Galois theory

Leandro Vendramin

Contents

Introduction

The notes correspond to the bachelor course *Galois theory* of the Vrije Universiteit Brussel, Faculty of Sciences, Department of Mathematics and Data Sciences. The course is divided into twelve two-hour lectures.

The material is somewhat standard. Basic texts on fields and Galois theory are for example [**3**] and [**4**].

As usual, we also mention a set of great expository papers by Keith Conrad, the notes are extremely well-written and useful at every stage of a mathematical career.

Several chapters contain optional paragraphs that give examples of how to apply OSCAR Computer Algebra System to concrete problems in Galois theory.

Thanks go to Wouter Appelmans, Luca Descheemaeker, Alejandro de la Cueva Merino, Wannes Malfait, Manet Michiels, Silvia Properzi, Lukas Simons.

This version was compiled on June 18, 2024 at 12:56.

Department of Mathematics and Data Science, Vrije Universiteit Brussel, Pleinlaan 2, 1050 Brussel
E-mail address: `Leandro.Vendramin@vub.be`.

Lecture 1. 12/02/2024

§ 1.1. Fields. Recall that a **field** is a commutative ring such that $1 \neq 0$ and every non-zero element is invertible. Examples of (infinite) fields are $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$. If p is a prime number, then $\mathbb{Z}/p$ is a field.

EXAMPLE 1.1. The abelian group $\mathbb{Z}/2 \times \mathbb{Z}/2$ is a field with multiplication

$$(a,b)(c,d) = (ac+bd, ad+bc+bd).$$

EXAMPLE 1.2. $\mathbb{Q}[i] = \{a+bi : a,b \in \mathbb{Q}\}$ and $\mathbb{Q}[\sqrt{2}]$ are fields.

$$\begin{tikzcd}
\mathbb{C} \arrow[rr, dash] \arrow[d, dash] & & \mathbb{R} \arrow[d, dash] \\
\mathbb{Q}[i] \arrow[dr, dash] & & \mathbb{Q}[\sqrt{2}] \arrow[dl, dash] \\
& \mathbb{Q} &
\end{tikzcd}$$

EXERCISE 1.3. Prove that $\mathbb{Q}[i]$ and $\mathbb{Q}[\sqrt{2}]$ are not isomorphic as fields.

If R is a ring, there exists a unique ring homomorphism $\mathbb{Z} \to R$, $m \mapsto m1$. The image

$$\{m1 : m \in \mathbb{Z}\}$$

of this homomorphism is a subring of R and it is known as the **ring of integers** of R. The kernel is a subgroup of $\mathbb{Z}$ generated by some $t \geq 0$. The integer t is the **characteristic** of the ring R.

EXERCISE 1.4. The characteristic of a field is either zero or a prime number.

EXAMPLE 1.5. The characteristic of the field of Example 1.1 is two. Why?

Recall that a commutative ring R is an **integral domain** if $xy = 0 \implies x = 0$ or $y = 0$. Fields are integral domains.

EXERCISE 1.6. Let K be a field. Prove that the following statements are equivalent:

1) K is of characteristic zero.
2) The additive order of 1 is infinite.
3) The additive order of each $x \neq 0$ is infinite.
4) The ring of integers of K is isomorphic to $\mathbb{Z}$.

Exercise 1.7. Let K be a field. Prove that the following statements are equivalent:

1) K is of characteristic p.
2) The additive order of 1 is p.
3) The additive order of each $x \neq 0$ is p.
4) The ring of integers of K is isomorphic to $\mathbb{Z}/p$.

Proposition 1.8. *Let K be a field. Then any finite subgroup of $K^\times = K \setminus \{0\}$ is cyclic.*

Proof. Let G be a finite subgroup of $K^\times$. Let $x \in G$ of maximal order N. We will show that any element of G is a power of x. Let $y \in G$ and $n = |y|$. Note that $X^n - 1 \in K[X]$ has at most n roots in K.

We claim that n divides N. If not, there exists a prime number p and a power $q = p^\beta$ of p such that $q \mid n$ and $q \nmid N$. Let $z = xy^{n/q}$. Since G is abelian,

$$|z| = \operatorname{lcm}\{N, q\} = Nq > N,$$

a contradiction.

The polynomial $X^n - 1 \in K[X]$ has n distinct roots in K, these are the elements $x^{kN/n}$ for $k \in \{0, \dots, n-1\}$. Since y has order n, it has to be one of these roots. Thus $y = x^{kN/n}$ for some $k \in \{0, \dots, n-1\}$. □

Definition 1.9. A **subfield** of a ring R is a subring of R that is also a field.

Note that if K is a subfield of E, then the characteristic of K coincides with the characteristic of E. Moreover, if $K \to L$ is a field homomorphism, then K and L have the same characteristic.

Exercise 1.10. Let K be a field of characteristic p. Prove that $K \to K$, $x \mapsto x^{p^n}$, is a field homomorphism for all $n \in \mathbb{Z}_{\geq 0}$.

Note that finite fields are of prime characteristic.

Let K be a subfield of a field E. Then E is a K-vector space with the usual scalar multiplication $K \times E \to E$, $(\lambda, x) \mapsto \lambda x$.

Definition 1.11. A field K is **prime** if there are no proper subfields of K.

Examples of prime fields are $\mathbb{Q}$ and $\mathbb{Z}/p$ for a prime number p.

Proposition 1.12. *Let K be a field. The following statements hold:*

1) *K contains a unique prime field, it is known as the **prime subfield** of K.*
2) *The prime subfield of K is either isomorphic to $\mathbb{Q}$ if the characteristic of K is zero, or it is isomorphic to $\mathbb{Z}/p$ for some prime number p if the characteristic of K is p.*

Proof. To prove the first claim let L be the intersection of all the subfields of K. Then L is a subfield of K. If F is a subfield of L, then F is a subfield of K. Thus $L \subseteq F$ and hence $F = L$, which proves that L is prime. If L_1 is a subfield of K and L_1 is prime, then $L \subseteq L_1$ and hence $L = L_1$.

Let K_0 be the prime field of K. Suppose that K is of characteristic $p > 0$. Then the ring $K_\mathbb{Z}$ of integers of K is a field isomorphic to $\mathbb{Z}/p$ and hence $K_0 \simeq K_\mathbb{Z}$. Suppose now that the characteristic of K is zero. Let $E = \{m1/n1 : m, n \in \mathbb{Z}, n \neq 0\}$. We claim that $K_0 = E$. Since $K_\mathbb{Z} \subseteq K_0$, it follows that $E \subseteq K_0$. Hence $E = K_0$, as E is a subfield of K. □

DEFINITION 1.13. Let E be a field and K be a subfield of E. Then E is a **field extension** of K. We will use the notation E/K.

If E is an extension of K, then E is a K-vector space.

DEFINITION 1.14. The **degree** of an extension E of K is the integer $\dim_K E$. It will be denoted by $[E:K]$.

We say that E is a **finite extension** of K if $[E:K]$ is finite.

EXAMPLE 1.15. Let K be a field. Then $[K:K]=1$. Conversely, if E is an extension of K and $[E:K]=1$, then $K=E$. If not, let $x \in E \setminus K$. We claim that $\{1,x\}$ is linearly independent over K. Indeed, if $a1+bx=0$ for some $a,b \in K$, then $bx=-a$. If $b \neq 0$, then $x=-a/b \in K$, a contradiction. If $b=0$, then $a=0$.

We know that $[\mathbb{C}:\mathbb{R}]=2$.

EXAMPLE 1.16. A basis of $\mathbb{Q}[\sqrt{2}]$ over $\mathbb{Q}$ is given by $\{1,\sqrt{2}\}$. Then $[\mathbb{Q}[\sqrt{2}]:\mathbb{Q}]=2$. The calculations can be easily done by computer:

```
julia> E, a = quadratic_field(2)
(Real quadratic field defined by x^2 - 2, sqrt(2))

julia> characteristic(E)
0

julia> K = prime_field(E)
Rational Field

julia> degree(E)
2

julia> basis(E)
2-element Vector{nf_elem}:
 1
 sqrt(2)

julia> one(K)==one(E)
true

julia> zero(K)==zero(E)
true
```

EXAMPLE 1.17. Since $\mathbb{Q}$ is numerable and $\mathbb{R}$ is not, $[\mathbb{R}:\mathbb{Q}] > \aleph_0$. If $\{x_i : i \in \mathbb{Z}_{>0}\}$ is a numerable basis of $\mathbb{R}$ over $\mathbb{Q}$, for each n consider the $\mathbb{Q}$-vector space V_n generated by $\{x_1,\dots,x_n\}$. Then

$$\mathbb{R} = \bigcup_{n\geq 1} V_n,$$

is numerable, as each V_n is numerable, a contradiction.

If E is an extension of K and E is finite, then $[E:K]$ is finite.

PROPOSITION 1.18. *Let K be a finite field. Then $|K|=p^m$ for some prime number p and some $m\geq 1$.*

PROOF. We know the prime subfield K_0 of K is isomorphic to $\mathbb{Z}/p$. In particular, $|K_0| = p$. Since K is finite, $[K : K_0] = m$ for some m. If $\{x_1,\dots,x_m\}$ is a basis of K over K_0, then each element of K can be written uniquely as $\sum_{i=1}^m a_i x_i$ for some $a_1,\dots,a_m \in K_0$. Then there is a bijection between K and K_0^m and hence $|K| = |K_0^m| = p^m$. □

We now perform some basic calculations with a finite field of eight elements:

```
julia> E, x = FiniteField(2, 3, "x")
(Finite field of degree 3 over F_2, x)

julia> characteristic(E)
2

julia> prime_field(E)
Galois field with characteristic 2

julia> degree(E)
3

julia> size(E)
8

julia> [z for z in E]
8-element Vector{fq_nmod}:
 0
 1
 x
 x + 1
 x^2
 x^2 + 1
 x^2 + x
 x^2 + x + 1
```

DEFINITION 1.19. Let E be an extension of K. A **subextension** F/K of E/K is a subfield F of E that contains K, that is $K \subseteq F \subseteq E$.

DEFINITION 1.20. Let E and E_1 be extensions over K. An **extension homomorphism**

$$E/K \to E_1/K$$

is a field homomorphism $\sigma\colon E \to E_1$ such that $\sigma(x) = x$ for all $x \in K$.

To describe the homomorphism $\sigma\colon E/K \to E_1/K$ of the extensions over K one typically writes the commutative diagram

$$\begin{tikzcd} K \arrow[r, equal] \arrow[d, hook] & K \arrow[d, hook] \\ E \arrow[r, "\sigma"] & E_1 \end{tikzcd}$$

We write $\mathrm{Hom}(E/K, E_1/K)$ to denote the set of homomorphism $E/K \to E_1/K$ of extensions of K. Note that if $\sigma \in \mathrm{Hom}(E/K, E_1/K)$, then σ is a K-linear map, as

$$\sigma(\lambda x) = \sigma(\lambda)\sigma(x) = \lambda\sigma(x)$$

for all $\lambda \in K$ and $x \in E$.

EXAMPLE 1.21. The conjugation map $\mathbb{C} \to \mathbb{C}$, $z \mapsto \bar{z}$, is an endomorphism of $\mathbb{C}$ as an extension over $\mathbb{R}$. Let $\varphi \in \mathrm{Hom}(\mathbb{C}/\mathbb{R}, \mathbb{C}/\mathbb{R})$. Then

$$\varphi(x+iy) = \varphi(x) + \varphi(i)\varphi(y) = x + \varphi(i)y$$

for all $x, y \in \mathbb{R}$. Since $\varphi(i)^2 = \varphi(i^2) = \varphi(-1) = -1$, it follows that $\varphi(i) \in \{-i, i\}$. Thus either $\varphi(x+iy) = x+iy$ or $\varphi(x+iy) = x-iy$.

EXERCISE 1.22. Let K be a field, K_0 be its prime field and $\sigma\colon K \to K$ be a field homomorphism. Prove that $\sigma \in \mathrm{Hom}(K/K_0, K/K_0)$.

If E/K is an extension, then

$$\begin{aligned}\mathrm{Aut}(E/K) &= \{\sigma\colon E/K \to E/K \text{ is a bijective extension homomorphism}\}\\ &= \{\sigma\colon E \to E : \sigma \text{ is a bijective field homomorphism with } \sigma|_K = \mathrm{id}\}\end{aligned}$$

is a group with composition.

DEFINITION 1.23. Let E/K be an extension. The **Galois group** of E/K is the group $\mathrm{Aut}(E/K)$ and it will be denoted by $\mathrm{Gal}(E/K)$.

A typical example: $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \simeq \mathbb{Z}/2$.

As an example, we show with the computer that $\mathrm{Gal}(\mathbb{Q}[\sqrt{2}]/\mathbb{Q}) \simeq \mathbb{Z}/2$:

```
julia> E, x = quadratic_field(2)
(Real quadratic field defined by x^2 - 2, sqrt(2))
julia> characteristic(E)
0
julia> G, C = galois_group(E);
julia> describe(G)
"C2"
julia> order(G)
2
```

EXAMPLE 1.24. Let $\theta = \sqrt[3]{2}$ and let $E = \{a + b\theta + c\theta^2 : a, b, c \in \mathbb{Q}\}$. Note that

$$a + b\theta + c\theta^2 = 0 \Longleftrightarrow a = b = c = 0.$$

Then E is an extension of $\mathbb{Q}$ such that $[E : \mathbb{Q}] = 3$. We claim that $\mathrm{Gal}(E/\mathbb{Q})$ is trivial. If $\sigma \in \mathrm{Gal}(E/\mathbb{Q})$ and $z = a + b\theta + c\theta^2$, then $\sigma(z) = a + b\sigma(\theta) + c\sigma^2(\theta)$. Since

$$\sigma(\theta)^3 = \sigma(\theta^3) = \sigma(2) = 2,$$

it follows that $\sigma(\theta) = \theta$ and therefore $\sigma = \mathrm{id}$.

EXERCISE 1.25. Prove that the polynomial $X^3 - 2$ is irreducible in $\mathbb{Q}[X]$.

The previous exercise can easily be solved using computers:

```
julia> R, x = PolynomialRing(QQ, "x");
julia> is_irreducible(x^3-2)
true
```

The following exercise is known as the *Eisenstein's irreducibility criterion*:

Exercise 1.26. Let A be a unique factorization domain and K be its fraction field. Let $f = \sum_{i=0}^{n} a_i X^i \in A[X]$ be a polynomial of degree $n > 0$. Assume that there exists a prime element $p \in A$ such that $p \mid a_i$ for all $i \in \{0, 1, \dots, n-1\}$, $p \nmid a_n$ and $p^2 \nmid a_0$. Then f is irreducible in $K[X]$.

Exercise 1.27. Prove that the polynomials

$$
\begin{aligned}
f &= X^{10} + 60X^7 + 82X^6 - 36X^3 + 2, \\
g &= 3X^{10} + 15X^2 - 45,
\end{aligned}
$$

are irreducible in $\mathbb{Q}[X]$.

Exercise 1.28. Is the polynomial $f = 3(X^{10} + 5X^2 - 15)$ irreducible in $\mathbb{Z}[X]$?

If E/K is an extension and S is a subset of E, then there exists a unique smallest subextension F/K of E/K such that $S \subseteq F$. In fact,

$$F = \bigcap \{T : T \text{ is a subfield of } E \text{ that contains } K \cup S\}$$

If L/K is a subextension of E/K such that $S \subseteq L$, then $F \subseteq L$ by definition. The extension F is known as the **subextension generated by** S and it will be denoted by $K(S)$. If $S = \{x_1, \dots, x_n\}$ is finite, then $K(S) = K(x_1, \dots, x_n)$ is said to be of **finite type**.

Example 1.29. If $\{e_1, \dots, e_n\}$ is a basis of E over K, then $E = K(e_1, \dots, e_n)$.

Example 1.30. The field $\mathbb{Q}(\sqrt{2})$ is precisely the extension of $\mathbb{R}/\mathbb{Q}$ generated by $\sqrt{2}$.

Let E/K be an extension and S and T be subsets of E. Then

$$K(S \cup T) = K(S)(T) = K(T)(S).$$

If, moreover, $S \subseteq T$, then $K(S) \subseteq K(T)$.

§ 1.2. Algebraic extensions.

Definition 1.31. Let E/K be an extension. An element $x \in E$ is **algebraic** over K if there exists a non-zero polynomial $f(X) \in K[X]$ such that $f(x) = 0$. If x is not algebraic over K, then it is called **transcendental** over K.

Definition 1.32. An extension E/K is **algebraic** if every $x \in E$ is algebraic over K.

If K is a field, every $x \in K$ is algebraic over K, as x is a root of $X - x \in K[X]$. In particular, K/K is an algebraic extension.

Example 1.33. $\mathbb{C}/\mathbb{R}$ is an algebraic extension. If $z \in \mathbb{C} \setminus \mathbb{R}$, then z is a root of the polynomial $X^2 - (z + \bar{z})X + |z|^2 \in \mathbb{R}[X]$.

If F/K is an extension $x \in E$ is algebraic over K for some field $E \supseteq F$, then x is algebraic over F.

Example 1.34. $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is algebraic, as the number $a + b\sqrt{2}$ is a root of the polynomial $X^2 - 2aX + (a^2 - 2b^2) \in \mathbb{Q}[X]$.

The extension $\mathbb{C}/\mathbb{Q}$ is not algebraic. For example, Hermite proved that e is transcendental over $\mathbb{Q}$; see [**4**, Theorem 24.4]. Lindemann's theorem states that π is not algebraic over $\mathbb{Q}$; see [**4**, Theorem 24.5].

Example 1.35. Let $a=\sqrt{2}$ and $b=\sqrt[3]{3}$. Both a and b are algebraic numbers over $\mathbb{Q}$. Let us show that $a+b$ is also algebraic. Let $f(X)=X^3-3\in\mathbb{Q}[X]$. Then $f(b)=0$. Note that the polynomial

$$g(X)=f(X-a)=X^3-3aX^2+3aX-a^3-3\in\mathbb{Q}(a)[X]$$

is such that $g(a+b)=0$. How can we find a polynomial with coefficients in $\mathbb{Q}$ that vanishes on $a+b$? We do the "conjugation" trick:

$$h(X)=f(X-a)f(X+a)=X^6-6X^4-6X^3+12X^2-36X+1\in\mathbb{Q}[X].$$

Note that $h(a+b)=0$. How can you prove that ab is also algebraic over $\mathbb{Q}$?

Lecture 2. 19/02/2024

If E/K is an extension and $x \in E$ is algebraic over K, then the evaluation homomorphism $K[X] \to E$, $p \mapsto p(x)$, is not injective. In particular, its kernel is a non-zero ideal. Hence it is generated by a monic polynomial f.

Definition 2.1. Let E/K be an extension and $x \in E$ be an algebraic element. The monic polynomial that generates the kernel of $K[X] \to E$, $f \mapsto f(x)$, is known as the **minimal polynomial** of x over K and it will be denoted by $f(x,K)$. The **degree** of x over K is then $\deg f(x,K)$.

Some basic properties of the minimal polynomial of an algebraic element:

Proposition 2.2. *Let E/K be an extension and $x \in E$. Assume that x is algebraic over K.*

1) *If $g \in K[X] \setminus \{0\}$ is such that $g(x) = 0$, then $f(x,K)$ divides g and $\deg f(x,K) \leq \deg g$.*
2) *$f(x,K)$ is irreducible in $K[X]$.*
3) *If F/K is a subextension of E/K, then $f(x,F)$ divides $f(x,K)$.*

Proof. Write $f = f(x,K)$ to denote the minimal polynomial of x. To prove 1) note that $g(x) = 0$ implies that g belongs to the kernel of the evaluation map, so g is a multiple of f. To prove 2) note that if $f = pq$ for some $p, q \in K[X]$ such that $0 < \deg p, \deg q < \deg f$, then $f(x) = 0$ implies that either $p(x) = 0$ or $q(x) = 0$, a contradiction. Finally, we prove 3). Since $f \in K[X] \subseteq F[X]$ and $f(x) = 0$, it follows from 1) that $f(x,F)$ divides f. ☐

Some easy examples: $f(i,\mathbb{R}) = X^2 + 1$, $f(i,\mathbb{C}) = X - i$ and $f(\sqrt[3]{2},\mathbb{Q}) = X^3 - 2$:

```
julia> E, x = radical_extension(3, QQ(2), "x");

julia> minpoly(x)
x^3 - 2

julia> F, y = quadratic_field(-1);

julia> minpoly(y)
x^2 + 1
```

Example 2.3. Let us compute $f(\sqrt{2}+\sqrt{3},\mathbb{Q})$. Let $\alpha = \sqrt{2}+\sqrt{3}$. Then

$$\alpha - \sqrt{2} = \sqrt{3} \implies (\alpha - \sqrt{2})^2 = 3 \implies \alpha^2 - 2\sqrt{2}\alpha + 2 = 3$$
$$\implies \alpha^2 - 1 = 2\sqrt{2}\alpha \implies (\alpha^2 - 1)^2 = 8\alpha^2 \implies \alpha^4 - 10\alpha^2 + 1 = 0.$$

Thus α is a root of $g = X^4 - 10X^2 + 1$. To prove that $g = f(\alpha,\mathbb{Q})$ it is enough to prove that g is irreducible in $\mathbb{Q}[X]$. First note that the roots of g are $\sqrt{2}+\sqrt{3}$, $\sqrt{2}-\sqrt{3}$, $-\sqrt{2}+\sqrt{3}$ and $-\sqrt{2}-\sqrt{3}$. This means that if g is not irreducible, then $g = hh_1$ for some polynomials $h, h_1 \in \mathbb{Q}[X]$ such that $\deg h = \deg h_1 = 2$. This is not possible, as $(\sqrt{2}+\sqrt{3}) + (\sqrt{2}-\sqrt{3}) = 2\sqrt{2} \notin \mathbb{Q}$, $(\sqrt{2}+\sqrt{3}) + (-\sqrt{2}+\sqrt{3}) = 2\sqrt{3} \notin \mathbb{Q}$ and $(\sqrt{2}+\sqrt{3})(-\sqrt{2}-\sqrt{3}) = -5 - 2\sqrt{6} \notin \mathbb{Q}$.

Proposition 2.4. *Let F/K be a subextension of E/K. Then*

$$[E:K] = [E:F][F:K].$$

Proof. Let $\{e_i : i \in I\}$ be a basis of E over F and $\{f_j : j \in J\}$ be a basis of F over K. If $x \in E$, then $x = \sum_i \lambda_i e_i$ (finite sum) for some $\lambda_i \in F$. For each i, $\lambda_i = \sum_j a_{ij} f_j$ (finite sum) for some $a_{ij} \in K$. Then $x = \sum_i \sum_j a_{ij}(f_j e_i)$. This means that $\{f_j e_i : i \in I, j \in J\}$ generates E as a K-vector space. Let us prove that $\{f_je_i : i\in I, j\in J\}$ is linearly independent. If $\sum_i\sum_j a_{ij}(f_je_i)=0$ (finite sum) for some $a_{ij}\in K$, then

$$\begin{aligned} 0=\sum_i\left(\sum_j a_{ij}f_j\right)e_i &\implies \sum_j a_{ij}f_j=0 \text{ for all } i\in I\\ &\implies a_{ij}=0 \text{ for all } i\in I \text{ and } j\in J. \end{aligned}$$

□

We state a lemma:

Lemma 2.5. *If A is a finite-dimensional commutative algebra over K and A is an integral domain, then A is a field.*

Proof. Let $a\in A\setminus\{0\}$. We need to prove that there exists $b\in A$ such that $ab=1$. Let $\theta\colon A\to A$, $x\mapsto ax$. Note that θ is K-linear transformation, as

$$\theta(x+y)=a(x+y)=ax+ay=\theta(x)+\theta(y),\quad \theta(\lambda x)=a(\lambda x)=\lambda(ax)=\lambda\theta(x),$$

for all $x,y\in A$ and $\lambda\in K$. It is injective since A is an integral domain. Since $\dim_K A<\infty$, it follows that θ is an isomorphism. In particular, $\theta(A)=A$, which implies that there exists $b\in A$ such that $1=ab$. □

Let E/K be an extension and $x\in E$. Then

$$K[x]=\{f(x):f\in K[X]\}$$

is a subring of E that contains K. Note that $K[x]$ is a K-vector space.

More generally, if $x_1,\dots,x_n\in E$, then

$$K[x_1,\dots,x_n]=\{f(x_1,\dots,x_n):f\in K[X_1,\dots,X_n]\}$$

is a subring of E. Note that $K[x_1,\dots,x_n]$ is a K-vector space. Clearly, $K[x_1,\dots,x_n]$ is a domain and

$$K(x_1,\dots,x_n)=\left\{\frac{f(x_1,\dots,x_n)}{g(x_1,\dots,x_n)}:f,g\in K[X_1,\dots,X_n] \text{ with } g(x_1,\dots,x_n)\neq 0\right\}$$

is the extension of K generated by $x_1,\dots,x_n$. Note that

$$K(x_1,\dots,x_n)=(K(x_1,\dots,x_{n-1}))(x_n).$$

The previous construction can be generalized. Let I be a non-empty set. For each $i\in I$, let X_i be a variable. Consider the polynomial ring $K[\{X_i:i\in I\}]$ and let $S=\{x_i:i\in I\}$ be a subset of E. There exists a unique algebra homomorphism

$$K[\{X_i:i\in I\}]\to E$$

such that $X_i\mapsto x_i$ for all $i\in I$. The image is denoted by $K[S]$. In particular, an element $z\in K[S]$ is of the form

$$z=h(x_1,\dots,x_n)$$

for a polynomial $h\in K[X_1,\dots,X_n]$ in finitely many variables $X_1,\dots,X_n$ and $x_1,\dots,x_n\in S$.

Exercise 2.6. Prove that $\mathbb{Q}[\sqrt{2}]=\mathbb{Q}(\sqrt{2})$.

The exercise is not an accident.

Theorem 2.7. *Let E/K be an extension and $x\in E\setminus K$. The following statements are equivalent:*

1) *x is algebraic over K.*

2) $\dim_K K[x] < \infty$.
3) $K[x]$ *is a field.*
4) $K[x] = K(x)$.

Proof. We first prove 1) $\implies$ 2). Let $z \in K[x]$, say $z = h(x)$ for some $h \in K[X]$. There exists $g \in K[X]$ such that $g \neq 0$ and $g(x) = 0$. Divide h by g to obtain polynomials $q, r \in K[X]$ such that $h = gq + r$, where $r = 0$ or $\deg r < \deg g$. This implies that

$$z = h(x) = g(x)q(x) + r(x) = r(x).$$

If $\deg g = m$, then $r = \sum_{i=0}^{m-1} a_i X^i$ for some $a_0, \dots, a_{m-1} \in K$. Thus

$$z = \sum_{i=0}^{m-1} a_i x^i$$

and hence $K[x] \subseteq \langle 1, x, \dots, x^{m-1} \rangle$.

The previous lemma proves that 2) $\implies$ 3).

It is trivial that 3) $\implies$ 4).

It remains to prove that 4) $\implies$ 1). Since $x \neq 0$, $1/x \in K(x) = K[x]$. There exists $a_0, \dots, a_n \in K$ such that $1/x = a_0 + a_1 x + \cdots + a_n x^n$. Thus

$$a_n x^{n+1} + \cdots + a_1 x^2 + a_0 x - 1 = 0,$$

and hence x is a root of $a_n X^{n+1} + \cdots + a_0 X - 1 \in K[X] \setminus \{0\}$. □

Note that if x is algebraic over K, then $K[x] \simeq K[X]/(f(x,K))$.

Exercise 2.8. Let E/K be an extension and $x \in E$ be an algebraic element over K. Prove that the degree of x over K is equal to $[K(x) : K]$.

Corollary 2.9. *If E/K is finite, then E/K is algebraic.*

Proof. Let $n = [E : K]$ and $x \in E \setminus K$. The set $\{1, x, \dots, x^n\}$ has $n+1$ elements, so it is linearly dependent. There exist $a_0, \dots, a_n \in K$, not all zero, such that

$$a_0 + a_1 x + \cdots + a_n x^n = 0.$$

Thus x is a root of the non-zero polynomial $a_0 + a_1 X + \cdots + a_n X^n \in K[X]$. □

In Example 1.35 we proved that $\sqrt{2} + \sqrt[3]{3}$ and $\sqrt{2}\sqrt[3]{3}$ are algebraic over $\mathbb{Q}$. This can be easily proved now with Corollary 2.9.

Exercise 2.10. Let E/K be an extension and a and b be algebraic over K. Prove that $a + b$ and ab are algebraic over K.

We note that the converse of Corollary 2.9 result does not hold.

Corollary 2.11. *If E/K is an extension and $x_1, \dots, x_n \in E$ are algebraic over K, then $K(x_1, \dots, x_n)/K$ is finite and $K(x_1, \dots, x_n) = K[x_1, \dots, x_n]$.*

Proof. We proceed by induction on n. The case $n=1$ follows immediately from the theorem. So assume the result holds for some $n\geq 1$. Since the extensions $K(x_1,\dots,x_n)/K(x_1,\dots,x_{n-1})$ and $K(x_1,\dots,x_{n-1})/K$ are both finite, it follows that $K(x_1,\dots,x_n)/K$ is finite. Moreover,

$$\begin{aligned}K(x_1,\dots,x_n) &= K(x_1,\dots,x_{n-1})(x_n)\\ &= K(x_1,\dots,x_{n-1})[x_n] = K[x_1,\dots,x_{n-1}][x_n] = K[x_1,\dots,x_n].\end{aligned}$$ □

Corollary 2.12. *Let $E=K(S)$ for some set S. Then E/K is algebraic if and only if x is algebraic over K for all $x\in S$.*

Proof. Let us prove the non-trivial implication. Let $z\in K(S)$. In particular, there exists a finite subset $T\subseteq S$ such that $z\in K(T)$. The previous result implies that $K(T)/K$ is algebraic, and hence z is algebraic. □

If E/K is an extension, let

$$\overline{K}_E = \{x\in E : x \text{ is algebraic over } K\}.$$

Corollary 2.13. *If E/K is an extension, then $\overline{K}_E$ is a subfield of E that contains K. Moreover, $K(\overline{K}_E)=\overline{K}_E$ and $K(\overline{K}_E)/K$ is algebraic.*

Proof. By definition, $K(\overline{K}_E)/K$ is algebraic. Thus $K(\overline{K}_E)\subseteq\overline{K}_E$. From this, it follows that $K(\overline{K}_E)=\overline{K}_E$. □

The following exercise is now almost trivial:

Exercise 2.14. Let E/K be an extension of finite type; this means that $E=K(S)$ for some finite set S. Prove that E/K is algebraic if and only if E/K is finite.

Let $\overline{\mathbb{Q}}=\{\alpha\in\mathbb{C}:\alpha \text{ is algebraic over } \mathbb{Q}\}$. Then $\overline{\mathbb{Q}}$ is the field of algebraic numbers. Can you compute $[\overline{\mathbb{Q}}:\mathbb{Q}]$?

Exercise 2.15. Prove that $[\mathbb{Q}[\sqrt[3]{2}]:\mathbb{Q}]=3$.

For the previous exercise, you may use Eisenstein's criterion.

Exercise 2.16. Let $E=\mathbb{Q}[i,\sqrt{2}]=\mathbb{Q}[\sqrt{2}][i]$. Prove that $[E:\mathbb{Q}]=4$.

Exercise 2.17. Let $E=\mathbb{Q}[\sqrt{2},\sqrt[3]{5}]$.

1) Compute $[E:\mathbb{Q}]$.
2) Prove that $E=\mathbb{Q}[\sqrt{2}+\sqrt[3]{5}]$.
3) Find the minimal polynomial of $\sqrt{2}+\sqrt[3]{5}$ over $\mathbb{Q}$.

Exercise 2.18. Find the minimal polynomials of $\sqrt[4]{3}i$ over $\mathbb{Q}[i]$ and over $\mathbb{Q}[\sqrt{3}]$.

Exercise 2.19. Find the minimal polynomial of $\sqrt{2}+\sqrt[3]{5}i$ over $\mathbb{Q}[i]$.

Algebraic field extensions form a nice class of extensions. The same happens with finite field extensions.

Proposition 2.20. *Let F/K be a subextension of E/K. Then E/K is algebraic if and only if E/F and F/K are algebraic.*

Proof. If E/K is algebraic, then E/F and F/K are both algebraic, as $K \subseteq F \subseteq E$. Let us assume that E/F and F/K are both algebraic. Let $x \in E$ and let L be the subextension over K generated by the coefficients of $f(x,F)$, the minimal polynomial of x over F. Then L/K is finite, since it is generated by finitely many algebraic elements. Moreover, x is algebraic over L. Since

$$[L(x):K] = [L(x):L][L:K] < \infty,$$

$L(x)/K$ is algebraic. In particular, x is algebraic over K. □

Exercise 2.21. Let F/K be a subextension of E/K. Prove that E/K is finite if and only if E/F and F/K are finite.

Let K be a field and $K \subseteq F \subseteq L$ and $K \subseteq F \subseteq L$ be fields. The **composite** of E and F is defined as

$$EF = K(E \cup F) = F(E) = E(F)$$

and it is equal to the smallest field that contains E and F. Here is the picture:

$$\begin{tikzcd}[column sep=small]
 & L \arrow[d, dash] & \\
 & EF \arrow[dl, dash] \arrow[dr, dash] & \\
E \arrow[dr, dash] & & F \arrow[dl, dash] \\
 & K &
\end{tikzcd}$$

Exercise 2.22. Let E/K and F/K be algebraic field extensions. Prove that

$$EF = \left\{ \sum_{i=1}^{m} e_i f_i : m \in \mathbb{Z}_{>0}, e_i \in E, f_i \in F \text{ for all } i \in \{1,\dots,m\} \right\}.$$

Exercise 2.23. If $E = \mathbb{Q}(\sqrt{2})$ and $F = \mathbb{Q}(\sqrt{3})$, then $EF = \mathbb{Q}(\sqrt{2},\sqrt{3})$. Compute $[\mathbb{Q}(\sqrt{2},\sqrt{3}) : \mathbb{Q}]$ and $\mathbb{Q}(\sqrt{2}) \cap \mathbb{Q}(\sqrt{3})$.

Exercise 2.24. Let $\xi \in \mathbb{C}$ be a primitive cubic root of one. If $E = \mathbb{Q}(\sqrt[3]{2})$ and $F = \mathbb{Q}(\xi)$, then $EF = \mathbb{Q}(\sqrt[3]{2},\xi)$. Compute $[\mathbb{Q}(\sqrt[3]{2},\xi) : \mathbb{Q}]$ and $\mathbb{Q}(\sqrt[3]{2}) \cap \mathbb{Q}(\xi)$.

Exercise 2.25. Let E/K and F/K be extensions, where both E and F are subfields of a field L. If F/K is algebraic, then EF/E is algebraic.

Exercise 2.26. Let E/K and F/K be extensions, where both E and F are subfields of a field L. If F/K is finite, then EF/E is finite.

The solution to the previous exercise shows, in particular, that $[EF : E] \leq [F : K]$.

Lecture 3. 26/02/2024

LEMMA 3.1. *Let $\sigma\colon K\to L$ be a field homomorphism. Then there exists an extension E/K and a field isomorphism $\varphi\colon E\to L$ such that $\varphi|_K=\sigma$.*

PROOF. Note that $\sigma\colon K\to\sigma(K)$ is bijective. Let A be a set in bijection with $L\setminus\sigma(K)$ and disjoint with K. Let $E=K\cup A$. If $\theta\colon A\to L\setminus\sigma(K)$ is bijective, then let

$$\varphi\colon E\to L,\quad \varphi(x)=\begin{cases}\sigma(x) & \text{if } x\in K,\\ \theta(x) & \text{if } x\in A.\end{cases}$$

Then φ is a bijective map such that $\varphi|_K=\sigma$. Transport the operations of L onto E, that is to define binary operations on E as follows:

$$(x,y)\mapsto x\oplus y=\varphi^{-1}(\varphi(x)+\varphi(y)),\qquad (x,y)\mapsto x\odot y=\varphi^{-1}(\varphi(x)\varphi(y)).$$

Then, for example,

$$x\oplus y=\varphi^{-1}(\varphi(x)+\varphi(y))=\varphi^{-1}(\sigma(x)+\sigma(y))=\varphi^{-1}(\sigma(x+y))=\varphi^{-1}(\varphi(x+y))=x+y$$

for all $x,y\in K$. □

If $\sigma\colon A\to B$ is a ring homomorphism, then σ induces a ring homomorphism $\overline{\sigma}\colon A[X]\to B[X]$, $\sum_i a_iX^i\mapsto\sum_i\sigma(a_i)X^i$.

THEOREM 3.2. *Let K be a field and $f\in K[X]$ be such that $\deg f>0$. Then there exists an extension E/K such that f admits a root in E.*

PROOF. We may assume that f is irreducible over K. Let $L=K[X]/(f)$ and $\pi\colon K[X]\to L$ be the canonical map. Then L is a field (the reader should explain why). Let $\sigma\colon K\to L$, $a\mapsto\pi(aX^0)$, and $g=\overline{\sigma}(f)\in L[X]$.

We claim that $\pi(X)$ is a root of g in L. Suppose that $f=\sum_i a_iX^i$. Then

$$\begin{aligned}g(\pi(X))&=\overline{\sigma}(f)(\pi(X))\\&=\sum_i\sigma(a_i)\pi(X)^i=\sum_i\pi(a_iX^0)\pi(X^i)=\pi(\sum a_iX^i)=\pi(f)=0.\end{aligned}$$

The previous lemma states that there exists an extension E/K and an isomorphism $\varphi\colon E\to L$ such that $\varphi|_K=\sigma$. Note that $\varphi(x)=0$ if and only if $x=0$. If $u=\pi(X)$, then $\varphi^{-1}(u)$ is a root of f in E, as

$$\begin{aligned}\varphi(f(\varphi^{-1}(u)))&=\varphi\left(\sum_i a_i\varphi^{-1}(u)^i\right)=\varphi\left(\sum_i a_i\varphi^{-1}(u^i)\right)\\&=\sum_i\varphi(a_i)u^i=\sum_i\sigma(a_i)u^i=g(u)=0.\end{aligned}$$

□

As a corollary, if K is a field and $f_1,\dots,f_n\in K[X]$ are polynomials of positive degree, then there exists an extension E/K such that each f_i admits a root in E. This is proved by induction on n.

DEFINITION 3.3. A field K is **algebraically closed** if each $f\in K[X]$ of positive degree admits a root in K.

The *fundamental theorem of algebra* states that $\mathbb{C}$ is algebraically closed. A typical proof uses complex analysis. Later we will give a proof of this result using Galois theory.

PROPOSITION 3.4. *The following statements are equivalent:*

1) *K is algebraically closed.*
2) *If $f \in K[X]$ is irreducible, then* $\deg f = 1$.
3) *If $f \in K[X]$ is non-zero, then f decomposes linearly in $K[X]$, that is*

$$f = a\prod_{i=1}^{n}(X-\alpha_i)^{m_i}$$

for some $a \in K$ and $\alpha_1,\dots,\alpha_n \in K$.
4) *If E/K is algebraic, then $E = K$.*

Proof. $1) \implies 2 \implies 3)$ are exercises.
Let us prove that $3) \implies 4)$. Let $x \in E$. Decompose $f(x,K)$ linearly in $K[X]$ as

$$f(x,K) = a\prod_{i=1}^{n}(X-\alpha_i)^{m_i}$$

and evaluate on x to obtain that $x = \alpha_j$ for some j.

To prove that $4) \implies 1)$ let $f \in K[X]$ be such that $\deg f > 0$. There exists an extension E/K such that f has a root x in E. The extension $K(x)/K$ is algebraic and hence $K(x) = K$, so $x \in K$. □

§ 3.1. Artin's theorem.

Definition 3.5. An **algebraic closure** of a field K is an algebraic extension C/K such that C is algebraically closed.

For example, $\mathbb{C}/\mathbb{R}$ is an algebraic closure but $\mathbb{C}/\mathbb{Q}$ is not.

Proposition 3.6. *Let C be algebraically closed and $\sigma\colon K \to C$ be a field homomorphism. If E/K is algebraic, then there exists a field homomorphism $\varphi\colon E \to C$ such that $\varphi|_K = \sigma$.*

Proof. Suppose first that $E = K(x)$ and let $f = f(x,K)$. Let $\overline{\sigma}(f) \in C[X]$ and let $y \in C$ be a root of $\overline{\sigma}(f)$. If $z \in E$, then $z = g(x)$ for some $g \in K[X]$. Let $\varphi\colon E \to C$, $z \mapsto \overline{\sigma}(g)(y)$.

The map φ is well-defined. If $z = h(x)$ for some $h \in K[X]$, then

$$0 = g(x) - h(x) = (g-h)(x)$$

and thus f divides $g-h$. In particular, $\overline{\sigma}(f)$ divides $\overline{\sigma}(g-h) = \overline{\sigma}(g) - \overline{\sigma}(h)$ and hence

$$(\overline{\sigma}(g) - \overline{\sigma}(h))(y) = 0.$$

It is an exercise to show that the map φ is a ring homomorphism.
Let $a \in K$. It follows that $\varphi|_K = \sigma$, as

$$\varphi(a) = \overline{\sigma}(aX^0)(y) = \sigma(a)$$

Let us now prove the proposition in full generality. Let X be the set of pairs (F,τ), where F is a subfield of E that contains K and $\tau\colon F \to C$ is a field homomorphism such that $\tau|_K = \sigma$. Note that $(K,\sigma) \in X$, so X is non-empty. Moreover, X is partially ordered by

$$(F,\tau) \leq (F_1,\tau_1) \iff F \subseteq F_1 \text{ and } \tau_1|_F = \tau.$$

If $\{(F_i,\tau_i) : i \in I\}$ is a chain in X, then $F = \cup_{i\in I} F_i$ is a subfield of E that contains K. Moreover, if $z \in F$, then $z \in F_i$ for some $i \in I$ and then one defines $\tau(z) = \tau_i(z)$. It is an exercise to prove that τ is well-defined. Since $(F,\tau) \in X$ is an upper bound, Zorn's lemma implies that there exists a maximal element $(E_1,\theta) \in X$. We claim that $E = E_1$. If not, let $z \in E \setminus E_1$. Since we know the proposition is true for the extension $E_1(z)/E_1$, let $\rho\colon E_1(z) \to C$ be a field homomorphism such that $\rho|_{E_1} = \theta$.

Then, in particular, $\rho|_K = \sigma$. This implies that $(E_1(z), \rho) \in X$ and hence $(E_1, \theta) < (E_1(z), \rho)$, a contradiction to the maximality of (E_1, θ). $\square$

Lecture 4. 04/03/2024

The previous proposition will be used to prove that the algebraic closure always exists.

THEOREM 4.1 (Artin). *Let K be a field. Then K admits an algebraic closure C/K. If C_1/K is an algebraic closure, then the extensions C/K and C_1/K are isomorphic.*

PROOF. Let us first prove the uniqueness. The previous proposition implies the existence of an extension homomorphism $\varphi\colon C \to C_1$. Let $y \in C_1$ and $f = f(y,K)$ be the minimal polynomial of y in K. Since f admits a factorization

$$f = \lambda \prod (X-\alpha_i)^{m_i}$$

in $C[X]$, it follows that

$$f = \overline{\varphi}(f) = \varphi(\lambda) \prod (X-\varphi(\alpha_i))^{m_i}$$

Since $0 = f(y)$, we conclude that $y = \varphi(\alpha_j)$ for some j. In particular, φ is surjective and hence φ is bijective.

We now prove the existence. Let us assume that K admits an extension E/K with E algebraically closed. We will prove later that this extension indeed exists; at the moment, we only want to get an algebraic extension from this setting. Let

$$F = \{x \in E : x \text{ is algebraic over } K\}.$$

Then F/K is algebraic. Let $g \in F[X]$ be such that $\deg g > 0$. Since E is algebraically closed, g admits a root α in E. In particular, α is algebraic over F and hence α is algebraic over K. This implies that $\alpha \in F$, thus F is algebraically closed. This proves that F/K is an algebraic closure.

Let us prove that there exists an extension E_1/K such that every polynomial $f \in K[X]$ with $\deg f > 0$ has a root in E_1. Let $\{f_i : i \in I\}$ be the family of monic irreducible polynomials with coefficients in K. We may think that $f_i = f_i(X_i)$. Let $R = K[\{X_i : i \in I\}]$ and let J be the ideal of R generated by the $f_i(X_i)$. We claim that $J \neq R$. If not, $1 \in J$, so

$$1 = \sum_{j=1}^{m} g_j f_{i_j}(X_{i_j})$$

for some $g_1, \dots, g_m \in R$. There exists an extension F/K such that f_{i_j} has a root α_j in F for all j. Let

$$\tau\colon R \to F, \quad \tau(X_k) = \begin{cases} \alpha_j & \text{if } k = i_j, \\ 0 & \text{if } k \notin \{i_1, \dots, i_m\}. \end{cases}$$

Then τ is a ring homomorphism and

$$1 = \tau(1) = \sum_{j=1}^{m} \tau(g_j) f_{i_j}(\alpha_j) = 0,$$

a contradiction.

Since J is a proper ideal, it is contained in a maximal ideal M. Let $L = R/M$ and let $\sigma\colon K \to L$ be the composition $K \hookrightarrow R \to R/M = L$, where $\pi\colon R \to R/M$ is the canonical map. As we did before, $\pi(X_i)$ is a root of $\overline{\sigma}(f_i)$ for all i. And there exists an extension E_1/K such that every f_i has a root in E_1. Proceeding in this way, we construct a sequence

$$E_1 \subseteq E_2 \subseteq \cdots$$

of fields such that every polynomial of positive degree and coefficients in E_k admits a root in E_{k+1}. Let $E = \cup E_k$. We claim that E is algebraically closed. In fact, let $g \in E[X]$ be such that $\deg g > 0$. Then, since $g \in E_r[X]$ for some r, it follows that g has a root in $E_{r+1} \subseteq E$. □

§ 4.1. Decomposition fields.

Definition 4.2. Let K be a field and $f \in K[X]$ be such that $\deg f > 0$. A **decomposition field** of f over K is a field E that contains K and that satisfies the following properties:

1) f factorizes linearly in $E[X]$.
2) If F is a field such that $K \subseteq F \subseteq E$ and f factorizes linearly in $F[X]$, then $F = E$.

Easy examples:

Example 4.3. $\mathbb{C}$ is a decomposition field of $X^2 + 1 \in \mathbb{R}[X]$.

Example 4.4. $\mathbb{Q}[\sqrt{2}]$ is a decomposition field of $X^2 - 2 \in \mathbb{Q}[X]$.

Example 4.5. The decomposition field of $f = X^2 - 2$ over $\mathbb{Z}/7$ is precisely $\mathbb{Z}/7$, as 3 and 4 are the roots of f in $\mathbb{Z}/7$.

Example 4.6. $\mathbb{Q}(\sqrt[3]{2})$ is not a decomposition field of $X^3 - 2 \in \mathbb{Q}[X]$. However, if ω is a primitive cubic root of one, then $\mathbb{Q}(\sqrt[3]{2}, \omega)$ is a decomposition field of the polynomial $X^3 - 2 \in \mathbb{Q}[X]$.

Proposition 4.7. *E is a decomposition field of $f \in K[X]$ if and only if f factorizes linearly in $E[X]$ and $E = K(x_1, \dots, x_n)$, where $x_1, \dots, x_n$ are the roots of f.*

Proof. Let $f = a \prod_{i=1}^{r} (X - x_i)^{n_i}$ and $F = K(x_1, \dots, x_r)$ with $x_1, \dots, x_r \in E$. Since f factorizes linearly in $F[X]$, it follows that $F = E$. Conversely, let $E = K(x_1, \dots, x_r)$ and assume that f factorizes linearly in $F[X]$. Then, in particular, $x_1, \dots, x_r \in F$. Hence $E \subseteq F$ and $F = E$. □

One immediately obtains the following consequence: If E is a decomposition field of $f \in K[X]$, then E/K is finite.

Theorem 4.8. *Let $f \in K[X]$ be such that* $\deg f > 0$. *There exists a (unique up to extension isomorphism) decomposition field of f over K.*

Proof. Let C/K be an algebraic closure of K. Write

$$f = a \prod_{i=1}^{r} (X - x_i)^{n_i}$$

in $C[X]$. Then $E = K(x_1, \dots, x_r)$ is a decomposition field of f over K.

Let us prove the uniqueness: if E_1/K is a decomposition field of f over K, then E_1/K is algebraic and thus Proposition 3.6 implies that there exists $\varphi \in \operatorname{Hom}(E_1/K, C/K)$, that is $\varphi \colon E_1 \to C$ is a field homomorphism such that $\varphi|_K$ is the identity. Factorize f linearly in $E_1[X]$ and apply $\overline{\varphi}$:

$$f = a \prod_{j=1}^{s} (X - y_j)^{m_j} \implies f = \overline{\varphi}(f) = \varphi(a) \prod_{j=1}^{s} (X - \varphi(y_j))^{m_j}$$

so f factorizes linearly in $\varphi(E_1)[X]$. Moreover, $E_1 = K(y_1, \dots, y_s)$ and

$$\varphi(E_1) = K(\varphi(y_1), \dots, \varphi(y_s)).$$

Thus $\varphi(E_1)$ is a decomposition field of f. Since $\varphi(E_1) \subseteq C$, it follows that $\varphi(E_1) = E$. □

Exercise 4.9. If C is an algebraic closure of K and $\varphi \in \mathrm{Hom}(C/K, C/K)$, then φ is an isomorphism.

Let C be an algebraic closure of K and $G = \mathrm{Gal}(C/K)$. The group G acts on C

$$\sigma \cdot x = \sigma(x), \quad \sigma \in G, x \in C.$$

The orbits are of the form

$$O_G(x) = \{\sigma(x) : \sigma \in G\} = \{y \in C : y = \sigma(x) \text{ for some } \sigma \in G\}$$

The elements $x, y \in C$ are **conjugate** if $y = \sigma(x)$ for some $\sigma \in G$.

Proposition 4.10. *Let C be an algebraic closure of K and $x, y \in C$. Then x and y are conjugate if and only if $f(x,K) = f(y,K)$. In particular, $O_G(x)$ is finite.*

Proof. Let $G = \mathrm{Gal}(C/K)$. If x and y are conjugate, say $y = \sigma(x)$ for some $\sigma \in G$, let us write $g = f(x,K)$ as

$$g = X^n + \sum_{i=0}^{n-1} a_i X^i$$

for some $n \geq 1$ and $a_0, \dots, a_{n-1} \in K$. Then $0 = g(x) = x^n + \sum_{i=0}^{n-1} a_i x^i$ and hence y is a root of g, as

$$\begin{aligned} 0 = \sigma\left(x^n + \sum_{i=0}^{n-1} a_i x^i\right) &= \sigma(x)^n + \sum_{i=0}^{n-1} \sigma(a_i)\sigma(x)^i \\ &= \sigma(x)^n + \sum_{i=0}^{n-1} a_i \sigma(x)^i = y^n + \sum_{i=0}^{n-1} a_i y^i. \end{aligned}$$

Thus $f(y,K) = g$.

Conversely, assume that $f(x,K) = f(y,K)$. Let $g = f(x,K) = f(y,K)$ and let

$$\varphi \colon K[x] \to K[y], \quad h(x) \mapsto h(y).$$

Let us show that the map φ is well-defined: we need to show that if $h_1(x) = h_2(x)$, then

$$h_1(y) = \varphi(h_1(x)) = \varphi(h_2(x)) = h_2(y).$$

If $h_1(x) = h_2(x)$, then

$$(h_1 - h_2)(x) = h_1(x) - h_2(x) = 0.$$

This implies that g divides $h_1 - h_2$. In particular, $h_1(y) = h_2(y)$.

A straightforward calculation shows that φ is a field homomorphism such that $\varphi|_K = \mathrm{id}$, this means that φ is an extension homomorphism such that $\varphi(x) = y$. There exists $\sigma \in \mathrm{Hom}(C/K, C/K)$ such that $\sigma|_{K[x]} = \varphi$. Since σ is bijective (this is left as an exercise, you did something similar before), $\sigma(x) = \varphi(x) = y$ and hence $O_G(x) = O_G(y)$. $\square$

Proposition 4.11. *Let C be an algebraic closure of K and $x \in C$. Then*

$$f(x,K) = \prod_{y \in O_G(x)} (X - y)^m$$

for some m.

PROOF. For each $y \in O_G(x)$ let m_y be the multiplicity of y in $f(x,K)$. Then, for example, $f(x,K) = (X-x)^{m_x} g$ for some g. If $y \in O_G(x)$, then $y = \sigma(x)$ for some $\sigma \in \mathrm{Gal}(C/K)$. Since

$$\overline{\sigma}(f(x,K)) = f(x,K) = (X-y)^{m_x}\overline{\sigma}(g),$$

it follows that $m_y \geq m_x$. By symmetry, we conclude that $m_x = m_y$. □

The previous proposition shows, in particular, that all the roots of an irreducible polynomial $f \in K[X]$ in an algebraic closure C of K have the same multiplicity. This is not true if f is not irreducible. Find an example.

DEFINITION 4.12. Let K be a field and $\{f_i : i \in I\}$ be a non-empty family of polynomials of positive degree with coefficients in K. A **decomposition field** of $\{f_i : i \in I\}$ is an extension E/K such that every f_i factorizes linearly in $E[X]$ and if F/K is a sub extension of E/K such that every f_i factorizes linearly in $F[X]$, then $F = E$.

EXERCISE 4.13. Prove that E/K is a decomposition field of $\{f_i : i \in I\}$ if and only if every f_i factorizes linearly in $E[X]$ and $E = K(S)$ where $S = \{\text{roots of } f_i \text{ for all } i\}$.

EXERCISE 4.14. Prove that if E/K is a decomposition field of $\{f_i : i \in I\}$, then E/K is algebraic. If, moreover, I is finite, then E/K is a decomposition field of $\prod_{i \in I} f_i$.

EXERCISE 4.15. Prove that there exists a decomposition field of $\{f_i : i \in I\}$ and it is unique up to extension isomorphism.

EXERCISE 4.16. Let $f = X^3 - X - 1 \in (\mathbb{Z}/3)[X]$ and E be a decomposition field of f. Compute $[E : \mathbb{Z}/3]$.

What about the decomposition field of $f = X^3 - X - 1 \in \mathbb{Q}[X]$?

EXERCISE 4.17. Let $f = X^4 - 5x^2 + 5 \in \mathbb{Q}[X]$ and E be a decomposition field of f. Compute $[E : \mathbb{Q}]$ and $\mathrm{Gal}(E/\mathbb{Q})$.

Lecture 5. 11/03/2024

§ 5.1. Normal extensions.

PROPOSITION 5.1. *Let E/K be an algebraic extension and $\sigma \in \operatorname{Hom}(E/K, E/K)$. Then σ is bijective.*

PROOF. It is enough to prove that σ is surjective. Why? Let $x \in E$ and C be an algebraic closure of K that contains E. By Proposition 3.6, there exists a field homomorphism $\varphi \colon C \to C$ such that $\varphi|_E = \sigma$. Thus $\varphi|_K = \sigma|_K = \mathrm{id}_K$. Let $G = \operatorname{Gal}(C/K)$. Then $\varphi \in G$. If $z \in O_G(x)$, then $z = \tau(x)$ for some $\tau \in G$ and hence

$$\varphi(z) = \varphi(\tau(x)) = (\varphi\tau)(x).$$

This implies that $\varphi(z) \in O_G(x)$ and $\varphi(O_G(x)) = O_G(x)$. The restriction $\sigma|_{E \cap O_G(x)}$ is injective. Then

$$\begin{aligned}\sigma(E \cap O_G(x)) &= \varphi(E \cap O_G(x)) \\ &= \varphi(E) \cap \varphi(O_G(x)) = \sigma(E) \cap O_G(x) \subseteq E \cap O_G(x).\end{aligned}$$

Since $|E \cap O_G(x)| < \infty$, it follows that $E \cap O_G(x) = \sigma(E \cap O_G(x))$ and hence x belongs to the image of σ. □

DEFINITION 5.2. Let E/K be an algebraic extension and C be an algebraic closure of K containing E. Then E/K is **normal** if $\sigma(E) \subseteq E$ for all $\sigma \in \operatorname{Hom}(E/K, C/K)$.

Note that $\sigma(E) \subseteq E$ in the previous definition is equivalent to $\sigma(E) = E$.

EXAMPLE 5.3. The extension $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is not normal. Why?

Some trivial examples of normal extensions: K/K is normal and if C is an algebraic closure of K, then C/K is normal.

EXAMPLE 5.4. The extension $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is normal. Every extension generated by algebraic elements of degree two is normal.

EXERCISE 5.5. Let ξ be a primitive cubic root of one. Then $\mathbb{Q}(\sqrt[3]{2}, \xi)/\mathbb{Q}$ is normal.

The following result is practical but technical. That is why we leave the proof as an exercise.

EXERCISE 5.6. Prove that the previous definition depends only on E (and not on the algebraic closure C).

Some properties:

PROPOSITION 5.7. *Let E/K be a normal extension and $f \in K[X]$ be an irreducible polynomial that admits a root x in E. Then f factorizes linearly in E.*

PROOF. We may assume that f is monic. Let C/K be an algebraic closure of K containing E. Let y be a root of f in C. Since $f = f(x,K) = f(y,K)$, it follows that $y = \sigma(x)$ for some $\sigma \in \operatorname{Gal}(C/K)$. Since E/K is normal, $\sigma|_E \colon E \to C$ is an automorphism of E/K, that is $\sigma(E) \subseteq E$. In particular, $y \in E$. □

Let $K \subseteq F \subseteq E$ be a tower of fields. If E/K is normal, then E/F is normal. However, Note that E/K normal does not imply F/K normal, as this would imply that every extension is normal. Moreover, E/F normal and F/K normal do not imply E/K normal.

Example 5.8. The extensions $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ are both normal, but $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$ is not normal, as the roots of X^4-2 are $\sqrt[4]{2}$, $-\sqrt[4]{2}$, $\sqrt[4]{2}i$ and $-\sqrt[4]{2}i$.

Recall that if C is an algebraic closure of K and $x \in C$, then

$$f(x,K) = \prod (X-y)^m,$$

where the product is taken over all $y \in O_{\mathrm{Gal}(C/K)}(x)$. If E/K is normal and $x \in E$, then there exists m such that

$$f(x,K) = \prod (X-y)^m,$$

where the product is taken over all $y \in O_{\mathrm{Gal}(E/K)}(x)$.

Proposition 5.9. *Let E/K and F/K be extensions. If F/K is normal, then EF/E is normal.*

Proof. Let C be an algebraic closure of E containing EF (this exists because EF/E is algebraic). Let $\sigma \in \mathrm{Hom}(EF/E, C/E)$. We claim that $\sigma(EF) = EF$. Let

$$\overline{K} = \{x \in C : x \text{ is algebraic over } K\}.$$

Then $\overline{K}$ is an algebraic closure over K and $F \subseteq \overline{K}$. Since F/K is normal and $\sigma|_F \in \mathrm{Hom}(F/K, \overline{K}/K)$, it follows that $\sigma(F) = F$. If $z \in EF$, then $z = \sum_{i=1}^m e_i f_i$ for some $e_1, \dots, e_m \in E$ and $f_1, \dots, f_m \in F$. Since $\sigma(e_i) = e_i$ for all i,

$$\sigma(z) = \sum_{i=1}^m \sigma(e_i)\sigma(f_i) = \sum_{i=1}^m e_i \sigma(f_i) \in EF. \qquad \square$$

What is the relation between normal extensions and decomposition fields? The notions look deeply related. The following proposition serves as an explanation:

Proposition 5.10. *Let E/K be an algebraic extension. Then E/K is normal if and only if E/K is the decomposition field of a family of polynomials of $K[X]$ of positive degree.*

Proof. Assume first that E/K is a normal extension. Let $G = \mathrm{Gal}(E/K)$. If $x \in E$ and $f(x,K) = \prod_{y \in O_G(x)} (X-y)^m$, then $f(x,K)$ factorizes linearly in $E[X]$. Thus E/K is a decomposition field of the family $\{f(x,K) : x \in E\}$.

Conversely, assume that E/K is a decomposition field of the family $\{f_i : i \in I\}$. Then $E = K(S)$ where S is the set of roots of the polynomials f_i. Let C/K be an algebraic closure of K that contains E and let $\sigma \in \mathrm{Hom}(E/K, C/K)$. Let $x \in S$. Then x is a root of some $f_j = \sum a_k X^k$. Since $f_j(x) = 0$, it follows that $\sigma(x)$ is a root of f_j, as

$$f_j(\sigma(x)) = \sum a_k \sigma(x)^k = \sum \sigma(a_k)\sigma(x^k) = \sigma\left(\sum a_k x^k\right) = \sigma(0) = 0.$$

Hence $\sigma(E) \subseteq E$. $\square$

Exercise 5.11. Let $E = \mathbb{Q}[\sqrt[4]{7} + \sqrt{2}]$.

1) Prove that $E/\mathbb{Q}$ is not normal.
2) Compute $[E : \mathbb{Q}]$.
3) Compute $\mathrm{Gal}(E/\mathbb{Q})$.

§ 5.2. Dedekind's theorem. Note that every extension homomorphism $E/K \to F/K$ is, in particular, a K-linear map $E \to F$, that is

$$\operatorname{Hom}(E/K, F/K) \subseteq \operatorname{Hom}_K(E, F).$$

If F/K is an extension and V is a K-vector space, the set $\operatorname{Hom}_K(V, F)$ of K-linear maps is a vector space over F with $(a \cdot f)(v) = af(v)$ for $a \in F$, $f \in \operatorname{Hom}_K(V, F)$ and $v \in V$.

Exercise 5.12. Let V be a K-vector space. Prove that $\dim_F \operatorname{Hom}_K(V, F) \geq \dim_K V$. Moreover, if $\dim_K V < \infty$, then $\dim_F \operatorname{Hom}_K(V, F) = \dim_K V$.

If V is a vector space and S is a (possibly infinite) subset of V, then S is linearly independent if every finite subset of S is linearly independent.

Theorem 5.13 (Dedekind). *Let E/K and F/K be extensions and let $\{\varphi_i : i \in I\}$ be a subset of $\operatorname{Hom}(E/K, F/K)$, i.e. a family of extension homomorphisms. Assume that $\varphi_i \neq \varphi_j$ if $i \neq j$. Then the subset $\{\varphi_i : i \in I\} \subseteq \operatorname{Hom}_K(E, F)$ is linearly independent over F.*

Proof. Assume it is not. Let $\{\varphi_1, \dots, \varphi_n\}$ be linearly dependent over F with n minimal. Clearly, $n > 1$. Without loss of generality, we may assume that

$$\sum_{i=1}^{n} a_i \varphi_i = 0 \tag{5.1}$$

for some $a_1, \dots, a_n \in F$ all different from zero. Let $z \in E \setminus \{0\}$ be such that $\varphi_1(z) \neq \varphi_2(z)$. If $x \in E$, then

$$0 = \left(\sum_{i=1}^{n} a_i \varphi_i\right)(xz) = \sum_{i=1}^{n} a_i \varphi_i(xz) = \sum_{i=1}^{n} a_i \varphi_i(x)\varphi_i(z) = \left(\sum_{i=1}^{n} (a_i \varphi_i(z))\varphi_i\right)(x).$$

Thus

$$\sum_{i=1}^{n} (a_i \varphi_i(z))\varphi_i = 0.$$

Since $\varphi_1(z) \neq 0$,

$$a_1 \varphi_1 + a_2 \frac{\varphi_2(z)}{\varphi_1(z)} \varphi_2 + \cdots + a_n \frac{\varphi_n(z)}{\varphi_1(z)} \varphi_n = 0. \tag{5.2}$$

Thus, subtracting (5.1) and (5.2),

$$\left(a_2 - a_2 \frac{\varphi_2(z)}{\varphi_1(z)}\right) \varphi_2 + \cdots + \left(a_n - a_n \frac{\varphi_n(z)}{\varphi_1(z)}\right) \varphi_n = 0.$$

Since $a_n \neq 0$ and $\varphi_2(z) \neq \varphi_1(z)$, the scalar $a_2 - a_2 \frac{\varphi_2(z)}{\varphi_1(z)} \neq 0$ and hence $\{\varphi_2, \dots, \varphi_n\}$ is linearly dependent, a contradiction. $\square$

If E/K and F/K are extensions, let $\gamma(E/K, F/K) = |\operatorname{Hom}(E/K, F/K)|$.

Exercise 5.14. Prove the following statements:

1) $\gamma(E/K, F/K) \leq \dim_F \operatorname{Hom}_K(E, F)$.
2) If $[E : K] < \infty$, then $\gamma(E/K, F/K) \leq [E : K]$.
3) If x is algebraic over K, then $\gamma(K(x)/K, F/K) \leq \deg f(x, K)$.

If C is an algebraic closure of K, then we define $\gamma(E/K) = \gamma(E/K, C/K)$. This definition does not depend on the algebraic closure.

Exercise 5.15. If C and C_1 are algebraic closures of K, then

$$|\operatorname{Hom}(E/K, C/K)| = |\operatorname{Hom}(E/K, C_1/K)|.$$

Proposition 5.16. *Let C be an algebraic closure of K and $G = \operatorname{Gal}(C/K)$. If $x \in C$, then $\gamma(K(x)/K) = |O_G(x)|$.*

Proof. If $\sigma \in \operatorname{Hom}(K(x)/K, C/K)$, then there exists $\phi \in G$ such that $\phi|_{K(x)} = \sigma$. Thus

$$\sigma(x) = \phi(x) \in O_G(x).$$

Conversely, if $y \in O_G(x)$, then there exists $\tau \in G$ such that $y = \tau(x)$. Hence

$$\tau|_{K(x)} \in \operatorname{Hom}(K(x)/K, C/K)$$

and $\tau|_{K(x)}(x) = y$. Since our sets are then in bijective correspondence, the claim follows. $\square$

Exercise 5.17. If E/K is finite, then $|\operatorname{Gal}(E/K)| \leq [E : K]$. Moreover, E/K is normal if and only if $|\operatorname{Gal}(E/K)| = \gamma(E/K)$.

Lecture 6. 18/03/2024

If $t\colon A \to B$ is a surjective map, then $a \sim a_1 \iff t(a) = t(a_1)$ defines an equivalence relation on A. The set $\overline{A}$ of equivalence classes is in bijective correspondence with B, $\overline{A} \to B$, $\overline{a} \mapsto t(a)$. Moreover, if $|t^{-1}(\{b\})| = m$ for all $b \in B$, then $|A| = m|\overline{A}| = m|B|$.

Proposition 6.1. *Let E/K be algebraic and F/K be a subextension such that E/F is finite. Then $\gamma(E/K) = \gamma(E/F)\gamma(F/K)$.*

Proof. Assume first that $E = F(x)$. Let C be an algebraic closure of K containing E and $G = \mathrm{Gal}(C/F)$. Let $f = f(x,F) = \sum b_i X^i$.

The map
$$\lambda\colon \mathrm{Hom}(E/K,C/K) \to \mathrm{Hom}(F/K,C/K), \quad \sigma \mapsto \sigma|_F,$$
is well-defined. It is surjective: if $\varphi \in \mathrm{Hom}(F/K,C/K)$, then $\varphi\colon F \to C$ is, in particular, a field homomorphism. Since E/F is algebraic, by Proposition 3.6 there exists a field homomorphism $\sigma\colon E \to C$ such that $\sigma|_F = \varphi$. Since $\sigma|_K = \varphi|_K = \mathrm{id}$, in particular $\sigma \in \mathrm{Hom}(E/K,C/K)$.

For $\varphi \in \mathrm{Hom}(F/K,C/K)$,
$$\lambda^{-1}(\{\varphi\}) = \{\sigma \in \mathrm{Hom}(E/K,C/K) : \sigma|_F = \varphi\}$$
and let R_φ be the set of roots (in C) of the polynomial $\overline{\varphi}(f) = \sum \varphi(b_i)X^i$.

Claim. The map $\alpha\colon \lambda^{-1}(\{\varphi\}) \to R_\varphi$, $\sigma \mapsto \sigma(x)$, is well-defined.

We need to show that $\sigma(x)$ is a root of $\overline{\varphi}(f)$:
$$\begin{aligned}\overline{\varphi}(f)(\sigma(x)) &= \sum \varphi(b_i)\sigma(x)^i = \sum \sigma(b_i)\sigma(x^i)\\ &= \sum \sigma(b_i x^i) = \sigma\left(\sum b_i x^i\right) = \sigma(f(x)) = \sigma(0) = 0.\end{aligned}$$

Claim. The map $\beta\colon R_\varphi \to \lambda^{-1}(\{\varphi\})$, $y \mapsto \sigma_y$, where $\sigma_y(z) = \overline{\varphi}(h)(y)$ if $z = h(x)$, is well-defined.

We need to show that if $z = h(x)$ and $z = h_1(x)$ for some $h, h_1 \in F[X]$, then $\overline{\varphi}(h)(y) = \overline{\varphi}(h_1)(y)$. The assumptions imply that $(h - h_1)(x) = 0$ and hence f divides $h - h_1$. Since $\overline{\varphi}$ is a ring homomorphism, $\overline{\varphi}(f)$ divides $\overline{\varphi}(h) - \overline{\varphi}(h_1)$. This implies $(\overline{\varphi}(h) - \overline{\varphi}(h_1))(y) = 0$. We also need to show that $\sigma_y|_F = \varphi$: if $a \in F$, then write $a = aX^0 \in F[X]$. Thus $\sigma_y(a) = \overline{\varphi}(aX^0)(y) = \varphi(a) \in C$. It is now an exercise to prove that $\sigma_y \in \mathrm{Hom}(E/K,C/K)$.

Claim. $|\lambda^{-1}(\{\varphi\})| = |R_\varphi|$.

For this we need to show that β is the inverse of α, that is $\alpha \circ \beta = \mathrm{id}$ and $\beta \circ \alpha = \mathrm{id}$. To prove that $\beta \circ \alpha = \mathrm{id}$ let σ be such that $\sigma|_F = \varphi$. Then $y = \sigma(x) \in R_\varphi$. Let $z = h(x) = \sum a_i x^i \in F[x] = E$. Then
$$\overline{\varphi}(h)(y) = \sum \varphi(a_i)y^i = \sum \sigma(a_i)y^i = \sigma\left(\sum a_i x^i\right) = \sigma(z).$$
Conversely, if $y \in R_\varphi$, then
$$\alpha(\sigma_y) = \sigma_y(x) = y,$$
as $\sigma_y(x) = \overline{\varphi}(X)(y) = y$.

Claim. If $\phi \in \mathrm{Gal}(C/K)$ is such that $\phi|_F = \varphi$, then $|\phi^{-1}(R_\varphi)| = |R_\varphi|$ and
$$O_G(x) = \phi^{-1}(R_\varphi).$$

Let us first prove $O_G(x) \supseteq \phi^{-1}(R_\varphi)$. If $y \in R_\varphi$, then

$$\begin{aligned} f(\phi^{-1}(y)) &= \sum b_i \phi^{-1}(y^i) = \phi^{-1}\left(\sum \phi(b_i) y^i\right) \\ &= \phi^{-1}(\overline{\varphi}(f)(y)) = \phi^{-1}(0) = 0. \end{aligned}$$

Then $f(x,F) = f(\phi^{-1}(y),F)$. By Proposition 4.10, $\phi^{-1}(y) \in O_G(x)$.

Now we prove $O_G(x) \subseteq \phi^{-1}(R_\varphi)$. Let $z \in O_G(x)$. Then $\overline{\varphi}(f)(\phi(z)) = 0$, as

$$\begin{aligned} \overline{\varphi}(f)(\phi(z)) &= \sum \varphi(b_i)\phi(z^i) \\ &= \sum \phi(b_i)\phi(z^i) = \phi\left(\sum b_i z^i\right) = \phi(f(z)) = \phi(0) = 0. \end{aligned}$$

Thus $\phi(z) \in R_\varphi$ and hence $z \in \phi^{-1}(R_\varphi)$. It follows that $|\lambda^{-1}(\{\varphi\})| = |O_G(x)|$ for all φ. By using the argument before the proposition,

$$\begin{aligned} \gamma(E/K) &= |\operatorname{Hom}(E/K, C/K)| \\ &= |O_G(x)||\operatorname{Hom}(F/K, C/K)| \\ &= |O_G(x)|\gamma(F/K). \end{aligned}$$

Since $\gamma(E/F) = \gamma(F(x)/F) = |O_G(x)|$ by Proposition 5.16, the claim follows.

For the general case, we assume that $E = F(x_1,\dots,x_n)$. We proceed by induction on n. If $n = 0$, then $E = F$ and the result is trivial. If $n > 0$, let $L = F[x_1,\dots,x_{n-1}]$ and $E = L(x_n)$. The case proved implies that $\gamma(E/F) = \gamma(E/L)\gamma(L/F)$. By the inductive hypothesis, $\gamma(L/K) = \gamma(L/F)\gamma(F/K)$. Thus

$$\gamma(E/F)\gamma(F/K) = \gamma(E/L)\gamma(L/F)\gamma(F/K) = \gamma(E/L)\gamma(L/K) = \gamma(E/K),$$

again using the previous case. □

§ 6.1. Separable extensions.

Definition 6.2. Let E/K be an extension and $x \in E$ an algebraic element over K. Then x is **separable** over K if x is a simple root of $f(x,K)$.

An algebraic extension E/K is **separable** if every $x \in E$ is separable over K. Then K/K is separable.

Exercise 6.3. Prove that an element x is separable over K if and only if x is a simple root of a polynomial with coefficients in K.

If F/K is a subextension of E/K and $x \in E$ is separable over K, then x is separable over F.

Exercise 6.4. If C is an algebraic closure of K, $x \in C$ and $G = \operatorname{Gal}(C/K)$. Prove that the following statements are equivalent:

1) x is separable over K.
2) Every $y \in O_G(x)$ is separable over K.
3) $\gamma(K(x)/K) = [K(x):K] = \deg f(x,K)$.

Let K be any field and $g \in K[X]$. Let z be a root of g. Then z is a multiple root of g if and only if z is a root of g'.

Exercise 6.5. Prove that if K has characteristic zero or K is finite, then every algebraic extension of K is separable.

Example 6.6. Let $E=\mathbb{Q}(\sqrt{2},\sqrt{3})$. Then $[E:\mathbb{Q}]=4$ and $\mathrm{Gal}(E/Q)\simeq C_2\times C_2$. The extension E/Q is normal, as it is the decomposition field of $(X^2-2)(X^2-3)$ and it is separable as $\mathbb{Q}$ has characteristic zero.

Example 6.7. Let E be a decomposition field of X^4-2 over $\mathbb{Q}$. Then $E/\mathbb{Q}$ is normal and separable. Note that $E=\mathbb{Q}(\sqrt[4]{2},i)$, so

$$[E:\mathbb{Q}]=8=|\mathrm{Gal}(E/\mathbb{Q})|.$$

Let us compute $\mathrm{Gal}(E/\mathbb{Q})$. If $\sigma\in\mathrm{Gal}(E/\mathbb{Q})$, then $\sigma(\sqrt[4]{2})\in\{\sqrt[4]{2},-\sqrt[4]{2},\sqrt[4]{2}i,-\sqrt[4]{2}i\}$ and $\sigma(i)\in\{-i,i\}$. Two examples are

$$\alpha\colon\begin{cases}\sqrt[4]{2}\mapsto\sqrt[4]{2}i,\\ i\mapsto i,\end{cases}\qquad \beta\colon\begin{cases}\sqrt[4]{2}\mapsto\sqrt[4]{2},\\ i\mapsto -i.\end{cases}$$

It follows that $\mathrm{Gal}(E/\mathbb{Q})$ is isomorphic to the group $\langle\alpha,\beta\rangle$, which turns out to be isomorphic to the dihedral group of eight elements.

Another consequence: If $E=K(S)$, then E/K is separable if and only if every $x\in S$ is separable over K. One first does the case $E=K(x)$ and then proceeds by induction.

Exercise 6.8. Let $K\subseteq F\subseteq E$ be a tower of fields. Prove that E/K is separable if and only if F/K and E/F are separable.

Exercise 6.9. Let E/K and F/K be extensions. Prove that if F/K is separable, then EF/E is separable.

If E/K is algebraic, then

$$F=\{x\in E: x \text{ is separable over } K\}$$

is a subfield of E that contains K. It is known as the **separable closure** of K with respect to E. Note that $F=K(F)$, as $K(F)$ is separable because it is generated by separable elements. Moreover, F/K is separable and E/F is a **purely inseparable** extension, meaning that for every $x\in E\setminus F$, the polynomial $f(x,F)$ is not separable.

Proposition 6.10. *If E/K is separable and finite, then $E=K(x)$ for some $x\in E$.*

Proof. Let us assume that K is finite. Then E is finite and hence the multiplicative group $E^\times=E\setminus\{0\}$ is cyclic, say $E^\times=\langle x\rangle$. It follows that $E=K(x)$.

Let us now assume that K is infinite. We first consider the case $E=K(x,y)$. The general case $E=K(x_1,\dots,x_n)$ is left as an exercise, one needs to proceed by induction. Let $n=[E:K]$ and C be an algebraic closure of K containing E. Write $\mathrm{Hom}(E/K,C/K)=\{\sigma_1,\dots,\sigma_n\}$. Let

$$f=\prod_{1\le i<j\le n}\big((\sigma_i(y)-\sigma_j(y))+X(\sigma_i(x)-\sigma_j(x))\big)\in C[X].$$

Then $f \neq 0$, as f is a product of non-zero polynomials. Since K is infinite, there exists a non-zero $c \in K$ such that $f(c) \neq 0$. For any $r, s \in \{1, \ldots, n\}$ with $r \neq s$,

$$\sigma_r(y) - \sigma_s(y) + c(\sigma_r(x) - \sigma_s(x)) \neq 0,$$

as $f(c) \neq 0$. It follows that $\sigma_r(y+cx) \neq \sigma_s(y+cx)$. Thus $\gamma(K(y+cx)/K) \geq n$. Now

$$n \geq [K(y+cx) : K] = \gamma(K(y+cx)/K) \geq n,$$

so $[K(y+cx) : K] = n$ and hence $K(y+cx) = E$. □

For example, $\mathbb{Q}(\sqrt{2}, i) = \mathbb{Q}(\sqrt{2} + i)$.

Lecture 7. 24/03/2024

THEOREM 7.1 (Steinitz). *Let E/K be a finite extension. Then $E=K(x)$ for some $x\in E$ if and only if E/K admits finitely many subextensions.*

PROOF. We may assume that K is infinite; otherwise, the result is trivial. We first prove $\implies$. Let us assume that $E=K(x)$ for some x. We claim that the map

$$\begin{aligned}\Psi\colon \{F : K\subseteq F\subseteq E\} &\to \{g\in K[X] : g \text{ is a monic divisor of } f(x,K)\},\\ F &\mapsto f(x,F),\end{aligned}$$

is injective. Take F_0 such that $K\subseteq F_0\subseteq F\subseteq E$ and $f(x,F)=f(x,F_0)$. Then

$$[E:F_0]=[F_0(x):F_0]=\deg f(x,F_0)=m=[F(x):F]=[E:F]$$

and hence $F=F_0$.

In general, let F_1 and F_2 be such that $K\subseteq F_1,F_2\subseteq E$ and $f(x,F_1)=f(x,F_2)$. Let $F_0=F_1\cap F_2$. Then $f=f(x,F_1)=f(x,F_2)\in F_0[X]$ and hence $f(x,F_0)=f$. Hence we can apply what we proved before to $F_0\subseteq F_1$ and $F_0\subseteq F_2$, to obtain that $F_1=F_0=F_2$. It follows that Ψ is injective and hence there are finitely many fields between K and E.

Let us prove $\impliedby$. Let us assume that $E=K(x,y)$. For each $a\in K$, we consider the extension $K(ay+x)/K$. By assumption, there exist $a,b\in K$ such that $a\neq b$ and

$$K(x+ay)=K(x+by)=L.$$

We claim that $L=E$. Note that $x+ay\in L$ and $x+by\in L$, so $(a-b)y\in L$ and hence, since $K\subseteq L$, it follows that $y\in L$. Thus $x\in L$ and therefore $L=E$. □

As a consequence, if E/K is finite and separable, then E/K admits finitely many subextensions.

§ 7.1. Galois extensions. Let E/K be an algebraic extension. Assume that $E=K(S)$ and let C be an algebraic closure of K containing E. Let

$$T=\{y\in C : y \text{ is a root of } f(x,K) \text{ for } x\in S\}$$

and let $L=K(T)$. Then $E\subseteq L$, as $S\subseteq T$. The extension L/K is normal, as L/K is a decomposition field of the family $\{f(x,K) : x\in S\}$. Moreover, L is the smallest normal extension of K containing E. The field L is the **normal closure** of E (with respect to C).

EXERCISE 7.2. If E/K is finite, then L/K is finite

EXERCISE 7.3. If E/K is separable, then L/K is separable.

Let E/K be an extension and $S\subseteq \mathrm{Gal}(E/K)$ be a subset. the set

$${}^{S}E=\{x\in E : \sigma(x)=x \text{ for all } \sigma\in S\}$$

is a subfield of E that contains K. The subfield ${}^{S}E$ is known as the **fixed field** of S.

DEFINITION 7.4. Let E/K be an algebraic extension and $G=\mathrm{Gal}(E/K)$. Then E/K is a **Galois extension** if ${}^{G}E=K$.

Clearly, K/K is a Galois extension. Note that $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is not a Galois extension. Why?

Exercise 7.5. Prove that $\mathbb{Q}(\sqrt{2},\sqrt{3})/\mathbb{Q}$ is a Galois extension.

Exercise 7.6. If the characteristic of K is different from two, then every quadratic extension of K is a Galois extension.

Exercise 7.7. Let E/K be an algebraic extension and $G=\mathrm{Gal}(E/K)$. Let $F={}^GE$. Prove that $\mathrm{Gal}(E/F)=G$ and hence E/F is a Galois extension.

Proposition 7.8. *Let E/K be an algebraic extension. Then E/K is a Galois extension if and only if E/K is normal and separable.*

Proof. Let $G=\mathrm{Gal}(E/K)$. Let us first assume that E/K is Galois. For $x\in E$ let

$$f_x=\prod_{y\in O_G(x)}(X-y)=\sum a_iX^i\in E[X].$$

If $\varphi\in G$, then

$$\overline{\varphi}(f_x)=\prod_{y\in O_G(x)}(X-\varphi(y))=f_x,$$

as if $O_G(x)=\{\sigma_1(x),\dots,\sigma_r(x)\}$, then $\varphi(\sigma_i(x))=(\varphi\sigma_i)(x)=\sigma_j(x)$ for some j. Since

$$\sum a_iX^i=f_x=\overline{\varphi}(f_x)=\sum\varphi(a_i)X^i,$$

it follows that $a_i\in {}^GE=K$ for all i. Thus $f_x\in K[X]$ and E/K is a decomposition field of the family $\{f_x:x\in E\}$. In particular, E/K is normal. Moreover, x is a simple root of $f_x\in K[X]$ and hence x is separable over K.

Conversely, let $x\in {}^GE$. Since E/K is normal, then $f(x,K)=\prod_{y\in O_G(x)}(X-y)^m$ for some m. Since E/K is separable, $m=1$. Moreover $x\in {}^GE$, so $O_G(x)=\{x\}$. Thus $f(x,K)=\prod_{y\in O_G(x)}(X-y)=X-x$ and $x\in K$. $\square$

Definition 7.9. Let K be a field and $f\in K[X]$. Then f is **separable** if all roots of f are simple (in some algebraic closure of K).

Proposition 7.10. *Let E/K be a finite extension. Then E/K is a Galois extension if and only if E is a decomposition field over K of a separable polynomial $f\in K[X]$.*

Proof. Let us assume first that E/K is a Galois extension. Since E/K is finite and separable, $E=K(x)$ by Proposition 6.10. Then E/K is a decomposition field of $f(x,K)$ since E/K is normal. Since E/K is separable, x is separable over K. Thus x is a simple root of $f(x,K)$ and hence $f(x,K)$ is separable. Conversely, let $x_1,\dots,x_r$ be the roots of a separable polynomial $f\in K[X]$. Then $E=K(x_1,\dots,x_r)$ is separable and normal. $\square$

In the previous case, $\mathrm{Gal}(E/K)$ is known as the **Galois group** of the polynomial f. The notation is $\mathrm{Gal}(f,K)$. If $n=\deg f$ and $x_1,\dots,x_n$ are the roots of f, then any $\varphi\in\mathrm{Gal}(f,K)$ permutes the roots of f, that is φ permutes the set $\{x_1,\dots,x_n\}$. In particular, $\mathrm{Gal}(f,K)$ is isomorphic to a subgroup of $\mathbb{S}_n$ and hence $|\mathrm{Gal}(f,K)|$ divides $n!$.

Proposition 7.11. *Let E/K be a normal extension and F be the separable closure of K with respect to E. Then F/K is a Galois extension.*

PROOF. Let C/K be an algebraic closure such that $E \subseteq C$. Let $\sigma \in \operatorname{Hom}(F/K, C/K)$. and let $\varphi \in \operatorname{Hom}(E/K, C/K)$ be such that $\varphi|_F = \sigma$. Since E/K is normal, $\varphi(E) = E$. Let $x \in F$. Then $\sigma(x) = \varphi(x) \in E$. Thus $f(\sigma(x), K) = f(x, K)$ and $\sigma(x)$ is separable over K, which implies that $\sigma(x) \in F$. Thus F/K is normal. Since F/K is separable, it follows that F/K is a Galois extension by Proposition 7.8. □

Some easy facts.

EXERCISE 7.12. Let E/K be a separable extension and L/K be the normal closure of E in some algebraic closure C that contains E. Prove that L/K is a Galois extension.

EXERCISE 7.13. Let E/K be a finite extension. Prove that E/K is Galois if and only if $[E:K] = |\operatorname{Gal}(E/K)|$.

For the previous exercise, note that if E/K is a finite extension, then

$$|\operatorname{Gal}(E/K)| \leq \gamma(E/K) \leq [E:K].$$

The first inequality is equality if and only if E/K is normal. The second inequality is equality if and only if E/K is separable.

EXERCISE 7.14. Let E/K be a Galois extension and F/K be a subextension of E/K. Prove that E/F is a Galois extension.

THEOREM 7.15 (Artin). *Let E be a field and G be a finite group of automorphisms of E. If $K = {}^G E$, then E/K is a Galois extension, $[E:K] = |G|$ and $\operatorname{Gal}(E/K) = G$.*

Before proving the theorem, we need a lemma.

LEMMA 7.16. *Let E/K be a separable extension such that $\deg f(x,K) \leq m$ for all $x \in E$. Then E/K is finite and $[E:K] \leq m$.*

PROOF. Let $z \in E$ be of maximal degree. If $x \in E$, then $K(x,z)/K$ is separable. There exists y such that $K(x,z) = K(y)$. Then

$$K(z) \subseteq K(x,z) = K(y).$$

Since $\deg f(z,K) \leq \deg f(y,K)$, $\deg f(z,K) = \deg f(y,K)$. Hence $K(y) = K(z)$. In particular, $x \in K(z)$ and therefore $E = K(z)$. □

Now we are ready to prove Artin's theorem:

PROOF OF THEOREM 7.15. Note that $G \subseteq \operatorname{Gal}(E/K)$. Let $x \in E$ and

$$f_x = \prod_{y \in O_G(x)} (X - y).$$

Since $f_x \in K[X]$, the extension E/K is normal and separable (as it is a decomposition field of a family of separable polynomials), so E/K is a Galois extension. Moreover,

$$\deg f(x,K) \leq \deg f_x = |O_G(x)| \leq |G|.$$

By the previous lemma, E/K is finite and $[E:K] \leq |G|$. This implies that $|\operatorname{Gal}(E/K)| = [E:K] \leq |G|$ and hence $|\operatorname{Gal}(E/K)| = |G|$. □

EXAMPLE 7.17. Let $E = K(X,Y)$ and $\sigma\colon K[X,Y] \to E$ be the ring homomorphism given by $\sigma(X) = Y$ and $\sigma(Y) = X$. Note that σ is bijective, as $\sigma^2 = \mathrm{id}$. The map σ induces a field homomorphism $\overline{\sigma}\colon E \to E$ such that $\overline{\sigma}^2 = \mathrm{id}$. Recall that such a homomorphism is given by $f/g \mapsto \sigma(f)/\sigma(g)$. Let $G = \langle\overline{\sigma}\rangle$. Then $|G| = 2$. We claim that ${}^{G}E = K(X+Y, XY)$. Let $F = K(X+Y, XY)$. We only prove that ${}^{G}E \subseteq F$, as the other inclusion is trivial. Artin's theorem implies that $[E : {}^{G}E] = 2$ and $E = F(X)$, as X is a root of the polynomial $Z^2 - (X+Y)Z + XY$. Then $[E : F] \leq 2$ and $[{}^{G}E : F] = 1$.

Lecture 8. 15/04/2024

§ 8.1. Galois' correspondence. A **partially order set** (or *poset*) is a pair $(X,\leq)$, where X is a non-empty set and $\leq$ is a reflexive, antisymmetric and transitive relation on X. This means:

1) $x \leq x$ for all $x \in X$,
2) $x \leq y$ and $y \leq x$ imply $x = y$, and
3) $x \leq y$ and $y \leq z$ imply $x \leq z$.

Let $(X,\leq)$ be a partially ordered set and $x,y \in X$. An element z of a poset $(X,\leq)$ is an **upper bound** of x and y if $x \leq z$ and $y \leq z$. And ξ is a **least upper bound** of x and y if it is an upper bound with $\xi \leq z$ for every upper bound z of x and y. Similarly, one defines lower bounds and greatest lower bounds.

Definition 8.1. A **lattice** if a partially ordered set L in which each pair of elements $x,y \in L$ has a least upper bound $x \vee y$ and a greatest lower bound $x \wedge y$.

The basic example is the following. Let X be a set and $\mathscr{P}(X)$ be the collection of all subsets of X. The relation $A \leq B \Longleftrightarrow A \subseteq B$ turns $\mathscr{P}(X)$ into a lattice with $A \vee B = A \cup B$ and $A \wedge B = A \cap B$.

Example 8.2. Let G be a group and $L(G)$ be the collection of subgroups of G. The relation

$$H \leq K \Longleftrightarrow H \subseteq K$$

turns $L(G)$ into a lattice with $H \vee K = \langle H, K \rangle$ and $H \wedge K = H \cap K$.

Example 8.3. Let E/K be a field extension and $L(E/K)$ be the collection of intermediate fields. The relation

$$F \leq L \Longleftrightarrow F \subseteq L$$

turns $L(E/K)$ into a lattice with $F \vee L = FL$ and $F \wedge L = F \cap L$.

A map $f\colon L \to L_1$ between two lattices is said to be **order-reversing** if $x \leq y$ implies $f(y) \leq f(x)$. We shall need an exercise.

Exercise 8.4. Let L_1 and L_2 be lattices and $f\colon L_1 \to L_2$ be a bijection such that f and its inverse are both order reversing. Then

$$f(x \vee y) = f(x) \wedge f(y), \quad f(x \wedge y) = f(x) \vee f(y)$$

for all $x, y \in L_1$.

Theorem 8.5 (Galois). *Let E/K be a finite Galois extension and $G = \mathrm{Gal}(E/K)$. There exists a bijective correspondence*

$$L(E/K) = \{F : K \subseteq F \subseteq E \textit{ subfields}\} \leftrightarrow \{S : S \textit{ is a subgroup of } G\} = L(G).$$

The correspondence is given by $\alpha\colon F \mapsto \mathrm{Gal}(E/F)$ and $\beta\colon {}^{S}E \leftarrow\!\shortmid S$. Moreover, the following conditions hold:

1) *α and β are order-reversing bijections.*
2) *$[F : K] = (G : \mathrm{Gal}(E/F))$ and $(G : S) = [{}^{S}E : K]$.*
3) *F/K is a Galois extension if and only if $\mathrm{Gal}(E/F)$ is a normal subgroup of G.*

$$\begin{tikzcd}
& & E \arrow[dl, dash] \arrow[dr, dash] & \\
& F \arrow[dl, dash] \arrow[dr, dash] & & \{1\} \arrow[dl, dash] \\
K \arrow[dr, dash] & & S \arrow[dl, dash] & \\
& G & &
\end{tikzcd}$$

Proof. Let $\alpha\colon L(E/K) \to L(G)$, $\alpha(F) = \mathrm{Gal}(E/F)$, and $\beta\colon L(G) \to L(E/K)$, $\beta(S) = {}^{S}E$. A routine exercise shows that α and β are well-defined. We first note that

$$\beta(\alpha(F)) = \beta(\mathrm{Gal}(E/F)) = {}^{\mathrm{Gal}(E/F)}E = F$$

since E/F is a Galois extension. Moreover,

$$\alpha(\beta(S)) = \alpha({}^{S}E) = \mathrm{Gal}(E/{}^{S}E) = S$$

by Artin's theorem, as S is finite.

It is straightforward to check that α and β are order-reversing bijections.

Let F be a subfield of E containing K and $S = \alpha(F)$. Then

$$[F:K] = \frac{[E:K]}{[E:F]} = \frac{|G|}{|S|} = (G:S).$$

Let C be an algebraic closure of K that contains E. If $S = \mathrm{Gal}(E/F)$, then $F = {}^{S}E$.

We need to prove that F/K is normal if and only if S is normal in G. Let us first prove $\implies$. Let $\tau \in S$ and $\sigma \in G$. Since F/K is normal, $\sigma|_F \in \mathrm{Aut}(F)$. Thus $\sigma^{-1}(F) = F$. In particular, if $x \in F$, then $\sigma^{-1}(x) \in F$ and

$$\sigma\tau\sigma^{-1}(x) = \sigma\sigma^{-1}(x) = x.$$

Conversely, let $\varphi \in \mathrm{Hom}(F/K, C/K)$. There exists $\Phi\colon E \to C$ such that $\Phi|_F = \varphi$. Since E/K is normal, $\Phi(E) = E$ and hence $\Phi \in G$. We claim that $\varphi(x) \in F$ for all $x \in F$. Note that $F = {}^{S}E$, so

$$\tau\varphi(x) = \tau\Phi(x) = \Phi\Phi^{-1}\tau\Phi(x) = \Phi(x) = \varphi(x)$$

for all $\tau \in S$, as $\Phi^{-1}\tau\Phi \in S$. This means that $\varphi(x) \in {}^{S}E = F$.

Let us compute $\mathrm{Gal}(F/K)$. Since F/K is normal, the map $\lambda\colon G \to \mathrm{Gal}(F/K)$, $\sigma \mapsto \sigma|_F$, is a surjective group homomorphism such that $\ker\lambda = S$. The first isomorphism theorem implies that $\mathrm{Gal}(F/K) \simeq G/S$. $\square$

Some easy consequences.

Exercise 8.6. If E/K is a Galois extension of degree n and p is a prime number dividing n, then E/K admits a subextension of degree n/p.

Exercise 8.7. If E/K is a Galois extension of degree $p^{\alpha}m$ with p a prime number coprime with m, then E/K admits a subextension of degree m.

Definition 8.8. An extension E/K is **abelian** if E/K is a Galois extension with $\mathrm{Gal}(E/K)$ abelian.

Exercise 8.9. If E/K is an abelian extension of degree n and d divides n, then E/K admits a subextension of degree d.

Definition 8.10. An extension E/K is **cyclic** if E/K is a Galois extension with $\mathrm{Gal}(E/K)$ cyclic.

Example 8.11. The extension $\mathbb{Q}(\sqrt{2},\sqrt{3})/\mathbb{Q}$ admits exactly three non-trivial subextensions:

$$\mathbb{Q}(\sqrt{2})/\mathbb{Q},\quad \mathbb{Q}(\sqrt{3})/\mathbb{Q},\quad \mathbb{Q}(\sqrt{6})/\mathbb{Q},$$

as $\mathrm{Gal}(\mathbb{Q}(\sqrt{2},\sqrt{3})/Q)\simeq C_2\times C_2$.

Example 8.12. Let $\omega\in\mathbb{C}\setminus\{1\}$ be such that $\omega^5=1$. Then

$$f(\omega,\mathbb{Q})=1+X+X^2+X^3+X^4$$

and $\mathbb{Q}(\omega)/\mathbb{Q}$ has degree four. Moreover, $\mathbb{Q}(\omega)/\mathbb{Q}$ is a Galois extension and $\mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})\simeq C_4$. If $\sigma\in\mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})$, then $\sigma(\omega)=\omega^i$ for some $i\in\{1,\dots,4\}$. Moreover, for every $i\in\{1,\dots,4\}$ the map $\omega\mapsto\omega^i$ induces an automorphism of $\mathbb{Q}(\omega)/\mathbb{Q}$. Thus $|\mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})|=4$. Now

$$\sigma_i^k=\mathrm{id}\iff\omega^{i^k}=\sigma_i^k(\omega)=\omega\iff i^k\equiv 1 \bmod 5.$$

Thus the map σ_2 given by $\omega\mapsto\omega^2$ has order four.

Since $\mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})=\langle\sigma\rangle$, where $\sigma(\omega)=\omega^2$, is cyclic of order four, the extension $\mathbb{Q}(\omega)/\mathbb{Q}$ has a unique degree-two subtextension $F/\mathbb{Q}$. Note that $|\langle\sigma^2\rangle|=2$ and $\sigma^2(\omega)=\omega^4=\omega^{-1}$. Thus $F={}^{\langle\sigma^2\rangle}\mathbb{Q}(\omega)$. Let $\theta=\omega+\omega^{-1}$. Then

$$\theta^2=\omega^2+\omega^3+2=-(1+\omega+\omega^{-1})+2=1-\theta$$

and hence θ is a root of X^2+X-1. It follows that

$$\theta\in\{(-1+\sqrt{5})/2,(-1-\sqrt{5})/2\}.$$

Therefore $F=\mathbb{Q}(\sqrt{5})$.

Let us mention some other consequences (of the fact that the correspondence depends on order-reversing bijections).

Exercise 8.13. Let E/K be a finite Galois extension and $G=\mathrm{Gal}(E/K)$. If S and T are subgroups of G, then ${}^{\langle S,T\rangle}E={}^{S}E\cap{}^{T}E$ and ${}^{S\cap T}E={}^{S}E\,{}^{T}E$.

Exercise 8.14. Let E/K be a finite Galois extension and $F,L\in L(E/K)$. Prove that $\mathrm{Gal}(E/FL)=\mathrm{Gal}(E/F)\cap\mathrm{Gal}(E/L)$ and $\mathrm{Gal}(E/F\cap L)=\langle\mathrm{Gal}(E/F),\mathrm{Gal}(E/L)\rangle$.

Exercise 8.15. Let E/K be a finite Galois extension and $G=\mathrm{Gal}(E/K)$. Assume that G is the direct product $G=S\times T$ of the groups S and T. Let $F={}^{S}E$ and $L={}^{T}E$. Then $F\cap L=K$ and $FL=E$.

Proposition 8.16. *Let $E_1/K, E_2/K$ be Galois extensions. If $E = E_1E_2$, then E/K is a Galois extension. If, moreover, E_1/K and E_2/K are finite, then*

$$\theta\colon \operatorname{Gal}(E/K) \to \operatorname{Gal}(E_1/K) \times \operatorname{Gal}(E_2/K), \quad \sigma \mapsto (\sigma|_{E_1}, \sigma|_{E_2}),$$

is an injective group homomorphism.

Proof. Since E_1/K is algebraic, then E_1E_2/E_2 is algebraic. Since E_2/K is algebraic, E_1E_2/K is algebraic. Similarly, E_1E_2/K is separable.

Let C/K be an algebraic closure such that $E_1E_2 \subseteq C$. If $\sigma \in \operatorname{Hom}(E_1E_2/K, C/K)$, then $\sigma(E_1E_2) \subseteq \sigma(E_1)\sigma(E_2) = E_1E_2$ (do this calculation as an exercise using the fact that E_1/K and E_2/K are normal extensions). Thus E_1E_2/K is normal.

If both E_1/K and E_2/K are finite, then E_1E_2/K is finite.

Then θ is a group homomorphism. We claim that the map θ is injective. Let $\sigma \in \ker\theta$. Then $\sigma|_{E_i} = \mathrm{id}_{E_i}$ for all $i \in \{1,2\}$. Let $S = \langle\sigma\rangle \subseteq \operatorname{Gal}(E/K)$ and $F = {}^{S}E$. Then $E_i \subseteq F$ for all $i \in \{1,2\}$ and hence $E \subseteq F$. It follows that $F = E = {}^{\{\mathrm{id}\}}E$ and therefore $S = \{\mathrm{id}\}$, so $\sigma = \mathrm{id}$. □

Exercise 8.17. Let $E_1/K, \dots, E_r/K$ be finite Galois extensions such that for each j one has $E_j \cap (E_1 \cdots E_{j-1}E_{j+1} \cdots E_r) = K$. Then

$$\operatorname{Gal}(E/K) \simeq \operatorname{Gal}(E_1/K) \times \cdots \times \operatorname{Gal}(E_r/K).$$

In this case, $[E : K] = \prod_{i=1}^{r} [E_i : K]$.

§ 8.2. The fundamental theorem of algebra. We now present an easy proof of the fundamental theorem of algebra based on the ideas of Galois Theory. We need the following well-known facts:

1) Every real polynomial of odd degree admits a real root. This means that $\mathbb{R}$ does not admit extension of odd degree > 1.
2) Every complex number admits a square root in $\mathbb{C}$. This means that $\mathbb{C}$ does not admit degree-two extensions.

Theorem 8.18. *The field $\mathbb{C}$ is algebraically closed.*

Proof. Let $E/\mathbb{C}$ be an algebraic finite extension. Then $E/\mathbb{R}$ is finite separable extension of even degree. There exists a Galois extension $L/\mathbb{R}$ such that $E \subseteq L$, so $[L:\mathbb{R}]$ is even. Let $G = \operatorname{Gal}(L/\mathbb{R})$. Then $|G| = 2^m s$ for some odd number s. If T is a 2-Sylow subgroup of G, then there exists a subextension $F/\mathbb{R}$ of degree s. Since $\mathbb{R}$ does not admit extensions of odd degree > 1, $s = 1$ and hence G is a 2-group. Since $L/\mathbb{R}$ is a Galois extension, $L/\mathbb{C}$ is a Galois extension. In particular, $|\operatorname{Gal}(L/\mathbb{C})| = 2^{m-1}$. If $m > 1$, let U be a subgroup of $\operatorname{Gal}(L/\mathbb{C})$ of order 2^{m-2}. Then U corresponds to a subextension $L_1/\mathbb{C}$ of degree two, a contradiction. Hence $m = 1$ and $[L:\mathbb{C}] = 1$, so $L = \mathbb{C}$ and $E = \mathbb{C}$. □

§ 8.3. Purely inseparable extensions. Let E/K be an algebraic extension. In page 6.1 we defined the **separable closure** of K with respect to E as the field

$$F = \{x \in E : x \text{ is separable over } K\}.$$

Note that $K \subseteq F \subseteq E$ and $F = K(F)$. Moreover, F/K is separable and E/F is a **purely inseparable** extension, meaning that for every $x \in E \setminus F$, the polynomial $f(x,F)$ is not separable.

The number $[E:K]_{\text{ins}} = [E:F]$ is known as the **degree of inseparability** of E/K. Clearly, E/K is separable if and only if $[E:K]_{\text{ins}} = 1$ and E/K is purely inseparable if and only if $[E:K]_{\text{ins}} = [E:K]$.

Exercise 8.19. Let K be a field of characteristic $p>0$ and $f \in K[X]$ be irreducible. If f is not separable, then $f = g(X^p)$ for some $g \in K[X]$.

Proposition 8.20. *Let K be a field of characteristic $p>0$ and E/K be an algebraic extension. The following statements are equivalent:*

1) *E/K is purely inseparable.*
2) *If $x \in E$, then $x^{p^m} \in K$ for some $m \geq 0$.*
3) *If $x \in E$, then $f(x,K) = X^{p^m} - a$ for some $a \in K$ and $m \geq 0$.*
4) *$\gamma(E/K) = 1$.*

Proof. We first prove $1) \implies 2)$. Let $x \in E$ and $f = f(x,K)$. Assume x is not separable. Then $f(x) = 0$ and $f'(x) = 0$, as x is not a simple root. By Exercise 8.19, $f = g(X^p)$ for some $g \in K[X]$. We now proceed by induction on the degree of x. The result is true for elements of degree one. So assume the result holds for the element of degree $\leq n$ for some $n \geq 1$. If $x \in E$ is such that $\deg f(x,K) = n+1$, then, since $f(x,K) = g(X^p)$, the element x^p has degree $\leq n$. By the inductive hypothesis, $x^{p^{m+1}} = (x^p)^{p^m} \in K$.

We now prove $2) \implies 3)$. Let $x \in E$ and m be the minimal positive integer such that $x^{p^m} \in K$. Then x is a root of $X^{p^m} - x^{p^m} \in K[X]$. Since $X^{p^m} - x^{p^m} = (X-x)^{p^m}$, it follows that

$$f(x,K) = (X-x)^r = X^r + \cdots + (-1)^r x^r$$

for some $r \in \{1,\dots,p^m\}$. Write $r = p^s t$ for some integer t coprime with p and s such that $0 \leq s \leq m$. Let $a,b \in \mathbb{Z}$ be such that $ar + bp^m = p^s$. Then

$$x^{p^s} = x^{ar+bp^m} = (x^r)^a \left(x^{p^m}\right)^b \in K.$$

The minimality of m implies that $s \geq m$ and hence $s = m$. Now $p^m t = p^s t = r \leq p^m$, so $t = 1$. This means $f(x,K) = X^{p^m} - x^{p^m}$.

We now prove $3) \implies 4)$. Let C/K be an algebraic closure of K containing E and $x \in E$. Let $\sigma \in \operatorname{Hom}(E/K, C/K)$. We claim that $\sigma(x) = x$. Since $f(x,K) = X^{p^m} - a$,

$$(\sigma(x))^{p^m} = \sigma\left(x^{p^m}\right) = \sigma(a) = a = x^{p^m}.$$

It follows that $\sigma(x)$ is a root of $X^{p^m} - x^{p^m} = (X-x)^{p^m}$. Thus $\sigma(x) = x$.

Finally, we prove that $4) \implies 1)$. Let C be an algebraic closure of K containing E. Then $\operatorname{Gal}(E/K) = \operatorname{Hom}(E/K, C/K) = \{\mathrm{id}\}$, as $\gamma(E/K) = 1$. If $x \in E$ is separable over K, then

$$f(x,K) = \prod_{y \in O_{\operatorname{Gal}(E/K)}(x)} (X-y) = X - x \in K[X].$$

Thus $x \in K$ and hence E/K is purely inseparable. □

Some consequences:

Exercise 8.21. Let K be a field of characteristic $p>0$ and E/K be finite and purely inseparable. Then $[E:K] = p^s$ for some prime number p and some s. Moreover, $x^{[E:K]} \in K$.

For the first part of the previous exercise, write $E = K(x_1,\dots,x_n)$ and proceed by induction on n.

EXERCISE 8.22. Let K be of characteristic $p > 0$ and E/K be a finite extension such that $[E : K]$ is not divisible by p. Then E/K is separable.

Let K be of characteristic $p > 0$, E/K be finite and F be the separable closure of K in E. Since

$$\gamma(E/K) = \gamma(E/F)\gamma(F/K) = \gamma(F/K),$$

it follows that

$$[E : K] = [E : F]\gamma(E/K) = [E : K]_{\text{ins}}\gamma(E/K).$$

Lecture 9. 22/04/2024

§ 9.1. Norm and trace.

DEFINITION 9.1. Let E/K be a finite extension and C/K be an algebraic closure that contains E. Let $A = \operatorname{Hom}(E/K, C/K)$. For $x \in E$ we define the **trace** of x in E/K as

$$\operatorname{trace}_{E/K}(x) = [E:K]_{\text{ins}} \sum_{\varphi \in A} \varphi(x)$$

and the **norm** of x in E/K as

$$\operatorname{norm}_{E/K}(x) = \left(\prod_{\varphi \in A} \varphi(x) \right)^{[E:K]_{\text{ins}}}.$$

As an optional exercise, one can show that these definitions do not depend on the algebraic closure.

We collect some basic properties as an exercise:

EXERCISE 9.2. Let E/K be a finite extension. The following statements hold:

1) If E/K is not separable, then $\operatorname{trace}_{E/K}(x) = 0$ for all $x \in E$.
2) If $x \in K$, then $\operatorname{trace}_{E/K}(x) = [E:K]x$.
3) $\operatorname{trace}_{E/K}(x) \in K$ for all $x \in E$.
4) $\operatorname{norm}_{E/K}(x) = 0$ if and only if $x = 0$.
5) If $x \in K$, then $\operatorname{norm}_{E/K}(x) = x^{[E:K]}$.
6) $\operatorname{norm}_{E/K}(x) \in K$ for all $x \in E$.

One proves, moreover, that $\operatorname{trace}_{E/K} \colon E \to K$ satisfies

$$\operatorname{trace}_{E/K}(x + \lambda y) = \operatorname{trace}_{E/K}(x) + \lambda \operatorname{trace}_{E/K}(y)$$

for all $x, y \in E$ and $\lambda \in K$, that is to say that $\operatorname{trace}_{E/K} \colon E \to K$ is a linear form in E The norm $\operatorname{norm}_{E/K} \colon E^\times \to K^\times$ is a group homomorphism.

EXERCISE 9.3. Let E/K be a finite extension and $x \in E$ be algebraic. If

$$f(x,K) = X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0,$$

then $\operatorname{norm}_{E/K}(x) = ((-1)^n a_0)^{[E:K(x)]}$ and $\operatorname{trace}_{E/K}(x) = -[E:K(x)]a_{n-1}$.

EXAMPLE 9.4. Let $E = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Then

$$\operatorname{trace}_{E/\mathbb{Q}}(\sqrt{2}) = 0, \qquad \operatorname{norm}_{E/\mathbb{Q}}(\sqrt{2}) = 4,$$
$$\operatorname{trace}_{E/\mathbb{Q}(\sqrt{2})}(\sqrt{2}) = 2\sqrt{2}, \qquad \operatorname{norm}_{E/\mathbb{Q}(\sqrt{2})}(\sqrt{2}) = 2.$$

EXAMPLE 9.5. If E/K is a finite Galois extension, then

$$\operatorname{trace}_{E/K}(x) = \sum_{\sigma \in \operatorname{Gal}(E/K)} \sigma(x) \quad \text{and} \quad \operatorname{norm}_{E/K}(x) = \prod_{\sigma \in \operatorname{Gal}(E/K)} \sigma(x)$$

for all $x \in E$. In particular, since $E = K(y)$ for some y by Proposition 6.10,

$$\operatorname{trace}_{E/K}(y) = -a_{n-1} \quad \text{and} \quad \operatorname{norm}_{E/K}(y) = (-1)^n a_0,$$

where $f(y,K) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$.

§ 9.2. Finite fields. In this section, p will be a prime number.

Proposition 9.6. *Let m be a positive integer. Up to isomorphism, there exists a unique field F_m of size p^m.*

Proof. Let C be an algebraic closure of the field $\mathbb{Z}/p$ and let $F_m = \{x \in C : x^{p^m} = x\}$ be the set of roots of $X^{p^m} - X$. Since the polynomial $X^{p^m} - X$ has no multiple roots, $|F_m| = p^m$. Moreover, F_m is the unique subfield of C of size p^m.

To prove the uniqueness, it is enough to note that if K is a field of p^m elements, then K is the decomposition field of $X^{p^m} - X$ over $\mathbb{Z}/p$. □

Let $K = \mathbb{Z}/p$ and C be an algebraic closure of K. We claim that $C = \cup_k F_k$. If $x \in C$, then x is algebraic over K. Since $K(x)/K$ is finite, $K(x)$ is a finite field, say $|K| = p^r$ for some r. Then $x^{p^r} = x$ and hence $x \in F_r$.

Exercise 9.7. Prove the following statements:

1) If $x \in F_r$, then $x^{p^{rk}} = x$ for all $k \geq 0$.
2) $F_m \subseteq F_n$ if and only if $m \mid n$.
3) $F_m \cap F_n = F_{\gcd(m,n)}$.

Proposition 9.8. *Every finite extension of a finite field is cyclic.*

Proof. Let $K = \mathbb{Z}/p$. It is enough to show that F_n/F_m is cyclic if m divides n.

We first prove that F_n/K is cyclic. Let

$$\sigma\colon F_n \to F_n, \quad x \mapsto x^p.$$

Then $\sigma \in \mathrm{Gal}(F_n/K)$ (it is bijective because all field homomorphisms are injective and F_n is finite).

Note that F_n/K is a Galois extension, as F_n is the splitting field over K of the separable polynomial $X^{p^n} - X \in K[X]$. Thus $|\mathrm{Gal}(F_n/K)| = [F_n : K] = n$.

We claim that σ generated $\mathrm{Gal}(F_n/K)$. Since $\sigma^i(x) = x^{p^i}$ for all $i \geq 0$, in particular,

$$\sigma^n(x) = x^{p^n} = x.$$

Thus $\sigma^n = \mathrm{id}$ and hence $|\sigma|$ divides n. Let $s = |\sigma|$. We know that $F_n^\times = F_n \setminus \{0\}$ is cyclic, say $F_n^\times = \langle g \rangle$. Since $|g| = p^n - 1$,

$$g = \sigma^s(g) = g^{p^s}$$

and hence $p^s \equiv 1 \bmod (p^n - 1)$. Thus $p^n - 1$ divides $p^s - 1$ and hence n divides s. Therefore $n = s$ and $\mathrm{Gal}(F_n/K) = \langle \sigma \rangle$.

For the general case note that if m divides n, then $\mathrm{Gal}(F_n/F_m)$ is a subgroup of $\mathrm{Gal}(F_n/K)$. Since $\mathrm{Gal}(F_n/K)$ is cyclic, the claim follows. □

If $K = \mathbb{Z}/p$ and m divides n, the subextension F_m corresponds to the unique subgroup of index m of $\mathrm{Gal}(F_n/K) = \langle \sigma \rangle$. This subgroup is $\langle \sigma^m \rangle$, where

$$\sigma^m(x) = x^{p^m} = x^{|F_m|}.$$

Note that $\mathrm{Gal}(F_n/F_m) = \langle \sigma^m \rangle$. The map σ^m is known as the **Frobenius automorphism**.

Exercise 9.9. Let E/K be an extension of finite fields. Then E/K is cyclic. Moreover, $\mathrm{Gal}(E/K) = \langle \tau \rangle$, where $\tau(x) = x^{|K|}$.

§ 9.3. Cyclotomic extensions. For $n \geq 1$ let $G_n(K) = \{x \in K : x^n = 1\}$ be the set of n-roots of one in K. Note that $G_n(K)$ is a cyclic subgroup of $K^\times$ and that $|G_n(K)|$ divides n.

Example 9.10. $G_n(\mathbb{R}) = \{-1, 1\}$ if n is odd and $G_n(\mathbb{R}) = \{1\}$ if n is even.

Exercise 9.11. Let K be a field of characteristic $p > 0$. Let $n = p^s m$ for some m not divisible by p. Then $G_n(K) = G_m(K)$.

Exercise 9.12. Let q be a prime number. Then $G_n(\mathbb{Z}/q) \simeq \mathbb{Z}/\gcd(n, q-1)$.

Similarly, one can prove that if K is a finite field, then $G_n(K)$ is a cyclic group of order $\gcd(n, |K^\times|)$.

Example 9.13. If C is algebraically closed of characteristic coprime with n, then $G_n(C)$ is cyclic of order n, as $X^n - 1$ has all its roots in C and does not contain multiple roots.

Let K be an algebraically closed field and n be such that n is coprime with the characteristic of K. The set of **primitive n-roots** is defined as

$$H_n(K) = \{x \in G_n(K) : |x| = n\}.$$

Definition 9.14. Let K be an algebraically closed field and n be such that n is coprime with the characteristic of K. The **n-th cyclotomic polynomial** is defined as

$$\Phi_n = \prod_{x \in H_n(K)} (X - x) \in K[X].$$

For $n \geq 1$ the Euler's function is defined as

$$\varphi(n) = |\{k : 1 \leq k \leq n,\ \gcd(k, n) = 1\}|.$$

For example, $\varphi(4) = 2$, $\varphi(8) = \varphi(10) = 4$ and $\varphi(p) = p - 1$ for every prime p.

Proposition 9.15. *Let K be an algebraically closed field and n be such that n is coprime with the characteristic of K. Let A be the ring of integers of K.*

1) $\deg \Phi_n = \varphi(n)$.
2) $\Phi_n \in A[X]$.

Proof. The first statement is clear. Let us prove 2) by induction on n. The case $n = 1$ is trivial, as $\Phi_1 = X - 1$. Assume that $\Phi_d \in A[X]$ for all d such that $d < n$. In particular,

$$\gamma = \prod_{\substack{d \mid n \\ d \neq n}} \Phi_d \in A[X].$$

Since γ is monic, it follows that $\frac{X^n - 1}{\gamma} \in A[X]$. Now the claim follows from

$$X^n - 1 = \prod_{d \mid n} \Phi_d = \Phi_n \prod_{\substack{d \mid n \\ d \neq n}} \Phi_d = \Phi_n \gamma. \qquad \square$$

By taking degree in the equality $X^n - 1 = \prod_{d|n} \Phi_d$ one gets

$$n = \sum_{d|n} \varphi(d).$$

Definition 9.16. Let $n \geq 2$ and K be a field of characteristic coprime with n. A **cyclotomic extension** of K of index n is a decomposition field of $X^n - 1$ over K.

Let C be an algebraic closure of K and $n \geq 2$ be coprime with the characteristic of K. If follows from Definition 9.16 that a cyclotomic extension of index n is of the form $K(\omega)/K$ for some $\omega \in H_n(K)$.

Proposition 9.17. *A cyclotomic extension of index n is abelian and of degree a divisor of* $\varphi(n)$.

Proof. Let C be an algebraic closure of K and $n \geq 2$ be coprime with the characteristic of K. Let $\omega \in H_n(C)$ and $K(\omega)/K$ be a cyclotomic extension. Then $K(\omega)/K$ is a Galois extension, as it is a decomposition field of a separable polynomial. Let $U = \mathscr{U}(\mathbb{Z}/n)$ be the group of units of $\mathbb{Z}/n$ and

$$\lambda\colon \operatorname{Gal}(K(\omega)/K) \to U, \quad \sigma \mapsto m_\sigma,$$

where m_σ is such that $\sigma(\omega) = \omega^{m_\sigma}$. The map λ is well-defined and it is a group homomorphism, as if $\sigma, \tau \in \operatorname{Gal}(K(\omega)/K)$, then, since

$$(\tau\sigma)(\omega) = \tau(\sigma(\omega)) = \tau(\omega^{m_\sigma}) = (\omega^{m_\sigma})^{m_\tau} = \omega^{m_\sigma m_\tau},$$

it follows that $\lambda(\sigma)\lambda(\tau) = \lambda(\sigma\tau)$. Since λ is injective, $\operatorname{Gal}(K(\omega)/K)$ is isomorphic to a subgroup of the abelian group U. Hence $\operatorname{Gal}(K(\omega)/K)$ is abelian. Moreover, $[K(\omega) : K] = |\operatorname{Gal}(K(\omega)/K)|$ is a divisor of $|U| = \varphi(n)$. □

Exercise 9.18. Prove that a cyclotomic extension $K(\omega)/K$ has degree $\varphi(n)$ if and only if Φ_n is irreducible over K.

Note that Φ_n is irreducible over $\mathbb{Q}$. Some concrete examples:

$$\Phi_1 = X - 1, \quad \Phi_2 = X + 1, \quad \Phi_3 = X^2 + X + 1, \quad \Phi_6 = X^2 - X + 1.$$

If p is a prime number, then $\Phi_p = X^{p-1} + \cdots + X + 1$.

Example 9.19. Φ_5 is irreducible over $\mathbb{Z}/2$. First note that $\Phi_5 = X^4 + \cdots + X + 1$ does not have roots in $\mathbb{Z}/2$. If Φ_5 is reducible, then, since $X^2 + X + 1$ is the unique degree-two monic irreducible polynomial over $\mathbb{Z}/2$, it follows that

$$\Phi_5 = (X^2 + X + 1)(X^2 + X + 1) = (X^2 + X + 1)^2 = X^4 + X^2 + 1,$$

a contradiction.

Exercise 9.20. Prove that $\Phi_{12} = X^4 - X^2 + 1$ is not irreducible over $\mathbb{Z}/5$.

§ 9.4. Hilbert's theorem 90.

Theorem 9.21 (Hilbert). *Let E/K be a cyclic extension. Assume that* $\operatorname{Gal}(E/K)$ *is generated by τ. For $a \in E$,* $\operatorname{norm}_{E/K}(a) = 1$ *if and only if $a = b/\tau(b)$ for some $b \in E \setminus \{0\}$.*

PROOF. Let $n=|G|$. We first prove $\Longleftarrow$. If $a=b/\tau(b)$ and $b\neq 0$, then

$$\operatorname{norm}_{E/K}(a)=a\tau(a)\tau^2(a)\cdots\tau^{n-1}(a)=\frac{b}{\tau(b)}\frac{\tau(b)}{\tau^2(b)}\cdots\frac{\tau^{n-1}(b)}{\tau^n(b)}=1.$$

Now we prove $\Longrightarrow$. Let $a\in E$ be such that $\operatorname{norm}_{E/K}(a)=1$. For $c\in E$ let

$$\begin{aligned}
d_0 &= ac,\\
d_1 &= a\tau(a)\tau(c),\\
d_2 &= a\tau(a)\tau^2(a)\tau^2(c),\\
&\vdots\\
d_{n-1} &= \underbrace{a\tau(a)\cdots\tau^{n-1}(a)}_{=\operatorname{norm}_{E/K}(a)}\tau^{n-1}(c)=\tau^{n-1}(c).
\end{aligned}$$

Then

$$a\tau(d_j)=a\tau(a)\cdots\tau^{j+1}(a)\tau^{j+1}(c)=d_{j+1}$$

for all $j\in\{0,\dots,n-2\}$. Let $b=d_0+\cdots+d_{n-1}$. We claim that $b\neq 0$ for some c. Suppose this is not true, say $b=0$ for all c. Then

$$0=ac+(a\tau(a))\tau(c)+\cdots+(a\tau(a)\cdots\tau^{n-1}(a))\tau^{n-1}(c)$$

for every $c\in E$. This implies that $a=0$ by Dedekind's theorem, a contradiction.

So let $c\in E$ be such that $b\neq 0$. Then

$$\begin{aligned}
\tau(b) &= \tau(d_0)+\cdots+\tau(d_{n-1})\\
&= \tau(ac)+\tau(a\tau(c))+\cdots+\tau(\tau^{n-1}(c))\\
&= \frac{1}{a}(d_1+\cdots+d_{n-1})+\tau^n(c)\\
&= \frac{1}{a}(d_0+\cdots+d_{n-1})\\
&= b/a.
\end{aligned}$$

□

EXERCISE 9.22. Let E/K be a cyclic extension. Assume that $\operatorname{Gal}(E/K)$ is generated by τ. Prove that for $a\in E$, $\operatorname{trace}_{E/K}(a)=0$ if an only if $a=b-\tau(b)$ for some $b\in L\setminus\{0\}$.

Lecture 10. 29/04/2024

COROLLARY 10.1. *Let $a,b,c \in \mathbb{Z}$ be such that $a^2+b^2=c^2$. Then*

$$(a,b,c)=\lambda(r^2-s^2,-2rs,r^2+s^2)$$

for some $r,s \in \mathbb{Z}$ and some $\lambda \in \mathbb{Z}$.

PROOF. We work with the extension $\mathbb{Q}(i)/\mathbb{Q}$. Note that $\mathrm{Gal}(\mathbb{Q}(i),\mathbb{Q})=\{\mathrm{id},\gamma\}$ is cyclic, where $\gamma\colon \mathbb{Q}(i)\to\mathbb{Q}(i)$, $z\mapsto\bar{z}$, is the complex conjugation. We may assume that $c\neq 0$, otherwise $a=b=0$ and the result is trivial. Write $(a/c)^2+(b/c)^2=1$ and let $\alpha=(a/c)+(b/c)i\in\mathbb{Q}(i)$. Then $\mathrm{norm}_{\mathbb{Q}(i)/\mathbb{Q}}(\alpha)=1$. By Hilbert's theorem, there exists $\beta\in\mathbb{Q}(i)\setminus\{0\}$ such that

$$\alpha=a+bi=\frac{\gamma(\beta)}{\beta}.$$

Note that if $m\in\mathbb{Z}\setminus\{0\}$, then $\frac{\gamma(m\beta)}{m\beta}=\frac{\gamma(\beta)}{\beta}$. There exists $m\in\mathbb{Z}\setminus\{0\}$ such that $m\beta\in\mathbb{Z}[i]$, say $m\beta=r+is$ with $r,s\in\mathbb{Z}$. Then

$$\alpha=\frac{\gamma(\beta)}{\beta}=\frac{\gamma(m\beta)}{m\beta}=\frac{r-is}{r+is}=\frac{r^2-s^2-2rsi}{r^2+s^2}.$$

From this the claim follows. □

EXERCISE 10.2. Let $A,B\in\mathbb{Z}$ be such that A^2-4B is not a square. Prove that a solution $(x,y,z)\in\mathbb{Z}^3$ of $x^2+Axy+By^2=z^2$ is proportional to

$$(r^2-Bs^2,2rs+As^2,r^2+Ars+Bs^2).$$

PROPOSITION 10.3. *Let $n\geq 2$ and K be a field containing a primitive n-root of one. If $a\in K^\times$ and E/K is a decomposition field of $f=X^n-a$, then E/K is cyclic of degree d, where d divides n. Moreover,*

$$d=\min\{k : a^k\in K^n\},$$

where $K^n=\{x\in K : x=y^n \text{ for some } y\in K\}$. Conversely, if E/K is cyclic of degree n, then E/K is a decomposition field of an irreducible polynomial of the form X^n-a for some $a\in K^\times$.

PROOF. A decomposition field of f over K is of the form $K(\alpha)$, where $\alpha^n=a$. Thus $K(\alpha)/K$ is a Galois extension. If $\sigma\in\mathrm{Gal}(K(\alpha)/K)$, then $\sigma(\alpha)$ is a root of f, so $\sigma(\alpha)=\omega_\sigma\alpha$, where $\omega_\sigma\in G_n(K)$. This means that there exists an injective map

$$\lambda\colon\ \mathrm{Gal}(K(\alpha)/K)\to G_n(K),\quad \sigma\mapsto\omega_\sigma.$$

Moreover, λ is a group homomorphism, as

$$\sigma\tau(\alpha)=\sigma(\tau(\alpha))=\sigma(\omega_\tau\alpha)=\omega_\tau\sigma(\alpha)=\omega_\tau\omega_\sigma\alpha.$$

Therefore $\mathrm{Gal}(K(\alpha)/K)$ is isomorphic to a subgroup of $G_n(K)$. In particular, $\mathrm{Gal}(K(\alpha)/K)$ is cyclic and $|\mathrm{Gal}(K(\alpha)/K)|$ divides n.

Let $d=|\mathrm{Gal}(K(\alpha)/K)|$. Since $a=\alpha^n$,

$$\mathrm{norm}_{K(\alpha)/K}(\alpha)^n=\mathrm{norm}_{K(\alpha)/K}(a)=a^d.$$

Thus $a^d\in K^n$, as $\mathrm{norm}_{K(\alpha)/K}(\alpha)\in K$. If $a^k\in K^n$, say $a^k=c^n$ for some $c\in K$, then

$$c^n=a^k=(\alpha^n)^k=(\alpha^k)^n \implies \alpha^k=c\omega\in K$$

for some $\omega \in G_n(K)$. Thus α is a root of $X^k - \alpha^k \in K[X]$ and hence $k \geq d$.

Note that $f(\alpha, K) = X^d - \alpha^d$.

Let E/K be cyclic of degree n. Assume that $\mathrm{Gal}(E/K) = \langle \sigma \rangle$. If ω is a primitive n-root of one,

$$\mathrm{norm}_{E/K}(\omega) = \omega^n = 1.$$

By Hilbert's theorem 90, there exists $b \in E^\times$ such that $\omega = \sigma(b)/b$. Thus $\sigma(b) = \omega b$ and hence $\sigma^i(b) = \omega^i b$ for all $i \geq 0$. Since $|\{b, \sigma(b), \dots, \sigma^{n-1}(b)\}| = n$, it follows that $E = K(b)$. Moreover,

$$\sigma(b^n) = \sigma(b)^n = (\omega b)^n = b^n$$

and hence $b^n \in K$. This means that E/K is a decomposition field of $X^n - b^n$. Note that $X^n - b^n$ is irreducible, as $[E : K] = [K(b) : K] = n$. $\square$

Proposition 10.4. *Let K be a field of characteristic $p > 0$.*

1) *Let $a \in K$ and $f = X^p - X - a$. Then f is irreducible over K or all the roots of f belong to K. In the first case, if b is a root of f, then $K(b)/K$ is a cyclic extension of degree p.*

2) *Every cyclic extension of degree p is a decomposition field of an irreducible polynomial of the form $X^p - X - a$.*

Proof. We first prove 1). Let K_0 be the prime field of K. Note that $K_0 \simeq \mathbb{Z}/p$. Let b be a root of f and let $x \in K_0$. Then

$$f(b+x) = (b+x)^p - (b+x) - a = (b^p - b - a) + (x^p - x) = 0$$

and thus $\{b + x : x \in K_0\}$ is the set of roots of f. Note that $f' = -1$, so f has no multiple roots.

We claim that if $b \notin K$, then f is irreducible. If f is not irreducible, then $f = gh$ for some $g, h \in K[X]$ such that $0 < \deg g < p$. There exists a subset S of K_0 such that $g = \prod_{x \in S}(X - (b + x))$ and hence

$$|S|b + \sum_{x \in S} x = \sum_{x \in S}(b + x) \in K.$$

This implies that $|S|b \in K$ and hence, since $|S| \in K^\times$, it follows that $b \in K$.

Since $K(b)/K$ is a decomposition field of a separable polynomial, $K(b)/K$ is a Galois extension. Moreover, $|\mathrm{Gal}(K(b)/K)| = [K(b) : K] = p$ and hence $\mathrm{Gal}(K(b)/K)$ is cyclic.

We now prove 2). Let E/K be cyclic of degree p. Assume that $\mathrm{Gal}(E/K) = \langle \sigma \rangle$. Since $\mathrm{trace}_{E/K}(1) = p = 0$, Hilbert's theorem implies that there exists $b \in E$ such that $\sigma(b) = b + 1$. In particular, $b \notin K$ and thus $E = K(b)$. Moreover, since

$$\sigma(b^p - b) = \sigma(b)^p - \sigma(b) = (b+1)^p - (b+1) = b^p - b,$$

it follows that $b^p - b \in K$. Thus $f(b, K) = X^p - X - (b^p - b) \in K[X]$. $\square$

§ 10.1. Symmetric polynomials. Let K be a field and $\{t_1, \dots, t_n\}$ be a commuting set of independent variables. Let $E = K(t_1, \dots, t_n)$ and $f = \prod_{i=1}^n (X - t_i) \in E[X]$. Then

$$f = X^n + \sum_{i=1}^n (-1)^i s_i X^{n-i},$$

where

$$\begin{aligned}
s_1 &= t_1+t_2+\cdots+t_n,\\
s_2 &= \sum_{1\leq i<j\leq n} t_it_j,\\
&\vdots\\
s_n &= t_1t_2\cdots t_n.
\end{aligned}$$

For example,

$$(X-t_1)(X-t_2)(X-t_3) = X^3-(t_1+t_2+t_3)X^2+(t_1t_2+t_2t_3+t_1t_3)X-t_1t_2t_3.$$

The polynomials $s_1,s_2,\dots,s_n$ are known as the **elementary symmetric polynomials** in the variables $t_1,\dots,t_n$. Note that $\deg s_i = i$.

Let $\sigma\in\mathbb{S}_n$ and

$$\alpha_\sigma\colon K[t_1,\dots,t_n]\to K[t_1,\dots,t_n],\quad t_i\mapsto t_{\sigma(i)}\quad\text{for all } i.$$

Then α_σ is a bijective homomorphism of K-algebras. In fact, $\alpha_\sigma^{-1}=\alpha_{\sigma^{-1}}$. Note that

$$\alpha_\sigma(h(t_1,\dots,t_n)) = h(t_{\sigma(1)},\dots,t_{\sigma(n)}).$$

Since α_σ is injective, it induces an element $\widehat{\sigma}\in\mathrm{Gal}(E/K)$ given by

$$\widehat{\sigma}\left(\frac{h}{g}\right)=\frac{\alpha_\sigma(h)}{\alpha_\sigma(h)}.$$

The map $\mathbb{S}_n\to\mathrm{Gal}(E/K)$, $\sigma\mapsto\widehat{\sigma}$, is an injective group homomorphism. Thus $\{\widehat{\sigma}:\sigma\in\mathbb{S}_n\}\simeq\mathbb{S}_n$.

Definition 10.5. Let $g\in K[t_1,\dots,t_n]$. Then g is **symmetric** if $\widehat{\sigma}(g)=g$ for all $\sigma\in\mathbb{S}_n$.

We write P to denote the set of symmetric polynomials in $K[t_1,\dots,t_n]$. Clearly, P is a subalgebra of $K[t_1,\dots,t_n]$. The following statements hold:

1) $K\subseteq P$.
2) $\sum_{i=1}^n t_i^r\in P$ for all $r\geq 1$.
3) $s_i\in P$ for all i.
4) $K(P)\subseteq {}^GE$, where $G=\{\widehat{\sigma}:\sigma\in\mathbb{S}_n\}$.

Let $F=K(s_1,s_2,\dots,s_n)$. Then E/F is a Galois extension, as it is a decomposition field of f.

Proposition 10.6. $[E:F]\leq n!$.

Proof. We proceed by induction on n. The case $n=1$ is clear, as $E=F$. Assume that $n>1$. Let $u_1,\dots,u_{n-1}$ be the elementary symmetric polynomials in $t_1,\dots,t_{n-1}$. Then

$$s_i=u_i+t_nu_{i-1}$$

for all $i\in\{1,\dots,n\}$, where $u_0=1$ and $u_n=0$. Note that $u_1=s_1-t_n$ and $u_i=s_i-t_nu_{i-1}$ for all i. Since $K(s_1,\dots,s_n,t_n)=K(u_1,\dots,u_{n-1},t_n)$,

$$F(t_n)=K(u_1,\dots,u_{n-1},t_n)=K(t_n)(u_1,\dots,u_{n-1})$$

and

$$[E:F]=[E:F(t_n)][F(t_n):F]\leq n[E:F(t_n)].$$

Note that $E=K(t_1,\dots,t_n)=K(t_n)(t_1,\dots,t_{n-1})$. By the inductive hypothesis,

$$[E:F(t_n)]\leq(n-1)!$$

and hence $[E:F] \leq n!$. □

THEOREM 10.7. ${}^{G}E = F$.

PROOF. By Artin's theorem,

$$\left[{}^{G}E : F\right] = \frac{[E:F]}{\left[E : {}^{G}E\right]} \leq \frac{n!}{\left[E : {}^{G}E\right]} = 1$$

and hence ${}^{G}E = F$. □

EXERCISE 10.8. Prove that $\mathrm{Gal}(E/F) \simeq \mathbb{S}_n$.

EXERCISE 10.9. Prove that $\{s_1, \dots, s_n\}$ is algebraically independent over K.

EXERCISE 10.10. Prove that every symmetric polynomial in $t_1, \dots, t_n$ can be written as a rational fraction in $s_1, \dots, s_n$.

§ 10.2. Solvable groups. Let G be a group. If $x, y \in G$ we define the **commutator** of x and y as

$$[x,y] = xyx^{-1}y^{-1}.$$

Note that $[x,y] = 1$ if and only if $xy = yx$. Moreover, $[x,y]^{-1} = [y,x]$. The **commutator (or derived) subgroup** $[G,G]$ of G is defined as the subgroup of G generated by all commutators, i.e.

$$[G,G] = \langle [x,y] : x,y \in G \rangle.$$

This means that every element of $[G,G]$ is a finite product of commutators, so every element of $[G,G]$ is of the form $\prod_{i=1}^{m}[x_i,y_i]$. In general, the commutator subgroup is not equal to the set of commutators!

EXAMPLE 10.11. This example is taken from the book [**1**] of Carmichael. Let G be the subgroup of $\mathbb{S}_{16}$ generated by the permutations

$$\begin{aligned} a &= (13)(24), & b &= (57)(68), \\ c &= (9\,11)(10\,12), & d &= (13\,15)(14\,16), \\ e &= (13)(57)(9\,11), & f &= (12)(34)(13\,15), \\ g &= (56)(78)(13\,14)(15\,16), & h &= (9\,10)(11\,12). \end{aligned}$$

Then $[G,G]$ has order 16. However, the set $\{[x,y] : x,y \in G\}$ of commutators has 15 elements:

```
julia> a = @perm (1,3)(2,4);

julia> b = @perm (5,7)(6,8);

julia> c = @perm (9,11)(10,12);

julia> d = @perm (13,15)(14,16);

julia> e = @perm (1,3)(5,7)(9,11);

julia> f = @perm (1,2)(3,4)(13,15);

```

```
julia> g = @perm (5,6)(7,8)(13,14)(15,16);

julia> h = @perm (9,10)(11,12);

julia> S16 = symmetric_group(16);

julia> G = sub(S16, [a,b,c,d,e,f,g,h])[1];

julia> commutators = G -> Set(comm(x,y) for x in G, y in G);

julia> length(commutators(G))
15

julia> order(derived_subgroup(G)[1])
16
```

EXERCISE 10.12. Let G be a group. Prove the following facts:

1) G is abelian if and only if $[G,G] = \{1\}$.
2) $[G,G]$ is a normal subgroup of G.
3) $G/[G,G]$ is abelian.
4) If H is a subgroup of G and $[G,G] \subseteq H$, then H is normal in G.
5) If H is a normal subgroup of G, then G/H is abelian if and only if $[G,G] \subseteq H$.

DEFINITION 10.13. Let G be a group. The **derived series** of G is defined as $G^{(0)} = G$ and $G^{(k+1)} = [G^{(k)}, G^{(k)}]$ for $k \geq 0$.

EXERCISE 10.14. Prove that $G^{(k)}$ is normal in G for all k.

Why derived series? We cannot explain this here, but let us use the following notation. We write $G' = [G,G]$, $G'' = [G',G']$... Note that

$$G \supseteq G' \supseteq G'' \supseteq \cdots$$

EXERCISE 10.15. Let $n \geq 3$. Prove that $[\mathbb{S}_n, \mathbb{S}_n] = \mathbb{A}_n$.

EXAMPLE 10.16. Let $K = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\}$. Then K is a normal subgroup of $\mathbb{A}_4$. One proves that $[\mathbb{A}_4, \mathbb{A}_4] = K$.

A group G is said to be **simple** if there are no proper non-trivial normal subgroups of G. If p is a prime number, then the group $\mathbb{Z}/p$ of integers modulo p is a simple group. We will prove later that $\mathbb{A}_n$ is simple if $n \geq 5$.

EXAMPLE 10.17. Let $n \geq 5$. Since $\mathbb{A}_n$ is a non-abelian simple group, $[\mathbb{A}_n, \mathbb{A}_n] = \mathbb{A}_n$.

Let us show that $\mathbb{A}_5$ is a non-abelian simple group. Hence it is not solvable:

```
julia> A5 = alternating_group(5)
Alt( [ 1 .. 5 ] )

julia> is_abelian(A5)
false
```

```
julia> is_simple(A5)
true

julia> is_solvable(A5)
false
```

Definition 10.18. A group G is **solvable** if and only if $G^{(m)} = \{1\}$ for some m.

Every abelian group is solvable.

Exercise 10.19. Prove that $\mathbb{S}_n$ is solvable if and only if $n \leq 4$.

Let us compute (with the computer software Oscar) the derived series of the symmetric group $\mathbb{S}_4$. The calculation shows that $\mathbb{S}_4$ is solvable:

```
julia> G = symmetric_group(4);

julia> derived_series(G)
4-element Vector{PermGroup}:
 Sym( [ 1 .. 4 ] )
 Alt( [ 1 .. 4 ] )
 Group([ (1,4)(2,3), (1,2)(3,4) ])
 Group(())

julia> [order(x) for x in derived_series(G)]
4-element Vector{fmpz}:
 24
 12
 4
 1

julia> is_solvable(G)
true
```

Proposition 10.20. *Let G be a group and H be a subgroup of G. The following statements hold:*

1) *If G is solvable, then H is solvable.*
2) *If H is normal in G and G is solvable, then G/H is solvable.*
3) *If H is normal in G and H and G/H are solvable, then G is solvable.*

Proof. The first statement follows from the fact that $H^{(i)} \subseteq G^{(i)}$ holds for all i.

Assume now that H is normal in G. Let $Q = G/H$ and $\pi\colon G \to Q$ be the canonical map. By induction one proves that $\pi(G^{(i)}) = Q^{(i)}$ for all $i \geq 0$. The case where $i = 0$ is trivial, as π is surjective. If the result holds for some $i \geq 0$, then

$$\pi(G^{(i+1)}) = \pi([G^{(i)}, G^{(i)}]) = [\pi(G^{(i)}), \pi(G^{(i)})] = [Q^{(i)}, Q^{(i)}] = Q^{(i+1)}.$$

We now prove 2). Since G is solvable, $G^{(n)} = \{1\}$ for some n. Thus Q is solvable, as $Q^n = \pi(G^{(n)}) = \pi(\{1\}) = \{1\}$.

We finally prove 3). Since Q is solvable, $Q^{(n)} = \{1\}$ for some n. Moreover, since $\pi(G^{(n)}) = Q^{(n)} = \{1\}$, it follows that $G^{(n)} \subseteq H$. Since H is solvable,

$$G^{(n+m)} \subseteq (G^{(n)})^{(m)} \subseteq H^{(m)} = \{1\}$$

for some m. Thus G is solvable. □

An application:

Proposition 10.21. *Let G be a finite p-group. Then G is solvable.*

Proof. Assume the result is not true. Let G be a finite p-group of minimal order that is not solvable. Since G is a p-group, $Z(G) \neq \{1\}$. Since $|G|$ is minimal, $G/Z(G)$ is a solvable p-group. Since $Z(G)$ is abelian, $Z(G)$ is solvable. Now G is solvable by Proposition 10.20. □

Let G be a group. A subgroup N of G is said to be **maximal normal** if N is a normal subgroup of G and there is no other normal subgroup of G containing N.

Exercise 10.22. If a subgroup N of G is maximal (for the inclusion) and normal, then it is maximal normal. Show that the converse does not hold.

The following result is a direct consequence of the correspondence theorem:

Exercise 10.23. Let G be a group and N be a normal subgroup of G. Prove that N is maximal normal if and only if G/N is simple.

Maximal normal subgroups always exist in finite groups (they could be trivial). We can compute maximal normal subgroups as follows:

```
julia> maximal_normal_subgroups(symmetric_group(3))
1-element Vector{PermGroup}:
 Group([ (1,2,3) ])

julia> maximal_normal_subgroups(quaternion_group(8))
3-element Vector{PcGroup}:
 Group([ y2, x ])
 Group([ y2, y ])
 Group([ y2, x*y ])

julia> maximal_normal_subgroups(alternating_group(4))
1-element Vector{PermGroup}:
 Group([ (1,4)(2,3), (1,2)(3,4) ])
```

Exercise 10.24. Let G be a finite solvable group. Prove that if G is simple, then G is cyclic of prime order.

The following result will be important later:

Proposition 10.25. *Every finite solvable group contains a normal subgroup of prime index.*

Proof. Let G be a finite solvable group. Let M be a maximal normal subgroup of G (there is at least one, as G is finite). Since G/M is simple and solvable (see Proposition 10.20), G/M is cylic of prime order by Exercise 10.24. □

We finish this discussion with two important theorems (without proof) about finite solvable groups.

Theorem 10.26 (Burnside). *Let p and q be prime numbers. If G is a group of order $p^a q^b$, then G is solvable.*

The proof appears in courses on the representation theory of finite groups.

Theorem 10.27 (Feit–Thompson). *Every finite group of odd order is solvable.*

The proof of the theorem is extremely hard. It occupies a full volume of *Pacific Journal of Mathematics*, see [**2**].

Lecture 11. 06/05/2024

§ 11.1. Simplicity of the alternating simple group. We will present a family of non-abelian simple groups. We start with some exercises.

EXERCISE 11.1. Let G be a group. Prove that G is simple if and only if $\{(g,g) : g \in G\}$ is a maximal subgroup of $G \times G$.

EXERCISE 11.2. Prove that $\mathbb{A}_n$ is generated by 3-cycles.

EXERCISE 11.3. Compute the commutator subgroup of $\mathbb{A}_n$ for $n \geq 2$.

Note that $\mathbb{A}_2$ and $\mathbb{A}_3$ are abelian. For $\mathbb{A}_4$, one proves that

$$[\mathbb{A}_4, \mathbb{A}_4] = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\}.$$

Finally, $[\mathbb{A}_n, \mathbb{A}_n] = \mathbb{A}_n$ for $n \geq 5$.

Let us compute some commutator subgroups (and the inclusion group homomorphism) with the computer:

```
julia> derived_subgroup(symmetric_group(3))
(Alt( [ 1 .. 3 ] ), Group homomorphism from
Alt( [ 1 .. 3 ] )
to
Sym( [ 1 .. 3 ] ))
```

EXERCISE 11.4. Let $n \geq 3$. Prove that $[\mathbb{S}_n, \mathbb{S}_n] = \mathbb{A}_n$.

Recall that every normal subgroup is a union of conjugacy classes. The group $\mathbb{A}_5$ has conjugacy classes of sizes 1, 15, 20, 12 and 12. It follows that the only possible normal subgroups of $\mathbb{A}_5$ are $\{\mathrm{id}\}$ and $\mathbb{A}_5$.

```
julia> A5 = alternating_group(5);

julia> [length(c) for c in conjugacy_classes(A5)]
5-element Vector{ZZRingElem}:
 1
 15
 20
 12
 12
```

THEOREM 11.5 (Jordan). *Let $n \geq 5$. Then $\mathbb{A}_n$ is simple.*

Before proving the theorem, we need some preliminary results.

Every permutation $\rho \in \mathbb{S}_n$ decomposes as a product of disjoint cycles, say

$$\rho = (a_1 \cdots a_r)(b_1 \cdots b_s) \cdots (c_1 \cdots c_t)$$

where, by convention, we do not write cycles of length one. The cyclic structure of ρ is, by definition, the ordered sequence of integers $r, s, \dots t$, where, again by convention, we omit fixed points. For example, the cyclic structure of the transposition (ab) is 2, of $(abc)(d)$ is 3 and of $(123)(45)(789a)(bcd)(d)$ is 2,3,3,4.

LEMMA 11.6. *If both $\rho_1 \in \mathbb{S}_n$ and $\rho_2 \in \mathbb{S}_n$ have the same cyclic structure, then $\rho_2 = \sigma\rho_1\sigma^{-1}$ for some $\sigma \in \mathbb{S}_n$.*

PROOF. Assume that

$$\rho_1 = (a_1 \cdots a_r)(b_1 \cdots b_s) \cdots (c_1 \cdots c_t),$$
$$\rho_2 = (x_1 \cdots x_r)(y_1 \cdots y_s) \cdots (z_1 \cdots z_t).$$

Let

$$\mathrm{Fix}(\rho_1) = \{x \in \{1, \dots, n\} : \rho_1(x) = x\} = \{k_1, \dots, k_m\}, \qquad \mathrm{Fix}(\rho_2) = \{l_1, \dots, l_m\}$$

be the fixed points of the permutations ρ_1 and ρ_2, respectively. Then

$$\sigma(x) = \begin{cases} x_j & \text{if } x = a_j \text{ for some } j, \\ y_j & \text{if } x = b_j \text{ for some } j, \\ \vdots & \\ z_j & \text{if } x = c_j \text{ for some } j, \\ l_j & \text{if } x = k_j \text{ for some } j, \end{cases}$$

is such that $\sigma\rho_1\sigma^{-1} = \rho_2$. □

What happens with the alternating group?

LEMMA 11.7. *If $\rho_1, \rho_2 \in \mathbb{S}_n$ are conjugate in $\mathbb{S}_n$ and $|\mathrm{Fix}(\rho_1)| \geq 2$, then $\mu\rho_1\mu^{-1} = \rho_2$ for some $\mu \in \mathbb{A}_n$.*

PROOF. Assume that $\rho_2 = \sigma\rho_1\sigma^{-1}$ for some $\sigma \in \mathbb{S}_n$. There are $a, b \in \{1, \dots, n\}$ such that $\rho_1(a) = a$, $\rho_1(b) = b$ and $a \neq b$. Let

$$\mu = \begin{cases} \sigma & \text{if } \sigma \in \mathbb{A}_n, \\ \sigma(ab) & \text{otherwise.} \end{cases}$$

Then $\mu \in \mathbb{A}_n$ and $\mu\rho_1\mu^{-1} = \rho_2$, as (ab) commutes with ρ_1. □

Let us discuss some examples.

EXAMPLE 11.8. If $\rho_1 = (23)(156)$ and $\rho_2 = (45)(123)$, then $\rho_2 = \sigma\rho_1\sigma^{-1}$ for

$$\sigma = \begin{pmatrix} 123456 \\ 145623 \end{pmatrix}.$$

EXAMPLE 11.9. The permutations $\rho_1 = (123) \in \mathbb{S}_3$ and $\rho_2 = (132) \in \mathbb{S}_3$ are conjugate in $\mathbb{S}_3$, as $(123) = \sigma(132)\sigma^{-1}$ if $\sigma = (23)$. However, ρ_1 and ρ_2 are not conjugate in $\mathbb{A}_3$.

Now we are ready to prove the theorem.

PROOF OF THEOREM 11.5. Let $N \neq \{\mathrm{id}\}$ be a normal subgroup of $\mathbb{A}_n$. If $(abc) \in N$, then every 3-cycle belongs to N, because all 3-cycles are conjugate in $\mathbb{S}_n$, and the previous lemma states that $(ijk) = \mu(abc)\mu^{-1} \in N$ for some $\mu \in \mathbb{A}_n$. Thus $N = \mathbb{A}_n$.

We claim that N contains a 3-cycle. Since $N \neq \{\mathrm{id}\}$, there exists $\sigma \in N \setminus \{\mathrm{id}\}$. Let $m = |\sigma|$ and let p be a prime number dividing m. Then $\tau = \sigma^{m/p}$ has order p and hence $\tau = \rho_1 \cdots \rho_s$, where the ρ_j's are disjoint p-cycles.

If $p = 2$, then $1 = \mathrm{sign}(\tau) = (-1)^s$. Thus s is even. Write

$$\tau = (ab)(cd)\rho_3 \cdots \rho_s.$$

Since $\rho_3 \cdots \rho_s$ commutes with (abc) and (acb),

$$\underbrace{(abc)\tau(abc)^{-1}\tau^{-1}}_{\in N} = (abc)(ab)(cd)(acb)(ab)(cd) = (ac)(bd).$$

Hence $(ac)(bd) \in N$. Let $e \in \{1, \dots, n\} \setminus \{a, b, c, d\}$. Then

$$(ae)(bd) = (aec)\underbrace{(ac)(bd)}_{\in N}(aec)^{-1} \in N$$

and therefore

$$(aec) = (ac)(ae) = (ac)(bd)(ae)(bd) \in N.$$

If $p = 3$, without loss of generality, we may assume that $s \geq 2$ (otherwise, τ would be a 3-cycle). Then $\tau = (abc)(def)\rho_3 \cdots \rho_s$. Since (bcd) commutes with $\rho_3 \cdots \rho_s$ and N is normal in $\mathbb{A}_n$,

$$\underbrace{(bcd)\tau(bcd)^{-1}\tau^{-1}}_{\in N} = (bcd)(abc)(def)(bdc)(acb)(dfe) = (adbce)$$

and therefore

$$(adc) = (adb)(adbce)(adb)^{-1}(adbce)^{-1} \in N.$$

If $p > 3$, then $\tau = (abcd \cdots z)\rho_2 \cdots \rho_s$. In particular, (abc) commutes with $\rho_2 \cdots \rho_s$. Then

$$(abd) = (abc)\tau(abc)^{-1}\tau^{-1} \in N.$$

□

As an application, we compute the normal subgroups of the symmetric group $\mathbb{S}_n$.

EXERCISE 11.10. Compute the list of normal subgroups of $\mathbb{S}_n$ for $n \geq 2$.

§ 11.2. Radical extensions.

DEFINITION 11.11. An extension E/K is said to be **pure** of type m if $E = K(x)$ for some x such that $x^m \in K$.

Note that if $E = K(x)$ is a pure extension of type m and K contains m-th roots of one, then E/K is a splitting field of $X^m - x^m$.

DEFINITION 11.12. The sequence $K = R_0 \subseteq R_1 \subseteq \cdots \subseteq R_m$ of fields is said to be a **radical tower** if each R_{i+1}/R_i is pure. In this case, R_m/K is a **radical extension**.

Note that radical extensions are finite.

EXAMPLE 11.13. Let E be a decomposition field of $X^4 - 2$ over $\mathbb{Q}$. Then $E/\mathbb{Q}$ is radical, as $E = \mathbb{Q}(\sqrt[4]{2}, i)$.

EXAMPLE 11.14. Let $\alpha, \beta \in \mathbb{C}$ be such that $\alpha^2 = 2$ and $\beta^5 = 1 + \alpha$. The number $\sqrt[5]{1+\sqrt{2}}$ belongs to the radical extension $\mathbb{Q}(\alpha, \beta)/\mathbb{Q}$.

Theorem 11.15. *Let K be of characteristic zero and R/K be a radical extension. If E/K is a subextension of R/K, then* $\mathrm{Gal}(E/K)$ *is solvable.*

Proof. Without loss of generality, we may assume that E/K is a Galois extension. To prove this fact, let $G = \mathrm{Gal}(E/K)$ and $F = {}^{G}E$. Then E/F is a Galois extension and $\mathrm{Gal}(E/F) = G$ by Artin's theorem. Thus, replacing K by F if needed, we may assume that E/K is Galois.

Let L be the normal closure of R in some algebraic closure C that contains R. Note that if $R = K(x_1,\dots,x_m)$, then

$$L = K(\{\sigma_i(x_j) : 1 \le i \le s,\, 1 \le j \le m\}),$$

where $\mathrm{Hom}(R/K, C/K) = \{\sigma_1,\dots,\sigma_s\}$.

Claim. L/K is radical.

Since $x_j^{a_j} \in K(x_1,\dots,x_{j-1})$ for some integer a_j,

$$\sigma_i(x_j)^{a_j} = \sigma_i\left(x_j^{a_j}\right) \in \sigma_i(K(x_1,\dots,x_{j-1}) = K(\sigma_i(x_1),\dots,\sigma_i(x_{j-1}))$$

Thus L/K is radical and Galois.

We may assume then that E/K and R/K are both Galois.

Since $\mathrm{Gal}(E/K) \simeq \mathrm{Gal}(R/K)/\mathrm{Gal}(R/E)$, we only need to prove that $\mathrm{Gal}(R/K)$ is solvable.

For a positive integer n, let ξ be a primitive n-th root of one (in some algebraic closure of K that contains R). Consider the diagram

$$\begin{tikzcd}
 & R(\xi) \arrow[dl, dash] \arrow[dr, dash] & \\
R \arrow[dr, dash] & & K(\xi) \arrow[dl, dash] \\
 & K &
\end{tikzcd}$$

Then

1) $K(\xi)/K$ and $R(\xi)/R$ are abelian.
2) $R(\xi)/K$ is Galois.
3) $\mathrm{Gal}(R/K) \simeq \mathrm{Gal}(R(\xi)/K)/\mathrm{Gal}(R(\xi)/R)$.
4) $\mathrm{Gal}(K(\xi)/K) \simeq \mathrm{Gal}(R(\xi)/K)/\mathrm{Gal}(R(\xi)/K(\xi))$.

The third item implies that we need to show that $\mathrm{Gal}(R(\xi)/K)$ is solvable. By the fourth item, it suffices to show that $\mathrm{Gal}(R(\xi)/K(\xi))$ is solvable (because $\mathrm{Gal}(K(\xi)/K)$ is abelian and hence solvable).

Since $R = K(x_1,\dots,x_m)$,

$$R(\xi) = K(x_1,\dots,x_m,\xi) = K(\xi)(x_1,\dots,x_m)$$

and hence $R(\xi)/K(\xi)$ is radical. This means that without loss of generality, we may assume that K contains primitive n-roots of one. For example, if $R = K(x_1,\dots,x_m)$ and $x_i^{a_i} \in K(x_1,\dots,x_{i-1})$, then we may assume that K contains a primitive a_i-root of one. We proceed by induction on m. The case $m = 0$ is trivial. Assume that the claim holds for some $m \ge 0$. Let $L = K(x_1)$. Then L/K is a decomposition field of $X^{a_1} - x_1^{a_1}$, and hence L/K is a cyclic extension. Thus $\mathrm{Gal}(L/K)$ is cyclic (and hence, in particular, solvable). Let H be the subgroup that corresponds to L, that is $H = \mathrm{Gal}(R/L)$ (here, we use Galois' correspondence). Then H is normal in $\mathrm{Gal}(R/K)$.

Since $R = K(x_1,\dots,x_m) = L(x_2,\dots,x_m)$, R/L is radical and Galois. By the inductive hypothesis, $\mathrm{Gal}(R/L)$ is solvable. Since

$$\mathrm{Gal}(L/K) \simeq \mathrm{Gal}(R/K)/\mathrm{Gal}(R/L),$$

it follows that $\mathrm{Gal}(R/K)$ is solvable. □

Definition 11.16. Let $f \in K[X]$ and E be a decomposition field of f over K. We say that f is **solvable by radicals** if there is a radical extension R/K such that $E \subseteq R$.

The general polynomial of degree two is solvable by radicals, as its Galois group is solvable (in fact, isomorphic to $\mathbb{S}_2$).

Exercise 11.17. Prove that $f = X^2 - s_1X + s_2 \in \mathbb{Q}[X]$ is solvable by radicals.

Theorem 11.15 translates into the following result:

Exercise 11.18. Let K be a field of characteristic zero. If $f \in K[X]$ is solvable by radicals, then $\mathrm{Gal}(f,K)$ is solvable.

As a consequence, the general polynomial of degree $n \geq 5$ is not solvable by radicals, as its Galois group is isomorphic to $\mathbb{S}_5$.

Example 11.19. Let p be a prime number and $f = X^5 - 2pX + p \in \mathbb{Q}[X]$. We claim that f is not solvable by radicals.

By Gauss' theorem, one proves that f has no rational roots.

Note that $f' = 5X^4 - 2p$. Then $\alpha = \sqrt[4]{2p/5}$ and $\beta = -\sqrt[4]{2p/5}$ are are critical points. Since $f(\alpha) < 0$ and $f(\beta) > 0$, it follows that f has exactly three real roots. Let $x_1, x_2 \in \mathbb{C} \setminus \mathbb{R}$ and $x_3, x_4, x_5 \in \mathbb{R}$ be the roots of f.

By Eisenstein's theorem, f is irreducible.

Let $E/\mathbb{Q}$ be a decomposition field of f. Then $\mathrm{Gal}(f,\mathbb{Q}) = \mathrm{Gal}(E/\mathbb{Q})$ is isomorphic to a subgroup G of $\mathbb{S}_5$. Since f is irreducible, 5 divides $[E:\mathbb{Q}] = |G|$. In particular, by Cauchy's theorem, G contains an element σ of order five. This element is a 5-cycle, so without loss of generality, we may assume that $\sigma = (x_1x_2x_3x_4x_5)$. Note that $(x_1x_2) \in G$. Thus $G \simeq \mathbb{S}_5$ and hence G is not solvable.

Exercise 11.20. Let $f = X^6 + 2X^5 - 5X^4 + 9X^3 - 5X^2 + 2X + 1 \in \mathbb{Q}[X]$. Prove that f is solvable by radicals.

It is now time to prove Galois' great theorem on solvability of polynomials.

Theorem 11.21 (Galois). *Let K be a field of characteristic zero and $f \in K[X]$. Then f is solvable by radicals if and only if* $\mathrm{Gal}(f,K)$ *is solvable.*

We proved in Theorem 11.15 that solvable polynomials have solvable Galois groups. For the converse, we need two auxiliary results.

Lemma 11.22. *Let E/K be a Galois extension of prime degree p. Assume that K admits a primitive p-root of one. Then $E = K(\beta)$ where $\beta^p \in K$.*

PROOF. Assume that $\operatorname{Gal}(E/K) = \langle\sigma\rangle$. Let $\omega \in K$ be a primitive p-root of one. Then $\operatorname{norm}_{E/K}(\omega) = \omega^p = 1$. By Hilbert's theorem, $\omega = \beta/\sigma(\beta)$ for some $\beta \in E$. Note that $\beta \notin K$, as $\omega \neq 1$. Moreover,

$$\sigma(\beta^p) = (\beta\omega^{-1})^p = \beta^p \in {}^{\operatorname{Gal}(E/K)}E = K.$$

Since $K \subseteq K(\beta) \subseteq E$ and $[E:K] = p$, we conclude that $E = K(\beta)$ with $\beta^p \in K$. $\square$

EXERCISE 11.23. Let E/K be a decomposition field of $f \in K[X]$ and K^*/K be an extension. If E^*/K^* is a decomposition field of f containing E, then

$$\operatorname{Gal}(E^*/K^*) \to \operatorname{Gal}(E/K), \quad \sigma \mapsto \sigma|_E,$$

is an injective group homomorphism.

Now Theorem 11.21 will follow from the following theorem.

THEOREM 11.24. *Let K be a field of characteristic zero and E/K be a Galois extension. If $\operatorname{Gal}(E/K)$ is solvable, then E can be embedded in a radical extension.*

PROOF. Let $G = \operatorname{Gal}(E/K)$. Since G is solvable, by Proposition 10.25, there exists a normal subgroup H of G of prime index p. Let ω be a primitive p-th root of one. (It exists because K is a field of characteristic zero.)

We proceed by induction on $[E:K]$.

If $[E:K] = 1$, there is nothing to prove. So assume that $[E:K] > 1$.

We first assume that $\omega \in K$. The group $\operatorname{Gal}(E/{}^HE)$ is solvable, as it is a subgroup of G. Moreover, since

$$[E:{}^HE] < [E:K],$$

the inductive hypothesis implies that ${}^HE/E$ can be embedded in a radical extension, so there exists a radical tower is

$$ {}^HE \subseteq R_1 \subseteq R_2 \subseteq \cdots \subseteq R_m, \tag{11.1}$$

where $E \subseteq R_m$. Now $E/{}^HE$ is a Galois extension, as H is normal in G. Moreover,

$$[{}^HE:K] = (G:H) = p.$$

Since $\omega \in K$, Lemma 11.22 implies that ${}^HE = K(\beta)$ for some β such that $\beta^p \in K$. The radical tower 11.1 can be extended by adding $K \subseteq {}^HE$.

For the general case, let $K^* = K(\omega)$ and $E^* = E(\omega)$. Then E^*/K^* is a Galois extension with Galois group $\operatorname{Gal}(E^*/K^*)$. By Exercise 11.23, $\operatorname{Gal}(E^*/K^*)$ is solvable. By the previous part, E^* and E can be embedded in a radical extension R^*/K^*, so there exists a radical tower

$$K^* \subseteq R_1^* \subseteq R_2^* \subseteq \cdots \subseteq R_n^*. \tag{11.2}$$

Since $K^* = K(\omega)$ is a pure extension, the radical tower (11.2) can be extended by adding $K \subseteq K^*$. $\square$

Lecture 12.

§ 12.1. The inverse Galois problem.

§ 12.2. Group cohomology. Let G be a group and A be a **(left) G-module**. This means that A is an abelian group together with a map

$$G\times A\to A,\quad (g,a)\mapsto g\cdot a$$

such that $1\cdot a=a$ for all $a\in A$, $(gh)\cdot a=g\cdot(h\cdot a)$ for all $g,h\in G$ and $a\in A$ and $g\cdot(a+b)=g\cdot a+g\cdot b$ for all $g\in G$ and $a,b\in A$.

Example 12.1. The group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on $\mathbb{C}$ and $\mathbb{C}^\times$. Moreover, it acts trivially on $\mathbb{R}$ and $\mathbb{R}^\times$.

More generally, if E/K is a finite Galois extension, then the Galois group $\mathrm{Gal}(E/K)$ acts on E and $E^\times$.

Definition 12.2. Let G be a group and M and N be G-modules. A map $f\colon M\to N$ is a **homomorphism** of G-modules if $f(\sigma\cdot m)=\sigma\cdot f(m)$ for all $m\in M$ and $\sigma\in G$.

Definition 12.3. Let G be a group and M be a G-module. The submodule of **G-invariants** is defined as

$$M^G=\{m\in M:\sigma\cdot m=m\text{ for all }\sigma\in G\}.$$

Note that M^G is the largest submodule of the G-module M where G acts trivially. For example, if $G=\mathrm{Gal}(E/K)$, then $E^G=K$.

Proposition 12.4. *Let G be a group. If the sequence of G-modules and G-module homomorphism*

$$0\longrightarrow P\xrightarrow{\alpha}M\xrightarrow{\beta}N\longrightarrow 0$$

is exact, then

$$0\longrightarrow P^G\xrightarrow{\alpha^0}M^G\xrightarrow{\beta^0}N^G$$

is exact, where α^0 is the restriction $\alpha|_{P^G}$ of α to P^G and β^0 is the restriction $\beta|_{M^G}$ of β to M^G.

Proof. Since α is injective, the restriction α^0 is injective.

Note that $\ker\beta^0=\ker\beta\cap M^G\subseteq\ker\beta$.

We claim that $\alpha^0(P^G)=\alpha(P)\cap M^G$. If $m\in\alpha(P)\cap M^G$, then $\alpha(p)=m$ for some $p\in P$ and $\sigma\cdot m=m$. Since

$$\alpha(p)=m=\sigma\cdot m=\sigma\cdot\alpha(p)=\alpha(\sigma\cdot p),$$

$\sigma\cdot p-p\in\ker\alpha=\{0\}$. Hence $\sigma\cdot p=p$ and $p\in P^G$. Conversely, if $m\in\alpha^0(P^G)$, then $m=\alpha(p)$ for some $p\in P^G$. If $\sigma\in G$, then

$$\sigma\cdot m=\sigma\cdot\alpha(p)=\alpha(\sigma\cdot p)=\alpha(p)=m.$$

Hence $m\in M^G\cap\alpha(P)$.

Now

$$\alpha^0(P^G)=\alpha(P)\cap M^G=\ker\beta\cap M^G=\ker\beta^0.\qquad\square$$

Note that in the previous proposition, we did not prove that the map $\beta|_{M^G}$ is surjective.

EXAMPLE 12.5. Let $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Consider the following exact sequence of G-modules:

$$1 \longrightarrow \{-1,1\} \longrightarrow \mathbb{C}^\times \xrightarrow{\beta} \mathbb{C}^\times \longrightarrow 1$$

where $\beta(z) = z^2$. Note that β is surjective. Take invariants to obtain the sequence

$$0 \longrightarrow \{-1,1\} \longrightarrow \mathbb{R}^\times \xrightarrow{\beta^0} \mathbb{R}^\times$$

where $\beta^0(x) = x^2$. Note that β^0 is not surjective!

DEFINITION 12.6. Let G be a group and N be a G-module. We define

$$\begin{aligned}
&H^0(G,M) = M^G,\\
&C^1(G,M) = \{\phi\colon G \to M : \phi \text{ is a map}\},\\
&Z^1(G,M) = \{\phi \in C^1(G,M) : \phi(\sigma\tau) = \phi(\sigma) + \sigma\cdot\phi(\tau) \text{ for all } \sigma,\tau \in G\},
\end{aligned}$$

Note that $Z^1(G,M)$ is an abelian group with the operation

$$(\phi + \phi_1)(\sigma) = \phi(\sigma) + \phi_1(\sigma).$$

Moreover, if $\phi \in Z^1(G,M)$, then $\phi(1_G) = 0_M$. To prove this fact, note that

$$\phi(1_G) = \phi(1_G 1_G) = \phi(1_G) + 1_G\cdot\phi(1_G) = \phi(1_G) + \phi(1_G)$$

implies that $\phi(1_G) = 0_M$.

EXAMPLE 12.7. Let G be a group and M be a G-module. Fix $m \in M$. Then the map $\phi\colon G \to M$, $\phi(\sigma) = \sigma\cdot m - m$, is an element of $Z^1(G,M)$, because

$$\begin{aligned}
\phi(\sigma\tau) &= (\sigma\tau)\cdot m - m\\
&= (\sigma\tau)\cdot m - \sigma\cdot m + \sigma\cdot m - m\\
&= \sigma\cdot(\tau\cdot m - m) + \sigma\cdot m - m\\
&= \sigma\cdot\phi(\tau) + \phi(\sigma)
\end{aligned}$$

for all $\sigma,\tau \in G$.

DEFINITION 12.8. Let G be a group and M be a G-module. The set $B^1(G,M)$ of **coboundaries** is the set of elements $\phi \in C^1(G,M)$ such that there is a fixed $m \in M$ such that $\phi(\sigma) = \sigma\cdot m = m$ for all $\sigma \in G$.

We proved in Example 12.7 that $B^1(G,M) \subseteq Z^1(G,M)$. A direct calculation shows that, in fact, $B^1(G,M)$ is a subgroup of $Z^1(G,M)$.

DEFINITION 12.9. Let G be a group and M be a G-module. The **first cohomology group** of G with coefficients in M is defined as the quotient

$$H^1(G,M) = Z^1(G,M)/B^1(G,M).$$

EXAMPLE 12.10. If G acts trivially on M, then

$$H^0(G,M) = M^G = M, \quad B^1(G,M) = \{0\}, \quad Z^1(G,M) = \mathrm{Hom}(G,M).$$

Hence $H^1(G,M) \simeq \mathrm{Hom}(G,M)$.

Example 12.11. Let $G=\mathrm{Gal}(\mathbb{C}/\mathbb{R})=\{\mathrm{id},\gamma\}$, where $\gamma\colon \mathbb{C}\to\mathbb{C}$, $z\mapsto\bar{z}$, is the complex conjugation. Then

$$H^0(G,\mathbb{R}^\times)=\left(\mathbb{R}^\times\right)^G=\mathbb{R}^\times.$$

Since G acts trivially on $\mathbb{R}^\times$,

$$H^1(G,\mathbb{R}^\times)=\mathrm{Hom}(G,\mathbb{R}^\times)\simeq\mathrm{Hom}(G,\{-1,1\})\simeq\mathbb{Z}/2.$$

The following lemma will be useful.

Lemma 12.12. *Let G be a group and $\alpha\colon M\to N$ be a homomorphism of G-modules. Then*

$$\alpha^1\colon H^1(G,M)\to H^1(G,N),\quad \phi+B^1(G,M)\mapsto\alpha\circ\phi+B^1(G,N),$$

is a group homomorphism.

Proof. Let us prove that the map α^1 is well-defined. If $\phi-\phi'\in B^1(G,M)$, then there exists a fixed $m\in M$ such that $(\phi-\phi')(\sigma)=\sigma\cdot m-m$ for all $\sigma\in G$. Let $n=\alpha(m)\in N$. For $\sigma\in G$,

$$\alpha((\phi-\phi')(\sigma))=\alpha(\sigma\cdot m-m)=\sigma\cdot\alpha(m)-\alpha(m)=\sigma\cdot n-n.$$

Thus $\alpha\circ\phi-\alpha\circ\phi'\in B^1(G,N)$.

We now prove that α^1 is group homomorphism. If $\phi,\phi'\in Z^1(G,M)$, then

$$\begin{aligned}\alpha^1(\phi+B^1(G,M)+\phi'+B^1(G,M))&=\alpha^1(\phi+\phi'+B^1(G,M))\\&=\alpha\circ(\phi+\phi')+B^1(G,N)\\&=\alpha\circ\phi+\alpha\circ\phi'+B^1(G,N)\\&=\alpha\circ\phi+B^1(G,N)+\alpha\circ\phi'+B^1(G,N)\\&=\alpha^1(\phi+B^1(G,M))+\alpha^1(\phi'+B^1(G,M)).\end{aligned}$$ □

We will provide a detailed proof of the upcoming result. The theorem will be established by applying a **diagram chasing** technique, a widely used tool in homological algebra. The proof is tedious and may seem intricate, but the method essentially involves “chasing” elements around a (commutative) diagram until we attain the desired result.

Theorem 12.13. *Let G be a group and*

$$0\longrightarrow P\xrightarrow{\alpha}M\xrightarrow{\beta}N\longrightarrow 0$$

*be an exact sequence of G-modules and G-module homomorphism. Then there exists a **connection homomorphism** δ such that the sequence*

$$\begin{tikzcd} 0 \arrow[r] & H^0(G,P) \arrow[r, "\alpha^0"] & H^0(G,M) \arrow[r, "\beta^0"] & H^0(G,N) \arrow[dll, rounded corners, "\delta"] \\ & H^1(G,P) \arrow[r, "\alpha^1"] & H^1(G,M) \arrow[r, "\beta^1"] & H^1(G,N) \end{tikzcd} \tag{12.1}$$

of abelian groups and group homomorphisms is exact.

Proof. By Proposition 12.4, the sequence is exact at $H^0(G,P)=P^G$, $H^0(G,M)=M^G$ and $H^0(G,N)=N^G$. Note that, in particular, $\alpha\colon P\to\alpha(P)$ is a bijective group homomorphism.

Let us construct the connecting homomorphism $\delta\colon H^0(G,N)\to H^1(G,P)$. For $n\in N^G$, let $m\in M$ be such that $\beta(m)=n$. We define $\delta(n)=\phi+B^1(G,P)$, where

$$\phi(\sigma)=\alpha^{-1}(\sigma\cdot m-m).$$

Note that $\sigma\cdot m-m\in\operatorname{im}\alpha=\ker\beta$, as

$$\beta(\sigma\cdot m-m)=\sigma\cdot\beta(m)-\beta(m)=\sigma\cdot n-n=0.$$

Let us prove that the map δ is well-defined: if $m,m'\in M$ are such that $\beta(m)=\beta(m')=n$, then $m-m'\in\ker\beta=\alpha(P)$. For $\sigma\in G$, write $\phi'(\sigma)=\sigma\cdot m'-m'$. Since $m-m'=\alpha(p)$ for some $p\in P$ and α^{-1} is a homomorphism of G-modules,

$$\begin{aligned}\phi(\sigma)-\phi'(\sigma)&=\alpha^{-1}(\sigma\cdot m-m)-\alpha^{-1}(\sigma\cdot m'-m')\\&=\alpha^{-1}(\sigma\cdot(m-m'))-\alpha^{-1}(m-m')\\&=\alpha^{-1}(\sigma\cdot\alpha(p)-\alpha(p))\\&=\sigma\cdot p-p.\end{aligned}$$

Thus $\phi-\phi'\in B^1(G,P)$.

Note that $\phi\in Z^1(G,P)$, because

$$\begin{aligned}\phi(\sigma\tau)&=\alpha^{-1}((\sigma\tau)\cdot m-m)\\&=\alpha^{-1}((\sigma\tau)\cdot m-\sigma\cdot m+\sigma\cdot m-m)\\&=\alpha^{-1}(\sigma\cdot(\tau\cdot m-m))+\alpha^{-1}(\sigma\cdot m-m)\\&=\sigma\cdot\phi(\tau)+\phi(\sigma)\end{aligned}$$

holds for all $\sigma,\tau\in G$.

We now prove that the sequence (12.1) is exact at $H^0(G,N)=N^G$. We need to prove that $\ker\delta=\operatorname{im}\beta^0$. To prove $\supseteq$ note that if $m\in M^G$ is such that $\delta(\beta(m))=\phi+B^1(G,P)$, then

$$\phi(\sigma)=\alpha^{-1}(\sigma\cdot m-m)=0.$$

Conversely, if $n\in\ker\delta$, then there exists $m\in M$ such that $\beta(m)=n$ and $\delta(\beta(m))=\phi+B^1(G,P)$, where $\phi\in B^1(G,P)$ and $\phi(\sigma)=\alpha^{-1}(\sigma\cdot m-m)$ for all $\sigma\in G$. Since $\phi\in B^1(G,P)$, there exists $p\in P$ such that $\phi(\sigma)=\sigma\cdot p-p$ for all $\sigma\in G$. Note that

$$\beta(m-\alpha(p))=\beta(m)-\beta(\alpha(p))=\beta(m)=n.$$

Moreover, $m-\alpha(p)\in M^G$, as $\sigma\cdot(m-\alpha(p))=m-\alpha(p)$. Hence $n\in\operatorname{im}\beta^0$.

We now prove that (12.1) is exact at $H^1(G,P)$, that is $\operatorname{im}\delta=\ker\alpha^1$. To prove $\subseteq$ note that for $n\in N^G$, $\delta(n)=\phi+B^1(G,P)$, where $\phi(\sigma)=\alpha^{-1}(\sigma\cdot m-m)$ for all $\sigma\in G$ and some $m\in M$ such that $\beta(m)=n$. In particular, $\alpha\circ\phi\in B^1(G,M)$. Then

$$\alpha^1(\phi+B^1(G,P))=\alpha\circ\phi+B^1(G,M)=B^1(G,M).$$

To prove $\supseteq$, let $\phi+B^1(G,P)\in\ker\alpha^1$. Then $\alpha\circ\phi\in B^1(G,M)$, that is $\alpha(\phi(\sigma))=\sigma\cdot m-m$ for all $\sigma\in G$ and some $m\in M$. Note that

$$\delta(\beta(m))=\psi+B^1(G,P),$$

where $\psi(\sigma)=\alpha^{-1}(\sigma\cdot m-m)$. This implies that $\alpha(\psi(\sigma))=\alpha(\phi(\sigma))$ for all $\sigma\in G$. Since α is injective, $\psi=\phi$. Therefore $\phi+B^1(G,P)$ belongs to the image of δ.

Finally, we prove that the sequence (12.1) is exact at $H^1(G,M)$, that is $\operatorname{im}\alpha^1 = \ker\beta^1$. To prove $\subseteq$ note that

$$\beta^1(\alpha^1(\phi + B^1(G,P))) = \beta^1(\alpha\circ\phi + B^1(G,M)) = (\beta\circ\alpha)\circ\phi + B^1(G,N) = B^1(G,N).$$

Conversely, let $\phi + B^1(G,M) \in \ker\beta_1$. Then $\beta\circ\phi \in B^1(G,N)$. Thus there exists $n \in N$ such that $\beta(\phi(\sigma)) = \sigma\cdot n - n$ for all $\sigma \in G$. Since β is surjective, $n = \beta(m)$ for some $m \in M$. Hence

$$\beta(\phi(\sigma)) = \sigma\cdot n - n = \sigma\cdot\beta(m) - \beta(m) = \beta(\sigma\cdot m - m).$$

For each $\sigma \in G$, $\phi(\sigma) - (\sigma\cdot m - m) \in \ker\beta = \operatorname{im}\alpha$. and therefore $\phi(\sigma) - (\sigma\cdot m - m) = \alpha(\rho_\sigma)$. This defines a map $\rho\colon G \to P$, $\sigma \mapsto \rho_\sigma$. We claim that $\rho \in Z^1(G,P)$. If $\sigma,\tau \in G$, then

$$\begin{aligned}\alpha(\rho_{\sigma\tau}) &= \phi(\sigma\tau) - ((\sigma\tau)\cdot m - m)\\ &= \phi(\sigma) + \sigma\cdot\phi(\tau) - (\sigma\cdot(\tau\cdot m - m) + \sigma\cdot m - m)\\ &= \alpha(\rho_\sigma) + \sigma\cdot\alpha(\rho_\tau).\end{aligned}$$

Since α is injective, it follows that $\rho \in Z^1(G,P)$. Now

$$\alpha_1(\rho + B^1(G,P)) = \alpha\circ\rho + B^1(G,M) = \phi + B^1(G,M).$$

□

Theorem 12.14. *Let G be a finite group and M be a G-module. Then*

$$|G|H^1(G,M) = \{0\}.$$

Proof. Let $n = |G|$. It is enough to prove that if $\phi \in Z^1(G,M)$, then $n\phi \in B^1(G,M)$. Let $\phi \in Z^1(G,M)$ and $\sigma \in G$. Then

$$\phi(\sigma\tau) = \phi(\sigma) + \sigma\cdot\phi(\tau)$$

for all $\tau \in G$. Summing over all possible $\tau \in G$, we obtain that

$$\sum_{\tau\in G}\phi(\tau) = \sum_{\tau\in G}\phi(\sigma\tau) = \sum_{\tau\in G}\sigma\cdot\phi(\tau) + \sum_{\tau\in G}\phi(\sigma) = n\phi(\sigma). \tag{12.2}$$

Let $m = -\sum_{\tau\in G}\phi(\tau) \in M$. Then (12.2) can be rewritten as

$$-m = \sum_{\tau\in G}\phi(\tau) = \sigma\cdot\sum_{\tau\in G}\phi(\tau) + n\phi(\sigma) = -\sigma\cdot m + n\phi(\sigma).$$

Thus $n\phi(\sigma) = \sigma\cdot m - m$ and hence $n\phi \in B^1(G,M)$. □

Exercise 12.15. Let G be a finite group and M be a finite G-module of size coprime to $|G|$. Prove that $H^1(G,M) = \{0\}$.

Exercise 12.16. Let G be a finite group and M be a finitely generated G-module. Prove that $H^1(G,M)$ is finite.

Some solutions

1.3. Assume that $\mathbb{Q}[i]$ and $\mathbb{Q}[\sqrt{2}]$ were isomorphic and let $\varphi : \mathbb{Q}[i] \to \mathbb{Q}[\sqrt{2}]$ be a field isomorphism. Then

$$\varphi(i)^2 = \varphi(i^2) = \varphi(-1) = -\varphi(1) = -1.$$

But $\varphi(i) \in \mathbb{Q}[\sqrt{2}]$ and $\mathbb{Q}[\sqrt{2}] \subseteq \mathbb{R}$ where every square is positive, a contradiction.

1.4. Let $t > 0$ be the characteristic of a field K and let $\varphi : \mathbb{Z} \to K$, $x \mapsto x1$. Then, by definition, $\ker\varphi$ is the ideal generated by t. On the other hand, the image $\varphi(\mathbb{Z})$ is a domain, being a subring of a field and is isomorphic to $\mathbb{Z}/\ker\varphi$. Therefore, $\ker\varphi$ is a prime ideal of $\mathbb{Z}$, i.e. t is a prime number.

1.6. Let $\varphi : \mathbb{Z} \to K$, $x \mapsto x1$.

We first prove that **1)** implies all the other properties. So suppose that the characteristic of K is zero, i.e. $\ker\varphi = \{0\}$.

Then $m1 = 0$ if and only if $m = 0$, i.e. the order of 1 is infinite.

Let $0 \neq x \in K$. If $mx = 0$, then $0 = mx = (m1)x$. But K is a field and $x \neq 0$, hence $m1 = 0$, so $m \in \ker\varphi = \{0\}$. Hence the order of x is infinite.

By definition, the ring of integers of K is the image of φ, which so it is isomorphic to $\mathbb{Z}/\ker\varphi = \mathbb{Z}$.

Finally, we prove that **4)** implies **1)**. Take $m \in \ker\varphi$. Then $m1 = 0$, but 1 has infinite order, hence $m = 0$. Therefore $\ker\varphi = \{0\}$, i.e. K has characteristic 0.

1.7. Let $\varphi : \mathbb{Z} \to K$, $x \mapsto x1$.

We first prove that **1)** implies all the other properties. So suppose that the characteristic of K is $p > 0$ i.e. $\ker\varphi$ is the ideal generated by p.

Then $m1 = 0$ if and only if p divides m, i.e. the order of 1 is p.

Let $0 \neq x \in K$. If $mx = 0$, then $0 = mx = (m1)x$. But K is a field and $x \neq 0$, hence $m1 = 0$, so p divides m. Hence x has order p.

By definition, the ring of integers of K is the image of φ, which so it is isomorphic to $\mathbb{Z}/\ker\varphi \cong \mathbb{Z}/p$.

Finally, we prove that **4)** implies **1)**. Take $m \in \ker\varphi$. Then $m1 = 0$, but 1 has order p, hence p divides m. Therefore $\ker\varphi$ is generated by p, i.e. K has characteristic 0.

1.10. Let $\Phi : K \to K$ be the map $x \mapsto x^p$. Since the map $x \mapsto x^{p^n}$ is exactly Φ^n, it is enough to prove that Φ is a field homomorphism.

As K is commutative under multiplication, for all $x, y \in K$

$$\Phi(xy) = (xy)^p = x^p y^p = \Phi(x)\Phi(y).$$

Moreover, for all $x, y \in K$

$$\Phi(x+y) = (x+y)^p \sum_{k=0}^{p} \binom{p}{k} x^p y^{p-k} = x^p + y^p + \sum_{k=1}^{p-1} \binom{p}{k} x^k y^{p-k},$$

where $\binom{p}{k} = \frac{p!}{k!(p-k)!}$, which can also be written as

$$p! = \binom{p}{k} \cdot k! \cdot (p-k)!$$

But p divides $p!$, so p has to divide at least one factor on the right side. But p doesn't divide i for $1 \le i \le p-1$, therefore if $k \le p-1$, p doesn't divide $k!$ and if $1 \le k$, p doesn't divide $(p-k)!$. Hence, if $1 \le k \le p-1$, p has to divide $\binom{p}{k}$ and

$$\sum_{k=1}^{p-1} \binom{p}{k} x^k y^{p-k} = 0.$$

Therefore, Φ is a field homomorphism.

1.22. By definition $K_0 = \{m1 \colon m \in \mathbb{Z}\}$ and $\sigma : K \to K$ is a field homomorphism, so $\sigma(1) = 1$. Hence, for every $m \in \mathbb{Z}$

$$\sigma(m1) = m\sigma(1) = m1,$$

i.e. $\sigma|_{K_0}$ is the identity.

1.25. If $X^3 - 2$ were reducible, since it has degree 3, it would have a linear factor in the decomposition in irreducibles. Therefore it would have a rational root. But the roots of $X^3 - 2$ are $\sqrt[3]{2}, \sqrt[3]{2}\xi, \sqrt[3]{2}\xi^2$, where ξ is a primitive third root of unity, So all the roots are not in $\mathbb{Q}$, a contradiction.

1.26. Recall first the following:

Lemma 12.17 (Gauss' Lemma). *Let A be a unique factorization domain and K be its fraction field. A non-constant polynomial $f \in A[X]$ is irreducible if and only if it is primitive and irreducible in $K[X]$.*

Suppose that f is reducible in $K[X]$. Then $g = c^{-1}f$, where c is the content of f, would be reducible and primitive. Hence, by Gauss' Lemma, g is also reducible in $A[X]$. So $c^{-1}f = g = hl$, for some non-constant polynomials $h, l \in A[X]$. Now consider $\pi : A \to A/(p)$, $a \mapsto \overline{a}$ the natural surjection. We know that $\overline{a_i} = 0$ for all $i \in \{0, 1, \ldots, n-1\}$ and $\overline{a_n} \neq 0$. Therefore

$$\overline{\pi}(ch)\overline{\pi}(l) = \overline{c}\,\overline{\pi}(h)\overline{\pi}(l) = \overline{\pi}(f) = \overline{a_n}X^n \in A/(p)[X].$$

But $A/(p)[X]$ is a UFD so the only possibility is that $\overline{\pi}(ch) = \overline{d}X^t$ and $\overline{\pi}(l) = \overline{f}X^s$, for some $f, d \in A/(p) \setminus \{\overline{0}\}$ and $t, s \in \{1, \ldots, n-1\}$. In particular, $\overline{\pi}(ch)$ and $\overline{\pi}(l)$ have both constant term equal to 0. Hence p divides $ch(0)$ and $l(0)$ in A. Therefore p^2 divides $ch(0)l(0) = f(0)$, a contradiction.

1.27. It is easy to see that f satisfies the Eisenstein criterion for $p = 2$ and g satisfies it for $p = 5$.

1.28. $f = 3(X^{10} + 5X^2 - 15)$ is a product of 3 and $(X^{10} + 5X^2 - 15)$, which are both non-invertible elements of $\mathbb{Z}[X]$. Hence f is reducible.

2.6. Clearly for every field extension L/K and every $\alpha \in L$ we have that $K[\alpha] \subseteq K(\alpha)$.

Vice versa take $\frac{a+\sqrt{2}b}{c+\sqrt{2}d} \in \mathbb{Q}(\sqrt{2})$, then we can write:

$$\frac{a+\sqrt{2}b}{c+\sqrt{2}d} = \frac{(a+\sqrt{2}b)(c-\sqrt{2}d)}{(c+\sqrt{2}d)(c-\sqrt{2}d)} = \frac{ac-2bd+(bc-ad)\sqrt{2}}{c^2-2d^2}.$$

Hence

$$\frac{a+\sqrt{2}b}{c+\sqrt{2}d} = \frac{ac-2bd}{c^2-2d^2} + \frac{bc-ad}{c^2-2d^2}\sqrt{2} \in \mathbb{Q}[\sqrt{2}].$$

2.8. Let $f = f(x,K)$ be the minimal polynomial of x over K of degree $\deg(f) = n$. We claim that $\{1,x,\dots,x^{n-1}\}$ is a basis of $K(x)$ as a K-vector space.

To prove that $\{1,x,\dots,x^{n-1}\}$ is a generating set, recall that $K(x) = K[x]$, since x is algebraic over K. Let $z \in K(x) = K[x]$, say $z = h(x)$ for some $h \in K[X]$. Divide h by f to obtain polynomials $q,r \in K[X]$ such that $h = fq + r$, where either $r = 0$ or $\deg r < \deg f = n$. Then

$$z = h(x) = f(x)q(x) + r(x) = r(x).$$

Write $r = \sum_{i=0}^{n-1} c_i X^i$ for some $c_0,\dots,c_{n-1} \in K$. Thus $z = \sum_{i=0}^{n-1} a_i x^i \in \langle 1,x,\dots,x^{n-1}\rangle$.

We now prove that $\{1,x,\dots,x^{n-1}\}$ is linearly independent. If not, there exists a linear combination $0 = \sum_{i=0}^{n-1} a_i x^i$ with $a_0,\dots,a_{n-1} \in K$ not all zero. Then $h(X) = \sum_{i=0}^{n-1} a_i X^i \in K[X] \setminus \{0\}$ has x as a root and

$$n = \deg(f) \le \deg(h) \le n-1,$$

a contradiction.

2.10. a is algebraic over K, so, by Theorem 2.7, it has finite degree over K and $K[a] = K(a)$. b is algebraic over K, so it is also algebraic over $K(a)$, hence it has finite degree over $K(a)$ and $K(a)[b] = K(a,b)$. This implies that the extension $K(a,b)/K$ is a finite extension since it is a tower of finite extensions. Hence, by Corollary 2.9, $K(a,b)/K$ is an algebraic extension. Therefore, since $a+b, ab \in K(a,b)$, this implies that $a+b$ and ab are algebraic over K.

2.14. Assume that $K(S)/K$ is algebraic, then, by Corollary 2.12, x is algebraic over K for all $x \in S$. By Corollary 2.11, we conclude that $K(S)/K$ is finite.

On the other hand, if $K(S)/K$ is finite, then $K \subseteq K(x) \subseteq K(S)$ for all $x \in S$, so $K(x)/K$ is finite for all $x \in S$. Then, by Theorem 2.7, x is algebraic over K for all $x \in S$. Hence, by Corollary 2.12, $K(S)/K$ is algebraic.

2.15. $\sqrt[3]{2}$ is a root of the monic polynomial $f = X^3 - 2 \in \mathbb{Q}[X]$. Therefore $\sqrt[3]{2}$ is algebraic over $\mathbb{Q}$ and $\mathbb{Q}[\sqrt[3]{2}] = \mathbb{Q}(\sqrt[3]{2})$. In Exercise 1.25, we proved that f is irreducible in $\mathbb{Q}[X]$. Hence f is the minimal polynomial of $\sqrt[3]{2}$ over $\mathbb{Q}$ and, by Theorem 2.7, $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = \deg f = 3$.

2.16. i is a root of the monic polynomial $X^2 + 1 \in \mathbb{Q}[X]$ and $\sqrt{2}$ is a root of the monic polynomial of $X^2 - 2 \in \mathbb{Q}[X]$. So, by Corollary 2.11, $\mathbb{Q}[i,\sqrt{2}] = \mathbb{Q}(i,\sqrt{2})$ and it is algebraic over $\mathbb{Q}$. By Eisenstein's criterion with $p = 2$, $X^2 - 2$ is irreducible in $\mathbb{Q}[X]$, so $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$. Since i is a root of $X^2 + 1 \in \mathbb{Q}[X]$, then $[E : \mathbb{Q}(\sqrt{2})] \le 2$. Moreover, $i \notin \mathbb{R} \supseteq \mathbb{Q}(\sqrt{2})$. Therefore $[E : \mathbb{Q}(\sqrt{2})] = 2$ and, by Proposition 2.4,

$$[E : \mathbb{Q}] = [E : \mathbb{Q}(\sqrt{2})][\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 4.$$

2.17.

1) We know that $[\mathbb{Q}(\sqrt[3]{5}) : \mathbb{Q}]$ is the same as the degree of the minimal polynomial of $\sqrt[3]{5}$ over $\mathbb{Q}$. Clearly $\sqrt[3]{5}$ is a root of $X^3 - 5 \in \mathbb{Q}[X]$. Moreover, by Eisenstein's criterion with $p = 5$, we get that $X^3 - 5$ is irreducible in $\mathbb{Q}[X]$. Hence $f(\sqrt[3]{5},\mathbb{Q}) = X^3 - 5$ and $[\mathbb{Q}(\sqrt[3]{5} : \mathbb{Q}] = 3$. We also know that $X^2 - 2$ is the minimal polynomial of $\sqrt{2}$ over $\mathbb{Q}$. So $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$. Therefore we are in the following situation:

$$\begin{tikzcd}[column sep=small]
& E=\mathbb{Q}(\sqrt[3]{5},\sqrt{2}) \arrow[dl, dash, "\leq 2"'] \arrow[dr, dash] & \\
\mathbb{Q}(\sqrt[3]{5}) \arrow[dr, dash, "3"'] & & \mathbb{Q}(\sqrt{2}) \arrow[dl, dash, "2"] \\
& \mathbb{Q} &
\end{tikzcd}$$

so on the one hand

$$[E:\mathbb{Q}] = [E:\mathbb{Q}(\sqrt{2})][\mathbb{Q}(\sqrt{2}):\mathbb{Q}] = [E:\mathbb{Q}(\sqrt{2})]2$$

and on the other hand

$$[E:\mathbb{Q}] = [E:\mathbb{Q}(\sqrt[3]{5})][\mathbb{Q}(\sqrt[3]{5}):\mathbb{Q}] = [E:\mathbb{Q}(\sqrt[3]{5})]3 \leq 2\cdot 3 = 6.$$

Therefore 2 and 3 divide $[EF:\mathbb{Q}] \leq 6$. Hence the only possibility is that $[E:\mathbb{Q}] = 6$.

2) Clearly $\mathbb{Q}(\sqrt{2}+\sqrt[3]{5}) \subseteq E$. On the other hand, let $\alpha = \sqrt{2}+\sqrt[3]{5}$. Then

$$5 = (\alpha-\sqrt{2})^3 = \alpha^3 - 3\sqrt{2}\alpha^2 + 6\alpha - 2\sqrt{2},$$

which implies that

$$\sqrt{2} = \frac{6\alpha-5}{3\alpha^2+2} \in \mathbb{Q}(\alpha).$$

Moreover, $\sqrt[3]{5} = \alpha - \sqrt{2} \in \mathbb{Q}(\alpha)$, hence $E = \mathbb{Q}(\alpha)$.

3) From the previous part of this exercise we get that

$$\sqrt{2}(3\alpha^2+2) = \alpha^3 + 6\alpha - 5.$$

Hence, squaring both sides of the previous equality, we obtain

$$18\alpha^4 + 24\alpha^2 + 8 = \alpha^6 + 36\alpha^2 + 25 + 12\alpha^4 - 10\alpha^3 - 60\alpha.$$

Therefore α is a root of the polynomial $f(X) = X^6 - 6X^4 - 10X^3 + 12X^2 - 60X + 17$. Moreover, from the first part, we know that $[E:\mathbb{Q}] = 6 = \deg f$, hence $f(\alpha,\mathbb{Q}) = f(X)$.

2.18. Let $\alpha = \sqrt[4]{3}i$. Observe that $\alpha^2 = -\sqrt{3}$, so $\mathbb{Q}(\sqrt{3}) \subseteq \mathbb{Q}(\alpha)$ and

$$2 \geq \deg f(\alpha,\mathbb{Q}(\sqrt{3})) = [\mathbb{Q}(\alpha):\mathbb{Q}(\sqrt{3})].$$

Moreover $\alpha \notin \mathbb{R}$, while $\mathbb{Q}(\sqrt{3}) \subseteq \mathbb{R}$. So $[\mathbb{Q}(\alpha):\mathbb{Q}(\sqrt{3})] > 1$, hence $[\mathbb{Q}(\alpha):\mathbb{Q}(\sqrt{3})] = 2$ and

$$f(\alpha,\mathbb{Q}(\sqrt{3})) = X^2 + \sqrt{3}.$$

Note that the minimal polynomial of α over $\mathbb{Q}(i)$ has degree $[\mathbb{Q}(\sqrt[4]{3}i,i):\mathbb{Q}(i)]$. Moreover, $\mathbb{Q}(\sqrt[4]{3}i,i) = \mathbb{Q}(\sqrt[4]{3},i)$ and

$$f(\alpha,\mathbb{Q}) = f(\sqrt[4]{3},\mathbb{Q}) = X^4 + 3,$$

since X^4+3 is an irreducible (due to Eisenstein with $p=3$) monic polynomial that has α and $\sqrt[4]{3}$ as roots. Therefore $[\mathbb{Q}(\sqrt[4]{3}):\mathbb{Q}] = [\mathbb{Q}(\sqrt[4]{3}i):\mathbb{Q}] = 4$. Since $\mathbb{Q}(\sqrt[4]{3}) \subseteq \mathbb{R}$, while $\sqrt[4]{3}i \notin \mathbb{R}$, we obtain that $1 < [\mathbb{Q}(\sqrt[4]{3}i,i):\mathbb{Q}(\sqrt[4]{3})] \leq [\mathbb{Q}(i):\mathbb{Q}] = 2$.

$$\begin{tikzcd}
& \mathbb{Q}(\sqrt[4]{3}i,i)=\mathbb{Q}(\sqrt[4]{3},i) \arrow[dl, dash, "2"'] \arrow[dr, dash] & \\
\mathbb{Q}(\sqrt[4]{3}) \arrow[dr, dash, "4"'] & & \mathbb{Q}(i) \arrow[dl, dash, "2"] \\
& \mathbb{Q} &
\end{tikzcd}$$

Hence $[\mathbb{Q}(\sqrt[4]{3}i,i):\mathbb{Q}]=8$ and $[\mathbb{Q}(\sqrt[4]{3}i,i):\mathbb{Q}(i)]=4$, which means that $\deg f(\sqrt[4]{3}i,\mathbb{Q}(i))=4$. But $f(\sqrt[4]{3}i,\mathbb{Q}(i))$ divides $f(\alpha,\mathbb{Q})=X^4+3$, so

$$f(\sqrt[4]{3}i,\mathbb{Q}(i))=X^4+3.$$

2.19. Let $\alpha=\sqrt{2}+i\sqrt[3]{5}$ and $E=\mathbb{Q}(\alpha,i)$. The minimal polynomial of α over $\mathbb{Q}(i)$ has degree $[E:\mathbb{Q}(i)]$. Observe that, since $\alpha-\sqrt{2}=i\sqrt[3]{5}$, we have that $\alpha^3-3\sqrt{2}\alpha^2+6\alpha-2\sqrt{2}=-i5$. Hence

$$\sqrt{2}=\frac{\alpha^3\alpha^2+6\alpha+i5}{3\alpha^2+2}\in E$$

and so also $\sqrt[3]{5}=\frac{\alpha-\sqrt{2}}{i}\in E$. Therefore $E=\mathbb{Q}(\sqrt{2},\sqrt[3]{5},i)$. To compute $[E:\mathbb{Q}(i)]$ we first compute $[\mathbb{Q}(\sqrt{2},i):\mathbb{Q}(i)]$ and $[\mathbb{Q}(\sqrt[3]{5},i):\mathbb{Q}(i)]$.

We know that i has degree 2 over $\mathbb{Q}$, so $[\mathbb{Q}(\sqrt[3]{5},i):\mathbb{Q}(\sqrt[3]{5})]$ and $[\mathbb{Q}(\sqrt{2},i):\mathbb{Q}(\sqrt{2})]$ are both at most 2. Moreover $\mathbb{Q}(\sqrt[3]{5})$ and $Q(\sqrt{2})$ are contained in $\mathbb{R}$, while $i\notin\mathbb{R}$. Hence

$$[\mathbb{Q}(\sqrt[3]{5},i):\mathbb{Q}(\sqrt[3]{5})]=2=[\mathbb{Q}(\sqrt{2},i):\mathbb{Q}(\sqrt{2})].$$

$$\begin{tikzcd}
& \mathbb{Q}(\sqrt{2},i) \arrow[dl, dash, "2"'] \arrow[dr, dash] & \\
\mathbb{R}\supseteq\mathbb{Q}(\sqrt{2}) \arrow[dr, dash, "2"'] & & \mathbb{Q}(i) \arrow[dl, dash, "2"] \\
& \mathbb{Q} &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathbb{Q}(\sqrt[3]{5},i) \arrow[dl, dash] \arrow[dr, dash, "2"] & \\
\mathbb{Q}(i) \arrow[dr, dash, "2"'] & & \mathbb{Q}(\sqrt[3]{5})\subseteq\mathbb{R} \arrow[dl, dash, "3"] \\
& \mathbb{Q} &
\end{tikzcd}$$

Therefore $[\mathbb{Q}(\sqrt{2},i):\mathbb{Q}(i)]=2$ and $[\mathbb{Q}(\sqrt[3]{5},i):\mathbb{Q}(i)]=3$.

$$\begin{tikzcd}
& \mathbb{Q}(\alpha,i)=\mathbb{Q}(\sqrt{2},\sqrt[3]{5},i) \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{Q}(\sqrt{2},i) \arrow[dr, dash, "2"'] & & \mathbb{Q}(\sqrt[3]{5},i) \arrow[dl, dash, "3"] \\
& \mathbb{Q}(i) &
\end{tikzcd}$$

So $[\mathbb{Q}(\alpha,i):\mathbb{Q}(i)]$ is divisible by 2 and 3 and it is also at most 6. Therefore the degree of the minimal polynomial of α over $\mathbb{Q}(i)$ is $[\mathbb{Q}(\alpha,i):\mathbb{Q}(i)]=6$.

We already got $\alpha^3-3\sqrt{2}\alpha^2+6\alpha-2\sqrt{2}=-i5$, so $2(3\alpha^2+2)^2=(\alpha^3\alpha^2+6\alpha+i5)^2$, i.e.

$$\alpha^6-6\alpha^4+10i\alpha^3+12\alpha^2+60i\alpha-33=0.$$

This means that α is a root of the polynomial

$$f(X)=X^6-6X^4+10iX^3+12X^2+60iX-33\in\mathbb{Q}(i)[X].$$

Since f is also monic and of degree 6, we can deduce that $f(\alpha,\mathbb{Q}(i))=f$.

2.21. By Proposition 2.4, we know that $[E:K]=[E:F][F:K]$, so $[E:K]$ is finite if and only if $[E:F]$ and $[F:K]$ are finite.

2.22. Let P be the set $\{\sum_{i=1}^m e_if_i : m\in\mathbb{Z}_{>0}, e_i\in E, f_i\in F \text{ for all } i\in\{1,\dots,m\}\}$. If $\sum_{i=1}^m e_if_i\in P$, it is a E-linear combination of elements in F, so in particular it is an element in $E(F)=EF$. Hence $P\subseteq EF$.

Moreover, since E/K and F/K are algebraic extensions, every element in $E\cup F$ is algebraic over K. So $EF=K(E\cup F)=K[E\cup F]$. Let $x\in EF$, then $x=f(\alpha_1,\dots,\alpha_k)$, for some polynomial $f\in K[X_1,\dots X_k]$ and $\alpha_1,\dots,\alpha_k\in E\cup F$. We can then split the polynomial in $f=p+q$ so that $x=p(e'_1,\dots,e'_n)+q(f'_1,\dots,f'_m)$, where $e'_i\in E$ and $f'_j\in F$ and $p,q\in K[X_1,\dots X_k]$. Since E and F are fields, so closed under multiplication, we can write x as $x=\sum_{i=1}^N k_ie_i+\sum_{j=1}^M h_jf_j$, for some $k_i,h_j\in K$, $e_i\in E$ and $f_j\in F$. Then in particular $k_i\in K\subseteq F$ and $h_j\in K\subseteq E$, hence $x\in P$ and $EF\subseteq P$.

2.23. We know that the minimal polynomials over $\mathbb{Q}$ are $f(\sqrt{2},\mathbb{Q})=X^2-2$ and $f(\sqrt{3},\mathbb{Q})=X^2-3$. So $[\mathbb{Q}(\sqrt{2}):\mathbb{Q}]=[\mathbb{Q}(\sqrt{3}):\mathbb{Q}]=2$. Moreover, $[\mathbb{Q}(\sqrt{2},\sqrt{3}):\mathbb{Q}(\sqrt{3})]\le[\mathbb{Q}(\sqrt{2}):\mathbb{Q}]=2$. It remains to check whether $\sqrt{2}\in\mathbb{Q}(\sqrt{3})$ or not. A $\mathbb{Q}$ basis of $\mathbb{Q}(\sqrt{3})$ is $\{1,\sqrt{3}\}$. If $\sqrt{2}\in\mathbb{Q}(\sqrt{3})$, then $\sqrt{2}=a+b\sqrt{3}$ for some $a,b\in\mathbb{Q}$, so $2=a^2+2ab\sqrt{3}+3b^2$. Using the $\mathbb{Q}$-linear independence of $\{1,\sqrt{3}\}$, we get that $2ab=0$ and $2=a^2+3b^2$. Therefore either $a=0$ and $2/3=b^2$, or $b=0$ and $2=a^2$. But both cases are not possible because neither 2 nor $2/3$ are squares in $\mathbb{Q}$.

We conclude that $\sqrt{2}\notin\mathbb{Q}(\sqrt{3})$. So $[\mathbb{Q}(\sqrt{2},\sqrt{3}):\mathbb{Q}(\sqrt{3})]=2$ and $[\mathbb{Q}(\sqrt{2},\sqrt{3}):\mathbb{Q}]=4$.

$$\begin{tikzcd}
 & \mathbb{Q}(\sqrt{2},\sqrt{3}) \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{Q}(\sqrt{2}) \arrow[dr, dash] \arrow[ddr, dash, "2"'] & & \mathbb{Q}(\sqrt{3}) \arrow[dl, dash] \arrow[ddl, dash, "2"] \\
 & \mathbb{Q}(\sqrt{2})\cap\mathbb{Q}(\sqrt{3}) \arrow[d, dash] & \\
 & \mathbb{Q} &
\end{tikzcd}$$

Similarly to before $[\mathbb{Q}(\sqrt{2}):\mathbb{Q}(\sqrt{2})\cap\mathbb{Q}(\sqrt{3})]\le[\mathbb{Q}(\sqrt{2}):\mathbb{Q}]=2$, but $\sqrt{2}\notin\mathbb{Q}(\sqrt{2})\cap\mathbb{Q}(\sqrt{3})$, so $[\mathbb{Q}(\sqrt{2}):\mathbb{Q}(\sqrt{2})\cap\mathbb{Q}(\sqrt{3})]=2$. Hence $[\mathbb{Q}(\sqrt{2})\cap\mathbb{Q}(\sqrt{3}):\mathbb{Q}]=1$, i.e. $\mathbb{Q}(\sqrt{2})\cap\mathbb{Q}(\sqrt{3})=\mathbb{Q}$.

2.24. The minimal polynomial of $\sqrt[3]{2}$ is X^3-2 since it is monic, irreducible (by Eisenstein) and has $\sqrt[3]{2}$ as a root. Hence $[\mathbb{Q}(\sqrt[3]{2}):\mathbb{Q}]=3$. $\xi\neq1$ and it is a root of the polynomial

$$X^3-1=(X-1)(X^2+X+1),$$

so it is a root of X^2+X+1, which is monic and irreducible (it is of degree 2 and the roots are not in $\mathbb{Q}$). Hence $f(\xi,\mathbb{Q}) = X^2+X+1$ and $[\mathbb{Q}(\xi):\mathbb{Q}] = 2$. We also have that $[\mathbb{Q}(\sqrt[3]{2},\xi):\mathbb{Q}] \leq 6$.

$$\begin{tikzcd}
& \mathbb{Q}(\sqrt[3]{2},\xi) & \\
\mathbb{Q}(\sqrt[3]{2}) \arrow[ur, dash] & & \mathbb{Q}(\xi) \arrow[ul, dash] \\
& \mathbb{Q}(\sqrt[3]{2})\cap\mathbb{Q}(\xi) \arrow[ul, dash] \arrow[ur, dash] & \\
& \mathbb{Q} \arrow[u, dash] \arrow[uul, dash, "3"] \arrow[uur, dash, "2"'] &
\end{tikzcd}$$

By multiplicity of the degree of extensions, we obtain that 6 has to divide $[\mathbb{Q}(\sqrt[3]{2},\xi):\mathbb{Q}] \leq 6$ and $[\mathbb{Q}(\sqrt[3]{2})\cap\mathbb{Q}(\xi):\mathbb{Q}]$ has to divide 2 and 3. Therefore

$$[\mathbb{Q}(\sqrt[3]{2},\xi):\mathbb{Q}] = 6 \text{ and } [\mathbb{Q}(\sqrt[3]{2})\cap\mathbb{Q}(\xi):\mathbb{Q}] = 1,$$

which means that $\mathbb{Q}(\sqrt[3]{2})\cap\mathbb{Q}(\xi) = \mathbb{Q}$.

2.25. By definition $EF = E(F)$. If F/K is algebraic, then EF is generated by algebraic elements over K, so also over E. Hence EF/E is an algebraic extension.

2.26. If F/K is finite, then F is generated by a finite number of algebraic elements over K. The same elements are algebraic over E and generate $EF = E(F)$ over E. Hence EF/E is a finite extension and $[EF:E] \leq [F:K]$.

4.17. Note that, since $f = X^4-5X^2+5$ is an even polynomial if $\alpha\in\mathbb{C}$ is a root of f, then also $-\alpha$ is a root of f. Hence, given two roots $\alpha,\beta\in\mathbb{C}$ such that $\beta\neq-\alpha$, we have that the decomposition field of f over $\mathbb{Q}$ is $E = \mathbb{Q}(\alpha,-\alpha,\beta,-\beta)$. But $-\alpha,-\beta\in\mathbb{Q}(\alpha,\beta)\subseteq E$ and so

$$E = \mathbb{Q}(\alpha,-\alpha,\beta,-\beta)\subseteq\mathbb{Q}(\alpha,\beta)\subseteq\mathbb{Q}(\alpha,-\alpha,\beta,-\beta) = E,$$

which means that $E = \mathbb{Q}(\alpha,\beta)$. Moreover we can decompose f in $\mathbb{C}[X]$ as

$$(X-\alpha)(X+\alpha)(X-\beta)(X+\beta) = (X^2-\alpha^2)(X^2-\beta^2) = X^4-(\alpha^2+\beta^2)X^2+\alpha^2\beta^2.$$

This implies in particular that $\alpha^2\beta^2 = 5$, hence $\beta = \pm\frac{\sqrt{5}}{\alpha}\in\mathbb{Q}(\alpha,\sqrt{5})$.

Therefore $E = \mathbb{Q}(\alpha,\beta)\subseteq\mathbb{Q}(\alpha,\sqrt{5})$. On the other hand $\sqrt{5} = \pm\alpha\beta\in\mathbb{Q}(\alpha,\beta)$, hence $\mathbb{Q}(\alpha,\sqrt{5})\subseteq\mathbb{Q}(\alpha,\beta) = E$. So we can conclude that $E = \mathbb{Q}(\alpha,\sqrt{5})$. Using the multiplicative of the degree of finite extension we get that

$$[E:\mathbb{Q}] = [E:\mathbb{Q}(\alpha)][\mathbb{Q}(\alpha):\mathbb{Q}].$$

But $[\mathbb{Q}(\alpha):\mathbb{Q}] = \deg(f(\alpha,\mathbb{Q}))$. Using Eisenstein criterion (Exercise 1.26) with $p=5$, we have that f is irreducible (and monic), so $f = f(\alpha,\mathbb{Q})$. Thus $[\mathbb{Q}(\alpha):\mathbb{Q}] = \deg f = 4$. It remains to compute $[E:\mathbb{Q}(\alpha)] = [\mathbb{Q}(\alpha,\sqrt{5}):\mathbb{Q}(\alpha)]$. We have the following situation:

$$\begin{tikzcd}
& E=\mathbb{Q}(\alpha,\sqrt{5}) \arrow[dl, dash, "\leq 2"'] \arrow[dr, dash] & \\
\mathbb{Q}(\alpha) \arrow[dr, dash, "4"'] & & \mathbb{Q}(\sqrt{5}) \arrow[dl, dash, "2"] \\
& \mathbb{Q} &
\end{tikzcd}$$

Observe that $\mathbb{Q}(\alpha,\sqrt{5})$ is equal to the composite of $\mathbb{Q}(\alpha)$ and $\mathbb{Q}(\sqrt{5})$. We can use the property of composite extension, $[LF:L] \leq [F:K]$, to deduce that

$$[\mathbb{Q}(\alpha,\sqrt{5}):\mathbb{Q}(\alpha)] \leq [\mathbb{Q}(\sqrt{5}):\mathbb{Q}] = 2.$$

The last equality is because $f(\sqrt{5},\mathbb{Q}) = X^2-5$, as it is monic has $\sqrt{5}$ as a root and it's irreducible (due to Eisenstein's criterion or because it is of degree 2 with 2 non-rational roots). Finally, we want to understand whether $[\mathbb{Q}(\alpha,\sqrt{5}):\mathbb{Q}(\alpha)]$ is 1 or 2. Note that $\alpha^4-5\alpha^2+5=0$, so we can solve the equation for α^2 as it is a root of X^2-5X+5, i.e.

$$\alpha^2 = \frac{5\pm\sqrt{25-20}}{2} = \frac{5\pm\sqrt{5}}{2},$$

hence $\sqrt{5} = \pm(2\alpha^2-5) \in \mathbb{Q}(\alpha)$. So $\mathbb{Q}(\alpha,\sqrt{5}) \subseteq \mathbb{Q}(\alpha) \subseteq \mathbb{Q}(\alpha,\sqrt{5})$, which means that $E=\mathbb{Q}(\alpha)$ and $[E:\mathbb{Q}] = [\mathbb{Q}(\alpha):\mathbb{Q}] = 4$.

5.5. First of all, note that $\sqrt[3]{2}$ is a root of the polynomial $f(X) = X^3-2$. To prove that $\mathbb{Q}(\sqrt[3]{2},\xi)$ is a normal extension we use Proposition 5.10, so it is enough to prove that $\mathbb{Q}(\sqrt[3]{2},\xi)$ is the decomposition field of f. We know that the decomposition field E of f over $\mathbb{Q}$ is $\mathbb{Q}$ extended with the roots of f, i.e. $E=\mathbb{Q}(\sqrt[3]{2},\sqrt[3]{2}\xi,\sqrt[3]{2}\xi^2)$. But it's easy to see that actually

$$\mathbb{Q}(\sqrt[3]{2},\xi) = \mathbb{Q}(\sqrt[3]{2},\sqrt[3]{2}\xi,\sqrt[3]{2}\xi^2) = E.$$

The inclusion $\subseteq$ is because $\sqrt[3]{2}, \xi = \frac{\sqrt[3]{2}\xi}{\sqrt[3]{2}} \in E$. Vice versa $\supseteq$ is due to the fact that the roots of f are products of $\sqrt[3]{2}$ and ξ, elements in $\mathbb{Q}(\sqrt[3]{2},\xi)$.

5.11. Let $\alpha = \sqrt[4]{7}+\sqrt{2}$. Then $(\alpha-\sqrt{2})^4-7=0$. By expanding the left side, we get

$$0 = \alpha^4-4\sqrt{2}\alpha^3+12\alpha^2-8\sqrt{2}\alpha-3 = (\alpha^4+12\alpha^2-3)-(4\alpha^3+8\alpha)\sqrt{2}.$$

But $4\alpha^3+8\alpha = 4\alpha(\alpha^2+2) \neq 0$, otherwise $\alpha \in \{0,\pm i\sqrt{2}\}$. Therefore $\sqrt{2} = \frac{\alpha^4+12\alpha^2-3}{4\alpha^3+8\alpha} \in \mathbb{Q}(\alpha)$.

This allows us to prove that $\mathbb{Q}(\sqrt{2},\sqrt[4]{7}) = \mathbb{Q}(\alpha)$. From the definition of α it's clear that $\mathbb{Q}(\alpha) \subseteq \mathbb{Q}(\sqrt{2},\sqrt[4]{7})$. On the other hand, we just proved that $\sqrt{2}\in\mathbb{Q}(\alpha)$. As $\sqrt[4]{7} = \alpha-\sqrt{2} \in \mathbb{Q}(\alpha)$, we also see that $\sqrt[4]{7}\in\mathbb{Q}(\alpha)$. It follows that $\mathbb{Q}(\sqrt{2},\sqrt[4]{7}) \subseteq \mathbb{Q}(\alpha)$.

Moreover, $\sqrt{2}\notin\mathbb{Q}(\sqrt[4]{7})$. Otherwise, as $[\mathbb{Q}(\sqrt[4]{7}):\mathbb{Q}] = 4$ and $[\mathbb{Q}(\sqrt{2}):\mathbb{Q}] = 2$ we would get that $[\mathbb{Q}(\sqrt[4]{7}):\mathbb{Q}(\sqrt{2})] = 2$. Let $f(\sqrt[4]{7},\mathbb{Q}(\sqrt{2})) = X^2+\beta X+\gamma$, with $\beta,\gamma\in\mathbb{Q}(\sqrt{2})$. So

$$0 = f(\sqrt[4]{7}) = \sqrt{7}+\beta\sqrt[4]{7}+\gamma.$$

Therefore

$$\beta^2\sqrt{7} = (-\sqrt{7}-\gamma)^2 = 7+2\gamma\sqrt{7}+\gamma^2.$$

So $(\beta^2-2\gamma)\sqrt{7}=\gamma^2+7$. But $\beta^2-2\gamma\neq 0$ because $\gamma^2+\beta\gamma+\frac{\beta^2}{2}=0$ holds only for $\gamma=\frac{\beta}{2}(-1\pm i)\in\mathbb{C}\setminus\mathbb{R}$, which is clearly not in $\mathbb{Q}(\sqrt{2})$. Thus we would have $\sqrt{7}=\frac{\gamma^2+7}{\beta^2-2\gamma}\in\mathbb{Q}(\sqrt{2})$, which is a contradiction.

To sum up we have that $\sqrt{2}\notin\mathbb{Q}(\sqrt[4]{7})$ and

$$\begin{tikzcd}
 & E=\mathbb{Q}(\alpha)=\mathbb{Q}(\sqrt{2},\sqrt[4]{7}) \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{Q}(\sqrt{2}) & & \mathbb{Q}(\sqrt[4]{7}) \\
 & \mathbb{Q} \arrow[ul, dash, "2"] \arrow[ur, dash, "4"'] &
\end{tikzcd}$$

1) We know that $\sqrt[4]{7}\in\mathbb{Q}(\alpha)$ which has minimal polynomial $f(\sqrt[4]{7},\mathbb{Q})=x^4-7$. One root of this polynomial is $i\sqrt[4]{7}$. This root is not in $\mathbb{Q}(\alpha)\subseteq\mathbb{R}$ as it is in $\mathbb{C}\setminus\mathbb{R}$. Therefore $\mathbb{Q}(\alpha)/\mathbb{Q}$ is not normal by Proposition 5.7.

2) As $\sqrt{2}\notin\mathbb{Q}(\sqrt[4]{7})$, we see that $[\mathbb{Q}(\alpha):\mathbb{Q}(\sqrt[4]{7})]>1$. On the other hand,

$$[\mathbb{Q}(\alpha):\mathbb{Q}(\sqrt[4]{7})]\leq[\mathbb{Q}(\sqrt{2}:\mathbb{Q}]=2,$$

which proves that $[\mathbb{Q}(\alpha):\mathbb{Q}(\sqrt[4]{7})]=2$. Therefore,

$$[E:\mathbb{Q}]=[\mathbb{Q}(\alpha):\mathbb{Q}]=[\mathbb{Q}(\alpha):\mathbb{Q}(\sqrt[4]{7})]\cdot[\mathbb{Q}(\sqrt[4]{7}):\mathbb{Q}]=2\cdot 4=8.$$

3) Let $\sigma\in G=\mathrm{Gal}(E/\mathbb{Q})$. Since $E=\mathbb{Q}(\sqrt{2},\sqrt[4]{7})$ and $\sqrt{2}$ and $\sqrt[4]{7}$ are independent because $\sqrt{2}\notin\mathbb{Q}(\sqrt[4]{7})$, we know that σ is completely determined by $\sigma(\sqrt{2})$ and $\sigma(\sqrt[4]{7})$. By Proposition 4.10, $\sigma(\sqrt{2})\in E$ has to be a root of $f(\sqrt{2},\mathbb{Q})=X^2-2$ and $\sigma(\sqrt[4]{7})\in E$ has to be a root of $f(\sqrt[4]{7},\mathbb{Q})=X^4-7$. So $\sigma(\sqrt{2})=\pm\sqrt{2}$ and, since $E\subseteq\mathbb{R}$,

$$\sigma(\sqrt[4]{7})\in E\cap\{\sqrt[4]{7}i^j\mid j\in\{0,1,2,3\}\}=\{\pm\sqrt[4]{7}\}.$$

Therefore G contains 4 elements $\sigma_{k,l}$ for $k,l\in\mathbb{Z}/2$ such that $\sigma_{k,l}(\sqrt{2})=(-1)^k\sqrt{2}$ and $\sigma_{k,l}(\sqrt[4]{7})=(-1)^l\sqrt[4]{7}$. This gives directly the isomorphism between G and $\mathbb{Z}/2\times\mathbb{Z}/2$.

5.12. Let $\{v_i : i\in I\}$ be a basis of V over K. For each $i\in I$ let $f_i\colon V\to F$, $f_i(v_j)=\delta_{ij}$. Then $\{f_i : i\in I\}$ is linearly independent over F. In fact, let $\sum a_i f_i=0$, where each $a_i\in F$. Then $a_i=0$ for almost all i. If $j\in I$, then

$$0=\left(\sum a_i f_i\right)(v_j)=\sum a_i f_i(v_j)=a_j.$$

Now assume that $\dim_K V=n$. Let $\{v_1,\dots,v_n\}$ be a basis of V over K. We claim that $\{f_1,\dots,f_n\}$ is a basis of $\mathrm{Hom}_K(V,F)$ over F. If $g\in\mathrm{Hom}_K(V,F)$, then $g=\sum g(v_i)f_i$. If $1\leq k\leq n$, then

$$\left(\sum g(v_i)f_i\right)(v_k)=\sum g(v_i)f_i(v_k)=g(v_k).$$

5.15. We need to find a bijective map

$$\mathrm{Hom}(E/K,C/K)\to\mathrm{Hom}(E/K,C_1/K).$$

If $\sigma \in \mathrm{Hom}(E/K, C/K)$, then $\theta^{-1}\sigma \in \mathrm{Hom}(E/K, C_1/K)$. If $\varphi \in \mathrm{Hom}(E/K, C_1/K)$, then $\theta\varphi \in \mathrm{Hom}(E/K, C/K)$. The maps $\sigma \mapsto \theta^{-1}\sigma$ and $\varphi \mapsto \theta\varphi$ are inverse to each other.

8.4. We first prove that for every order reversing function φ and every element s,t in its domain,

$$\varphi(s \vee t) \leq \varphi(s) \wedge \varphi(t) \text{ and } \varphi(s) \vee \varphi(t) \leq \varphi(s \wedge t).$$

Since that $s \leq s \vee t$ and $t \leq s \vee t$ and φ is order reversing, we have that $\varphi(s \vee t) \leq \varphi(s)$ and $\varphi(s \vee t) \leq \varphi(t)$. Hence $\varphi(s \vee t) \leq \varphi(s) \wedge \varphi(t)$. Moreover $s \wedge t \leq s$ and $s \wedge t \leq t$. So $\varphi(s) \leq \varphi(s \wedge t)$ and $\varphi(t) \leq \varphi(s \wedge t)$. Hence $\varphi(s) \vee \varphi(t) \leq \varphi(s \wedge t)$.

We can now apply this result for $\varphi = f$, $s = x$ and $t = y$ obtaining that

$$f(x \vee y) \leq f(x) \wedge f(y), \quad f(x) \vee f(y) \leq f(x \wedge y).$$

On the other hand, for $\varphi = f^{-1}$, $s = f(x)$ and $t = f(y)$, we obtain that

$$f^{-1}(f(x) \vee f(y)) \leq f^{-1}(f(x)) \wedge f^{-1}(f(y)) = x \wedge y,$$

and

$$x \vee y = f^{-1}(f(x)) \vee f^{-1}(f(y)) \leq f^{-1}\big(f(x) \wedge f(y)\big).$$

Thus, applying f, which is order reversing, it implies that

$$f(x \wedge y) \leq f\big(f^{-1}(f(x) \vee f(y))\big) = f(x) \vee f(y)$$

and

$$f(x) \wedge f(y) = f\big(f^{-1}\big(f(x) \wedge f(y)\big)\big) \leq f(x \vee y).$$

8.6. Since E/K is a Galois extension, the order of $\mathrm{Gal}(E/K)$ is precisely $[E:K] = n$. So, by Cauchy's Theorem, there exists a subgroup S of $\mathrm{Gal}(E/K)$ of order p. Then, by Galois' Theorem, the subextension ${}^{S}E/K$ has degree equal to the index of S, which is n/p.

8.7. Since E/K is a Galois extension, $|\mathrm{Gal}(E/K)| = [E:K] = p^{\alpha}m$. So, by Sylow's Theorem, there exists a subgroup P of $\mathrm{Gal}(E/K)$ of order p^{α}. Then, by Galois' Theorem, the subextension ${}^{P}E/K$ has degree $(\mathrm{Gal}(E/K) : P) = m$.

8.19. Write $f = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$. Then

$$f' = nX^{n-1} + (n-1)a_{n-1}X^{n-2} + \cdots + 2a_2X + a_1.$$

Since f is not separable, $f' = 0$. Thus $n = ka_k = 0$ in K for all $k \in \{0, \dots, n-1\}$. This implies that p divides k whenever $a_k \neq 0$. This means that the only terms in f occur in degree that are multiples of p. In particular, $n = pm$ for some m. Hence

$$f = X^{pm} + a_{p(n-1)}X^{p(m-1)} + \cdots + a_pX^p + a_0 = g(X^p)$$

for some $g \in K[X]$.

10.24. If G is solvable, then $[G,G]$ is a proper normal subgroup of G. Since G is simple, $[G,G] = \{1\}$ and G is abelian. Thus G is cyclic of prime order.

11.1. Assume that G is simple. Let $A = G \times \{1\}$, $B = \{1\} \times G$ and $D = \{(x,x) : x \in G\}$ the diagonal subgroup of $G \times G$. Since

$$(g,h) = (g,1)(1,h) = (gh^{-1},1)(h,h)$$

it follows that $G = AB = AD$. Let M be a subgroup of $G \times G$ that contains D. Note that

$$M = M \cap (G \times G) = M \cap AD = (M \cap A)D.$$

Similarly, $M = (M \cap B)D$. Since A is normal in $G \times G$, $M \cap A$ is normal in $G \times G$ and $(M \cap A)B$ is normal in $MB = G \times G$. Using the second isomorphism theorem, we see that

$$M \cap A \simeq \frac{(M \cap A)B}{B}$$

is a normal subgroup of $(G \times G)/B \simeq A$. Since $A \simeq G$ is simple, either $M \cap A = \{1\}$ or $M \cap A = A$. Thus either $M = D$ or $BD = G \times G$. Therefore D is maximal.

References

[1] R. D. Carmichael. *Introduction to the theory of groups of finite order*. Dover Publications, Inc., New York, 1956.
[2] W. Feit and J. G. Thompson. Solvability of groups of odd order. *Pacific J. Math.*, 13:775–1029, 1963.
[3] J. Rotman. *Galois theory*. Universitext. Springer-Verlag, New York, second edition, 1998.
[4] I. Stewart. *Galois theory*. CRC Press, Boca Raton, FL, fourth edition, 2015.

Index