)$ given by

$$\widehat{\sigma}\left(\frac{h}{g}\right) = \frac{\alpha_\sigma(h)}{\alpha_\sigma(h)}.$$

The map $\mathbb{S}_n \to \mathrm{Gal}(E/K)$, $\sigma \mapsto \widehat{\sigma}$, is an injective group homomorphism. Thus $\{\widehat{\sigma} : \sigma \in \mathbb{S}_n\} \simeq \mathbb{S}_n$.

10.5. Definition. Let $g \in K[t_1, \dots, t_n]$. Then g is **symmetric** if $\widehat{\sigma}(g) = g$ for all $\sigma \in \mathbb{S}_n$.

We write P to denote the set of symmetric polynomials in $K[t_1, \dots, t_n]$. Clearly, P is a subalgebra of $K[t_1, \dots, t_n]$. The following statements hold:

1) $K \subseteq P$.
2) $\sum_{i=1}^n t_i^r \in P$ for all $r \ge 1$.
3) $s_i \in P$ for all i.
4) $K(P) \subseteq {}^G E$, where $G = \{\widehat{\sigma} : \sigma \in \mathbb{S}_n\}$.

Let $F = K(s_1, s_2, \dots, s_n)$. Then E/F is a Galois extension, as it is a decomposition field of f.

10.6. Proposition. $[E : F] \le n!$.

Proof. We proceed by induction on n. The case $n = 1$ is clear, as $E = F$. Assume that $n > 1$. Let $u_1, \dots, u_{n-1}$ be the elementary symmetric polynomials in $t_1, \dots, t_{n-1}$. Then

$$s_i = u_i + t_n u_{i-1}$$

for all $i \in \{1, \dots, n\}$, where $u_0 = 1$ and $u_n = 0$. Note that $u_1 = s_1 - t_n$ and $u_i = s_i - t_n u_{i-1}$ for all i. Since $K(s_1, \dots, s_n, t_n) = K(u_1, \dots, u_{n-1}, t_n)$,

$$F(t_n) = K(u_1, \dots, u_{n-1}, t_n) = K(t_n)(u_1, \dots, u_{n-1})$$

and

$$[E : F] = [E : F(t_n)][F(t_n) : F] \le n[E : F(t_n)].$$

Note that $E = K(t_1, \dots, t_n) = K(t_n)(t_1, \dots, t_{n-1})$. By the inductive hypothesis,

$$[E : F(t_n)] \leq (n-1)!$$

and hence $[E : F] \leq n!$. $\square$

10.7. Theorem. ${}^G E = F$.

Proof. By Artin's theorem,

$$\left[{}^G E : F\right] = \frac{[E : F]}{\left[E : {}^G E\right]} \leq \frac{n!}{\left[E : {}^G E\right]} = 1$$

and hence ${}^G E = F$. $\square$

10.8. Exercise. Prove that $\mathrm{Gal}(E/F) \simeq \mathbb{S}_n$.

10.9. Exercise. Prove that $\{s_1, \dots, s_n\}$ is algebraically independent over K.

10.10. Exercise. Prove that every symmetric polynomial in $t_1, \dots, t_n$ can be written as a rational fraction in $s_1, \dots, s_n$.

§ 10.2. Solvable groups. Let G be a group. If $x, y \in G$ we define the **commutator** of x and y as

$$[x, y] = xyx^{-1}y^{-1}.$$

Note that $[x, y] = 1$ if and only if $xy = yx$. Moreover, $[x, y]^{-1} = [y, x]$. The **commutator (or derived) subgroup** $[G, G]$ of G is defined as the subgroup of G generated by all commutators, i.e.

$$[G, G] = \langle [x, y] : x, y \in G \rangle.$$

This means that every element of $[G, G]$ is a finite product of commutators, so every element of $[G, G]$ is of the form $\prod_{i=1}^{m} [x_i, y_i]$. In general, the commutator subgroup is not equal to the set of commutators!

10.11. Example. This example is taken from the book [1] of Carmichael. Let G be the subgroup of $\mathbb{S}_{16}$ generated by the permutations

$$\begin{aligned} a &= (13)(24), & b &= (57)(68), \\ c &= (911)(10\,12), & d &= (13\,15)(14\,16), \\ e &= (13)(57)(9\,11), & f &= (12)(34)(13\,15), \\ g &= (56)(78)(13\,14)(15\,16), & h &= (9\,10)(11\,12). \end{aligned}$$

Then $[G, G]$ has order 16. However, the set $\{[x, y] : x, y \in G\}$ of commutators has 15 elements:

```
> S16 := Sym(16);
> a := S16 ! (1,3)(2,4);
> b := S16 ! (5,7)(6,8);
> c := S16 ! (9,11)(10,12);
> d := S16 ! (13,15)(14,16);
> e := S16 ! (1,3)(5,7)(9,11);
> f := S16 ! (1,2)(3,4)(13,15);
```

```
> g := S16 ! (5,6)(7,8)(13,14)(15,16);
> h := S16 ! (9,10)(11,12);
> G := PermutationGroup< 16 | a,b,c,d,e,f,g,h >;
> D := DerivedSubgroup(G);
> #D;
16
> #{ x*y*Inverse(x)*Inverse(y) : x in G, y in G };
15
> c*d in { x : x in D } \
> diff { u*v*Inverse(u)*Inverse(v) : u in G, v in G };
true
```

10.12. Exercise. Let G be a group. Prove the following facts:

1) G is abelian if and only if $[G,G]=\{1\}$.
2) $[G,G]$ is a normal subgroup of G.
3) $G/[G,G]$ is abelian.
4) If H is a subgroup of G and $[G,G]\subseteq H$, then H is normal in G.
5) If H is a normal subgroup of G, then G/H is abelian if and only if $[G,G]\subseteq H$.

10.13. Definition. Let G be a group. The **derived series** of G is defined as $G^{(0)}=G$ and $G^{(k+1)}=[G^{(k)},G^{(k)}]$ for $k\geq 0$.

10.14. Exercise. Prove that $G^{(k)}$ is normal in G for all k.

Why derived series? We cannot explain this here, but let us use the following notation. We write $G'=[G,G]$, $G''=[G',G']$... Note that

$$G\supseteq G'\supseteq G''\supseteq\cdots$$

10.15. Exercise. Let $n\geq 3$. Prove that $[\mathbb{S}_n,\mathbb{S}_n]=\mathbb{A}_n$.

10.16. Example. Let $K=\{\mathrm{id},(12)(34),(13)(24),(14)(23)\}$. Then K is a normal subgroup of $\mathbb{A}_4$. One proves that $[\mathbb{A}_4,\mathbb{A}_4]=K$.

A group G is said to be **simple** if there are no proper non-trivial normal subgroups of G. If p is a prime number, then the group $\mathbb{Z}/p$ of integers modulo p is a simple group. We will prove later that $\mathbb{A}_n$ is simple if $n\geq 5$.

10.17. Example. Let $n\geq 5$. Since $\mathbb{A}_n$ is a non-abelian simple group, $[\mathbb{A}_n,\mathbb{A}_n]=\mathbb{A}_n$.

Let us show that $\mathbb{A}_5$ is a non-abelian simple group. Hence it is not solvable:

```
> G := Alt(5);
> IsAbelian(G);
false
> IsSimple(G);
true
> IsSolvable(G);
false
```

10.18. Definition. A group G is **solvable** if and only if $G^{(m)} = \{1\}$ for some m.

Every abelian group is solvable.

10.19. Exercise. Prove that $\mathbb{S}_n$ is solvable if and only if $n \leq 4$.

Let us compute (with the computer software Oscar) the derived series of the symmetric group $\mathbb{S}_4$. The calculation shows that $\mathbb{S}_4$ is solvable:

```
> G := Sym(4);
> DerivedSeries(G);
[
    Symmetric group G acting on a set of cardinality 4
    Order = 24 = 2^3 * 3
        (1, 2, 3, 4)
        (1, 2),
    Permutation group acting on a set of cardinality 4
    Order = 12 = 2^2 * 3
        (1, 2, 3)
        (2, 3, 4),
    Permutation group acting on a set of cardinality 4
    Order = 4 = 2^2
        (1, 4)(2, 3)
        (1, 3)(2, 4),
    Permutation group acting on a set of cardinality 4
    Order = 1
]
> IsSolvable(G);
true
```

10.20. Proposition. *Let G be a group and H be a subgroup of G. The following statements hold:*

1) *If G is solvable, then H is solvable.*
2) *If H is normal in G and G is solvable, then G/H is solvable.*
3) *If H is normal in G and H and G/H are solvable, then G is solvable.*

Proof. The first statement follows from the fact that $H^{(i)} \subseteq G^{(i)}$ holds for all i.

Assume now that H is normal in G. Let $Q = G/H$ and $\pi \colon G \to Q$ be the canonical map. By induction one proves that $\pi(G^{(i)}) = Q^{(i)}$ for all $i \geq 0$. The case where $i = 0$ is trivial, as π is surjective. If the result holds for some $i \geq 0$, then

$$\pi(G^{(i+1)}) = \pi([G^{(i)}, G^{(i)}]) = [\pi(G^{(i)}), \pi(G^{(i)})] = [Q^{(i)}, Q^{(i)}] = Q^{(i+1)}.$$

We now prove 2). Since G is solvable, $G^{(n)} = \{1\}$ for some n. Thus Q is solvable, as $Q^n = \pi(G^{(n)}) = \pi(\{1\}) = \{1\}$.

We finally prove 3). Since Q is solvable, $Q^{(n)} = \{1\}$ for some n. Moreover, since $\pi(G^{(n)}) = Q^{(n)} = \{1\}$, it follows that $G^{(n)} \subseteq H$. Since H is solvable,

$$G^{(n+m)} \subseteq (G^{(n)})^{(m)} \subseteq H^{(m)} = \{1\}$$

for some m. Thus G is solvable. $\square$

An application:

10.21. Proposition. *Let G be a finite p-group. Then G is solvable.*

Proof. Assume the result is not true. Let G be a finite p-group of minimal order that is not solvable. Since G is a p-group, $Z(G) \neq \{1\}$. Since $|G|$ is minimal, $G/Z(G)$ is a solvable p-group. Since $Z(G)$ is abelian, $Z(G)$ is solvable. Now G is solvable by Proposition 10.20. □

Let G be a group. A subgroup N of G is said to be **maximal normal** if N is a normal subgroup of G and there is no other normal subgroup of G containing N.

10.22. Exercise. If a subgroup N of G is maximal (for the inclusion) and normal, then it is maximal normal. Show that the converse does not hold.

The following result is a direct consequence of the correspondence theorem:

10.23. Exercise. Let G be a group and N be a normal subgroup of G. Prove that N is maximal normal if and only if G/N is simple.

Maximal normal subgroups always exist in finite groups (they could be trivial). We can compute maximal normal subgroups as follows:

```
> MaximalNormalSubgroup(Sym(3));
Permutation group acting on a set of cardinality 3
Order = 3
    (1, 2, 3)
julia> maximal_normal_subgroups(quaternion_group(8))
3-element Vector{PcGroup}:
 Group([ y2, x ])
 Group([ y2, y ])
 Group([ y2, x*y ])

> MaximalNormalSubgroup(Alt(4));
Permutation group acting on a set of cardinality 4
Order = 4 = 2^2
    (1, 2)(3, 4)
    (1, 3)(2, 4)
```

10.24. Exercise. Let G be a finite solvable group. Prove that if G is simple, then G is cyclic of prime order.

The following result will be important later:

10.25. Proposition. *Every finite solvable group contains a normal subgroup of prime index.*

Proof. Let G be a finite solvable group. Let M be a maximal normal subgroup of G (there is at least one, as G is finite). Since G/M is simple and solvable (see Proposition 10.20), G/M is cyclic of prime order by Exercise 10.24. □

We finish this discussion with two important theorems (without proof) about finite solvable groups.

10.26. THEOREM (Burnside). *Let p and q be prime numbers. If G is a group of order $p^a q^b$, then G is solvable.*

The proof appears in courses on the representation theory of finite groups.

10.27. THEOREM (Feit–Thompson). *Every finite group of odd order is solvable.*

The proof of the theorem is extremely hard. It occupies a full volume of **Pacific Journal of Mathematics**, see [**2**].

11. Lecture – Week 11

§ 11.1. Simplicity of the alternating simple group. We will present a family of non-abelian simple groups. We start with some exercises.

11.1. Exercise. Let G be a group. Prove that G is simple if and only if $\{(g,g) : g \in G\}$ is a maximal subgroup of $G \times G$.

11.2. Exercise. Prove that $\mathbb{A}_n$ is generated by 3-cycles.

11.3. Exercise. Compute the commutator subgroup of $\mathbb{A}_n$ for $n \geq 2$.

Note that $\mathbb{A}_2$ and $\mathbb{A}_3$ are abelian. For $\mathbb{A}_4$, one proves that

$$[\mathbb{A}_4, \mathbb{A}_4] = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\}.$$

Finally, $[\mathbb{A}_n, \mathbb{A}_n] = \mathbb{A}_n$ for $n \geq 5$.

Let us compute some commutator subgroups (and the inclusion group homomorphism) with the computer:

```
> S3 := Sym(3);
> DerivedSubgroup(S3);
Permutation group acting on a set of cardinality 3
Order = 3
    (1, 2, 3)
```

11.4. Exercise. Let $n \geq 3$. Prove that $[\mathbb{S}_n, \mathbb{S}_n] = \mathbb{A}_n$.

Recall that every normal subgroup is a union of conjugacy classes. The group $\mathbb{A}_5$ has conjugacy classes of sizes 1, 15, 20, 12 and 12. It follows that the only possible normal subgroups of $\mathbb{A}_5$ are $\{\mathrm{id}\}$ and $\mathbb{A}_5$.

```
> A5 := Alt(5);
> ConjugacyClasses(A5);
Conjugacy Classes of group A5
-----------------------------
[1]     Order 1       Length 1
        Rep Id(A5)

[2]     Order 2       Length 15
        Rep (1, 2)(3, 4)

[3]     Order 3       Length 20
        Rep (1, 2, 3)

[4]     Order 5       Length 12
        Rep (1, 2, 3, 4, 5)

[5]     Order 5       Length 12
        Rep (1, 3, 4, 5, 2)
```

11.5. Theorem (Jordan). *Let $n \geq 5$. Then $\mathbb{A}_n$ is simple.*

Before proving the theorem, we need some preliminary results.

Every permutation $\rho \in \mathbb{S}_n$ decomposes as a product of disjoint cycles, say

$$\rho = (a_1 \cdots a_r)(b_1 \cdots b_s) \cdots (c_1 \cdots c_t)$$

where, by convention, we do not write cycles of length one. The cyclic structure of ρ is, by definition, the ordered sequence of integers $r, s, \dots t$, where, again by convention, we omit fixed points. For example, the cyclic structure of the transposition (ab) is 2, of $(abc)(d)$ is 3 and of $(123)(45)(789a)(bcd)(d)$ is 2,3,3,4.

11.6. Lemma. *If both $\rho_1 \in \mathbb{S}_n$ and $\rho_2 \in \mathbb{S}_n$ have the same cyclic structure, then $\rho_2 = \sigma\rho_1\sigma^{-1}$ for some $\sigma \in \mathbb{S}_n$.*

Proof. Assume that

$$\begin{aligned} \rho_1 &= (a_1 \cdots a_r)(b_1 \cdots b_s) \cdots (c_1 \cdots c_t), \\ \rho_2 &= (x_1 \cdots x_r)(y_1 \cdots y_s) \cdots (z_1 \cdots z_t). \end{aligned}$$

Let

$$\mathrm{Fix}(\rho_1) = \{x \in \{1, \dots, n\} : \rho_1(x) = x\} = \{k_1, \dots, k_m\}, \qquad \mathrm{Fix}(\rho_2) = \{l_1, \dots, l_m\}$$

be the fixed points of the permutations ρ_1 and ρ_2, respectively. Then

$$\sigma(x) = \begin{cases} x_j & \text{if } x = a_j \text{ for some } j, \\ y_j & \text{if } x = b_j \text{ for some } j, \\ \vdots & \\ z_j & \text{if } x = c_j \text{ for some } j, \\ l_j & \text{if } x = k_j \text{ for some } j, \end{cases}$$

is such that $\sigma\rho_1\sigma^{-1} = \rho_2$. □

What happens with the alternating group?

11.7. Lemma. *If $\rho_1, \rho_2 \in \mathbb{S}_n$ are conjugate in $\mathbb{S}_n$ and $|\mathrm{Fix}(\rho_1)| \geq 2$, then $\mu\rho_1\mu^{-1} = \rho_2$ for some $\mu \in \mathbb{A}_n$.*

Proof. Assume that $\rho_2 = \sigma\rho_1\sigma^{-1}$ for some $\sigma \in \mathbb{S}_n$. There are $a, b \in \{1, \dots, n\}$ such that $\rho_1(a) = a$, $\rho_1(b) = b$ and $a \neq b$. Let

$$\mu = \begin{cases} \sigma & \text{if } \sigma \in \mathbb{A}_n, \\ \sigma(ab) & \text{otherwise.} \end{cases}$$

Then $\mu \in \mathbb{A}_n$ and $\mu\rho_1\mu^{-1} = \rho_2$, as (ab) commutes with ρ_1. □

Let us discuss some examples.

11.8. Example. If $\rho_1 = (23)(156)$ and $\rho_2 = (45)(123)$, then $\rho_2 = \sigma\rho_1\sigma^{-1}$ for

$$\sigma = \begin{pmatrix} 123456 \\ 145623 \end{pmatrix}.$$

11.9. Example. The permutations $\rho_1 = (123) \in \mathbb{S}_3$ and $\rho_2 = (132) \in \mathbb{S}_3$ are conjugate in $\mathbb{S}_3$, as $(123) = \sigma(132)\sigma^{-1}$ if $\sigma = (23)$. However, ρ_1 and ρ_2 are not conjugate in $\mathbb{A}_3$.

Now we are ready to prove the theorem.

Proof of Theorem 11.5. Let $N \neq \{\mathrm{id}\}$ be a normal subgroup of $\mathbb{A}_n$. If $(abc) \in N$, then every 3-cycle belongs to N, because all 3-cycles are conjugate in $\mathbb{S}_n$, and the previous lemma states that $(ijk) = \mu(abc)\mu^{-1} \in N$ for some $\mu \in \mathbb{A}_n$. Thus $N = \mathbb{A}_n$.

We claim that N contains a 3-cycle. Since $N \neq \{\mathrm{id}\}$, there exists $\sigma \in N \setminus \{\mathrm{id}\}$. Let $m = |\sigma|$ and let p be a prime number dividing m. Then $\tau = \sigma^{m/p}$ has order p and hence $\tau = \rho_1 \cdots \rho_s$, where the ρ_j's are disjoint p-cycles.

If $p = 2$, then $1 = \mathrm{sign}(\tau) = (-1)^s$. Thus s is even. Write

$$\tau = (ab)(cd)\rho_3 \cdots \rho_s.$$

Since $\rho_3 \cdots \rho_s$ commutes with (abc) and (acb),

$$\underbrace{(abc)\tau(abc)^{-1}\tau^{-1}}_{\in N} = (abc)(ab)(cd)(acb)(ab)(cd) = (ac)(bd).$$

Hence $(ac)(bd) \in N$. Let $e \in \{1, \dots, n\} \setminus \{a, b, c, d\}$. Then

$$(ae)(bd) = (aec)\underbrace{(ac)(bd)}_{\in N}(aec)^{-1} \in N$$

and therefore

$$(aec) = (ac)(ae) = (ac)(bd)(ae)(bd) \in N.$$

If $p = 3$, without loss of generality, we may assume that $s \geq 2$ (otherwise, τ would be a 3-cycle). Then $\tau = (abc)(def)\rho_3 \cdots \rho_s$. Since (bcd) commutes with $\rho_3 \cdots \rho_s$ and N is normal in $\mathbb{A}_n$,

$$\underbrace{(bcd)\tau(bcd)^{-1}\tau^{-1}}_{\in N} = (bcd)(abc)(def)(bdc)(acb)(dfe) = (adbce)$$

and therefore

$$(adc) = (adb)(adbce)(adb)^{-1}(adbce)^{-1} \in N.$$

If $p > 3$, then $\tau = (abcd \cdots z)\rho_2 \cdots \rho_s$. In particular, (abc) commutes with $\rho_2 \cdots \rho_s$. Then

$$(abd) = (abc)\tau(abc)^{-1}\tau^{-1} \in N.$$

□

As an application, we compute the normal subgroups of the symmetric group $\mathbb{S}_n$.

11.10. Exercise. Compute the list of normal subgroups of $\mathbb{S}_n$ for $n \geq 2$.

§ 11.2. Radical extensions.

11.11. Definition. An extension E/K is said to be **pure** of type m if $E = K(x)$ for some x such that $x^m \in K$.

Note that if $E = K(x)$ is a pure extension of type m and K contains m-th roots of one, then E/K is a splitting field of $X^m - x^m$.

11.12. Definition. The sequence $K = R_0 \subseteq R_1 \subseteq \cdots \subseteq R_m$ of fields is said to be a **radical tower** if each R_{i+1}/R_i is pure. In this case, R_m/K is a **radical extension**.

Note that radical extensions are finite.

11.13. Example. Let E be a decomposition field of X^4-2 over $\mathbb{Q}$. Then $E/\mathbb{Q}$ is radical, as $E=\mathbb{Q}(\sqrt[4]{2},i)$.

11.14. Example. Let $\alpha,\beta\in\mathbb{C}$ be such that $\alpha^2=2$ and $\beta^5=1+\alpha$. The number $\sqrt[5]{1+\sqrt{2}}$ belongs to the radical extension $\mathbb{Q}(\alpha,\beta)/\mathbb{Q}$.

11.15. Theorem. *Let K be of characteristic zero and R/K be a radical extension. If E/K is a subextension of R/K, then* $\operatorname{Gal}(E/K)$ *is solvable.*

Proof. Without loss of generality, we may assume that E/K is a Galois extension. To prove this fact, let $G=\operatorname{Gal}(E/K)$ and $F={}^{G}E$. Then E/F is a Galois extension and $\operatorname{Gal}(E/F)=G$ by Artin's theorem. Thus, replacing K by F if needed, we may assume that E/K is Galois.

Let L be the normal closure of R in some algebraic closure C that contains R. Note that if $R=K(x_1,\dots,x_m)$, then

$$L=K(\{\sigma_i(x_j):1\leq i\leq s,\,1\leq j\leq m\}),$$

where $\operatorname{Hom}(R/K,C/K)=\{\sigma_1,\dots,\sigma_s\}$.

Claim. L/K is radical.

Since $x_j^{a_j}\in K(x_1,\dots,x_{j-1})$ for some integer a_j,

$$\sigma_i(x_j)^{a_j}=\sigma_i\left(x_j^{a_j}\right)\in\sigma_i(K(x_1,\dots,x_{j-1})=K(\sigma_i(x_1),\dots,\sigma_i(x_{j-1}))$$

Thus L/K is radical and Galois.

We may assume then that E/K and R/K are both Galois.

Since $\operatorname{Gal}(E/K)\simeq\operatorname{Gal}(R/K)/\operatorname{Gal}(R/E)$, we only need to prove that $\operatorname{Gal}(R/K)$ is solvable.

For a positive integer n, let ξ be a primitive n-th root of one (in some algebraic closure of K that contains R). Consider the diagram

$$\begin{tikzcd} & R(\xi) \arrow[dl, dash] \arrow[dr, dash] & \\ R \arrow[dr, dash] & & K(\xi) \arrow[dl, dash] \\ & K & \end{tikzcd}$$

Then

1) $K(\xi)/K$ and $R(\xi)/R$ are abelian.
2) $R(\xi)/K$ is Galois.
3) $\operatorname{Gal}(R/K)\simeq\operatorname{Gal}(R(\xi)/K)/\operatorname{Gal}(R(\xi)/R)$.
4) $\operatorname{Gal}(K(\xi)/K)\simeq\operatorname{Gal}(R(\xi)/K)/\operatorname{Gal}(R(\xi)/K(\xi))$.

The third item implies that we need to show that $\operatorname{Gal}(R(\xi)/K)$ is solvable. By the fourth item, it suffices to show that $\operatorname{Gal}(R(\xi)/K(\xi))$ is solvable (because $\operatorname{Gal}(K(\xi)/K)$ is abelian and hence solvable).

Since $R=K(x_1,\dots,x_m)$,

$$R(\xi)=K(x_1,\dots,x_m,\xi)=K(\xi)(x_1,\dots,x_m)$$

and hence $R(\xi)/K(\xi)$ is radical. This means that without loss of generality, we may assume that K contains primitive n-roots of one. For example, if $R = K(x_1, \dots, x_m)$ and $x_i^{a_i} \in K(x_1, \dots, x_{i-1})$, then we may assume that K contains a primitive a_i-root of one. We proceed by induction on m. The case $m = 0$ is trivial. Assume that the claim holds for some $m \geq 0$. Let $L = K(x_1)$. Then L/K is a decomposition field of $X^{a_1} - x_1^{a_1}$, and hence L/K is a cyclic extension. Thus $\mathrm{Gal}(L/K)$ is cyclic (and hence, in particular, solvable). Let H be the subgroup that corresponds to L, that is $H = \mathrm{Gal}(R/L)$ (here, we use Galois' correspondence). Then H is normal in $\mathrm{Gal}(R/K)$. Since $R = K(x_1, \dots, x_m) = L(x_2, \dots, x_m)$, R/L is radical and Galois. By the inductive hypothesis, $\mathrm{Gal}(R/L)$ is solvable. Since

$$\mathrm{Gal}(L/K) \simeq \mathrm{Gal}(R/K)/\mathrm{Gal}(R/L),$$

it follows that $\mathrm{Gal}(R/K)$ is solvable. □

11.16. Definition. Let $f \in K[X]$ and E be a decomposition field of f over K. We say that f is **solvable by radicals** if there is a radical extension R/K such that $E \subseteq R$.

The general polynomial of degree two is solvable by radicals, as its Galois group is solvable (in fact, isomorphic to $\mathbb{S}_2$).

11.17. Exercise. Prove that $f = X^2 - s_1X + s_2 \in \mathbb{Q}[X]$ is solvable by radicals.

Theorem 11.15 translates into the following result:

11.18. Exercise. Let K be a field of characteristic zero. If $f \in K[X]$ is solvable by radicals, then $\mathrm{Gal}(f, K)$ is solvable.

As a consequence, the general polynomial of degree $n \geq 5$ is not solvable by radicals, as its Galois group is isomorphic to $\mathbb{S}_5$.

11.19. Example. Let p be a prime number and $f = X^5 - 2pX + p \in \mathbb{Q}[X]$. We claim that f is not solvable by radicals.

By Gauss' theorem, one proves that f has no rational roots.

Note that $f' = 5X^4 - 2p$. Then $\alpha = \sqrt[4]{2p/5}$ and $\beta = -\sqrt[4]{2p/5}$ are are critical points. Since $f(\alpha) < 0$ and $f(\beta) > 0$, it follows that f has exactly three real roots. Let $x_1, x_2 \in \mathbb{C} \setminus \mathbb{R}$ and $x_3, x_4, x_5 \in \mathbb{R}$ be the roots of f.

By Eisenstein's theorem, f is irreducible.

Let $E/\mathbb{Q}$ be a decomposition field of f. Then $\mathrm{Gal}(f, \mathbb{Q}) = \mathrm{Gal}(E/\mathbb{Q})$ is isomorphic to a subgroup G of $\mathbb{S}_5$. Since f is irreducible, 5 divides $[E : \mathbb{Q}] = |G|$. In particular, by Cauchy's theorem, G contains an element σ of order five. This element is a 5-cycle, so without loss of generality, we may assume that $\sigma = (x_1x_2x_3x_4x_5)$. Note that $(x_1x_2) \in G$. Thus $G \simeq \mathbb{S}_5$ and hence G is not solvable.

11.20. Exercise. Let $f = X^6 + 2X^5 - 5X^4 + 9X^3 - 5X^2 + 2X + 1 \in \mathbb{Q}[X]$. Prove that f is solvable by radicals.

It is now time to prove Galois' great theorem on solvability of polynomials.

11.21. Theorem (Galois). *Let K be a field of characteristic zero and $f \in K[X]$. Then f is solvable by radicals if and only if $\mathrm{Gal}(f, K)$ is solvable.*

We proved in Theorem 11.15 that solvable polynomials have solvable Galois groups. For the converse, we need two auxiliary results.

11.22. LEMMA. *Let E/K be a Galois extension of prime degree p. Assume that K admits a primitive p-root of one. Then $E = K(\beta)$ where $\beta^p \in K$.*

PROOF. Assume that $\operatorname{Gal}(E/K) = \langle\sigma\rangle$. Let $\omega \in K$ be a primitive p-root of one. Then $\operatorname{norm}_{E/K}(\omega) = \omega^p = 1$. By Hilbert's theorem, $\omega = \beta/\sigma(\beta)$ for some $\beta \in E$. Note that $\beta \notin K$, as $\omega \neq 1$. Moreover,

$$\sigma(\beta^p) = (\beta\omega^{-1})^p = \beta^p \in {}^{\operatorname{Gal}(E/K)}E = K.$$

Since $K \subseteq K(\beta) \subseteq E$ and $[E : K] = p$, we conclude that $E = K(\beta)$ with $\beta^p \in K$. ☐

11.23. EXERCISE. Let E/K be a decomposition field of $f \in K[X]$ and K^*/K be an extension. If E^*/K^* is a decomposition field of f containing E, then

$$\operatorname{Gal}(E^*/K^*) \to \operatorname{Gal}(E/K), \quad \sigma \mapsto \sigma|_E,$$

is an injective group homomorphism.

Now Theorem 11.21 will follow from the following theorem.

11.24. THEOREM. *Let K be a field of characteristic zero and E/K be a Galois extension. If $\operatorname{Gal}(E/K)$ is solvable, then E can be embedded in a radical extension.*

PROOF. Let $G = \operatorname{Gal}(E/K)$. Since G is solvable, by Proposition 10.25, there exists a normal subgroup H of G of prime index p. Let ω be a primitive p-th root of one. (It exists because K is a field of characteristic zero.)

We proceed by induction on $[E : K]$.

If $[E : K] = 1$, there is nothing to prove. So assume that $[E : K] > 1$.

We first assume that $\omega \in K$. The group $\operatorname{Gal}(E/{}^HE)$ is solvable, as it is a subgroup of G. Moreover, since

$$[E : {}^HE] < [E : K],$$

the inductive hypothesis implies that ${}^HE/E$ can be embedded in a radical extension, so there exists a radical tower is

$$ {}^HE \subseteq R_1 \subseteq R_2 \subseteq \cdots \subseteq R_m, \tag{11.1}$$

where $E \subseteq R_m$. Now $E/{}^HE$ is a Galois extension, as H is normal in G. Moreover,

$$[{}^HE : K] = (G : H) = p.$$

Since $\omega \in K$, Lemma 11.22 implies that ${}^HE = K(\beta)$ for some β such that $\beta^p \in K$. The radical tower 11.1 can be extended by adding $K \subseteq {}^HE$.

For the general case, let $K^* = K(\omega)$ and $E^* = E(\omega)$. Then E^*/K^* is a Galois extension with Galois group $\operatorname{Gal}(E^*/K^*)$. By Exercise 11.23, $\operatorname{Gal}(E^*/K^*)$ is solvable. By the previous part, E^* and E can be embedded in a radical extension R^*/K^*, so there exists a radical tower

$$K^* \subseteq R_1^* \subseteq R_2^* \subseteq \cdots \subseteq R_n^*. \tag{11.2}$$

Since $K^* = K(\omega)$ is a pure extension, the radical tower (11.2) can be extended by adding $K \subseteq K^*$. ☐

12. Lecture – Week 12

§ 12.1. Group cohomology. Let G be a group and A be a **(left) G-module**. This means that A is an abelian group together with a map

$$G \times A \to A, \quad (g,a) \mapsto g \cdot a$$

such that $1 \cdot a = a$ for all $a \in A$, $(gh) \cdot a = g \cdot (h \cdot a)$ for all $g, h \in G$ and $a \in A$ and $g \cdot (a+b) = g \cdot a + g \cdot b$ for all $g \in G$ and $a, b \in A$.

12.1. Example. The group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on $\mathbb{C}$ and $\mathbb{C}^\times$. Moreover, it acts trivially on $\mathbb{R}$ and $\mathbb{R}^\times$.

More generally, if E/K is a finite Galois extension, then the Galois group $\mathrm{Gal}(E/K)$ acts on E and $E^\times$.

12.2. Definition. Let G be a group and M and N be G-modules. A map $f \colon M \to N$ is a **homomorphism** of G-modules if $f(\sigma \cdot m) = \sigma \cdot f(m)$ for all $m \in M$ and $\sigma \in G$.

12.3. Definition. Let G be a group and M be a G-module. The submodule of **G-invariants** is defined as

$$M^G = \{m \in M : \sigma \cdot m = m \text{ for all } \sigma \in G\}.$$

Note that M^G is the largest submodule of the G-module M where G acts trivially. For example, if $G = \mathrm{Gal}(E/K)$, then $E^G = K$.

12.4. Proposition. *Let G be a group. If the sequence of G-modules and G-module homomorphism*

$$0 \longrightarrow P \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$$

is exact, then

$$0 \longrightarrow P^G \xrightarrow{\alpha^0} M^G \xrightarrow{\beta^0} N^G$$

is exact, where α^0 is the restriction $\alpha|_{P^G}$ of α to P^G and β^0 is the restriction $\beta|_{M^G}$ of β to M^G.

Proof. Since α is injective, the restriction α^0 is injective.

Note that $\ker \beta^0 = \ker \beta \cap M^G \subseteq \ker \beta$.

We claim that $\alpha^0(P^G) = \alpha(P) \cap M^G$. If $m \in \alpha(P) \cap M^G$, then $\alpha(p) = m$ for some $p \in P$ and $\sigma \cdot m = m$. Since

$$\alpha(p) = m = \sigma \cdot m = \sigma \cdot \alpha(p) = \alpha(\sigma \cdot p),$$

$\sigma \cdot p - p \in \ker \alpha = \{0\}$. Hence $\sigma \cdot p = p$ and $p \in P^G$. Conversely, if $m \in \alpha^0(P^G)$, then $m = \alpha(p)$ for some $p \in P^G$. If $\sigma \in G$, then

$$\sigma \cdot m = \sigma \cdot \alpha(p) = \alpha(\sigma \cdot p) = \alpha(p) = m.$$

Hence $m \in M^G \cap \alpha(P)$.

Now

$$\alpha^0(P^G) = \alpha(P) \cap M^G = \ker \beta \cap M^G = \ker \beta^0. \qquad \square$$

Note that in the previous proposition, we did not prove that the map $\beta|_{M^G}$ is surjective.

12.5. Example. Let $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Consider the following exact sequence of G-modules:

$$1 \longrightarrow \{-1,1\} \longrightarrow \mathbb{C}^\times \xrightarrow{\beta} \mathbb{C}^\times \longrightarrow 1$$

where $\beta(z) = z^2$. Note that β is surjective. Take invariants to obtain the sequence

$$0 \longrightarrow \{-1,1\} \longrightarrow \mathbb{R}^\times \xrightarrow{\beta^0} \mathbb{R}^\times$$

where $\beta^0(x) = x^2$. Note that β^0 is not surjective!

12.6. Definition. Let G be a group and N be a G-module. We define

$H^0(G,M) = M^G$,
$C^1(G,M) = \{\phi\colon G \to M : \phi \text{ is a map}\}$,
$Z^1(G,M) = \{\phi \in C^1(G,M) : \phi(\sigma\tau) = \phi(\sigma) + \sigma\cdot\phi(\tau) \text{ for all } \sigma,\tau \in G\}$,

Note that $Z^1(G,M)$ is an abelian group with the operation

$$(\phi + \phi_1)(\sigma) = \phi(\sigma) + \phi_1(\sigma).$$

Moreover, if $\phi \in Z^1(G,M)$, then $\phi(1_G) = 0_M$. To prove this fact, note that

$$\phi(1_G) = \phi(1_G 1_G) = \phi(1_G) + 1_G \cdot \phi(1_G) = \phi(1_G) + \phi(1_G)$$

implies that $\phi(1_G) = 0_M$.

12.7. Example. Let G be a group and M be a G-module. Fix $m \in M$. Then the map $\phi\colon G \to M$, $\phi(\sigma) = \sigma\cdot m - m$, is an element of $Z^1(G,M)$, because

$$\begin{aligned}\phi(\sigma\tau) &= (\sigma\tau)\cdot m - m\\ &= (\sigma\tau)\cdot m - \sigma\cdot m + \sigma\cdot m - m\\ &= \sigma\cdot(\tau\cdot m - m) + \sigma\cdot m - m\\ &= \sigma\cdot\phi(\tau) + \phi(\sigma)\end{aligned}$$

for all $\sigma,\tau \in G$.

12.8. Definition. Let G be a group and M be a G-module. The set $B^1(G,M)$ of **coboundaries** is the set of elements $\phi \in C^1(G,M)$ such that there is a fixed $m \in M$ such that $\phi(\sigma) = \sigma\cdot m = m$ for all $\sigma \in G$.

We proved in Example 12.7 that $B^1(G,M) \subseteq Z^1(G,M)$. A direct calculation shows that, in fact, $B^1(G,M)$ is a subgroup of $Z^1(G,M)$.

12.9. Definition. Let G be a group and M be a G-module. The **first cohomology group** of G with coefficients in M is defined as the quotient

$$H^1(G,M) = Z^1(G,M)/B^1(G,M).$$

12.10. Example. If G acts trivially on M, then

$$H^0(G,M) = M^G = M, \quad B^1(G,M) = \{0\}, \quad Z^1(G,M) = \mathrm{Hom}(G,M).$$

Hence $H^1(G,M) \simeq \mathrm{Hom}(G,M)$.

12.11. Example. Let $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{\mathrm{id}, \gamma\}$, where $\gamma\colon \mathbb{C} \to \mathbb{C}$, $z \mapsto \overline{z}$, is the complex conjugation. Then

$$H^0(G, \mathbb{R}^\times) = \left(\mathbb{R}^\times\right)^G = \mathbb{R}^\times.$$

Since G acts trivially on $\mathbb{R}^\times$,

$$H^1(G, \mathbb{R}^\times) = \mathrm{Hom}(G, \mathbb{R}^\times) \simeq \mathrm{Hom}(G, \{-1, 1\}) \simeq \mathbb{Z}/2.$$

The following lemma will be useful.

12.12. Lemma. *Let G be a group and $\alpha\colon M \to N$ be a homomorphism of G-modules. Then*

$$\alpha^1\colon H^1(G, M) \to H^1(G, N), \quad \phi + B^1(G, M) \mapsto \alpha \circ \phi + B^1(G, N),$$

is a group homomorphism.

Proof. Let us prove that the map α^1 is well-defined. If $\phi - \phi' \in B^1(G, M)$, then there exists a fixed $m \in M$ such that $(\phi - \phi')(\sigma) = \sigma \cdot m - m$ for all $\sigma \in G$. Let $n = \alpha(m) \in N$. For $\sigma \in G$,

$$\alpha((\phi - \phi')(\sigma)) = \alpha(\sigma \cdot m - m) = \sigma \cdot \alpha(m) - \alpha(m) = \sigma \cdot n - n.$$

Thus $\alpha \circ \phi - \alpha \circ \phi' \in B^1(G, N)$.

We now prove that α^1 is a group homomorphism. If $\phi, \phi' \in Z^1(G, M)$, then

$$\begin{aligned}
\alpha^1(\phi + B^1(G, M) + \phi' + B^1(G, M)) &= \alpha^1(\phi + \phi' + B^1(G, M)) \\
&= \alpha \circ (\phi + \phi') + B^1(G, N) \\
&= \alpha \circ \phi + \alpha \circ \phi' + B^1(G, N) \\
&= \alpha \circ \phi + B^1(G, N) + \alpha \circ \phi' + B^1(G, N) \\
&= \alpha^1(\phi + B^1(G, M)) + \alpha^1(\phi' + B^1(G, M)). \qquad \square
\end{aligned}$$

We will provide a detailed proof of the upcoming result. The theorem will be established by applying a **diagram chasing** technique, a widely used tool in homological algebra. The proof is tedious and may seem intricate, but the method essentially involves “chasing” elements around a (commutative) diagram until we attain the desired result.

12.13. Theorem. *Let G be a group and*

$$0 \longrightarrow P \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$$

*be an exact sequence of G-modules and G-module homomorphism. Then there exists a **connection homomorphism** δ such that the sequence*

$$\begin{tikzcd}
0 \arrow[r] & H^0(G, P) \arrow[r, "\alpha^0"] & H^0(G, M) \arrow[r, "\beta^0"] & H^0(G, N) \arrow[dll, rounded corners, "\delta"] \\
& H^1(G, P) \arrow[r, "\alpha^1"] & H^1(G, M) \arrow[r, "\beta^1"] & H^1(G, N)
\end{tikzcd} \tag{12.1}$$

of abelian groups and group homomorphisms is exact.

Proof. By Proposition 12.4, the sequence is exact at $H^0(G, P) = P^G$, $H^0(G, M) = M^G$ and $H^0(G, N) = N^G$. Note that, in particular, $\alpha\colon P \to \alpha(P)$ is a bijective group homomorphism.

Let us construct the connecting homomorphism $\delta\colon H^0(G,N)\to H^1(G,P)$. For $n\in N^G$, let $m\in M$ be such that $\beta(m)=n$. We define $\delta(n)=\phi+B^1(G,P)$, where

$$\phi(\sigma)=\alpha^{-1}(\sigma\cdot m-m).$$

Note that $\sigma\cdot m-m\in\operatorname{im}\alpha=\ker\beta$, as

$$\beta(\sigma\cdot m-m)=\sigma\cdot\beta(m)-\beta(m)=\sigma\cdot n-n=0.$$

Let us prove that the map δ is well-defined: if $m,m'\in M$ are such that $\beta(m)=\beta(m')=n$, then $m-m'\in\ker\beta=\alpha(P)$. For $\sigma\in G$, write $\phi'(\sigma)=\sigma\cdot m'-m'$. Since $m-m'=\alpha(p)$ for some $p\in P$ and α^{-1} is a homomorphism of G-modules,

$$\begin{aligned}\phi(\sigma)-\phi'(\sigma)&=\alpha^{-1}(\sigma\cdot m-m)-\alpha^{-1}(\sigma\cdot m'-m')\\&=\alpha^{-1}(\sigma\cdot(m-m'))-\alpha^{-1}(m-m')\\&=\alpha^{-1}(\sigma\cdot\alpha(p)-\alpha(p))\\&=\sigma\cdot p-p.\end{aligned}$$

Thus $\phi-\phi'\in B^1(G,P)$.

Note that $\phi\in Z^1(G,P)$, because

$$\begin{aligned}\phi(\sigma\tau)&=\alpha^{-1}((\sigma\tau)\cdot m-m)\\&=\alpha^{-1}((\sigma\tau)\cdot m-\sigma\cdot m+\sigma\cdot m-m)\\&=\alpha^{-1}(\sigma\cdot(\tau\cdot m-m))+\alpha^{-1}(\sigma\cdot m-m)\\&=\sigma\cdot\phi(\tau)+\phi(\sigma)\end{aligned}$$

holds for all $\sigma,\tau\in G$.

We now prove that the sequence (12.1) is exact at $H^0(G,N)=N^G$. We need to prove that $\ker\delta=\operatorname{im}\beta^0$. To prove $\supseteq$ note that if $m\in M^G$ is such that $\delta(\beta(m))=\phi+B^1(G,P)$, then

$$\phi(\sigma)=\alpha^{-1}(\sigma\cdot m-m)=0.$$

Conversely, if $n\in\ker\delta$, then there exists $m\in M$ such that $\beta(m)=n$ and $\delta(\beta(m))=\phi+B^1(G,P)$, where $\phi\in B^1(G,P)$ and $\phi(\sigma)=\alpha^{-1}(\sigma\cdot m-m)$ for all $\sigma\in G$. Since $\phi\in B^1(G,P)$, there exists $p\in P$ such that $\phi(\sigma)=\sigma\cdot p-p$ for all $\sigma\in G$. Note that

$$\beta(m-\alpha(p))=\beta(m)-\beta(\alpha(p))=\beta(m)=n.$$

Moreover, $m-\alpha(p)\in M^G$, as $\sigma\cdot(m-\alpha(p))=m-\alpha(p)$. Hence $n\in\operatorname{im}\beta^0$.

We now prove that (12.1) is exact at $H^1(G,P)$, that is $\operatorname{im}\delta=\ker\alpha^1$. To prove $\subseteq$ note that for $n\in N^G$, $\delta(n)=\phi+B^1(G,P)$, where $\phi(\sigma)=\alpha^{-1}(\sigma\cdot m-m)$ for all $\sigma\in G$ and some $m\in M$ such that $\beta(m)=n$. In particular, $\alpha\circ\phi\in B^1(G,M)$. Then

$$\alpha^1(\phi+B^1(G,P))=\alpha\circ\phi+B^1(G,M)=B^1(G,M).$$

To prove $\supseteq$, let $\phi+B^1(G,P)\in\ker\alpha^1$. Then $\alpha\circ\phi\in B^1(G,M)$, that is $\alpha(\phi(\sigma))=\sigma\cdot m-m$ for all $\sigma\in G$ and some $m\in M$. Note that

$$\delta(\beta(m))=\psi+B^1(G,P),$$

where $\psi(\sigma)=\alpha^{-1}(\sigma\cdot m-m)$. This implies that $\alpha(\psi(\sigma))=\alpha(\phi(\sigma))$ for all $\sigma\in G$. Since α is injective, $\psi=\phi$. Therefore $\phi+B^1(G,P)$ belongs to the image of δ.

Finally, we prove that the sequence (12.1) is exact at $H^1(G,M)$, that is $\operatorname{im}\alpha^1 = \ker\beta^1$. To prove $\subseteq$ note that

$$\beta^1(\alpha^1(\phi + B^1(G,P))) = \beta^1(\alpha\circ\phi + B^1(G,M)) = (\beta\circ\alpha)\circ\phi + B^1(G,N) = B^1(G,N).$$

Conversely, let $\phi + B^1(G,M) \in \ker\beta_1$. Then $\beta\circ\phi \in B^1(G,N)$. Thus there exists $n \in N$ such that $\beta(\phi(\sigma)) = \sigma\cdot n - n$ for all $\sigma \in G$. Since β is surjective, $n = \beta(m)$ for some $m \in M$. Hence

$$\beta(\phi(\sigma)) = \sigma\cdot n - n = \sigma\cdot\beta(m) - \beta(m) = \beta(\sigma\cdot m - m).$$

For each $\sigma \in G$, $\phi(\sigma) - (\sigma\cdot m - m) \in \ker\beta = \operatorname{im}\alpha$. and therefore $\phi(\sigma) - (\sigma\cdot m - m) = \alpha(\rho_\sigma)$. This defines a map $\rho\colon G \to P$, $\sigma \mapsto \rho_\sigma$. We claim that $\rho \in Z^1(G,P)$. If $\sigma,\tau \in G$, then

$$\begin{aligned}\alpha(\rho_{\sigma\tau}) &= \phi(\sigma\tau) - ((\sigma\tau)\cdot m - m)\\ &= \phi(\sigma) + \sigma\cdot\phi(\tau) - (\sigma\cdot(\tau\cdot m - m) + \sigma\cdot m - m)\\ &= \alpha(\rho_\sigma) + \sigma\cdot\alpha(\rho_\tau).\end{aligned}$$

Since α is injective, it follows that $\rho \in Z^1(G,P)$. Now

$$\alpha_1(\rho + B^1(G,P)) = \alpha\circ\rho + B^1(G,M) = \phi + B^1(G,M).$$ □

12.14. Theorem. *Let G be a finite group and M be a G-module. Then*

$$|G|H^1(G,M) = \{0\}.$$

Proof. Let $n = |G|$. It is enough to prove that if $\phi \in Z^1(G,M)$, then $n\phi \in B^1(G,M)$. Let $\phi \in Z^1(G,M)$ and $\sigma \in G$. Then

$$\phi(\sigma\tau) = \phi(\sigma) + \sigma\cdot\phi(\tau)$$

for all $\tau \in G$. Summing over all possible $\tau \in G$, we obtain that

$$\sum_{\tau\in G}\phi(\tau) = \sum_{\tau\in G}\phi(\sigma\tau) = \sum_{\tau\in G}\sigma\cdot\phi(\tau) + \sum_{\tau\in G}\phi(\sigma) = n\phi(\sigma). \tag{12.2}$$

Let $m = -\sum_{\tau\in G}\phi(\tau) \in M$. Then (12.2) can be rewritten as

$$-m = \sum_{\tau\in G}\phi(\tau) = \sigma\cdot\sum_{\tau\in G}\phi(\tau) + n\phi(\sigma) = -\sigma\cdot m + n\phi(\sigma).$$

Thus $n\phi(\sigma) = \sigma\cdot m - m$ and hence $n\phi \in B^1(G,M)$. □

12.15. Exercise. Let G be a finite group and M be a finite G-module of size coprime to $|G|$. Prove that $H^1(G,M) = \{0\}$.

12.16. Exercise. Let G be a finite group and M be a finitely generated G-module. Prove that $H^1(G,M)$ is finite.

Some solutions

1.3. Assume that $\mathbb{Q}[i]$ and $\mathbb{Q}[\sqrt{2}]$ were isomorphic and let $\varphi : \mathbb{Q}[i] \to \mathbb{Q}[\sqrt{2}]$ be a field isomorphism. Then

$$\varphi(i)^2 = \varphi(i^2) = \varphi(-1) = -\varphi(1) = -1.$$

But $\varphi(i) \in \mathbb{Q}[\sqrt{2}]$ and $\mathbb{Q}[\sqrt{2}] \subseteq \mathbb{R}$ where every square is positive, a contradiction.

1.4. Let $t > 0$ be the characteristic of a field K and let $\varphi : \mathbb{Z} \to K$, $x \mapsto x1$. Then, by definition, $\ker\varphi$ is the ideal generated by t. On the other hand, the image $\varphi(\mathbb{Z})$ is a domain, being a subring of a field and is isomorphic to $\mathbb{Z}/\ker\varphi$. Therefore, $\ker\varphi$ is a prime ideal of $\mathbb{Z}$, i.e. t is a prime number.

1.6. Let $\varphi : \mathbb{Z} \to K$, $x \mapsto x1$.

We first prove that **1)** implies all the other properties. So suppose that the characteristic of K is zero, i.e. $\ker\varphi = \{0\}$.

Then $m1 = 0$ if and only if $m = 0$, i.e. the order of 1 is infinite.

Let $0 \neq x \in K$. If $mx = 0$, then $0 = mx = (m1)x$. But K is a field and $x \neq 0$, hence $m1 = 0$, so $m \in \ker\varphi = \{0\}$. Hence the order of x is infinite.

By definition, the ring of integers of K is the image of φ, which so it is isomorphic to $\mathbb{Z}/\ker\varphi = \mathbb{Z}$.

Finally, we prove that **4)** implies **1)**. Take $m \in \ker\varphi$. Then $m1 = 0$, but 1 has infinite order, hence $m = 0$. Therefore $\ker\varphi = \{0\}$, i.e. K has characteristic 0.

1.7. Let $\varphi : \mathbb{Z} \to K$, $x \mapsto x1$.

We first prove that **1)** implies all the other properties. So suppose that the characteristic of K is $p > 0$ i.e. $\ker\varphi$ is the ideal generated by p.

Then $m1 = 0$ if and only if p divides m, i.e. the order of 1 is p.

Let $0 \neq x \in K$. If $mx = 0$, then $0 = mx = (m1)x$. But K is a field and $x \neq 0$, hence $m1 = 0$, so p divides m. Hence x has order p.

By definition, the ring of integers of K is the image of φ, which so it is isomorphic to $\mathbb{Z}/\ker\varphi \cong \mathbb{Z}/p$.

Finally, we prove that **4)** implies **1)**. Take $m \in \ker\varphi$. Then $m1 = 0$, but 1 has order p, hence p divides m. Therefore $\ker\varphi$ is generated by p, i.e. K has characteristic 0.

1.10. Let $\Phi : K \to K$ be the map $x \mapsto x^p$. Since the map $x \mapsto x^{p^n}$ is exactly Φ^n, it is enough to prove that Φ is a field homomorphism.

As K is commutative under multiplication, for all $x, y \in K$

$$\Phi(xy) = (xy)^p = x^p y^p = \Phi(x)\Phi(y).$$

Moreover, for all $x, y \in K$

$$\Phi(x+y) = (x+y)^p \sum_{k=0}^{p} \binom{p}{k} x^p y^{p-k} = x^p + y^p + \sum_{k=1}^{p-1} \binom{p}{k} x^k y^{p-k},$$

where $\binom{p}{k} = \frac{p!}{k!(p-k)!}$, which can also be written as

$$p! = \binom{p}{k} \cdot k! \cdot (p-k)!$$

But p divides $p!$, so p has to divide at least one factor on the right side. But p doesn't divide i for $1 \leq i \leq p-1$, therefore if $k \leq p-1$, p doesn't divide $k!$ and if $1 \leq k$, p doesn't divide $(p-k)!$. Hence, if $1 \leq k \leq p-1$, p has to divide $\binom{p}{k}$ and

$$\sum_{k=1}^{p-1} \binom{p}{k} x^k y^{p-k} = 0.$$

Therefore, Φ is a field homomorphism.

1.22. By definition $K_0 = \{m1 \colon m \in \mathbb{Z}\}$ and $\sigma : K \to K$ is a field homomorphism, so $\sigma(1) = 1$. Hence, for every $m \in \mathbb{Z}$

$$\sigma(m1) = m\sigma(1) = m1,$$

i.e. $\sigma|_{K_0}$ is the identity.

1.25. If $X^3 - 2$ were reducible, since it has degree 3, it would have a linear factor in the decomposition in irreducibles. Therefore it would have a rational root. But the roots of $X^3 - 2$ are $\sqrt[3]{2}$, $\sqrt[3]{2}\xi$, $\sqrt[3]{2}\xi^2$, where ξ is a primitive third root of unity, So all the roots are not in $\mathbb{Q}$, a contradiction.

1.26. Recall first the following:

12.17. Lemma (Gauss' Lemma). *Let A be a unique factorization domain and K be its fraction field. A non-constant polynomial $f \in A[X]$ is irreducible if and only if it is primitive and irreducible in $K[X]$.*

Suppose that f is reducible in $K[X]$. Then $g = c^{-1}f$, where c is the content of f, would be reducible and primitive. Hence, by Gauss' Lemma, g is also reducible in $A[X]$. So $c^{-1}f = g = hl$, for some non-constant polynomials $h, l \in A[X]$. Now consider $\pi : A \to A/(p)$, $a \mapsto \overline{a}$ the natural surjection. We know that $\overline{a_i} = 0$ for all $i \in \{0, 1, \dots, n-1\}$ and $\overline{a_n} \neq 0$. Therefore

$$\overline{\pi}(ch)\overline{\pi}(l) = \overline{c}\,\overline{\pi}(h)\overline{\pi}(l) = \overline{\pi}(f) = \overline{a_n}X^n \in A/(p)[X].$$

But $A/(p)[X]$ is a UFD so the only possibility is that $\overline{\pi}(ch) = \overline{d}X^t$ and $\overline{\pi}(l) = \overline{f}X^s$, for some $f, d \in A/(p) \setminus \{\overline{0}\}$ and $t, s \in \{1, \dots, n-1\}$. In particular, $\overline{\pi}(ch)$ and $\overline{\pi}(l)$ have both constant term equal to 0. Hence p divides $ch(0)$ and $l(0)$ in A. Therefore p^2 divides $ch(0)l(0) = f(0)$, a contradiction.

1.27. It is easy to see that f satisfies the Eisenstein criterion for $p = 2$ and g satisfies it for $p = 5$.

1.28. $f = 3(X^{10} + 5X^2 - 15)$ is a product of 3 and $(X^{10} + 5X^2 - 15)$, which are both non-invertible elements of $\mathbb{Z}[X]$. Hence f is reducible.

2.6. Clearly for every field extension L/K and every $\alpha \in L$ we have that $K[\alpha] \subseteq K(\alpha)$. Vice versa take $\frac{a+\sqrt{2}b}{c+\sqrt{2}d} \in \mathbb{Q}(\sqrt{2})$, then we can write:

$$\frac{a+\sqrt{2}b}{c+\sqrt{2}d} = \frac{(a+\sqrt{2}b)(c-\sqrt{2}d)}{(c+\sqrt{2}d)(c-\sqrt{2}d)} = \frac{ac - 2bd + (bc - ad)\sqrt{2}}{c^2 - 2d^2}.$$

Hence

$$\frac{a+\sqrt{2}b}{c+\sqrt{2}d} = \frac{ac - 2bd}{c^2 - 2d^2} + \frac{bc - ad}{c^2 - 2d^2}\sqrt{2} \in \mathbb{Q}[\sqrt{2}].$$

2.8. Let $f = f(x,K)$ be the minimal polynomial of x over K of degree $\deg(f) = n$. We claim that $\{1, x, \dots, x^{n-1}\}$ is a basis of $K(x)$ as a K-vector space.

To prove that $\{1, x, \dots, x^{n-1}\}$ is a generating set, recall that $K(x) = K[x]$, since x is algebraic over K. Let $z \in K(x) = K[x]$, say $z = h(x)$ for some $h \in K[X]$. Divide h by f to obtain polynomials $q, r \in K[X]$ such that $h = fq + r$, where either $r = 0$ or $\deg r < \deg f = n$. Then

$$z = h(x) = f(x)q(x) + r(x) = r(x).$$

Write $r = \sum_{i=0}^{n-1} c_i X^i$ for some $c_0, \dots, c_{n-1} \in K$. Thus $z = \sum_{i=0}^{n-1} a_i x^i \in \langle 1, x, \dots, x^{n-1} \rangle$.

We now prove that $\{1, x, \dots, x^{n-1}\}$ is linearly independent. If not, there exists a linear combination $0 = \sum_{i=0}^{n-1} a_i x^i$ with $a_0, \dots, a_{n-1} \in K$ not all zero. Then $h(X) = \sum_{i=0}^{n-1} a_i X^i \in K[X] \setminus \{0\}$ has x as a root and

$$n = \deg(f) \leq \deg(h) \leq n - 1,$$

a contradiction.

2.10. a is algebraic over K, so, by Theorem 2.7, it has finite degree over K and $K[a] = K(a)$. b is algebraic over K, so it is also algebraic over $K(a)$, hence it has finite degree over $K(a)$ and $K(a)[b] = K(a,b)$. This implies that the extension $K(a,b)/K$ is a finite extension since it is a tower of finite extensions. Hence, by Corollary 2.9, $K(a,b)/K$ is an algebraic extension. Therefore, since $a+b, ab \in K(a,b)$, this implies that $a+b$ and ab are algebraic over K.

2.14. Assume that $K(S)/K$ is algebraic, then, by Corollary 2.12, x is algebraic over K for all $x \in S$. By Corollary 2.11, we conclude that $K(S)/K$ is finite.

On the other hand, if $K(S)/K$ is finite, then $K \subseteq K(x) \subseteq K(S)$ for all $x \in S$, so $K(x)/K$ is finite for all $x \in S$. Then, by Theorem 2.7, x is algebraic over K for all $x \in S$. Hence, by Corollary 2.12, $K(S)/K$ is algebraic.

2.15. $\sqrt[3]{2}$ is a root of the monic polynomial $f = X^3 - 2 \in \mathbb{Q}[X]$. Therefore $\sqrt[3]{2}$ is algebraic over $\mathbb{Q}$ and $\mathbb{Q}[\sqrt[3]{2}] = \mathbb{Q}(\sqrt[3]{2})$. In Exercise 1.25, we proved that f is irreducible in $\mathbb{Q}[X]$. Hence f is the minimal polynomial of $\sqrt[3]{2}$ over $\mathbb{Q}$ and, by Theorem 2.7, $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = \deg f = 3$.

2.16. i is a root of the monic polynomial $X^2 + 1 \in \mathbb{Q}[X]$ and $\sqrt{2}$ is a root of the monic polynomial of $X^2 - 2 \in \mathbb{Q}[X]$. So, by Corollary 2.11, $\mathbb{Q}[i, \sqrt{2}] = \mathbb{Q}(i, \sqrt{2})$ and it is algebraic over $\mathbb{Q}$. By Eisenstein's criterion with $p = 2$, $X^2 - 2$ is irreducible in $\mathbb{Q}[X]$, so $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$. Since i is a root of $X^2 + 1 \in \mathbb{Q}[X]$, then $[E : \mathbb{Q}(\sqrt{2})] \leq 2$. Moreover, $i \notin \mathbb{R} \supseteq \mathbb{Q}(\sqrt{2})$. Therefore $[E : \mathbb{Q}(\sqrt{2})] = 2$ and, by Proposition 2.4,

$$[E : \mathbb{Q}] = [E : \mathbb{Q}(\sqrt{2})][\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 4.$$

2.17.

1) We know that $[\mathbb{Q}(\sqrt[3]{5}) : \mathbb{Q}]$ is the same as the degree of the minimal polynomial of $\sqrt[3]{5}$ over $\mathbb{Q}$. Clearly $\sqrt[3]{5}$ is a root of $X^3 - 5 \in \mathbb{Q}[X]$. Moreover, by Eisenstein's criterion with $p = 5$, we get that $X^3 - 5$ is irreducible in $\mathbb{Q}[X]$. Hence $f(\sqrt[3]{5}, \mathbb{Q}) = X^3 - 5$ and $[\mathbb{Q}(\sqrt[3]{5} : \mathbb{Q}] = 3$. We also know that $X^2 - 2$ is the minimal polynomial of $\sqrt{2}$ over $\mathbb{Q}$. So $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$. Therefore we are in the following situation:

$$\begin{tikzcd}[column sep=small]
& E = \mathbb{Q}(\sqrt[3]{5}, \sqrt{2}) \arrow[dl, dash, "\leq 2"'] \arrow[dr, dash] & \\
\mathbb{Q}(\sqrt[3]{5}) \arrow[dr, dash, "3"'] & & \mathbb{Q}(\sqrt{2}) \arrow[dl, dash, "2"] \\
& \mathbb{Q} &
\end{tikzcd}$$

so on the one hand

$$[E:\mathbb{Q}] = [E:\mathbb{Q}(\sqrt{2})][\mathbb{Q}(\sqrt{2}):\mathbb{Q}] = [E:\mathbb{Q}(\sqrt{2})]2$$

and on the other hand

$$[E:\mathbb{Q}] = [E:\mathbb{Q}(\sqrt[3]{5})][\mathbb{Q}(\sqrt[3]{5}):\mathbb{Q}] = [E:\mathbb{Q}(\sqrt[3]{5})]3 \leq 2\cdot 3 = 6.$$

Therefore 2 and 3 divide $[EF:\mathbb{Q}] \leq 6$. Hence the only possibility is that $[E:\mathbb{Q}] = 6$.

2) Clearly $\mathbb{Q}(\sqrt{2}+\sqrt[3]{5}) \subseteq E$. On the other hand, let $\alpha = \sqrt{2}+\sqrt[3]{5}$. Then

$$5 = (\alpha - \sqrt{2})^3 = \alpha^3 - 3\sqrt{2}\alpha^2 + 6\alpha - 2\sqrt{2},$$

which implies that

$$\sqrt{2} = \frac{6\alpha - 5}{3\alpha^2 + 2} \in \mathbb{Q}(\alpha).$$

Moreover, $\sqrt[3]{5} = \alpha - \sqrt{2} \in \mathbb{Q}(\alpha)$, hence $E = \mathbb{Q}(\alpha)$.

3) From the previous part of this exercise we get that

$$\sqrt{2}(3\alpha^2 + 2) = \alpha^3 + 6\alpha - 5.$$

Hence, squaring both sides of the previous equality, we obtain

$$18\alpha^4 + 24\alpha^2 + 8 = \alpha^6 + 36\alpha^2 + 25 + 12\alpha^4 - 10\alpha^3 - 60\alpha.$$

Therefore α is a root of the polynomial

$$f(X) = X^6 - 6X^4 - 10X^3 + 12X^2 - 60X + 17.$$

Moreover, from the first part, we know that

$$[E:\mathbb{Q}] = 6 = \deg f.$$

Hence $f(\alpha, \mathbb{Q}) = f(X)$.

2.18. Let $\alpha = \sqrt[4]{3}i$. Observe that $\alpha^2 = -\sqrt{3}$, so $\mathbb{Q}(\sqrt{3}) \subseteq \mathbb{Q}(\alpha)$ and

$$2 \geq \deg f(\alpha, \mathbb{Q}(\sqrt{3})) = [\mathbb{Q}(\alpha):\mathbb{Q}(\sqrt{3})].$$

Moreover $\alpha \notin \mathbb{R}$, while $\mathbb{Q}(\sqrt{3}) \subseteq \mathbb{R}$. So $[\mathbb{Q}(\alpha):\mathbb{Q}(\sqrt{3})] > 1$, hence $[\mathbb{Q}(\alpha):\mathbb{Q}(\sqrt{3})] = 2$ and

$$f(\alpha, \mathbb{Q}(\sqrt{3})) = X^2 + \sqrt{3}.$$

Note that the minimal polynomial of α over $\mathbb{Q}(i)$ has degree $[\mathbb{Q}(\sqrt[4]{3}i, i):\mathbb{Q}(i)]$. Moreover, $\mathbb{Q}(\sqrt[4]{3}i, i) = \mathbb{Q}(\sqrt[4]{3}, i)$ and

$$f(\alpha, \mathbb{Q}) = f(\sqrt[4]{3}, \mathbb{Q}) = X^4 + 3,$$

since X^4+3 is an irreducible (due to Eisenstein with $p=3$) monic polynomial that has α and $\sqrt[4]{3}$ as roots. Therefore $[\mathbb{Q}(\sqrt[4]{3}):\mathbb{Q}] = [\mathbb{Q}(\sqrt[4]{3}i):\mathbb{Q}] = 4$. Since $\mathbb{Q}(\sqrt[4]{3}) \subseteq \mathbb{R}$, while $\sqrt[4]{3}i \notin \mathbb{R}$, we obtain that $1 < [\mathbb{Q}(\sqrt[4]{3}i, i) : \mathbb{Q}(\sqrt[4]{3})] \leq [\mathbb{Q}(i):\mathbb{Q}] = 2$.

$$\begin{tikzcd}[column sep=small]
& \mathbb{Q}(\sqrt[4]{3}i, i) = \mathbb{Q}(\sqrt[4]{3}, i) \arrow[dl, dash, "2"'] \arrow[dr, dash] & \\
\mathbb{Q}(\sqrt[4]{3}) \arrow[dr, dash, "4"'] & & \mathbb{Q}(i) \arrow[dl, dash, "2"] \\
& \mathbb{Q} &
\end{tikzcd}$$

Hence $[\mathbb{Q}(\sqrt[4]{3}i, i):\mathbb{Q}] = 8$ and $[\mathbb{Q}(\sqrt[4]{3}i, i):\mathbb{Q}(i)] = 4$, which means that $\deg f(\sqrt[4]{3}i, \mathbb{Q}(i)) = 4$. But $f(\sqrt[4]{3}i, \mathbb{Q}(i))$ divides $f(\alpha, \mathbb{Q}) = X^4+3$, so

$$f(\sqrt[4]{3}i, \mathbb{Q}(i)) = X^4 + 3.$$

2.19. Let $\alpha = \sqrt{2} + i\sqrt[3]{5}$ and $E = \mathbb{Q}(\alpha, i)$. The minimal polynomial of α over $\mathbb{Q}(i)$ has degree $[E:\mathbb{Q}(i)]$. Observe that, since $\alpha - \sqrt{2} = i\sqrt[3]{5}$,

$$\alpha^3 - 3\sqrt{2}\alpha^2 + 6\alpha - 2\sqrt{2} = -i5.$$

Hence

$$\sqrt{2} = \frac{\alpha^3\alpha^2 + 6\alpha + i5}{3\alpha^2 + 2} \in E$$

and so also $\sqrt[3]{5} = \frac{\alpha - \sqrt{2}}{i} \in E$. Therefore $E = \mathbb{Q}(\sqrt{2}, \sqrt[3]{5}, i)$. To compute $[E:\mathbb{Q}(i)]$ we first compute $[\mathbb{Q}(\sqrt{2}, i):\mathbb{Q}(i)]$ and $[\mathbb{Q}(\sqrt[3]{5}, i):\mathbb{Q}(i)]$.

We know that i has degree 2 over $\mathbb{Q}$, so $[\mathbb{Q}(\sqrt[3]{5}, i):\mathbb{Q}(\sqrt[3]{5})]$ and $[\mathbb{Q}(\sqrt{2}, i):\mathbb{Q}(\sqrt{2})]$ are both at most 2. Moreover $\mathbb{Q}(\sqrt[3]{5})$ and $Q(\sqrt{2})$ are contained in $\mathbb{R}$, while $i \notin \mathbb{R}$. Hence

$$[\mathbb{Q}(\sqrt[3]{5}, i):\mathbb{Q}(\sqrt[3]{5})] = 2 = [\mathbb{Q}(\sqrt{2}, i):\mathbb{Q}(\sqrt{2})].$$

$$\begin{tikzcd}[column sep=small]
& \mathbb{Q}(\sqrt{2}, i) \arrow[dl, dash, "2"'] \arrow[dr, dash] & & & \mathbb{Q}(\sqrt[3]{5}, i) \arrow[dl, dash] \arrow[dr, dash, "2"] & \\
\mathbb{R} \supseteq \mathbb{Q}(\sqrt{2}) \arrow[dr, dash, "2"'] & & \mathbb{Q}(i) \arrow[dl, dash, "2"] & \mathbb{Q}(i) \arrow[dr, dash, "2"'] & & \mathbb{Q}(\sqrt[3]{5}) \subseteq \mathbb{R} \arrow[dl, dash, "3"] \\
& \mathbb{Q} & & & \mathbb{Q} &
\end{tikzcd}$$

Therefore $[\mathbb{Q}(\sqrt{2}, i):\mathbb{Q}(i)] = 2$ and $[\mathbb{Q}(\sqrt[3]{5}, i):\mathbb{Q}(i)] = 3$.

$$\begin{tikzcd}[column sep=small]
 & \mathbb{Q}(\alpha, i) = \mathbb{Q}(\sqrt{2}, \sqrt[3]{5}, i) \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{Q}(\sqrt{2}, i) \arrow[dr, dash, "2"'] & & \mathbb{Q}(\sqrt[3]{5}, i) \arrow[dl, dash, "3"] \\
 & \mathbb{Q}(i) &
\end{tikzcd}$$

So $[\mathbb{Q}(\alpha, i) : \mathbb{Q}(i)]$ is divisible by 2 and 3 and it is also at most 6. Therefore the degree of the minimal polynomial of α over $\mathbb{Q}(i)$ is $[\mathbb{Q}(\alpha, i) : \mathbb{Q}(i)] = 6$.

We already got $\alpha^3 - 3\sqrt{2}\alpha^2 + 6\alpha - 2\sqrt{2} = -i5$, so $2(3\alpha^2 + 2)^2 = (\alpha^3\alpha^2 + 6\alpha + i5)^2$, i.e.

$$\alpha^6 - 6\alpha^4 + 10i\alpha^3 + 12\alpha^2 + 60i\alpha - 33 = 0.$$

This means that α is a root of the polynomial

$$f(X) = X^6 - 6X^4 + 10iX^3 + 12X^2 + 60iX - 33 \in \mathbb{Q}(i)[X].$$

Since f is also monic and of degree 6, we can deduce that $f(\alpha, \mathbb{Q}(i)) = f$.

2.21. By Proposition 2.4, we know that $[E : K] = [E : F][F : K]$, so $[E : K]$ is finite if and only if $[E : F]$ and $[F : K]$ are finite.

2.22. Let P be the set $\{\sum_{i=1}^m e_i f_i : m \in \mathbb{Z}_{>0}, e_i \in E, f_i \in F \text{ for all } i \in \{1, \dots, m\}\}$. If $\sum_{i=1}^m e_i f_i \in P$, it is a E-linear combination of elements in F, so in particular it is an element in $E(F) = EF$. Hence $P \subseteq EF$.

Moreover, since E/K and F/K are algebraic extensions, every element in $E \cup F$ is algebraic over K. So $EF = K(E \cup F) = K[E \cup F]$. Let $x \in EF$, then $x = f(\alpha_1, \dots, \alpha_k)$, for some polynomial $f \in K[X_1, \dots X_k]$ and $\alpha_1, \dots, \alpha_k \in E \cup F$. We can then split the polynomial in $f = p + q$ so that $x = p(e'_1, \dots, e'_n) + q(f'_1, \dots, f'_m)$, where $e'_i \in E$ and $f'_j \in F$ and $p, q \in K[X_1, \dots X_k]$. Since E and F are fields, so closed under multiplication, we can write x as $x = \sum_{i=1}^N k_i e_i + \sum_{j=1}^M h_j f_j$, for some $k_i, h_j \in K$, $e_i \in E$ and $f_j \in F$. Then in particular $k_i \in K \subseteq F$ and $h_j \in K \subseteq E$, hence $x \in P$ and $EF \subseteq P$.

2.23. We know that the minimal polynomials over $\mathbb{Q}$ are $f(\sqrt{2}, \mathbb{Q}) = X^2 - 2$ and $f(\sqrt{3}, \mathbb{Q}) = X^2 - 3$. So $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{3}) : \mathbb{Q}] = 2$. Moreover,

$$[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{3})] \leq [\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2.$$

It remains to check whether $\sqrt{2} \in \mathbb{Q}(\sqrt{3})$ or not. A $\mathbb{Q}$ basis of $\mathbb{Q}(\sqrt{3})$ is $\{1, \sqrt{3}\}$. If $\sqrt{2} \in \mathbb{Q}(\sqrt{3})$, then $\sqrt{2} = a + b\sqrt{3}$ for some $a, b \in \mathbb{Q}$, so $2 = a^2 + 2ab\sqrt{3} + 3b^2$. Using the $\mathbb{Q}$-linear independence of $\{1, \sqrt{3}\}$, we get that $2ab = 0$ and $2 = a^2 + 3b^2$. Therefore either $a = 0$ and $2/3 = b^2$, or $b = 0$ and $2 = a^2$. But both cases are not possible because neither 2 nor 2/3 are squares in $\mathbb{Q}$.

We conclude that $\sqrt{2} \notin \mathbb{Q}(\sqrt{3})$. So $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{3})] = 2$ and $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = 4$.

$$\begin{tikzcd}
 & \mathbb{Q}(\sqrt{2},\sqrt{3}) \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{Q}(\sqrt{2}) \arrow[dr, dash] \arrow[ddr, dash, "2"'] & & \mathbb{Q}(\sqrt{3}) \arrow[dl, dash] \arrow[ddl, dash, "2"] \\
 & \mathbb{Q}(\sqrt{2})\cap\mathbb{Q}(\sqrt{3}) \arrow[d, dash] & \\
 & \mathbb{Q} &
\end{tikzcd}$$

Similarly,

$$[\mathbb{Q}(\sqrt{2}):\mathbb{Q}(\sqrt{2})\cap\mathbb{Q}(\sqrt{3})] \leq [\mathbb{Q}(\sqrt{2}):\mathbb{Q}] = 2,$$

but $\sqrt{2}\notin\mathbb{Q}(\sqrt{2})\cap\mathbb{Q}(\sqrt{3})$. Thus

$$[\mathbb{Q}(\sqrt{2}):\mathbb{Q}(\sqrt{2})\cap\mathbb{Q}(\sqrt{3})] = 2$$

and hence $[\mathbb{Q}(\sqrt{2})\cap\mathbb{Q}(\sqrt{3}):\mathbb{Q}] = 1$. Therefore $\mathbb{Q}(\sqrt{2})\cap\mathbb{Q}(\sqrt{3}) = \mathbb{Q}$.

2.24. The minimal polynomial of $\sqrt[3]{2}$ is X^3-2 since it is monic, irreducible (by Eisenstein) and has $\sqrt[3]{2}$ as a root. Hence $[\mathbb{Q}(\sqrt[3]{2}):\mathbb{Q}]=3$. $\xi\neq 1$ and it is a root of the polynomial

$$X^3-1=(X-1)(X^2+X+1),$$

so it is a root of X^2+X+1, which is monic and irreducible (it is of degree 2 and the roots are not in $\mathbb{Q}$). Hence $f(\xi,\mathbb{Q})=X^2+X+1$ and $[\mathbb{Q}(\xi):\mathbb{Q}]=2$. We also have that $[\mathbb{Q}(\sqrt[3]{2},\xi):\mathbb{Q}]\leq 6$.

$$\begin{tikzcd}
 & \mathbb{Q}(\sqrt[3]{2},\xi) \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{Q}(\sqrt[3]{2}) \arrow[dr, dash] \arrow[ddr, dash, "3"'] & & \mathbb{Q}(\xi) \arrow[dl, dash] \arrow[ddl, dash, "2"] \\
 & \mathbb{Q}(\sqrt[3]{2})\cap\mathbb{Q}(\xi) \arrow[d, dash] & \\
 & \mathbb{Q} &
\end{tikzcd}$$

By multiplicity of the degree of extensions, we obtain that 6 has to divide $[\mathbb{Q}(\sqrt[3]{2},\xi):\mathbb{Q}]\leq 6$ and $[\mathbb{Q}(\sqrt[3]{2})\cap\mathbb{Q}(\xi):\mathbb{Q}]$ has to divide 2 and 3. Therefore

$$[\mathbb{Q}(\sqrt[3]{2},\xi):\mathbb{Q}]=6 \text{ and } [\mathbb{Q}(\sqrt[3]{2})\cap\mathbb{Q}(\xi):\mathbb{Q}]=1,$$

which means that $\mathbb{Q}(\sqrt[3]{2})\cap\mathbb{Q}(\xi)=\mathbb{Q}$.

2.25. By definition $EF=E(F)$. If F/K is algebraic, then EF is generated by algebraic elements over K, so also over E. Hence EF/E is an algebraic extension.

2.26. If F/K is finite, then F is generated by a finite number of algebraic elements over K. The same elements are algebraic over E and generate $EF = E(F)$ over E. Hence EF/E is a finite extension and $[EF : E] \leq [F : K]$.

4.17. Note that, since $f = X^4 - 5X^2 + 5$ is an even polynomial if $\alpha \in \mathbb{C}$ is a root of f, then also $-\alpha$ is a root of f. Hence, given two roots $\alpha, \beta \in \mathbb{C}$ such that $\beta \neq -\alpha$, we have that the decomposition field of f over $\mathbb{Q}$ is $E = \mathbb{Q}(\alpha, -\alpha, \beta, -\beta)$. But $-\alpha, -\beta \in \mathbb{Q}(\alpha, \beta) \subseteq E$ and so

$$E = \mathbb{Q}(\alpha, -\alpha, \beta, -\beta) \subseteq \mathbb{Q}(\alpha, \beta) \subseteq \mathbb{Q}(\alpha, -\alpha, \beta, -\beta) = E,$$

which means that $E = \mathbb{Q}(\alpha, \beta)$. Moreover we can decompose f in $\mathbb{C}[X]$ as

$$(X - \alpha)(X + \alpha)(X - \beta)(X + \beta) = (X^2 - \alpha^2)(X^2 - \beta^2) = X^4 - (\alpha^2 + \beta^2)X^2 + \alpha^2\beta^2.$$

This implies in particular that $\alpha^2\beta^2 = 5$, hence $\beta = \pm\frac{\sqrt{5}}{\alpha} \in \mathbb{Q}(\alpha, \sqrt{5})$.

Therefore $E = \mathbb{Q}(\alpha, \beta) \subseteq \mathbb{Q}(\alpha, \sqrt{5})$. On the other hand $\sqrt{5} = \pm\alpha\beta \in \mathbb{Q}(\alpha, \beta)$, hence $\mathbb{Q}(\alpha, \sqrt{5}) \subseteq \mathbb{Q}(\alpha, \beta) = E$. So we can conclude that $E = \mathbb{Q}(\alpha, \sqrt{5})$. Using the multiplicative of the degree of finite extension we get that

$$[E : \mathbb{Q}] = [E : \mathbb{Q}(\alpha)][\mathbb{Q}(\alpha) : \mathbb{Q}].$$

But $[\mathbb{Q}(\alpha) : \mathbb{Q}] = \deg(f(\alpha, \mathbb{Q}))$. Using Eisenstein criterion (Exercise 1.26) with $p = 5$, we have that f is irreducible (and monic), so $f = f(\alpha, \mathbb{Q})$. Thus $[\mathbb{Q}(\alpha) : \mathbb{Q}] = \deg f = 4$. It remains to compute $[E : \mathbb{Q}(\alpha)] = [\mathbb{Q}(\alpha, \sqrt{5}) : \mathbb{Q}(\alpha)]$. We have the following situation:

$$\begin{tikzcd}[column sep=small]
 & E = \mathbb{Q}(\alpha, \sqrt{5}) \arrow[dl, dash, "\leq 2"'] \arrow[dr, dash] & \\
\mathbb{Q}(\alpha) \arrow[dr, dash, "4"'] & & \mathbb{Q}(\sqrt{5}) \arrow[dl, dash, "2"] \\
 & \mathbb{Q} &
\end{tikzcd}$$

Observe that $\mathbb{Q}(\alpha, \sqrt{5})$ is equal to the composite of $\mathbb{Q}(\alpha)$ and $\mathbb{Q}(\sqrt{5})$. We can use the property of composite extension, $[LF : L] \leq [F : K]$, to deduce that

$$[\mathbb{Q}(\alpha, \sqrt{5}) : \mathbb{Q}(\alpha)] \leq [\mathbb{Q}(\sqrt{5}) : \mathbb{Q}] = 2.$$

The last equality is because $f(\sqrt{5}, \mathbb{Q}) = X^2 - 5$, as it is monic has $\sqrt{5}$ as a root and it's irreducible (due to Eisenstein's criterion or because it is of degree 2 with 2 non-rational roots). Finally, we want to understand whether $[\mathbb{Q}(\alpha, \sqrt{5}) : \mathbb{Q}(\alpha)]$ is 1 or 2. Note that $\alpha^4 - 5\alpha^2 + 5 = 0$, so we can solve the equation for α^2 as it is a root of $X^2 - 5X + 5$, i.e.

$$\alpha^2 = \frac{5 \pm \sqrt{25 - 20}}{2} = \frac{5 \pm \sqrt{5}}{2},$$

hence $\sqrt{5} = \pm(2\alpha^2 - 5) \in \mathbb{Q}(\alpha)$. So $\mathbb{Q}(\alpha, \sqrt{5}) \subseteq \mathbb{Q}(\alpha) \subseteq \mathbb{Q}(\alpha, \sqrt{5})$, which means that $E = \mathbb{Q}(\alpha)$ and $[E : \mathbb{Q}] = [\mathbb{Q}(\alpha) : \mathbb{Q}] = 4$.

5.5. First of all, note that $\sqrt[3]{2}$ is a root of the polynomial $f(X) = X^3 - 2$. To prove that $\mathbb{Q}(\sqrt[3]{2}, \xi)$ is a normal extension we use Proposition 5.10, so it is enough to prove that $\mathbb{Q}(\sqrt[3]{2}, \xi)$ is the decomposition field of f. We know that the decomposition field E of f over

$\mathbb{Q}$ is $\mathbb{Q}$ extended with the roots of f, i.e. $E = \mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{2}\xi, \sqrt[3]{2}\xi^2)$. But it's easy to see that actually

$$\mathbb{Q}(\sqrt[3]{2}, \xi) = \mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{2}\xi, \sqrt[3]{2}\xi^2) = E.$$

The inclusion $\subseteq$ is because $\sqrt[3]{2}, \xi = \frac{\sqrt[3]{2}\xi}{\sqrt[3]{2}} \in E$. Vice versa $\supseteq$ is due to the fact that the roots of f are products of $\sqrt[3]{2}$ and ξ, elements in $\mathbb{Q}(\sqrt[3]{2}, \xi)$.

5.11. Let $\alpha = \sqrt[4]{7} + \sqrt{2}$. Then $(\alpha - \sqrt{2})^4 - 7 = 0$. By expanding the left side, we get

$$0 = \alpha^4 - 4\sqrt{2}\alpha^3 + 12\alpha^2 - 8\sqrt{2}\alpha - 3 = (\alpha^4 + 12\alpha^2 - 3) - (4\alpha^3 + 8\alpha)\sqrt{2}.$$

But $4\alpha^3 + 8\alpha = 4\alpha(\alpha^2 + 2) \neq 0$, otherwise $\alpha \in \{0, \pm i\sqrt{2})$. Therefore $\sqrt{2} = \frac{\alpha^4+12\alpha^2-3}{4\alpha^3+8\alpha} \in \mathbb{Q}(\alpha)$.

This allows us to prove that $\mathbb{Q}(\sqrt{2}, \sqrt[4]{7}) = \mathbb{Q}(\alpha)$. From the definition of α it is clear that

$$\mathbb{Q}(\alpha) \subseteq \mathbb{Q}(\sqrt{2}, \sqrt[4]{7}).$$

On the other hand, we just proved that $\sqrt{2} \in \mathbb{Q}(\alpha)$. As $\sqrt[4]{7} = \alpha - \sqrt{2} \in \mathbb{Q}(\alpha)$, we also see that $\sqrt[4]{7} \in \mathbb{Q}(\alpha)$. It follows that $\mathbb{Q}(\sqrt{2}, \sqrt[4]{7}) \subseteq \mathbb{Q}(\alpha)$.

Moreover, $\sqrt{2} \notin \mathbb{Q}(\sqrt[4]{7})$. Otherwise, as $[\mathbb{Q}(\sqrt[4]{7}) : \mathbb{Q}] = 4$ and $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$ we would get that $[\mathbb{Q}(\sqrt[4]{7}) : \mathbb{Q}(\sqrt{2})] = 2$. Let $f(\sqrt[4]{7}, \mathbb{Q}(\sqrt{2})) = X^2 + \beta X + \gamma$, with $\beta, \gamma \in \mathbb{Q}(\sqrt{2})$. So

$$0 = f(\sqrt[4]{7}) = \sqrt{7} + \beta\sqrt[4]{7} + \gamma.$$

Therefore

$$\beta^2\sqrt{7} = (-\sqrt{7} - \gamma)^2 = 7 + 2\gamma\sqrt{7} + \gamma^2.$$

Thus

$$(\beta^2 - 2\gamma)\sqrt{7} = \gamma^2 + 7.$$

But $\beta^2 - 2\gamma \neq 0$ because $\gamma^2 + \beta\gamma + \frac{\beta^2}{2} = 0$ holds only for $\gamma = \frac{\beta}{2}(-1 \pm i) \in \mathbb{C} \setminus \mathbb{R}$, which is clearly not in $\mathbb{Q}(\sqrt{2})$. Thus $\sqrt{7} = \frac{\gamma^2+7}{\beta^2-2\gamma} \in \mathbb{Q}(\sqrt{2})$, a contradiction.

To sum up we have that $\sqrt{2} \notin \mathbb{Q}(\sqrt[4]{7})$ and

$$\begin{tikzcd}
 & E = \mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{2}, \sqrt[4]{7}) \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{Q}(\sqrt{2}) \arrow[dr, dash, "2"'] & & \mathbb{Q}(\sqrt[4]{7}) \arrow[dl, dash, "4"] \\
 & \mathbb{Q} &
\end{tikzcd}$$

1) We know that $\sqrt[4]{7} \in \mathbb{Q}(\alpha)$ which has minimal polynomial $f(\sqrt[4]{7}, \mathbb{Q}) = x^4 - 7$. One root of this polynomial is $i\sqrt[4]{7}$. This root is not in $\mathbb{Q}(\alpha) \subseteq \mathbb{R}$ as it is in $\mathbb{C} \setminus \mathbb{R}$. Therefore $\mathbb{Q}(\alpha)/\mathbb{Q}$ is not normal by Proposition 5.7.

2) As $\sqrt{2} \notin \mathbb{Q}(\sqrt[4]{7})$, we see that $[\mathbb{Q}(\alpha) : \mathbb{Q}(\sqrt[4]{7})] > 1$. On the other hand,

$$[\mathbb{Q}(\alpha) : \mathbb{Q}(\sqrt[4]{7})] \leq [\mathbb{Q}(\sqrt{2} : \mathbb{Q}] = 2,$$

which proves that $[\mathbb{Q}(\alpha) : \mathbb{Q}(\sqrt[4]{7})] = 2$. Therefore,

$$[E : \mathbb{Q}] = [\mathbb{Q}(\alpha) : \mathbb{Q}] = [\mathbb{Q}(\alpha) : \mathbb{Q}(\sqrt[4]{7})] \cdot [\mathbb{Q}(\sqrt[4]{7}) : \mathbb{Q}] = 2 \cdot 4 = 8.$$

3) Let $\sigma \in G = \mathrm{Gal}(E/\mathbb{Q})$. Since $E = \mathbb{Q}(\sqrt{2}, \sqrt[4]{7})$ and $\sqrt{2}$ and $\sqrt[4]{7}$ are independent because $\sqrt{2} \notin \mathbb{Q}(\sqrt[4]{7})$, we know that σ is completely determined by $\sigma(\sqrt{2})$ and $\sigma(\sqrt[4]{7})$. By Proposition 4.10, $\sigma(\sqrt{2}) \in E$ has to be a root of $f(\sqrt{2}, \mathbb{Q}) = X^2 - 2$ and $\sigma(\sqrt[4]{7}) \in E$ has to be a root of $f(\sqrt[4]{7}, \mathbb{Q}) = X^4 - 7$. So $\sigma(\sqrt{2}) = \pm\sqrt{2}$ and, since $E \subseteq \mathbb{R}$,

$$\sigma(\sqrt[4]{7}) \in E \cap \{\sqrt[4]{7}i^j \mid j \in \{0,1,2,3\}\} = \{\pm\sqrt[4]{7}\}.$$

Therefore G contains 4 elements $\sigma_{k,l}$ for $k, l \in \mathbb{Z}/2$ such that $\sigma_{k,l}(\sqrt{2}) = (-1)^k\sqrt{2}$ and $\sigma_{k,l}(\sqrt[4]{7}) = (-1)^l\sqrt[4]{7}$. This gives directly the isomorphism between G and $\mathbb{Z}/2 \times \mathbb{Z}/2$.

5.12. Let $\{v_i : i \in I\}$ be a basis of V over K. For each $i \in I$ let $f_i\colon V \to F$, $f_i(v_j) = \delta_{ij}$. Then $\{f_i : i \in I\}$ is linearly independent over F. In fact, let $\sum a_i f_i = 0$, where each $a_i \in F$. Then $a_i = 0$ for almost all i. If $j \in I$, then

$$0 = \left(\sum a_i f_i\right)(v_j) = \sum a_i f_i(v_j) = a_j.$$

Now assume that $\dim_K V = n$. Let $\{v_1, \dots, v_n\}$ be a basis of V over K. We claim that $\{f_1, \dots, f_n\}$ is a basis of $\mathrm{Hom}_K(V, F)$ over F. If $g \in \mathrm{Hom}_K(V, F)$, then $g = \sum g(v_i) f_i$. If $1 \leq k \leq n$, then

$$\left(\sum g(v_i) f_i\right)(v_k) = \sum g(v_i) f_i(v_k) = g(v_k).$$

5.15. We need to find a bijective map

$$\mathrm{Hom}(E/K, C/K) \to \mathrm{Hom}(E/K, C_1/K).$$

If $\sigma \in \mathrm{Hom}(E/K, C/K)$, then $\theta^{-1}\sigma \in \mathrm{Hom}(E/K, C_1/K)$. If $\varphi \in \mathrm{Hom}(E/K, C_1/K)$, then $\theta\varphi \in \mathrm{Hom}(E/K, C/K)$. The maps $\sigma \mapsto \theta^{-1}\sigma$ and $\varphi \mapsto \theta\varphi$ are inverse to each other.

8.4. We first prove that for every order reversing function φ and every element s, t in its domain,

$$\varphi(s \vee t) \leq \varphi(s) \wedge \varphi(t) \text{ and } \varphi(s) \vee \varphi(t) \leq \varphi(s \wedge t).$$

Since that $s \leq s \vee t$ and $t \leq s \vee t$ and φ is order reversing, we have that $\varphi(s \vee t) \leq \varphi(s)$ and $\varphi(s \vee t) \leq \varphi(t)$. Hence $\varphi(s \vee t) \leq \varphi(s) \wedge \varphi(t)$. Moreover $s \wedge t \leq s$ and $s \wedge t \leq t$. So $\varphi(s) \leq \varphi(s \wedge t)$ and $\varphi(t) \leq \varphi(s \wedge t)$. Hence $\varphi(s) \vee \varphi(t) \leq \varphi(s \wedge t)$.

We can now apply this result for $\varphi = f$, $s = x$ and $t = y$ obtaining that

$$f(x \vee y) \leq f(x) \wedge f(y), \quad f(x) \vee f(y) \leq f(x \wedge y).$$

On the other hand, for $\varphi = f^{-1}$, $s = f(x)$ and $t = f(y)$, we obtain that

$$f^{-1}(f(x) \vee f(y)) \leq f^{-1}(f(x)) \wedge f^{-1}(f(y)) = x \wedge y,$$

and

$$x \vee y = f^{-1}(f(x)) \vee f^{-1}(f(y)) \leq f^{-1}\big(f(x) \wedge f(y)\big).$$

Thus, applying f, which is order reversing, it implies that

$$f(x \wedge y) \leq f\big(f^{-1}(f(x) \vee f(y))\big) = f(x) \vee f(y)$$

and

$$f(x) \wedge f(y) = f\big(f^{-1}\big(f(x) \wedge f(y)\big)\big) \leq f(x \vee y).$$

8.6. Since E/K is a Galois extension, the order of $\operatorname{Gal}(E/K)$ is precisely $[E : K] = n$. So, by Cauchy's Theorem, there exists a subgroup S of $\operatorname{Gal}(E/K)$ of order p. Then, by Galois' Theorem, the subextension ${}^{S}E/K$ has degree equal to the index of S, which is n/p.

8.7. Since E/K is a Galois extension, $|\operatorname{Gal}(E/K)| = [E : K] = p^{\alpha}m$. So, by Sylow's Theorem, there exists a subgroup P of $\operatorname{Gal}(E/K)$ of order p^{α}. Then, by Galois' Theorem, the subextension ${}^{P}E/K$ has degree $(\operatorname{Gal}(E/K) : P) = m$.

8.19. Write $f = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$. Then

$$f' = nX^{n-1} + (n-1)a_{n-1}X^{n-2} + \cdots + 2a_2X + a_1.$$

Since f is not separable, $f' = 0$. Thus $n = ka_k = 0$ in K for all $k \in \{0, \dots, n-1\}$. This implies that p divides k whenever $a_k \neq 0$. This means that the only terms in f occur in degree that are multiples of p. In particular, $n = pm$ for some m. Hence

$$f = X^{pm} + a_{p(n-1)}X^{p(m-1)} + \cdots + a_pX^p + a_0 = g(X^p)$$

for some $g \in K[X]$.

10.24. If G is solvable, then $[G, G]$ is a proper normal subgroup of G. Since G is simple, $[G, G] = \{1\}$ and G is abelian. Thus G is cyclic of prime order.

11.1. Assume that G is simple. Let $A = G \times \{1\}$, $B = \{1\} \times G$ and $D = \{(x, x) : x \in G\}$ the diagonal subgroup of $G \times G$. Since

$$(g, h) = (g, 1)(1, h) = (gh^{-1}, 1)(h, h)$$

it follows that $G = AB = AD$. Let M be a subgroup of $G \times G$ that contains D. Note that

$$M = M \cap (G \times G) = M \cap AD = (M \cap A)D.$$

Similarly, $M = (M \cap B)D$. Since A is normal in $G \times G$, $M \cap A$ is normal in $G \times G$ and $(M \cap A)B$ is normal in $MB = G \times G$. Using the second isomorphism theorem, we see that

$$M \cap A \simeq \frac{(M \cap A)B}{B}$$

is a normal subgroup of $(G \times G)/B \simeq A$. Since $A \simeq G$ is simple, either $M \cap A = \{1\}$ or $M \cap A = A$. Thus either $M = D$ or $BD = G \times G$. Therefore D is maximal.

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