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Galois theory

Notes

Monday 13th March, 2023

Preface

The notes correspond to the bachelor course *Galois theory* of the Vrije Universiteit Brussel, Faculty of Sciences, Department of Mathematics and Data Sciences. The course is divided into thirteen two-hour lectures.

The material is somewhat standard. Basic texts on fields and Galois theory are for example [2] and [3].

As usual, we also mention a set of great expository papers by Keith Conrad, the notes are extremely well-written and useful at every stage of a mathematical career.

Several chapters contain optional paragraphs which give examples of how to apply OSCAR Computer Algebra System to concrete problems in Galois theory.

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Lecture 1

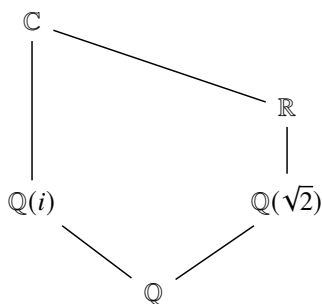
§1. Fields

Recall that a **field** is a commutative ring such that $1 \neq 0$ and that every non-zero element is invertible. Examples of (infinite) fields are \mathbb{Q} , \mathbb{R} and \mathbb{C} . If p is a prime number, then \mathbb{Z}/p is a field.

Example 1.1. The abelian group $\mathbb{Z}/2 \times \mathbb{Z}/2$ is a field with multiplication

$$(a, b)(c, d) = (ac + bd, ad + bc + bd).$$

Example 1.2. $\mathbb{Q}(i) = \{a + bi : a, b \in \mathbb{Q}\}$ and $\mathbb{Q}(\sqrt{2})$ are fields.



Exercise 1.3. Prove that $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{2})$ are not isomorphic as fields.

If R is a ring, there exists a unique ring homomorphism $\mathbb{Z} \rightarrow R$, $m \mapsto m1$. The image $\{m1 : m \in \mathbb{Z}\}$ of this homomorphism is a subring of R and it is known as the **ring of integers** of R . The kernel is a subgroup of \mathbb{Z} and is generated by some $t \geq 0$. The integer t is the **characteristic** of the ring R .

Exercise 1.4. The characteristic of a field is either zero or a prime number.

Recall that a commutative ring R is an **integral domain** if $xy = 0 \implies x = 0$ or $y = 0$. Fields are integral domains.

Exercise 1.5. Let K be a field. Prove that the following statements are equivalent:

- 1) K is of characteristic zero.
- 2) The additive order of 1 is infinite.
- 3) The additive order of each $x \neq 0$ is infinite.
- 4) The ring of integers of K is isomorphic to \mathbb{Z} .

Exercise 1.6. Let K be a field. Prove that the following statements are equivalent:

- 1) K is of characteristic p .
- 2) The additive order of 1 is p .
- 3) The additive order of each $x \neq 0$ is p .
- 4) The ring of integers of K is isomorphic to \mathbb{Z}/p .

Definition 1.7. A **subfield** of a ring R is a subring of R that is also a field.

Note that if K is a subfield of E , then the characteristic of K coincides with the characteristic of E . Moreover, if $K \rightarrow L$ is a field homomorphism, then K and L have the same characteristic.

Exercise 1.8. Let K be a field of characteristic p . Prove that $K \rightarrow K, x \mapsto x^{p^n}$, is a field homomorphism for all $n \in \mathbb{Z}_{\geq 0}$.

Note that finite fields are of characteristic p .

Let K be a subfield of a field E . Then E is a K -vector space with the usual scalar multiplication $K \times E \rightarrow E, (\lambda, x) \mapsto \lambda x$.

Definition 1.9. A field K is **prime** if there are no proper subfields of K .

Examples of prime fields are \mathbb{Q} and \mathbb{Z}/p for a prime number p .

Proposition 1.10. Let K be a field. The following statements hold:

- 1) K contains a unique prime field, it is known as the **prime subfield** of K .
- 2) The prime subfield of K is either isomorphic to \mathbb{Q} if the characteristic of K is zero, or it is isomorphic to \mathbb{Z}/p for some prime number p if the characteristic of K is p .

Proof. To prove the first claim let L be the intersection of all the subfields of K . Then L is a subfield of K . If F is a subfield of L , then F is a subfield of K . Thus $L \subseteq F$ and hence $F = L$, which proves that L is prime. If L_1 is a subfield of K and L_1 is prime, then $L \subseteq L_1$ and hence $L = L_1$.

Let K_0 be the prime field of K . Suppose that K is of characteristic $p > 0$. Then the ring $K_{\mathbb{Z}}$ of integers of K is a field isomorphic to \mathbb{Z}/p and hence $K_0 \simeq K_{\mathbb{Z}}$. Suppose now that the characteristic of K is zero. Let $E = \{m/1 : m \in \mathbb{Z}, 1 \neq 0\}$. We claim that $K_0 = E$. Since $K_{\mathbb{Z}} \subseteq K_0$, it follows that $E \subseteq K_0$. Hence $E = K_0$, as E is a subfield of K . \square

Definition 1.11. Let E be a field and K be a subfield of E . Then E is a **field extension** of K . We will use the notation E/K .

If E is an extension of K , then E is a K -vector space.

Definition 1.12. The degree of an extension E of K is the integer $\dim_K E$. It will be denoted by $[E : K]$.

We say that E is a finite extension of K if $[E : K]$ is finite.

Example 1.13. Let K be a field. Then $[K : K] = 1$. Conversely, if E is an extension of K and $[E : K] = 1$, then $K = E$. If not, let $x \in E \setminus K$. We claim that $\{1, x\}$ is linearly independent over K . Indeed, if $a + bx = 0$ for some $a, b \in K$, then $bx = -a$. If $b \neq 0$, then $x = -a/b \in K$, a contradiction. If $b = 0$, then $a = 0$.

We know that $[\mathbb{C} : \mathbb{R}] = 2$.

Example 1.14. A basis of $\mathbb{Q}(\sqrt{2})$ over \mathbb{Q} is given by $\{1, \sqrt{2}\}$. Then $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$. The calculations can be easily done by computer:

```
julia> E, a = quadratic_field(2)
(Real quadratic field defined by x^2 - 2, sqrt(2))
julia> characteristic(E)
0
julia> K = prime_field(E)
Rational Field
julia> degree(E)
2
julia> basis(E)
2-element Vector{nf_elem}:
 1
sqrt(2)
julia> one(K) == one(E)
true
julia> zero(K) == zero(E)
true
```

Example 1.15. Since \mathbb{Q} is numerable and \mathbb{R} is not, $[\mathbb{R} : \mathbb{Q}] > \aleph_0$. If $\{x_i : i \in \mathbb{Z}_{>0}\}$ is a numerable basis of \mathbb{R} over \mathbb{Q} , for each n consider the \mathbb{Q} -vector space V_n generated by $\{x_1, \dots, x_n\}$. Then

$$\mathbb{R} = \bigcup_{n \geq 1} V_n,$$

is numerable, as each V_n is numerable, a contradiction.

If E is an extension of K and E is finite, then $[E : K]$ is finite.

Proposition 1.16. Let K be a finite field. Then $|K| = p^m$ for some prime number p and some $m \geq 1$.

Proof. We know the prime subfield K_0 of K is isomorphic to \mathbb{Z}/p . In particular, $|K_0| = p$. Since K is finite, $[K : K_0] = m$ for some m . If $\{x_1, \dots, x_m\}$ is a basis of K over K_0 , then each element of K can be written uniquely as $\sum_{i=1}^m a_i x_i$ for some $a_1, \dots, a_m \in K_0$. Then $K \simeq K_0^m$ and hence $|K| = |K_0^m| = p^m$. \square

We now perform some basic calculations with a finite field of eight elements:

```
julia> E, x = FiniteField(2, 3, "x")
(Finite field of degree 3 over F_2, x)
julia> characteristic(E)
2
julia> prime_field(E)
Galois field with characteristic 2
julia> degree(E)
3
julia> size(E)
8
julia> [z for z in E]
8-element Vector{fq_nmod}:
 0
 1
 x
 x + 1
 x^2
 x^2 + 1
 x^2 + x
 x^2 + x + 1
```

Definition 1.17. Let E be an extension of K . A **subextension** F/K of E/K is a subfield F of E that contains K , that is $K \subseteq F \subseteq E$.

Definition 1.18. Let E and E_1 be extensions over K . An extension **homomorphism** $E \rightarrow E_1$ is a field homomorphism $\sigma: E \rightarrow E_1$ such that $\sigma(x) = x$ for all $x \in K$.

To describe the homomorphism $\sigma: E \rightarrow E_1$ of the extensions over K one typically writes the commutative diagram

$$\begin{array}{ccc} K & \xlongequal{\quad} & K \\ \downarrow & & \downarrow \\ E & \xrightarrow{\sigma} & E_1 \end{array}$$

We write $\text{Hom}(E/K, E_1/K)$ to denote the set of homomorphism $E \rightarrow E_1$ of extensions of K . Note that if $\sigma \in \text{Hom}(E/K, E_1/K)$, then σ is a K -linear map, as

$$\sigma(\lambda x) = \sigma(\lambda)\sigma(x) = \lambda\sigma(x)$$

for all $\lambda \in K$ and $x \in E$.

Example 1.19. The conjugation map $\mathbb{C} \rightarrow \mathbb{C}, z \mapsto \bar{z}$, is an endomorphism of \mathbb{C} as an extension over \mathbb{R} . Let $\varphi \in \text{Hom}(\mathbb{C}/\mathbb{R}, \mathbb{C}/\mathbb{R})$. Then

$$\varphi(x+iy) = \varphi(x) + \varphi(i)\varphi(y) = x + \varphi(i)y$$

for all $x, y \in \mathbb{R}$. Since $\varphi(i)^2 = \varphi(i^2) = \varphi(-1) = -1$, it follows that $\varphi(i) \in \{-i, i\}$. Thus either $\varphi(x+iy) = x+iy$ or $\varphi(x+iy) = x-iy$.

Exercise 1.20. Prove that if K is a field and $\sigma: K \rightarrow K$ is a field homomorphism, then $\sigma \in \text{Hom}(K/K_0, K/K_0)$.

If E/K is an extension, then

$$\text{Aut}(E/K) = \{\sigma : \sigma: E \rightarrow E \text{ is a bijective extension homomorphism}\}$$

is a group with composition.

Definition 1.21. Let E/K be an extension. The **Galois group** of E/K is the group $\text{Aut}(E/K)$ and it will be denoted by $\text{Gal}(E/K)$.

A typical example: $\text{Gal}(\mathbb{C}/\mathbb{R}) \simeq \mathbb{Z}/2$.

As an example, we show with the computer that $\text{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q}) \simeq \mathbb{Z}/2$:

```
julia> E, x = quadratic_field(2)
(Real quadratic field defined by x^2 - 2, sqrt(2))
julia> characteristic(E)
0
julia> G, C = galois_group(E);
julia> describe(G)
"C2"
julia> order(G)
2
```

Example 1.22. Let $\theta = \sqrt[3]{2}$ and let $E = \{a + b\theta + c\theta^2 : a, b, c \in \mathbb{Q}\}$. Note that

$$a + b\theta + c\theta^2 = 0 \iff a = b = c = 0.$$

Then E is an extension of \mathbb{Q} such that $[E : \mathbb{Q}] = 3$. We claim that $\text{Gal}(E/\mathbb{Q})$ is trivial. If $\sigma \in \text{Gal}(E/\mathbb{Q})$ and $z = a + b\theta + c\theta^2$, then $\sigma(z) = a + b\sigma(\theta) + c\sigma^2(\theta)$. Since $\sigma(\theta)^3 = \sigma(\theta^3) = \sigma(2) = 2$, it follows that $\sigma(\theta) = \theta$ and therefore $\sigma = \text{id}$.

Exercise 1.23. Prove that the polynomial $X^3 - 2$ is irreducible in $\mathbb{Q}[X]$.

The previous exercise can easily be solved using computers:

```
julia> R, x = PolynomialRing(QQ, "x");
julia> is_irreducible(x^3-2)
true
```

The following exercise is known as the *Eisenstein's irreducibility criterion*:

Exercise 1.24. Let A be a unique factorization domain and K be its fraction field. Let $f = \sum_{i=0}^n a_i X^i \in K[X]$ be a polynomial of degree $n > 0$. Assume that there exists a prime element $p \in A$ such that $p \mid a_i$ for all $i \in \{0, 1, \dots, n-1\}$, $p \nmid a_n$ and $p^2 \nmid a_0$. Then f is irreducible in $K[X]$.

Exercise 1.25. Prove that the polynomials $f = X^{10} + 60X^7 + 82X^6 - 36X^3 + 2$ and $g = 3X^{10} + 15X^2 - 45$ are irreducible in $\mathbb{Z}[X]$.

Exercise 1.26. Is the polynomial $f = 3(X^{10} + 5X^2 - 15)$ irreducible in $\mathbb{Z}[X]$?

If E/K is an extension and S is a subset of E , then there exists a unique smallest subextension F/K of E/K such that $S \subseteq F$. In fact,

$$F = \bigcap \{T : T \text{ is a subfield of } E \text{ that contains } K \cup S\}$$

If L/K is a subextension of E/K such that $S \subseteq L$, then $F \subseteq L$ by definition. The extension F is known as the **subextension generated by S** and it will be denoted by $K(S)$. If $S = \{x_1, \dots, x_n\}$ is finite, then $K(S) = K(x_1, \dots, x_n)$ is said to be of **finite type**.

Example 1.27. If $\{e_1, \dots, e_n\}$ is a basis of E over K , then $E = K(e_1, \dots, e_n)$.

Example 1.28. The field $\mathbb{Q}(\sqrt{2})$ is precisely the extension of \mathbb{R}/\mathbb{Q} generated by $\sqrt{2}$.

Let E/K be an extension and S and T be subsets of E . Then

$$K(S \cup T) = K(S)(T) = K(T)(S).$$

If, moreover, $S \subseteq T$, then $K(S) \subseteq K(T)$.

§2. Algebraic extensions

Definition 2.1. Let E/K be an extension. An element $x \in E$ is **algebraic** over K if there exists a non-zero polynomial $f(X) \in K[X]$ such that $f(x) = 0$. If x is not algebraic over K , then it is called **transcendental** over K .

If E/K is an extension, let

$$\overline{K}_E = \{x \in E : x \text{ is algebraic over } K\}.$$

Definition 2.2. An extension E/K is **algebraic** if every $x \in E$ is algebraic over K .

If K is a field, every $x \in K$ is algebraic over K , as x is a root of $X - x \in K[X]$. In particular, K/K is an algebraic extension.

§2 Algebraic extensions

Example 2.3. \mathbb{C}/\mathbb{R} is an algebraic extension. If $z \in \mathbb{C} \setminus \mathbb{R}$, then z is a root of the polynomial $X^2 - (z + \bar{z})X + |z|^2 \in \mathbb{R}[X]$.

If F/K is an algebraic extension $x \in E$ is algebraic over K for some field $E \supseteq F$, then x is algebraic over F .

Example 2.4. $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is algebraic, as the number $a + b\sqrt{2}$ is a root of the polynomial $X^2 - 2aX + (a^2 - 2b^2) \in \mathbb{Q}[X]$.

The extension \mathbb{C}/\mathbb{Q} is not algebraic. For example, Hermite proved that e is transcendental over \mathbb{Q} ; see [3, Theorem 24.4]. Lindemann's theorem states that π is not algebraic over \mathbb{Q} ; see [3, Theorem 24.5].

Example 2.5. Let $a = \sqrt{2}$ and $b = \sqrt[3]{3}$. Both a and b are algebraic numbers over \mathbb{Q} . Let us show that $a + b$ is also algebraic. Let $f(X) = X^3 - 3 \in \mathbb{Q}[X]$. Then $f(b) = 0$. Note that the polynomial

$$g(X) = f(X - a) = X^3 - 3aX^2 + 3aX - a^3 - 3 \in \mathbb{Q}(a)[X]$$

is such that $g(a + b) = 0$. How can we find a polynomial with coefficients in \mathbb{Q} that vanishes on $a + b$? We do the "conjugation" trick:

$$h(X) = f(X - a)f(X + a) = X^6 - 6X^4 - 6X^3 + 12X^2 - 36X + 1 \in \mathbb{Q}[X].$$

Note that $h(a + b) = 0$. How can you prove that ab is also algebraic over \mathbb{Q} ?

Lecture 2

If E/K is an extension and $x \in E$ is algebraic over K , then the evaluation homomorphism $K[X] \rightarrow E, p \mapsto p(x)$, is not injective. In particular, its kernel is a non-zero ideal. Hence it is generated by a monic polynomial f .

Definition 2.6. Let E/K be an extension and $x \in E$ be an algebraic element. The monic polynomial that generates the kernel of $K[X] \rightarrow E, f \mapsto f(x)$, is known as the **minimal polynomial** of x over K and it will be denoted by $f(x, K)$. The **degree** of x over K is then $\deg f(x, K)$.

Some basic properties of the minimal polynomial of an algebraic element:

Proposition 2.7. Let E/K be an extension and $x \in E$.

- 1) If $g \in K[X] \setminus \{0\}$ is such that $g(x) = 0$, then $f(x, K)$ divides g . In particular, $\deg f(x, K) \leq \deg g$.
- 2) $f(x, K)$ is irreducible in $K[X]$.
- 3) If F/K is a subextension of E/K , then $f(x, F)$ divides $f(x, K)$.

Proof. Write $f = f(x, K)$ to denote the minimal polynomial of x . To prove 1) note that $g(x) = 0$ implies that g belongs to the kernel of the evaluation map, so g is a multiple of f . To prove 2) note that if $f = pq$ for some $p, q \in K[X]$ such that $0 < \deg p, \deg q < \deg f$, then $f(x) = 0$ implies that either $p(x) = 0$ or $q(x) = 0$, a contradiction. Finally, we prove 3). Since $f \in K[X] \subseteq F[X]$ and $f(x) = 0$, it follows from 1) that $f(x, F)$ divides f . \square

Some easy examples: $f(i, \mathbb{R}) = X^2 + 1$, $f(i, \mathbb{C}) = X - i$ and $f(\sqrt[3]{2}, \mathbb{Q}) = X^3 - 2$:

```
julia> E, x = radical_extension(3, QQ(2), "x");
julia> minpoly(x)
x^3 - 2
julia> F, y = quadratic_field(-1);
julia> minpoly(y)
x^2 + 1
```

Example 2.8. Let us compute $f(\sqrt{2} + \sqrt{3}, \mathbb{Q})$. Let $\alpha = \sqrt{2} + \sqrt{3}$. Then

$$\begin{aligned} \alpha - \sqrt{2} = \sqrt{3} &\implies (\alpha - \sqrt{2})^2 = 3 \implies \alpha^2 - 2\sqrt{2}\alpha + 2 = 3 \\ &\implies \alpha^2 - 1 = 2\sqrt{2}\alpha \implies (\alpha^2 - 1)^2 = 8\alpha^2 \implies \alpha^4 - 10\alpha^2 + 1 = 0. \end{aligned}$$

Thus α is a root of $g = X^4 - 10X^2 + 1$. To prove that $g = f(\alpha, \mathbb{Q})$ it is enough to prove that g is irreducible in $\mathbb{Q}[X]$. First note that the roots of g are $\sqrt{2} + \sqrt{3}$, $\sqrt{2} - \sqrt{3}$, $-\sqrt{2} + \sqrt{3}$ and $-\sqrt{2} - \sqrt{3}$. This means that if g is not irreducible, then $g = hh_1$ for some polynomials $h, h_1 \in \mathbb{Q}[X]$ such that $\deg h = \deg h_1 = 2$. This is not possible, as $(\sqrt{2} + \sqrt{3}) + (\sqrt{2} - \sqrt{3}) = 2\sqrt{2} \notin \mathbb{Q}$, $(\sqrt{2} + \sqrt{3}) + (-\sqrt{2} + \sqrt{3}) = 2\sqrt{3} \notin \mathbb{Q}$ and $(\sqrt{2} + \sqrt{3})(-\sqrt{2} - \sqrt{3}) = -5 - 2\sqrt{6} \notin \mathbb{Q}$.

Proposition 2.9. Let F/K be a subextension and E/K . Then

$$[E : K] = [E : F][F : K].$$

Proof. Let $\{e_i : i \in I\}$ be a basis of E over F and $\{f_j : j \in J\}$ be a basis of F over K . If $x \in E$, then $x = \sum_i \lambda_i e_i$ (finite sum) for some $\lambda_i \in F$. For each i , $\lambda_i = \sum_j a_{ij} f_j$ (finite sum) for some $a_{ij} \in K$. Then $x = \sum_i \sum_j a_{ij} (f_j e_i)$. This means that $\{f_j e_i : i \in I, j \in J\}$ generates E as a K -vector space. Let us prove that $\{f_j e_i : i \in I, j \in J\}$ is linearly independent. If $\sum_i \sum_j a_{ij} (f_j e_i) = 0$ (finite sum) for some $a_{ij} \in K$, then

$$\begin{aligned} 0 = \sum_i \left(\sum_j a_{ij} f_j \right) e_i &\implies \sum_j a_{ij} f_j = 0 \text{ for all } i \in I \\ &\implies a_{ij} = 0 \text{ for all } i \in I \text{ and } j \in J. \quad \square \end{aligned}$$

We state a lemma:

Lemma 2.10. If A is a finite-dimensional commutative algebra over K and A is an integral domain, then A is a field.

Proof. Let $a \in A \setminus \{0\}$. We need to prove that there exists $b \in A$ such that $ab = 1$. Let $\theta : A \rightarrow A$, $x \mapsto ax$. Note that θ is K -linear transformation, as

$$\theta(x + y) = a(x + y) = ax + ay = \theta(x) + \theta(y), \quad \theta(\lambda x) = a(\lambda x) = \lambda(ax) = \lambda\theta(x),$$

for all $x, y \in A$ and $\lambda \in K$. It is injective, since A is an integral domain. Since $\dim_K A < \infty$, it follows that θ is an isomorphism. In particular, $\theta(A) = A$, which implies that there exists $b \in A$ such that $1 = ab$. \square

Let E/K be an extension and $x \in E$. Then

$$K[x] = \{y = f(x) : \text{for some } f \in K[X]\}$$

is a subring of E that contains K . Note that $K[x]$ is a K -vector space.

More generally, if $x_1, \dots, x_n \in E$, then

$$K[x_1, \dots, x_n] = \{f(x_1, \dots, x_n) : f \in K[X_1, \dots, X_n]\}$$

is a subring of E . Note that $K[x_1, \dots, x_n]$ is a K -vector space. Clearly, $K[x_1, \dots, x_n]$ is a domain and

$$K(x_1, \dots, x_n) = \left\{ \frac{f(x_1, \dots, x_n)}{g(x_1, \dots, x_n)} : f, g \in K[X_1, \dots, X_n] \text{ with } g(x_1, \dots, x_n) \neq 0 \right\}$$

is the extension of K generated by x_1, \dots, x_n . Note that

$$K(x_1, \dots, x_n) = (K(x_1, \dots, x_{n-1}))(x_n).$$

The previous construction can be generalized. Let I be a non-empty set. For each $i \in I$, let X_i be a variable. Consider the polynomial ring $K[\{X_i : i \in I\}]$ and let $S = \{x_i : i \in I\}$ be a subset of E . There exists a unique algebra homomorphism

$$K[\{X_i : i \in I\}] \rightarrow E$$

such that $X_i \mapsto x_i$ for all $i \in I$. The image is denoted by $K[S]$. In particular, an element $z \in K[S]$ is of the form

$$z = h(x_1, \dots, x_n)$$

for a polynomial $h \in K[X_1, \dots, X_n]$ in finitely many variables X_1, \dots, X_n and $x_1, \dots, x_n \in S$.

Exercise 2.11. Prove that $\mathbb{Q}[\sqrt{2}] = \mathbb{Q}(\sqrt{2})$.

The exercise is not an accident.

Theorem 2.12. Let E/K be an extension and $x \in E \setminus K$. The following statements are equivalent:

- 1) x is algebraic over K .
- 2) $\dim_K K[x] < \infty$.
- 3) $K[x]$ is a field.
- 4) $K[x] = K(x)$.

Proof. We first prove 1) \implies 2). Let $z \in K[x]$, say $z = h(x)$ for some $h \in K[X]$. There exists $g \in K[X]$ such that $g \neq 0$ and $g(x) = 0$. Divide h by g to obtain polynomials $q, r \in K[X]$ such that $h = gq + r$, where $r = 0$ or $\deg r < \deg g$. This implies that

$$z = h(x) = g(x)q(x) + r(x) = r(x).$$

If $\deg g = m$, then $r = \sum_{i=0}^{m-1} a_i X^i$ for some $a_0, \dots, a_{m-1} \in K$. Thus $z = \sum_{i=0}^{m-1} a_i x^i$, so $K[x] \subseteq \langle 1, x, \dots, x^{m-1} \rangle$.

The previous lemma proves that 2) \implies 3).

It is trivial that 3) \implies 4).

It remains to prove that 4) \implies 1). Since $x \neq 0$, $1/x \in K(x) = K[x]$. There exists $a_0, \dots, a_n \in K$ such that $1/x = a_0 + a_1x + \dots + a_nx^n$. Thus

$$a_nx^{n+1} + \dots + a_1x^2 + a_0x - 1 = 0,$$

and hence x is a root of $a_nX^{n+1} + \dots + a_0X - 1 \in K[X] \setminus \{0\}$. \square

Note that if x is algebraic over K , then $K[x] \simeq K[X]/(f(x, K))$.

Corollary 2.13. *If E/K is finite, then E/K is algebraic.*

Proof. Let $n = [E : K]$ and $x \in E \setminus K$. The set $\{1, x, \dots, x^n\}$ has $n+1$ elements, so it is linearly dependent. There exist $a_0, \dots, a_n \in K$, not all zero, such that

$$a_0 + a_1x + \dots + a_nx^n = 0.$$

Thus x is a root of the non-zero polynomial $a_0 + a_1X + \dots + a_nX^n \in K[X]$. \square

We note that the converse of the previous result does not hold.

Corollary 2.14. *If E/K is an extension and $x_1, \dots, x_n \in E$ are algebraic over K , then $K(x_1, \dots, x_n)/K$ is finite and $K(x_1, \dots, x_n) = K[x_1, \dots, x_n]$.*

Proof. We proceed by induction on n . The case $n = 1$ follows immediately from the theorem. So assume the result holds for some $n \geq 1$. Since the extensions $K(x_1, \dots, x_n)/K(x_1, \dots, x_{n-1})$ and $K(x_1, \dots, x_{n-1})/K$ are both finite, it follows that $K(x_1, \dots, x_n)/K$ is finite. Moreover,

$$\begin{aligned} K(x_1, \dots, x_n) &= K(x_1, \dots, x_{n-1})(x_n) \\ &= K(x_1, \dots, x_{n-1})[x_n] = K[x_1, \dots, x_{n-1}][x_n] = K[x_1, \dots, x_n]. \end{aligned} \quad \square$$

Corollary 2.15. *Let $E = K(S)$ for some set S . Then E/K is algebraic if and only if x is algebraic over K for all $x \in S$.*

Proof. Let us prove the non-trivial implication. Let $z \in K(S)$. In particular, there exists a finite subset $T \subseteq S$ such that $z \in K(T)$. The previous result implies that $K(T)/K$ is algebraic, and hence z is algebraic. \square

Corollary 2.16. *If E/K is an extension, then \overline{K}_E is a subfield of E that contains K . Moreover, $K(\overline{K}_E)/K$ is algebraic.*

Proof. By definition, $K(\overline{K}_E)/K$ is algebraic. Thus $K(\overline{K}_E) \subseteq \overline{K}_E$. From this, it follows that $K(\overline{K}_E) = \overline{K}_E$. \square

The following exercise is now almost trivial:

Exercise 2.17. Let E/K be an extension of finite type; this means that $E = K(S)$ for some finite set S . Prove that E/K is algebraic if and only if E/K is finite.

Let $\overline{\mathbb{Q}} = \{\alpha \in \mathbb{C} : \alpha \text{ is algebraic over } \mathbb{Q}\}$. Then $\overline{\mathbb{Q}}$ is the field of algebraic numbers. Can you compute $[\overline{\mathbb{Q}} : \mathbb{Q}]$?

Exercise 2.18. Prove that $[\mathbb{Q}[\sqrt[3]{2}] : \mathbb{Q}] = 3$.

For the previous exercise, you may use Eisenstein's criterion.

Exercise 2.19. Let $E = \mathbb{Q}[i, \sqrt{2}] = \mathbb{Q}[\sqrt{2}][i]$. Prove that $[E : \mathbb{Q}] = 4$.

Exercise 2.20. Let $E = \mathbb{Q}[\sqrt{2}, \sqrt[3]{5}]$.

- 1) Compute $[E : \mathbb{Q}]$.
- 2) Prove that $E = \mathbb{Q}[\sqrt{2} + \sqrt[3]{5}]$.
- 3) Find the minimal polynomial of $\sqrt{2} + \sqrt[3]{5}$ over \mathbb{Q} .

Exercise 2.21. Find the minimal polynomials of $\sqrt[4]{3}i$ over $\mathbb{Q}[i]$ and over $\mathbb{Q}[\sqrt{3}]$.

Exercise 2.22. Find the minimal polynomial of $\sqrt{2} + \sqrt[3]{5}i$ over $\mathbb{Q}[i]$.

Lecture 3

Algebraic field extensions form a nice class of extensions. The same happens with finite field extensions.

Proposition 2.23. *Let F/K be a subextension of E/K . Then E/K is algebraic if and only if E/F and F/K are algebraic.*

Proof. If E/K is algebraic, then E/F and F/K are both algebraic, as $K \subseteq F \subseteq E$. Let us assume that E/F and F/K are both algebraic. Let $x \in E$ and let L be the subextension over K generated by the coefficients of $f(x, F)$, the minimal polynomial of x over F . Then L/K is finite, since it is generated by finitely many algebraic elements. Moreover, x is algebraic over L . Since

$$[L(x) : K] = [L(x) : L][L : K] < \infty,$$

$L(x)/K$ is algebraic. In particular, x is algebraic over K . □

Exercise 2.24. Let F/K be a subextension of E/K . Prove that E/K is finite if and only if E/F and F/K are finite.

Let K be a field and $K \subseteq F \subseteq E$ and $K \subseteq L \subseteq E$ be fields. The composite of F and L is defined as

$$FL = K(F \cup L) = F(L) = L(F)$$

and it is equal to the smallest field that contains F and L .

Exercise 2.25. If $F = \mathbb{Q}(\sqrt{2})$ and $L = \mathbb{Q}(\sqrt{3})$, then $FL = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Compute $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}]$ and $\mathbb{Q}(\sqrt{2}) \cap \mathbb{Q}(\sqrt{3})$.

Exercise 2.26. Let $\xi \in \mathbb{C}$ be a primitive cubic root of one. If $F = \mathbb{Q}(\sqrt[3]{2})$ and $L = \mathbb{Q}(\xi)$, then $FL = \mathbb{Q}(\sqrt[3]{2}, \xi)$. Compute $[\mathbb{Q}(\sqrt[3]{2}, \xi) : \mathbb{Q}]$ and $\mathbb{Q}(\sqrt[3]{2}) \cap \mathbb{Q}(\xi)$.

Exercise 2.27. Let E/K and F/K be extensions, where both E and F are subfields of a field L . If F/K is algebraic, then EF/E is algebraic.

Exercise 2.28. Let E/K and F/K be extensions, where both E and F are subfields of a field L . If F/K is finite, then EF/E is finite.

The solution to the previous exercise shows, in particular, that $[EF : E] \leq [F : K]$.

Lemma 2.29. Let $\sigma : K \rightarrow L$ be a field homomorphism. Then there exists an extension E/K and a field isomorphism $\varphi : E \rightarrow L$ such that $\varphi|_K = \sigma$.

Proof. Note that σ is bijective. Let A be a set in bijection with $L \setminus \sigma(K)$ and disjoint with K . Let $E = K \cup A$. If $\theta : A \rightarrow L \setminus \sigma(K)$ is bijective, then let

$$\varphi : E \rightarrow L, \quad \varphi(x) = \begin{cases} \sigma(x) & \text{if } x \in K, \\ \theta(x) & \text{if } x \in A. \end{cases}$$

Then φ is a bijective map such that $\varphi|_K = \sigma$. Transport the operations of L onto E , that is to define binary operations on E as follows:

$$(x, y) \mapsto x \oplus y = \varphi^{-1}(\varphi(x) + \varphi(y)), \quad (x, y) \mapsto x \odot y = \varphi^{-1}(\varphi(x)\varphi(y)).$$

Then, for example,

$$x \oplus y = \varphi^{-1}(\varphi(x) + \varphi(y)) = \varphi^{-1}(\sigma(x) + \sigma(y)) = \varphi^{-1}(\sigma(x+y)) = \varphi^{-1}(\varphi(x+y)) = x+y$$

for all $x, y \in K$. \square

If $\sigma : A \rightarrow B$ is a ring homomorphism, then σ induces a ring homomorphism $\overline{\sigma} : A[X] \rightarrow B[X]$, $\sum_i a_i X^i \mapsto \sum_i \sigma(a_i) X^i$.

Theorem 2.30. Let K be a field and $f \in K[X]$ be such that $\deg f > 0$. Then there exists an extension E/K such that f admits a root in E .

Proof. We may assume that f is irreducible over K . Let $L = K[X]/(f)$ and $\pi : K[X] \rightarrow L$ be the canonical map. Then L is a field (the reader should explain why). Let $\sigma : K \rightarrow L$, $a \mapsto \pi(aX^0)$, and $g = \overline{\sigma}(f) \in L[X]$.

We claim that $\pi(X)$ is a root of g in L . Suppose that $f = \sum_i a_i X^i$. Then

$$\begin{aligned} g(\pi(X)) &= \overline{\sigma}(f)(\pi(X)) \\ &= \sum_i \sigma(a_i) \pi(X)^i = \sum_i \pi(a_i X^0) \pi(X^i) = \pi\left(\sum_i a_i X^i\right) = \pi(f) = 0. \end{aligned}$$

The previous lemma states that there exists an extension E/K and an isomorphism $\varphi : E \rightarrow L$ such that $\varphi|_K = \sigma$. Note that $\varphi(x) = 0$ if and only if $x = 0$. If $u = \pi(X)$, then $\varphi^{-1}(u)$ is a root of f in E , as

$$\begin{aligned} \varphi(f(\varphi^{-1}(u))) &= \varphi\left(\sum_i a_i \varphi^{-1}(u)^i\right) = \varphi\left(\sum_i a_i \varphi^{-1}(u^i)\right) \\ &= \sum_i \varphi(a_i) u^i = \sum_i \sigma(a_i) u^i = g(u) = 0. \end{aligned} \quad \square$$

§3 Artin's theorem

As a corollary, if K is a field and $f_1, \dots, f_n \in K[X]$ are polynomials of positive degree, then there exists an extension E/K such that each f_i admits a root in E . This is proved by induction on n .

Definition 2.31. A field K is **algebraically closed** if each $f \in K[X]$ of positive degree admits a root in K .

The *fundamental theorem of algebra* states that \mathbb{C} is algebraically closed. A typical proof uses complex analysis. Later we will give a proof of this result using Galois theory.

Proposition 2.32. *The following statements are equivalent:*

- 1) K is algebraically closed.
- 2) If $f \in K[X]$ is irreducible, then $\deg f = 1$.
- 3) If $f \in K[X]$ is non-zero, then f decomposes linearly in $K[X]$, that is

$$f = a \prod_{i=1}^n (X - \alpha_i)^{m_i}$$

for some $a \in K$ and $\alpha_1, \dots, \alpha_n \in K$.

- 4) If E/K is algebraic, then $E = K$.

Proof. 1) \implies 2 \implies 3) are exercises.

Let us prove that 3) \implies 4). Let $x \in E$. Decompose $f(x, K)$ linearly in $K[X]$ as $f(x, K) = a \prod_{i=1}^n (X - \alpha_i)^{m_i}$ and evaluate on x to obtain that $x = \alpha_j$ for some j .

To prove that 4) \implies 1) let $f \in K[X]$ be such that $\deg f > 0$. There exists an extension E/K such that f has a root x in E . The extension $K(x)/K$ is algebraic and hence $K(x) = K$, so $x \in K$. \square

§3. Artin's theorem

Definition 3.1. The **algebraic closure** of a field K is an algebraic extension C/K such that C is algebraically closed.

For example, \mathbb{C}/\mathbb{R} is an algebraic closure but \mathbb{C}/\mathbb{Q} is not.

Proposition 3.2. *Let C be algebraically closed and $\sigma: K \rightarrow C$ be a field homomorphism. If E/K is algebraic, then there exists a field homomorphism $\varphi: E \rightarrow C$ such that $\varphi|_K = \sigma$.*

Proof. Suppose first that $E = K(x)$ and let $f = f(x, K)$. Let $\overline{\sigma}(f) \in C[X]$ and let $y \in C$ be a root of $\overline{\sigma}(f)$. If $z \in E$, then $z = g(x)$ for some $g \in K[X]$. Let $\varphi: E \rightarrow C$, $z \mapsto \overline{\sigma}(g)(y)$.

The map φ is well-defined. If $z = h(x)$ for some $h \in K[X]$, then

$$0 = g(x) - h(x) = (g - h)(x)$$

and thus f divides $g - h$. In particular, $\overline{\sigma}(f)$ divides $\overline{\sigma}(g - h) = \overline{\sigma}(g) - \overline{\sigma}(h)$ and hence $(\overline{\sigma}(g) - \overline{\sigma}(h))(y) = 0$.

It is an exercise to show that the map φ is a ring homomorphism.

Let $a \in K$. It follows that $\varphi|_K = \sigma$, as

$$\varphi(a) = \overline{\sigma}(aX^0)(y) = \sigma(a)$$

Let us now prove the proposition in full generality. Let X be the set of pairs (F, τ) , where F is a subfield of E that contains K and $\tau: F \rightarrow C$ is a field homomorphism such that $\tau|_K = \sigma$. Note that $(K, \sigma) \in X$, so X is non-empty. Moreover, X is partially ordered by

$$(F, \tau) \leq (F_1, \tau_1) \iff F \subseteq F_1 \text{ and } \tau_1|_F = \tau.$$

If $\{(F_i, \tau_i) : i \in I\}$ is a chain in X , then $F = \cup_{i \in I} F_i$ is a subfield of E that contains K . Moreover, if $z \in F$, then $z \in F_i$ for some $i \in I$ and then one defines $\tau(z) = \tau_i(z)$. It is an exercise to prove that τ is well-defined. Since $(F, \tau) \in X$ is an upper bound, Zorn's lemma implies that there exists a maximal element $(E_1, \theta) \in X$. We claim that $E = E_1$. If not, let $z \in E \setminus E_1$. Since we know the proposition is true for the extension $E_1(z)/E_1$, let $\rho: E_1(z) \rightarrow C$ be a field homomorphism such that $\rho|_{E_1} = \theta$. Then, in particular, $\rho|_K = \sigma$. This implies that $(E_1(z), \rho) \in X$ and hence $(E_1, \theta) < (E_1(z), \rho)$, a contradiction to the maximality of (E_1, θ) . \square

Lecture 4

The previous proposition will be used to prove that the algebraic closure always exists.

Theorem 3.3 (Artin). *Let K be a field. Then K admits an algebraic closure C/K . If C_1/K is an algebraic closure, then the extensions C/K and C_1/K are isomorphic.*

Proof. Let us first prove the uniqueness. The previous proposition implies the existence of an extensions homomorphism $\varphi: C \rightarrow C_1$. Let $y \in C_1$ and $f = f(y, K)$ be the minimal polynomial of y in K . Since f admits a factorization

$$f = \lambda \prod (X - \alpha_i)^{m_i}$$

in $C[X]$, it follows that

$$f = \overline{\varphi}(f) = \varphi(\lambda) \prod (X - \varphi(\alpha_i))^{m_i}$$

Since $0 = f(y)$, we conclude that $y = \varphi(\alpha_j)$ for some j . In particular, φ is surjective and hence φ is bijective.

We now prove the existence. Let us assume that K admits an extension E/K with E algebraically closed. We will prove later that this extension indeed exists; at the moment, we only want to get an algebraic extension from this setting. Let

$$F = \{x \in E : x \text{ is algebraic over } K\}.$$

Then F/K is algebraic. Let $g \in F[X]$ be such that $\deg g > 0$. Since E is algebraically closed, g admits a root α in E . In particular, α is algebraic over F and hence α is algebraic over K . This implies that $\alpha \in F$, thus F is algebraically closed. This proves that F/K is an algebraic closure.

Let us prove that there exists an extension E_1/K such that every polynomial $f \in K[X]$ with $\deg f > 0$ has a root in E_1 . Let $\{f_i : i \in I\}$ be the family of monic irreducible polynomials with coefficients in K . We may think that $f_i = f_i(X_i)$. Let $R = K[\{X_i : i \in I\}]$ and let J be the ideal of R generated by the $f_i(X_i)$. We claim that $J \neq R$. If not, $1 \in J$, so

$$1 = \sum_{j=1}^m g_j f_{i_j}(X_{i_j})$$

for some $g_1, \dots, g_m \in R$. There exists an extension F/K such that f_{i_j} has a root α_j in F for all j . Let

$$\tau: R \rightarrow F, \quad \tau(X_k) = \begin{cases} \alpha_j & \text{if } k = i_j, \\ 0 & \text{if } k \notin \{i_1, \dots, i_m\}. \end{cases}$$

Then τ is a ring homomorphism and

$$1 = \tau(1) = \sum_{j=1}^m \tau(g_j) f_{i_j}(\alpha_j) = 0,$$

a contradiction.

Since J is a proper ideal, it is contained in a maximal ideal M . Let $L = R/M$ and let $\sigma: K \rightarrow L$ be the composition $K \hookrightarrow R \rightarrow R/M = L$, where $\pi: R \rightarrow R/M$ is the canonical map. As we did before, $\pi(X_i)$ is a root of $\overline{\sigma}(f_i)$ for all i . And there exists an extension E_1/K such that every f_i has a root in E_1 . Proceeding in this way, we construct a sequence

$$E_1 \subseteq E_2 \subseteq \dots$$

of fields such that every polynomial of positive degree and coefficients in E_k admits a root in E_{k+1} . Let $E = \cup E_k$. We claim that E is algebraically closed. In fact, let $g \in E[X]$ be such that $\deg g > 0$. Then, since $g \in E_r[X]$ for some r , it follows that g has a root in $E_{r+1} \subseteq E$. \square

§4. Decomposition fields

Definition 4.1. Let K be a field and $f \in K[X]$ be such that $\deg f > 0$. A **decomposition field** of f over K is field E that contains K and that satisfies the following properties:

- 1) f factorizes linearly in $E[X]$.
- 2) If F is a field such that $K \subseteq F \subseteq E$ and f factorizes linearly in $F[X]$, then $F = E$.

Easy examples:

Example 4.2. \mathbb{C} is a decomposition field of $X^2 + 1 \in \mathbb{R}[X]$.

Example 4.3. $\mathbb{Q}[\sqrt{2}]$ is a decomposition field of $X^2 - 2 \in \mathbb{Q}[X]$.

Example 4.4. The decomposition field of $f = X^2 - 2$ over $\mathbb{Z}/7$ is precisely $\mathbb{Z}/7$, as 3 and 4 are the roots of f in $\mathbb{Z}/7$.

Example 4.5. $\mathbb{Q}(\sqrt[3]{2})$ is not a decomposition field of $X^3 - 2 \in \mathbb{Q}[X]$. However, if ω is a primitive cubic root of one, then $\mathbb{Q}(\sqrt[3]{2}, \omega)$ is a decomposition field of the polynomial $X^3 - 2 \in \mathbb{Q}[X]$.

Proposition 4.6. E is a decomposition field of $f \in K[X]$ if and only if f factorizes linearly in $E[X]$ and $E = K(x_1, \dots, x_n)$, where x_1, \dots, x_n are the roots of f .

Proof. Let $f = a \prod_{i=1}^r (X - x_i)^{n_i}$ and $F = K(x_1, \dots, x_r)$ with $x_1, \dots, x_r \in E$. Since f factorizes linearly in $F[X]$, it follows that $F = E$. Conversely, let $E = K(x_1, \dots, x_r)$ and assume that f factorizes linearly in $F[X]$. Then, in particular, $x_1, \dots, x_r \in F$. Hence $E \subseteq F$ and $F = E$. \square

One immediately obtains the following consequence: If E is a decomposition field of $f \in K[X]$, then E/K is finite.

Theorem 4.7. Let $f \in K[X]$ be such that $\deg f > 0$. There exists a (unique up to extension isomorphism) decomposition field of f over K .

Proof. Let C/K be an algebraic closure. Write $f = a \prod_{i=1}^r (X - x_i)^{n_i}$ in $C[X]$. Then $E = K(x_1, \dots, x_r)$ is a decomposition field of f over K . Let us prove uniqueness: if E_1/K is a decomposition field of f over K , then E_1/K is algebraic and thus Proposition 3.2 implies that there exists $\varphi \in \text{Hom}(E_1/K, C/K)$, that is $\varphi: E_1 \rightarrow C$ is a field homomorphism such that $\varphi|_K$ is the identity. Factorize f linearly in $E_1[X]$ and apply $\bar{\varphi}$:

$$f = a \prod_{j=1}^s (X - y_j)^{m_j} \implies f = \bar{\varphi}(f) = \varphi(a) \prod_{j=1}^s (X - \varphi(y_j))^{m_j}$$

so f factorizes linearly in $\varphi(E_1)[X]$. Moreover, $E_1 = K(y_1, \dots, y_s)$ and it follows that $\varphi(E_1) = K(\varphi(y_1), \dots, \varphi(y_s))$. Thus $\varphi(E_1)$ is a decomposition field of f . Since $\varphi(E_1) \subseteq C$, it follows that $\varphi(E_1) = E$. \square

Exercise 4.8. If E/K is finite and $\varphi \in \text{Hom}(E/K, E/K)$, then φ is an isomorphism.

Let C be an algebraic closure of K and $G = \text{Gal}(C/K)$. The group G acts on C

$$\sigma \cdot x = \sigma(x), \quad \sigma \in G, x \in C.$$

The orbits are of the form

$$O_G(x) = \{\sigma(x) : \sigma \in G\} = \{y \in C : y = \sigma(x) \text{ for some } \sigma \in G\}$$

The elements $x, y \in C$ are **conjugate** if $y = \sigma(x)$ for some $\sigma \in G$.

Proposition 4.9. Let C be an algebraic closure of K and $x, y \in C$. Then x and y are conjugate if and only if $f(x, K) = f(y, K)$. In particular, $O_G(x)$ is finite.

Proof. Let $G = \text{Gal}(C/K)$. If x and y are conjugate, say $y = \sigma(x)$ for some $\sigma \in G$, let us write $g = f(x, K)$ as

$$g = X^n + \sum_{i=0}^{n-1} a_i X^i.$$

Then $0 = g(x) = x^n + \sum_{i=0}^{n-1} a_i x^i$ and hence y is a root of g , as

$$\begin{aligned} 0 &= \sigma \left(x^n + \sum_{i=0}^{n-1} a_i x^i \right) = \sigma(x)^n + \sum_{i=0}^{n-1} \sigma(a_i) \sigma(x)^i \\ &= \sigma(x)^n + \sum_{i=0}^{n-1} a_i \sigma(x)^i = y^n + \sum_{i=0}^{n-1} a_i y^i. \end{aligned}$$

Thus $f(y, K) = g$.

Conversely, assume that $f(x, K) = f(y, K)$. Let $g = f(x, K) = f(y, K)$ and let

$$\varphi: K[x] \rightarrow K[y], \quad h(x) \mapsto h(y).$$

Let us show that the map φ is well-defined: we need to show that if $h_1(x) = h_2(x)$, then $h_1(y) = \varphi(h_1(x)) = \varphi(h_2(x)) = h_2(y)$. If $h_1(x) = h_2(x)$, then

$$(h_1 - h_2)(x) = h_1(x) - h_2(x) = 0.$$

This implies that g divides $h_1 - h_2$. In particular, $h_1(y) = h_2(y)$.

A straightforward calculation shows that φ is a field homomorphism such that $\varphi|_K = \text{id}$, this means that φ is an extension homomorphism such that $\varphi(x) = y$. There exists $\sigma \in \text{Hom}(C/K, C/K)$ such that $\sigma|_{K[x]} = \varphi$. Since σ is bijective (this is left as an exercise, you did something similar before), $\sigma(x) = \varphi(x) = y$ and hence $O_G(x) = O_G(y)$. \square

Proposition 4.10. *Let C be an algebraic closure of K and $x \in C$. Then*

$$f(x, K) = \prod_{y \in O_G(x)} (X - y)^m$$

for some m .

Proof. For each $y \in O_G(x)$ let m_y be the multiplicity of y in $f(x, K)$. Then, for example, $f(x, K) = (X - x)^{m_x} g$ for some g . If $y \in O_G(x)$, then $y = \sigma(x)$ for some $\sigma \in \text{Gal}(C/K)$. Since

$$\overline{\sigma}(f(x, K)) = f(x, K) = (X - y)^{m_x} \overline{\sigma}(g),$$

it follows that $m_y \geq m_x$. By symmetry, we conclude that $m_x = m_y$. \square

The previous proposition shows, in particular, that all the roots of an irreducible polynomial $f \in K[X]$ in an algebraic closure C of K have the same multiplicity. This is not true if f is not irreducible. Find an example.

Definition 4.11. Let K be a field and $\{f_i : i \in I\}$ be a non-empty family of polynomials of positive degree with coefficients in K . A **decomposition field** of $\{f_i : i \in I\}$ is an extension E/K such that every f_i factorizes linearly in $E[X]$ and if F/K is a subextension of E/K such that every f_i factorizes linearly in $F[X]$, then $F = E$.

Exercise 4.12. Prove that E/K is a decomposition field of $\{f_i : i \in I\}$ if and only if every f_i factorizes linearly in $E[X]$ and $E = K(S)$ where $S = \{\text{roots of } f_i \text{ for all } i\}$.

Exercise 4.13. Prove that if E/K is a decomposition field of $\{f_i : i \in I\}$, then E/K is algebraic. If, moreover, I is finite, then E/K is a decomposition field of $\prod_{i \in I} f_i$.

Exercise 4.14. Prove that there exists a decomposition field of $\{f_i : i \in I\}$ and it is unique up to extension isomorphism.

Exercise 4.15. Let $f = X^3 - X - 1 \in (\mathbb{Z}/3)[X]$ and E be a decomposition field of f . Compute $[E : \mathbb{Z}/3]$.

What about the decomposition field of $f = X^3 - X - 1 \in \mathbb{Q}[X]$?

Exercise 4.16. Let $f = X^4 - 5x^2 + 5 \in \mathbb{Q}[X]$ and E be a decomposition field of f . Compute $[E : \mathbb{Q}]$ and $\text{Gal}(E/K)$.

Lecture 5

§5. Normal extensions

Proposition 5.1. *Let E/K be an algebraic extension and $\sigma \in \text{Hom}(E/K, E/K)$. Then σ is bijective.*

Proof. Let $x \in E$ and C be an algebraic closure of K that contains E . There exists $\varphi: C \rightarrow C$ such that $\varphi|_E = \sigma$. Thus $\varphi|_K = \sigma|_K = \text{id}_K$. Let $G = \text{Gal}(C/K)$. Then $\varphi \in G$. If $z \in O_G(x)$, then $z = \tau(x)$ for some $\tau \in G$ and hence

$$\varphi(z) = \varphi(\tau(x)) = (\varphi\tau)(x).$$

This implies that $\varphi(z) \in O_G(x)$ and $\varphi(O_G(x)) = O_G(x)$. The restriction $\sigma|_{E \cap O_G(x)}$ is injective. Then

$$\begin{aligned} \sigma(E \cap O_G(x)) &= \varphi(E \cap O_G(x)) \\ &= \varphi(E) \cap \varphi(O_G(x)) = \sigma(E) \cap O_G(x) \subseteq E \cap O_G(x). \end{aligned}$$

Since $|E \cap O_G(x)| < \infty$, it follows that $E \cap O_G(x) = \sigma(E \cap O_G(x))$ and hence x belongs to the image of σ . \square

Definition 5.2. Let E/K be an algebraic extension and C be an algebraic closure of K containing E . Then E/K is **normal** if $\sigma(E) \subseteq E$ for all $\sigma \in \text{Hom}(E/K, C/K)$.

Note that $\sigma(E) \subseteq E$ in the previous definition is equivalent to $\sigma(E) = E$.

Example 5.3. The extension $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is not normal. Why?

Some trivial examples of normal extensions: K/K is normal and if C is an algebraic closure of K , then C/K is normal.

Example 5.4. The extension $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is normal. Every extension generated by algebraic elements of degree two is normal.

Exercise 5.5. Let ξ be a primitive cubic root of one. Then $\mathbb{Q}(\sqrt[3]{2}, \xi)/\mathbb{Q}$ is normal.

The following result is practical but technical. That is why we leave the proof as an exercise.

Exercise 5.6. Prove that the previous definition depends only on E (and not on the algebraic closure C).

Some properties:

Proposition 5.7. *Let E/K be a normal extension and $f \in K[X]$ be an irreducible polynomial that admits a root x in E . Then f factorizes linearly in E .*

Proof. We may assume that f is monic. Let C/K be an algebraic closure of K containing E . Let y be a root of f in C . Since $f = f(x, K) = f(y, K)$, it follows that $y = \sigma(x)$ for some $\sigma \in \text{Gal}(C/K)$. Since E/K is normal, $\sigma|_E : E \rightarrow C$ is an automorphism of E/K , that is $\sigma(E) \subseteq E$. In particular, $y \in E$. \square

Let $K \subseteq F \subseteq E$ be a tower of fields. If E/K is normal, then E/F is normal. However, Note that E/K normal does not imply F/K normal, as this would imply that every extension is normal. Moreover, E/F normal and F/K normal do not imply E/K normal.

Example 5.8. The extensions $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ are both normal, but $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$ is not normal, as the roots of $X^4 - 2$ are $\sqrt{2}$, $-\sqrt{2}$, $\sqrt{2}i$ and $-\sqrt{2}i$.

Recall that if C is an algebraic closure of K and $x \in C$, then

$$f(x, K) = \prod (X - y)^m,$$

where the product is taken over all $y \in O_{\text{Gal}(C/K)}(x)$. If E/K is normal and $x \in E$, then there exists m such that

$$f(x, K) = \prod (X - y)^m,$$

where the product is taken over all $y \in O_{\text{Gal}(E/K)}(x)$.

Proposition 5.9. *Let E/K and F/K be extensions. If F/K is normal, then EF/E is normal.*

Proof. Let C be an algebraic closure of E containing EF . Let $\sigma \in \text{Hom}(EF/E, C/E)$. We claim that $\sigma(EF) = EF$. Let

$$\overline{K} = \{x \in C : x \text{ is algebraic over } K\}.$$

Then \overline{K} is an algebraic closure over K and $F \subseteq \overline{K}$. Since F/K is normal and $\sigma|_F \in \text{Hom}(F/K, \overline{K}/K)$, it follows that $\sigma(F) = F$. If $z \in EF$, then $z = \sum_{i=1}^m e_i f_i$ for some $e_1, \dots, e_m \in E$ and $f_1, \dots, f_m \in F$. Since $\sigma(e_i) = e_i$ for all i ,

$$\sigma(z) = \sum_{i=1}^m \sigma(e_i) \sigma(f_i) = \sum_{i=1}^m e_i \sigma(f_i) \in EF. \quad \square$$

What is the relation between normal extensions and decomposition fields? The notions look deeply related. The following explain serves as an explanation:

Proposition 5.10. *Let E/K be an algebraic extension. Then E/K is normal if and only if E/K is the decomposition field of a family of polynomials of $K[X]$ of positive degree.*

Proof. Assume first that E/K is a normal extension. Let $G = \text{Gal}(E/K)$. If $x \in E$ and $f(x, K) = \prod_{y \in O_G(x)} (X - y)^m$, then $f(x, K)$ factorizes linearly in $E[X]$. Thus E/K is a decomposition field of the family $\{f(x, K) : x \in E\}$.

Conversely, assume that E/K is a decomposition field of the family $\{f_i : i \in I\}$. Then $E = K(S)$ where S is the set of roots of the polynomials f_i . Let C/K be an algebraic closure of K that contains E and let $\sigma \in \text{Hom}(E/K, C/K)$. Let $x \in S$. Then x is a root of some $f_j = \sum a_k X^k$. Since $f_j(x) = 0$, it follows that $\sigma(x)$ is a root of f_j , as

$$f_j(\sigma(x)) = \sum a_k \sigma(x)^k = \sum \sigma(a_k) \sigma(x^k) = \sigma\left(\sum a_k x^k\right) = \sigma(0) = 0.$$

Hence $\sigma(E) \subseteq E$. □

Exercise 5.11. Let $E = \mathbb{Q}[\sqrt[4]{7} + \sqrt{2}]$.

- 1) Prove that E/\mathbb{Q} is not normal.
- 2) Compute $[E : \mathbb{Q}]$.
- 3) Compute $\text{Gal}(E/\mathbb{Q})$.

§6. Dedekind's theorem

Note that every extension homomorphism $E/K \rightarrow F/K$ is, in particular, a K -linear map $E \rightarrow F$, that is

$$\text{Hom}(E/K, F/K) \subseteq \text{Hom}_K(E, F).$$

If F/K is an extension and V is a K -vector space, the set $\text{Hom}_K(E, F)$ of K -linear maps is a vector space over F with $(a \cdot f)(v) = af(v)$ for $a \in F$, $f \in \text{Hom}_K(E, F)$ and $v \in V$.

Exercise 6.1. Prove that $\dim_F \text{Hom}_K(V, F) \geq \dim_K V$. Moreover, if $\dim_K V < \infty$, then $\dim_F \text{Hom}_K(V, F) = \dim_K V$.

If V is a vector space and S is a (possibly infinite) subset of V , then S is linearly independent if every finite subset of S is linearly independent.

Theorem 6.2 (Dedekind). *Let E/K and F/K be extensions and let $\{\varphi_i : i \in I\}$ be a subset of $\text{Hom}(E/K, F/K)$, i.e. a family of extension homomorphisms. Assume that $\varphi_i \neq \varphi_j$ if $i \neq j$. Then the subset $\{\varphi_i : i \in I\} \subseteq \text{Hom}_K(E, F)$ is linearly independent over F .*

Proof. Assume it is not. Let $\{\varphi_1, \dots, \varphi_n\}$ be linearly dependent over F with n minimal. Clearly, $n > 1$. We may assume that

$$\sum_{i=1}^n a_i \varphi_i = 0 \quad (5.1)$$

for some $a_1, \dots, a_n \in F$ all different from zero. Let $z \in E \setminus \{0\}$ be such that $\varphi_1(z) \neq \varphi_2(z)$. If $x \in E$, then

$$0 = \left(\sum_{i=1}^n a_i \varphi_i \right)(xz) = \sum_{i=1}^n a_i \varphi_i(xz) = \sum_{i=1}^n a_i \varphi_i(x) \varphi_i(z) = \left(\sum_{i=1}^n (a_i \varphi_i(z)) \varphi_i \right)(x).$$

Thus

$$\sum_{i=1}^n (a_i \varphi_i(z)) \varphi_i = 0. \quad (5.2)$$

Since $\sum_{i=1}^n a_i \varphi_i = 0$ and $\varphi_1(z) \neq 0$,

$$a_1 \varphi_1 + a_2 \frac{\varphi_2(z)}{\varphi_1(z)} \varphi_2 + \dots + a_n \frac{\varphi_n(z)}{\varphi_1(z)} \varphi_n = 0.$$

Thus, subtracting (5.1) and (5.2),

$$\left(a_2 - a_2 \frac{\varphi_2(z)}{\varphi_1(z)} \right) \varphi_2 + \dots + \left(a_n - a_n \frac{\varphi_n(z)}{\varphi_1(z)} \right) \varphi_n = 0.$$

Since $a_n \neq 0$ and $\varphi_2(z) \neq \varphi_1(z)$, the scalar $a_2 - a_2 \frac{\varphi_2(z)}{\varphi_1(z)} \neq 0$ and hence $\{\varphi_2, \dots, \varphi_n\}$ is linearly dependent, a contradiction. \square

If E/K and F/K are extensions, let $\gamma(E/K, F/K) = |\text{Hom}(E/K, F/K)|$.

Exercise 6.3. Prove the following statements:

- 1) $\gamma(E/K, F/K) \leq \dim_F \text{Hom}_K(E, F)$.
- 2) If $[E : K] < \infty$, then $\gamma(E/K, F/K) \leq [E : K]$.
- 3) If x is algebraic over K , then $\gamma(K(x)/K, F/K) \leq \deg(x, K)$.

If C is an algebraic closure of K , then we define $\gamma(E/K) = \gamma(E/K, C/K)$. This definition does not depend on the algebraic closure.

Exercise 6.4. If C and C_1 are algebraic closures of K , then

$$|\text{Hom}(E/K, C/K)| = |\text{Hom}(E/K, C_1/K)|.$$

Proposition 6.5. Let C be an algebraic closure of K and $G = \text{Gal}(C/K)$. If $x \in C$, then $\gamma(K(x)/K) = |O_G(x)|$.

Proof. If $\sigma \in \text{Hom}(K(x)/K, C/K)$, then there exists $\phi \in G$ such that $\phi|_{K(x)} = \sigma$. Thus $\sigma(x) = \phi(x) \in O_G(x)$. Conversely, if $y \in O_G(x)$, then there exists $\tau \in G$ such

that $y = \tau(x)$. Hence $\tau|_{K(x)} \in \text{Hom}(K(x)/K, C/K)$ and $\tau|_{K(x)}(x) = y$. In particular, $\gamma(K(x)/K)$ divides $\deg(x, K)$. \square

Exercise 6.6. If E/K is finite, then $|\text{Gal}(E/K)| \leq [E : K]$. Moreover, E/K is normal if and only if $|\text{Gal}(E/K)| = \gamma(E/K)$.

Lecture 6

If $t: A \rightarrow B$ is a surjective map, then $a \sim a_1 \iff t(a) = t(a_1)$ defines an equivalence relation on A . The set \bar{A} of equivalence classes is in bijective correspondence with B , $\bar{A} \rightarrow B, \bar{a} \mapsto t(a)$. Moreover, if $|t^{-1}(\{b\})| = m$ for all $b \in B$, then $|A| = m|\bar{A}| = m|B|$.

Proposition 6.7. *Let E/K be algebraic and F/K be a subextension such that E/F is finite. Then $\gamma(E/K) = \gamma(E/F)\gamma(F/K)$.*

Proof. Assume that $E = F(x)$. Let $f = f(x, F) = \sum b_i X^i$ and let $G = \text{Gal}(E/F)$. Let C be an algebraic closure of K containing E . The map

$$\lambda: \text{Hom}(E/K, C/K) \rightarrow \text{Hom}(F/K, C/K), \quad \sigma \mapsto \sigma|_F,$$

is well-defined. It is surjective: if $\varphi \in \text{Hom}(F/K, C/K)$, then $\varphi: F \rightarrow C$ is, in particular, a field homomorphism. Since E/F is algebraic, by Proposition 3.2 there exists a field homomorphism $\sigma: E \rightarrow C$ such that $\sigma|_F = \varphi$. Since $\sigma|_K = \varphi|_K = \text{id}$, in particular $\sigma \in \text{Hom}(E/K, C/K)$.

For $\varphi \in \text{Hom}(F/K, C/K)$,

$$\lambda^{-1}(\{\varphi\}) = \{\sigma \in \text{Hom}(E/K, C/K) : \sigma|_F = \varphi\}$$

and let R_φ be the set of roots (in C) of the polynomial $\bar{\varphi}(f) = \sum \varphi(b_i)X^i$.

Claim. The map $\alpha: \lambda^{-1}(\{\varphi\}) \rightarrow R_\varphi, \sigma \mapsto \sigma(x)$, is well-defined.

We need to show that $\sigma(x)$ is a root of $\bar{\varphi}(f)$:

$$\begin{aligned} \bar{\varphi}(f)(\sigma(x)) &= \sum \varphi(b_i)\sigma(x)^i = \sum \sigma(b_i)\sigma(x^i) \\ &= \sum \sigma(b_i x^i) = \sigma\left(\sum b_i x^i\right) = \sigma(f(x)) = \sigma(0) = 0. \end{aligned}$$

Claim. The map $\beta: R_\varphi \rightarrow \lambda^{-1}(\{\varphi\}), y \mapsto \sigma_y$, where $\sigma_y(z) = \bar{\varphi}(h)(y)$ if $z = h(x)$, is well-defined.

We need to show that if $z = h(x)$ and $z = h_1(x)$ for some $h, h_1 \in F[X]$, then $\bar{\varphi}(h)(y) = \bar{\varphi}(h_1)(y)$. The assumptions imply that $(h - h_1)(x) = 0$ and hence f divides $h - h_1$. Since $\bar{\varphi}$ is a ring homomorphism, $\bar{\varphi}(f)$ divides $\bar{\varphi}(h) - \bar{\varphi}(h_1)$. This implies $(\bar{\varphi}(h) - \bar{\varphi}(h_1))(y) = 0$. We also need to show that $\sigma_y|_F = \varphi$: if $a \in F$, then write $a = aX^0 \in F[X]$. Thus $\sigma_y(a) = \bar{\varphi}(aX^0)(y) = \varphi(a) \in C$. It is now an exercise to prove that $\sigma_y \in \text{Hom}(E/K, C/K)$.

Claim. $|\lambda^{-1}(\{\varphi\})| = |R_\varphi|$.

For this we need to show that β is the inverse of α , that is $\alpha \circ \beta = \text{id}$ and $\beta \circ \alpha = \text{id}$. To prove that $\beta \circ \alpha = \text{id}$ let σ be such that $\sigma|_F = \varphi$. Then $y = \sigma(x) \in R_\varphi$. Let $z = h(x) = \sum a_i x^i \in F[x] = E$. Then

$$\bar{\varphi}(h)(y) = \sum \varphi(a_i) y^i = \sum \sigma(a_i) y^i = \sigma\left(\sum a_i x^i\right) = \sigma(y).$$

Conversely, if $y \in R_\varphi$, then

$$\alpha(\sigma_y) = \sigma_y(x) = y,$$

as $\sigma_y(x) = \bar{\varphi}(X)(y) = y$.

Claim. If $\phi \in G$ is such that $\phi|_F = \varphi$, then $O_G(x) = \phi^{-1}(R_\varphi)$.

Let us first prove $O_G(x) \supseteq \phi^{-1}(R_\varphi)$. If $y \in R_\varphi$, then

$$\begin{aligned} f(\phi^{-1}(y)) &= \sum b_i \phi^{-1}(y^i) = \phi^{-1}\left(\sum \phi(b_i) y^i\right) \\ &= \phi^{-1}\left(\sum \varphi(b_i) y^i\right) = \phi^{-1} \bar{\varphi}(f)(y) = \phi^{-1}(0) = 0. \end{aligned}$$

Now we prove $O_G(x) \subseteq \phi^{-1}(R_\varphi)$. Let $z \in O_G(x)$ and $y \in C$ be such that $\phi^{-1}(y) = z$. Then $\bar{\varphi}(f)(y) = 0$, as

$$\begin{aligned} \bar{\varphi}(f)(y) &= \sum \varphi(b_i) y^i \\ &= \sum \varphi(b_i) \phi(z^i) = \sum \phi(b_i) \phi(z^i) = \phi\left(\sum b_i z^i\right) = \phi(f(z)) = \phi(0) = 0. \end{aligned}$$

It follows that $|\lambda^{-1}(\varphi)| = |O_G(x)|$ for all φ . By using the argument before the proposition,

$$\begin{aligned} \gamma(E/K) &= |\text{Hom}(E/K, C/K)| \\ &= |O_G(x)| |\text{Hom}(F/K, C/K)| \\ &= |O_G(x)| \gamma(F/K). \end{aligned}$$

Since $\gamma(K(x)/K) = |O_G(x)|$ by Proposition 6.5, the claim follows.

For the general case we assume that $E = F(x_1, \dots, x_n)$. We proceed by induction on n . If $n = 0$, then $E = F$ and the result is trivial. If $n > 0$, let $L = F[x_1, \dots, x_{n-1}]$

and $E = L(x_n)$. The case proved implies that $\gamma(E/F) = \gamma(E/L)\gamma(L/F)$. By the inductive hypothesis, $\gamma(L/K) = \gamma(L/F)\gamma(F/K)$. Thus

$$\gamma(E/F)\gamma(F/K) = \gamma(E/L)\gamma(L/F)\gamma(F/K) = \gamma(E/L)\gamma(L/K) = \gamma(E/K),$$

again using the previous case. \square

§7. Separable extensions

Definition 7.1. Let E/K be an algebraic extension and $x \in E$. Then x is **separable** over K if x is a simple root of $f(x, K)$.

An algebraic extension E/K is **separable** if every $x \in E$ is separable over K . Clearly, K/K is separable.

Exercise 7.2. Prove that an element x is separable over K if and only if x is a simple root of a polynomial with coefficients in K .

If F/K is a subextension of E/K and $x \in E$ is separable over K , then x is separable over F .

Exercise 7.3. If C is an algebraic closure of K , $x \in C$ and $G = \text{Gal}(C/K)$. Prove that the following statements are equivalent:

- 1) x is separable over K .
- 2) Every $y \in O_G(x)$ is separable over K .
- 3) $\gamma(K(x)/K) = [K(x) : K] = \deg f(x, K)$.

Let K be any field and $g \in K[X]$. Let z be a root of g . Then z is a multiple root of g if and only if z is a root of g' .

Exercise 7.4. Prove that if K has characteristic zero or K is finite, then every algebraic extension of K is separable.

A consequence: Let E/K be a finite extension. Then E/K is separable if and only if $\gamma(E/K) = [E : K]$.

Example 7.5. Let $E = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Then $[E : \mathbb{Q}] = 4$ and $\text{Gal}(E/\mathbb{Q}) \simeq C_2 \times C_2$. The extension E/\mathbb{Q} is normal, as it is the decomposition field of $(X^2 - 2)(X^2 - 3)$ and it is separable as \mathbb{Q} has characteristic zero.

Example 7.6. Let E be a decomposition field of $X^4 - 2$ over \mathbb{Q} . Then E/\mathbb{Q} is normal and separable. Note that $E = \mathbb{Q}(\sqrt[4]{2}, i)$, so $[E : \mathbb{Q}] = 8 = |\text{Gal}(E/\mathbb{Q})|$.

Let us compute $\text{Gal}(E/\mathbb{Q})$. If $\sigma \in \text{Gal}(E/\mathbb{Q})$, then $\sigma(\sqrt[4]{2}) \in \{\sqrt[4]{2}, -\sqrt[4]{2}, \sqrt[4]{2}i, -\sqrt[4]{2}i\}$ and $\sigma(i) \in \{-i, i\}$. Two examples are

$$\alpha: \begin{cases} \sqrt[4]{2} \mapsto \sqrt[4]{2}i, \\ i \mapsto i, \end{cases} \quad \beta: \begin{cases} \sqrt[4]{2} \mapsto \sqrt[4]{2}, \\ i \mapsto -i. \end{cases}$$

It follows that $\text{Gal}(E/\mathbb{Q})$ is isomorphic to the group $\langle \alpha, \beta \rangle$, which turns out to be isomorphic to the dihedral group of eight elements.

Another consequence: If $E = K(S)$, then E/K is separable if and only if every $x \in S$ is separable over K . One first does the case $E = K(x)$ and then proceeds by induction.

Exercise 7.7. Let $K \subseteq F \subseteq E$ be a tower of fields. Prove that if E/K is separable, then F/K and E/F are separable.

Exercise 7.8. Let E/K and F/K be extensions. Prove that if E/K is separable, then EF/E is separable.

Lecture 7

If E/K is algebraic, then

$$F = \{x \in E : x \text{ is separable over } K\}$$

is a subfield of E that contains K . It is known as the **separable closure** of K with respect to E . Note that $F = K(F)$, as $K(F)$ is separable because it is generated by separable elements. Moreover, F/K is separable and E/F is a **purely inseparable** extension, meaning that for every $x \in E \setminus F$, the polynomial $f(x, F)$ is not separable.

Proposition 7.9. *If E/K is separable and finite, then $E = K(x)$ for some $x \in E$.*

Proof. Let us assume that K is finite. Then E is finite and hence the multiplicative group $E^\times = E \setminus \{0\}$ is cyclic, say $E^\times = \langle x \rangle$. It follows that $E = K(x)$.

Let us now assume that K is infinite. We first consider the case $E = K(x, y)$. The general case $E = K(x_1, \dots, x_n)$ is left as an exercise, one needs to proceed by induction. Let $n = [E : K]$ and C be an algebraic closure of K containing E . Write $\text{Hom}(E/K, C/K) = \{\sigma_1, \dots, \sigma_n\}$. Let

$$f = \prod_{1 \leq i < j \leq n} ((\sigma_i(y) - \sigma_j(y)) + X(\sigma_i(x) - \sigma_j(x))) \in C[X].$$

Then $f \neq 0$, as f is a product of non-zero polynomials. Since K is infinite, there exists $c \in K$ such that $f(c) \neq 0$. For any $r, s \in \{1, \dots, n\}$ with $r \neq s$,

$$\sigma_r(y) - \sigma_s(y) + c(\sigma_r(x) - \sigma_s(x)) \neq 0,$$

as $c \in K$. It follows that $\sigma_r(y + cx) \neq \sigma_s(y + cx)$. Thus $\gamma(K(y + cx)/K) \geq n$. Now

$$n \geq [K(y + cx) : K] = \gamma(K(y + cx)/K) \geq n,$$

so $[K(y + cx) : K] = n$ and hence $K(y + cx) = E$. □

For example, $\mathbb{Q}(\sqrt{2}, i) = \mathbb{Q}(\sqrt{2} + i)$.

Proposition 7.10. *Let E/K be a finite extension. Then $E = K(x)$ for some $x \in E$ if and only if E/K admits finitely many subextensions.*

Proof. We first prove \implies . We may assume that K is infinite, otherwise the result is trivial. Let us assume that $E = K(x)$. We claim that the map

$$\Psi: \{F : K \subseteq F \subseteq E\} \rightarrow \{\text{monic divisors of } f(x, K)\}, \quad F \mapsto f(x, F),$$

is injective. Take $K \subseteq F_0 \subseteq F \subseteq E$ with $f(x, F) = f(x, F_0)$. Then

$$[E : F_0] = [F_0(x) : F_0] = \deg f(x, F_0) = m = [F(x) : F] = [E : F]$$

and hence $F = F_0$. It follows that Ψ is injective and therefore there are finitely many fields between K and E .

Let us prove \impliedby . As before let us assume that $E = K(x, y)$. For each $a \in K$ we consider the extension $K(ay + x)/K$. By assumption, there exist $a, b \in K$ such that $a \neq b$ and $K(x + ay) = K(x + by) = L$. We claim that $L = E$. Note that $x + ay \in L$ and $x + by \in L$, so $(a - b)y \in L$ and hence, since $K \subseteq L$, it follows that $y \in L$. Thus $x \in L$ and therefore $L = E$. \square

As a consequence, if E/K is finite and separable, then E/K admits finitely many subextensions.

§8. Galois extensions

Let E/K be an algebraic extension. Assume that $E = K(S)$ and let C be an algebraic closure of K containing E . Let

$$T = \{y \in C : y \text{ is a root of } f(x, K) \text{ for some } x \in S\}$$

and let $L = K(T)$. Then $E \subseteq L$, as $S \subseteq T$. The extension L/K is normal, as L/K is a decomposition field of the family $\{f(x, K) : x \in S\}$. Moreover, L is the smallest normal extension of K containing E . The field L is the **normal closure** of E (with respect to C).

Exercise 8.1. If E/K is finite, then L/K is finite

Exercise 8.2. If E/K is separable, then L/K is separable.

Let E/K be an extension and $S \subseteq \text{Gal}(E/K)$ be a subset. the set

$${}^S E = \{x \in E : \sigma(x) = x \text{ for all } \sigma \in S\}$$

is a subfield of E that contains K . The subfield ${}^S E$ is known as the **fixed field** of S .

Definition 8.3. Let E/K be an algebraic extension and $G = \text{Gal}(E/K)$. Then E/K is a **Galois extension** if ${}^G E = K$.

Clearly, K/K is a Galois extension. Note that $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is not a Galois extension. Why?

Exercise 8.4. Prove that $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$ is a Galois extension.

Exercise 8.5. If the characteristic of K is different from two, then every quadratic extension of K is a Galois extension.

Exercise 8.6. Let E/K be an algebraic extension and $G = \text{Gal}(E/K)$. Let $F = {}^G E$. Prove that $\text{Gal}(E/F) = G$ and hence E/F is a Galois extension.

Proposition 8.7. Let E/K be an algebraic extension. Then E/K is a Galois extension if and only if E/K is normal and separable.

Proof. Let $G = \text{Gal}(E/K)$. Let us first assume that E/K is Galois. For $x \in E$ let $f_x = \prod_{y \in O_G(x)} (X - y) = \sum a_i X^i \in E[X]$. If $\varphi \in G$, then

$$\bar{\varphi}(f_x) = \prod_{y \in O_G(x)} (X - \varphi(y)) = f_x,$$

as if $O_G(x) = \{\sigma_1(x), \dots, \sigma_r(x)\}$, then $\varphi(\sigma_i(x)) = (\varphi\sigma_i)(x) = \sigma_j(x)$ for some j . Since

$$\sum a_i X^i = f_x = \bar{\varphi}(f_x) = \sum \varphi(a_i) X^i,$$

it follows that $a_i \in {}^G E = K$ for all i . Thus $f_x \in K[X]$ and E/K is a decomposition field of the family $\{f_x : x \in E\}$. In particular, E/K is normal. Moreover, x is a simple root of $f_x \in K[X]$ and hence x is separable over K .

Conversely, let $x \in {}^G E$. Since E/K is normal, then $f(x, K) = \prod_{y \in O_G(x)} (X - y)^m$ for some m . Since E/K is separable, $m = 1$. Thus $f(x, K) = \prod_{y \in O_G(x)} (X - y) = X - x$ and $x \in K$. \square

Definition 8.8. Let K be a field and $f \in K[X]$. Then f is **separable** if all roots of f are simple (in some algebraic closure of K).

Proposition 8.9. Let E/K be a finite extension. Then E/K is a Galois extension if and only if E is a decomposition field over K of a separable polynomial $f \in K[X]$.

Proof. Let us assume first that E/K is a Galois extension. Since E/K is finite and separable, $E = K(x)$ by Proposition 7.9. Then E/K is a decomposition field of $f(x, K)$ since E/K is normal. Since E/K is separable, x is separable over K . Thus x is a simple root of $f(x, K)$ and hence $f(x, K)$ is separable.

Conversely, let x_1, \dots, x_r be the roots of a separable polynomial $f \in K[X]$. Then $E = K(x_1, \dots, x_r)$ is separable and normal. \square

In the previous case, $\text{Gal}(E/K)$ is known as the **Galois group** of the polynomial f . The notation is $\text{Gal}(f, K)$. If $n = \deg f$ and x_1, \dots, x_n are the roots of f , then any $\varphi \in \text{Gal}(f, K)$ permutes the roots of f , that is φ permutes the set $\{x_1, \dots, x_n\}$. In particular, $\text{Gal}(f, K)$ is isomorphic to a subgroup of \mathbb{S}_n and hence $|\text{Gal}(f, K)|$ divides $n!$.

Proposition 8.10. *Let E/K be a normal extension and F be the separable closure of K with respect to E . Then F/K is a Galois extension.*

Proof. Let C/K be an algebraic closure such that $E \subseteq C$. Let $\sigma \in \text{Hom}(F/K, C/K)$. and let $\varphi \in \text{Hom}(E/K, C/K)$ be such that $\varphi|_F = \sigma$. Since E/K is normal, $\varphi(E) = E$. Let $x \in F$. Then $\sigma(x) = \varphi(x) \in E$. Thus $f(\sigma(x), K) = f(x, K)$ and $\sigma(x)$ is separable over K , which implies that $\sigma(x) \in F$. Thus F/K is normal. Since F/K is separable, it follows that F/K is a Galois extension by Proposition 8.7. \square

Some easy facts.

Exercise 8.11. Let E/K be a separable extension and L/K be the normal closure of E in some algebraic closure C that contains E . Prove that L/K is a Galois extension.

Exercise 8.12. Let E/K be a finite extension. Prove that E/K is Galois if and only if $[E : K] = |\text{Gal}(E/K)|$.

Exercise 8.13. Let E/K be a Galois extension and F/K be a subextension of E/K . Prove that E/F is a Galois extension.

Lecture 8

Theorem 8.14 (Artin). *Let E be a field and G be a finite group of automorphisms of E . If $K = {}^G E$, then E/K is a Galois extension, $[E : K] = |G|$ and $\text{Gal}(E/K) = G$.*

Before proving the theorem, we need a lemma.

Lemma 8.15. *Let E/K be a separable extension such that $\deg(x, K) \leq m$ for all $x \in E$. Then E/K is finite and $[E : K] \leq m$.*

Proof. Let $z \in E$ be of maximal degree. If $x \in E$, then $K(x, z)/K$ is separable. Then $K(x, z) = K(y)$ for some y . It follows that

$$K(z) \subseteq K(x, z) = K(y).$$

Since $\deg(z, K) \leq \deg(y, K)$, $\deg(z, K) = \deg(y, K)$. Hence $K(y) = K(z)$. In particular, $x \in K(z)$ and therefore $E = K(z)$. \square

Now we are ready to prove Artin's theorem:

Proof of Theorem 8.14. Note that $G \subseteq \text{Gal}(E/K)$. Let $x \in E$ and

$$f_x = \prod_{y \in O_G(x)} (X - y).$$

Since $f_x \in K[X]$, it follows that the extension E/K is normal and separable (as it is a decomposition field of a family of separable polynomials), so E/K is a Galois extension. Moreover,

$$\deg(x, K) \leq \deg f_x = |O_G(x)| \leq |G|.$$

By the previous lemma, E/K is finite and $[E : K] \leq |G|$. This implies that $|G(E/K)| = [E : K] \leq |G|$ and hence $|G(E/K)| = |G|$. \square

Example 8.16. Let $E = K(X, Y)$ and $\sigma : K[X, Y] \rightarrow E$ be the ring homomorphism given by $\sigma(X) = Y$ and $\sigma(Y) = X$. Note that σ is bijective, as $\sigma^2 = \text{id}$. The map

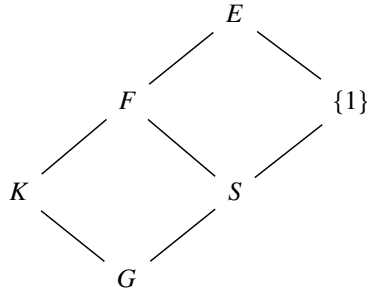
σ induces a field homomorphism $\bar{\sigma}: E \rightarrow E$ such that $\bar{\sigma}^2 = \text{id}$. Recall that such a homomorphism is given by $f/g \mapsto \sigma(f)/\sigma(g)$. Let $G = \langle \bar{\sigma} \rangle$. Then $|G| = 2$. We claim that ${}^G E = K(X+Y, XY)$. Let $F = K(X+Y, XY)$. We only prove that ${}^G E \subseteq F$, as the other inclusion is trivial. Artin's theorem implies that $[E : {}^G E] = 2$ and $E = F(X)$, as X is a root of the polynomial $Z^2 - (X+Y)Z + XY$. Then $[E : F] \leq 2$ and $[{}^G E : F] = 1$.

§9. Galois' correspondence

Theorem 9.1 (Galois). *Let E/K be a finite Galois extension and $G = \text{Gal}(E/K)$. There exists a bijective correspondence*

$$\{F : K \subseteq F \subseteq E \text{ subfields}\} \leftrightarrow \{\text{subgroups of } G\}$$

The correspondence is given by $F \mapsto G(E/F)$ and ${}^S E \mapsto S$. Moreover, normal subextensions of E/K correspond to normal subgroups of G .



Proof. Let α and β be the maps $\alpha(F) = G(E/F)$ and $\beta(S) = {}^S E$. A routine exercise shows that α and β are well-defined. We first note that

$$\beta(\alpha(F)) = \beta(G(E/F)) = {}^{G(E/F)} E = F$$

since E/F is a Galois extension. Moreover,

$$\alpha(\beta(S)) = \alpha({}^S E) = G(E/{}^S E) = S$$

by Artin's theorem, as S is finite.

Let F be a subfield of E containing K and $S = \alpha(F)$. Then

$$[F : K] = \frac{[E : K]}{[E : F]} = \frac{|G|}{|S|} = (G : S).$$

Let C be an algebraic closure of K that contains E . If $S = G(E/F)$, then $F = {}^S E$.

We need to prove that F/K is normal if and only if S is normal in G . Let us first prove \implies . Let $\tau \in S$ and $\sigma \in G$. Since F/K is normal, $\sigma|_F \in \text{Aut}(F)$. Thus

$\sigma^{-1}(F) = F$. In particular, if $x \in F$, then $\sigma^{-1}(x) \in F$ and

$$\sigma\tau\sigma^{-1}(x) = \sigma\sigma^{-1}(x) = x.$$

Conversely, let $\varphi \in \text{Hom}(F/K, C/K)$. There exists $\Phi: E \rightarrow C$ such that $\Phi|_F = \varphi$. Since E/K is normal, $\Phi(E) = E$ and hence $\Phi \in G$. We claim that $\varphi(x) \in F$ for all $x \in F$. Note that $F = {}^S E$, so

$$\tau\varphi(x) = \tau\Phi(x) = \Phi\Phi^{-1}\tau\Phi(x) = \Phi(x) = \varphi(x)$$

for all $\tau \in S$, as $\Phi^{-1}\tau\Phi \in S$. This means that $\varphi(x) \in {}^S E = F$.

Let us compute $\text{Gal}(F/K)$. Since F/K is normal, the map $\lambda: G \rightarrow \text{Gal}(F/K)$, $\sigma \mapsto \sigma|_F$, is a surjective group homomorphism such that $\ker \lambda = S$. The first isomorphism theorem implies that $\text{Gal}(F/K) \simeq G/S$. \square

Some easy consequences.

Exercise 9.2. If E/K is a Galois extension of degree n and p is a prime number dividing n , then E/K admits a subextension of degree n/p .

Exercise 9.3. If E/K is a Galois extension of degree $p^\alpha m$ with p a prime number coprime with m , then E/K admits a subextension of degree m .

Definition 9.4. An extension E/K is **abelian** if E/K is a Galois extension with $\text{Gal}(E/K)$ abelian.

Exercise 9.5. If E/K is an abelian extension of degree n and d divides n , then E/K admits a subextension of degree d .

Definition 9.6. An extension E/K is **cyclic** if E/K is a Galois extension with $\text{Gal}(E/K)$ cyclic.

Example 9.7. The extension $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$ admits exactly three non-trivial subextensions:

$$\mathbb{Q}(\sqrt{2})/\mathbb{Q}, \quad \mathbb{Q}(\sqrt{3})/\mathbb{Q}, \quad \mathbb{Q}(\sqrt{6})/\mathbb{Q},$$

as $\text{Gal}(\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}) \simeq C_2 \times C_2$.

Example 9.8. Let $\omega \in \mathbb{C} \setminus \{1\}$ be such that $\omega^5 = 1$. Then

$$f(\omega, \mathbb{Q}) = 1 + X + X^2 + X^3 + X^4$$

and $\mathbb{Q}(\omega)/\mathbb{Q}$ has degree four. Moreover, $\mathbb{Q}(\omega)/\mathbb{Q}$ is a Galois extension and $\text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) \simeq C_4$. If $\sigma \in \text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})$, then $\sigma(\omega) = \omega^i$ for some $i \in \{1, \dots, 4\}$. Moreover, for every $i \in \{1, \dots, 4\}$ the map $\omega \mapsto \omega^i$ induces an automorphism of $\mathbb{Q}(\omega)/\mathbb{Q}$. Thus $|\text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})| = 4$. Now

$$\sigma_i^k = \text{id} \iff \omega^{i^k} = \sigma_i^k(\omega) = \omega \iff i^k \equiv 1 \pmod{5}.$$

Thus the map σ_2 given by $\omega \mapsto \omega^2$ has order four.

Since $\text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) = \langle \sigma \rangle$, where $\sigma(\omega) = \omega^2$, is cyclic of order four, the extension $\mathbb{Q}(\omega)/\mathbb{Q}$ has a unique degree-two subextension F/\mathbb{Q} . Note that $|\langle \sigma^2 \rangle| = 2$ and $\sigma^2(\omega) = \omega^4 = \omega^{-1}$. Thus $F = \langle \sigma^2 \rangle \mathbb{Q}(\omega)$. Let $\theta = \omega + \omega^{-1}$. Then

$$\theta^2 = \omega^2 + \omega^3 + 2 = -(1 + \omega + \omega^{-1}) + 2 = 1 - \theta$$

and hence θ is a root of $X^2 + X - 1$. Since $\theta \notin \mathbb{Q}$, it follows that

$$\theta \in \{(-1 + \sqrt{5})/2, (-1 - \sqrt{5})/2\}.$$

Therefore $F = \mathbb{Q}(\sqrt{5})$.

Let us mention some other consequences.

Exercise 9.9. Let E/K be a finite Galois extension and F_1, \dots, F_n fields such that $K \subseteq F_i \subseteq E$ for all $i \in \{1, \dots, n\}$. For every i let $S_i = \text{Gal}(E/F_i)$. Then

$$\text{Gal}\left(E/\bigcap_{i=1}^n F_i\right) = \left\langle \bigcup_{i=1}^n S_i \right\rangle, \quad \text{Gal}\left(E/\prod_{i=1}^n F_i\right) = \bigcap_{i=1}^n S_i.$$

The following statement is a concrete application of the previous exercise.

Exercise 9.10. Let E/K be a finite Galois extension and $G = \text{Gal}(E/K)$. Assume that G is the direct product $G = S \times T$ of the groups S and T . Let $F = {}^S E$ and $L = {}^T E$. Then $F \cap L = K$ and $FL = E$.

Proposition 9.11. Let $E_1/K, \dots, E_r/K$ be Galois extensions. If $E = \prod_{i=1}^r E_i$, then E/K is a Galois extension. If, moreover, each E_i/K is finite, then

$$\theta: \text{Gal}(E/K) \rightarrow \text{Gal}(E_1/K) \times \cdots \times \text{Gal}(E_r/K), \quad \sigma \mapsto (\sigma|_{E_1}, \dots, \sigma|_{E_r}),$$

is an injective group homomorphism.

Proof. We only do the first part in the case $r = 2$, the general case is left as an exercise. Since E_1/K is algebraic, then $E_1 E_2/E_2$ is algebraic. Since E_2/K is algebraic, $E_1 E_2/K$ is algebraic. Similarly, $E_1 E_2/K$ is separable.

Let C/K be an algebraic closure such that $E_1 E_2 \subseteq C$. If $\sigma \in \text{Hom}(E_1 E_2/K, C/K)$, then $\sigma(E_1 E_2) \subseteq \sigma(E_1)\sigma(E_2) = E_1 E_2$ (do this calculation as an exercise). Thus $E_1 E_2/K$ is normal.

If both E_1/K and E_2/K are finite, then $E_1 E_2/K$ is finite.

Clearly, θ is a group homomorphism. We claim that the map θ is injective. Let $\sigma \in \ker \theta$. Then $\sigma|_{E_i} = \text{id}_{E_i}$ for all $i \in \{1, \dots, r\}$. Let $S = \langle \sigma \rangle \subseteq \text{Gal}(E/K)$ and $F = {}^S E$. Then $E_i \subseteq F$ for all $i \in \{1, \dots, r\}$ and hence $E \subseteq F$. It follows that $F = E = {}^{\{\text{id}\}} E$ and therefore $S = \{\text{id}\}$, so $\sigma = \text{id}$. \square

Exercise 9.12. Let $E_1/K, \dots, E_r/K$ be finite Galois extensions such that for each j one has $E_j \cap (E_1 \cdots E_{j-1} E_{j+1} \cdots E_r) = K$. Then

§9 Galois' correspondence

$$\mathrm{Gal}(E/K) \simeq \mathrm{Gal}(E_1/K) \times \cdots \times \mathrm{Gal}(E_r/K).$$

In this case, $[E : K] = \prod_{i=1}^r [E_i : K]$.

Lecture 9

§10. The fundamental theorem of algebra

We now present an easy proof of the fundamental theorem of algebra based on the ideas of Galois Theory. We need the following well-known facts:

- 1) Every real polynomial of odd degree admits a real root. This means that \mathbb{R} does not admit extension of odd degree > 1 .
- 2) Every complex number admits a square root in \mathbb{C} . This means that \mathbb{C} does not admit degree-two extensions.

Theorem 10.1. *The field \mathbb{C} is algebraically closed.*

Proof. Let E/\mathbb{C} be an algebraic finite extension. Then E/\mathbb{R} is finite separable extension of even degree. There exists a Galois extension L/\mathbb{R} such that $E \subseteq L$, so $[L : \mathbb{R}]$ is even. Let $G = \text{Gal}(L/\mathbb{R})$. Then $|G| = 2^m s$ for some odd number s . If T is a 2-Sylow subgroup of G , then there exists a subextension F/\mathbb{R} of degree s . Since \mathbb{R} does not admit extensions of odd degree > 1 , $s = 1$ and hence G is a 2-group. Since L/\mathbb{R} is a Galois extension, L/\mathbb{C} is a Galois extension. In particular, $|\text{Gal}(L/\mathbb{C})| = 2^{m-1}$. If $m > 1$, let U be a subgroup of $\text{Gal}(L/\mathbb{C})$ of order 2^{m-2} . Then U corresponds to a subextension L_1/\mathbb{C} of degree two, a contradiction. Hence $m = 1$ and $[L : \mathbb{C}] = 1$, so $L = \mathbb{C}$ and $E = \mathbb{C}$. \square

§11. Purely inseparable extensions

Let E/K be an algebraic extension. In page 7 we defined the **separable closure** of K with respect to E as the field

$$F = \{x \in E : x \text{ is separable over } K\}.$$

Note that $K \subseteq F \subseteq E$ and $F = K(F)$. Moreover, F/K is separable and E/F is a **purely inseparable** extension, meaning that for every $x \in E \setminus F$, the polynomial $f(x, F)$ is not separable.

The number $[E : F]$ is known as the **degree of inseparability** of E/K . We write $[E : K]_{\text{ins}} = [E : F]$. Clearly, E/K is separable if and only if $[E : K]_{\text{ins}} = 1$ and E/K is purely inseparable if and only if $[E : K]_{\text{ins}} = [E : K]$.

Proposition 11.1. *Let K be a field of characteristic $p > 0$ and E/K be an algebraic extension. The following statements are equivalent:*

- 1) E/K is purely inseparable.
- 2) If $x \in E$, then $x^{p^m} \in K$ for some $m \geq 0$.
- 3) If $x \in E$, then $f(x, K) = X^{p^m} - a$ for some $a \in K$ and $m \geq 0$.
- 4) $\gamma(E/K) = 1$.

Proof. We first prove 1) \implies 2). Let $x \in E$ and $f = f(x, K)$. Assume x is not separable. Then $f(x) = 0$ and $f'(x) = 0$, as x is not a simple root. Since $\deg f' < \deg f$ and f is the minimal polynomial of x , it follows that $f' = 0$. The coefficients of f' are of the form ka_k . Since E is a field, $a_k = 0$ if k is not divisible by p . If $a_k \neq 0$, then $k = pm$ for some $m \geq 0$. It follows that $f = g(X^p)$ for some $g \in K[X]$ with $\deg g < \deg f$. We now proceed by induction on the degree of x . The result is clearly true for elements of degree one. So assume the result holds for element of degree $\leq n$ for some $n \geq 1$. If $x \in E$ is such that $\deg(x, K) = n + 1$, then, since $f(x, K) = g(X^p)$, the element x^p has degree $\leq n$. By the inductive hypothesis, $x^{p^{m+1}} = (x^p)^{p^m} \in K$. We now prove 2) \implies 3). Let $x \in E$ and m be the minimal positive integer such that $x^{p^m} \in K$. Then x is a root of $X^{p^m} - x^{p^m} \in K[X]$. Since $X^{p^m} - x^{p^m} = (X - x)^{p^m}$, it follows that

$$f(x, K) = (X - x)^r = X^r + \cdots + (-1)^r x^r$$

for some $r \in \{1, \dots, p^m\}$. Write $r = p^s t$ for some integer t coprime with p and s such that $0 \leq s \leq m$. Let $a, b \in \mathbb{Z}$ be such that $ar + bp^m = p^s$. Then

$$x^{p^s} = x^{ar+bp^m} = (x^r)^a (x^{p^m})^b \in K.$$

The minimality of m implies that $s \geq m$ and hence $s = m$. Now $p^m t = p^s t = r \leq p^m$, so $t = 1$. This means $f(x, K) = X^{p^m} - x^{p^m}$.

We now prove 3) \implies 4). Let C/K be an algebraic closure that contains E and $\sigma \in \text{Hom}(E/K, C/K)$. Let $x \in E$. We claim that $\sigma(x) = x$. Since $f(x, K) = X^{p^m} - a$,

$$(\sigma(x))^{p^m} = \sigma(x^{p^m}) = \sigma(a) = a = x^{p^m}.$$

It follows that $\sigma(x)$ is a root of $X^{p^m} - x^{p^m} = (X - x)^{p^m}$. Thus $\sigma(x) = x$.

Finally, we prove that 4) \implies 1). Let C be an algebraic closure of K containing E . Then $\text{Gal}(E/K) = \text{Hom}(E/K, C/K) = \{\text{id}\}$, as $\gamma(E/K) = 1$. If $x \in E$ is separable over K , then

$$f(x, K) = \prod_{y \in \text{Gal}(E/K)(x)} (X - y) = X - x \in K[X].$$

Thus $x \in K$ and hence E/K is purely inseparable. \square

Some consequences:

Exercise 11.2. Let E/K be finite and purely inseparable. Then $[E : K] = p^s$ for some prime number p and some s . Moreover, $x^{[E:K]} \in K$.

For the first part of previous exercise write $E = K(x_1, \dots, x_n)$ and proceed by induction on n .

Exercise 11.3. Let K be of characteristic $p > 0$ and E/K be a finite extension such that $[E : K]$ is not divisible by p . Then E/K is separable.

Let E/K be finite and F be the separable closure of K in E . Since

$$\gamma(E/K) = \gamma(E/F)\gamma(F/K) = \gamma(F/K),$$

it follows that

$$[E : K] = [E : F]\gamma(E/K) = [E : K]_{\text{ins}}\gamma(E/K).$$

§12. Norm and trace

Definition 12.1. Let E/K be a finite extension and C/K be an algebraic closure that contains E . Let $A = \text{Hom}(E/K, C/K)$. For $x \in E$ we define the **trace** of x in E/K as

$$\text{trace}_{E/K}(x) = [E : K]_{\text{ins}} \sum_{\varphi \in A} \varphi(x)$$

and the **norm** of x in E/K as

$$\text{norm}_{E/K}(x) = \left(\prod_{\varphi \in A} \varphi(x) \right)^{[E:K]_{\text{ins}}}.$$

As an optional exercise, one can show that these definitions do not depend on the algebraic closure.

We collect some basic properties as an exercise:

Exercise 12.2. Let E/K be a finite extension. The following statements hold:

- 1) If E/K is not separable, then $\text{trace}_{E/K}(x) = 0$ for all $x \in E$.
- 2) If $x \in K$, then $\text{trace}_{E/K}(x) = [E : K]x$.
- 3) $\text{trace}_{E/K}(x) \in K$ for all $x \in E$.
- 4) $\text{norm}_{E/K}(x) = 0$ if and only if $x = 0$.
- 5) If $x \in K$, then $\text{norm}_{E/K}(x) = x^{[E:K]}$.
- 6) $\text{norm}_{E/K}(x) \in K$ for all $x \in E$.

One proves, moreover, that $\text{trace}_{E/K} : E \rightarrow K$ satisfies

$$\text{trace}_{E/K}(x + \lambda y) = \text{trace}_{E/K}(x) + \lambda \text{trace}_{E/K}(y)$$

for all $x, y \in E$ and $\lambda \in K$, that is to say that $\text{trace}_{E/K} : E \rightarrow K$ is a linear form in E . The norm $\text{norm}_{E/K} : E^\times \rightarrow K^\times$ is a group homomorphism.

Exercise 12.3. Let E/K be a finite extension and $x \in E$. If

$$f(x, K) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0,$$

then $\text{norm}_{E/K}(x) = ((-1)^n a_0)^{[E:K(x)]}$ and $\text{trace}_{E/K}(x) = -[E : K(x)]a_{n-1}$.

Example 12.4. Let $E = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Then

$$\begin{aligned} \text{trace}_{E/\mathbb{Q}}(\sqrt{2}) &= 0, & \text{norm}_{E/\mathbb{Q}}(\sqrt{2}) &= -4, \\ \text{trace}_{E/\mathbb{Q}(\sqrt{2})}(\sqrt{2}) &= 2\sqrt{2}, & \text{norm}_{E/\mathbb{Q}(\sqrt{2})}(\sqrt{2}) &= 2. \end{aligned}$$

Example 12.5. If E/K is a finite Galois extension, then

$$\text{trace}_{E/K}(x) = \sum_{\sigma \in \text{Gal}(E/K)} \sigma(x) \quad \text{and} \quad \text{trace}_{E/K}(x) = \prod_{\sigma \in \text{Gal}(E/K)} \sigma(x)$$

for all $x \in E$. In particular, since $E = K(y)$ for some y by Proposition 7.9,

$$\text{trace}_{E/K}(y) = -a_{n-1} \quad \text{and} \quad \text{norm}_{E/K}(y) = (-1)^n a_0,$$

where $f(y, K) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$.

§13. Finite fields

In this section, p will be a prime number.

Proposition 13.1. *Let m be a positive integer. Up to isomorphism, there exists a unique field F_m of size p^m .*

Proof. Let C be an algebraic closure of the field \mathbb{Z}/p and let $F_m = \{x \in C : x^{p^m} = x\}$ be the set of roots of $X^{p^m} - X$. Since the polynomial $X^{p^m} - X$ has no multiple roots, $|F_m| = p^m$. Moreover, F_m is the unique subfield of C of size p^m .

To prove the uniqueness it is enough to note that if K is a field of p^m elements, then K is the decomposition field of $X^{p^m} - X$ over \mathbb{Z}/p . \square

Let $K = \mathbb{Z}/p$ and C be an algebraic closure of K . We claim that $C = \cup_k F_k$. If $x \in C$, then x is algebraic over K . Since $K(x)/K$ is finite, $K(x)$ is a finite field, say $|K| = p^r$ for some r . Then $x^{p^r} = x$ and hence $x \in F_r$.

Exercise 13.2. Prove the following statements:

- 1) If $x \in F_r$, then $x^{p^{rk}} = x$ for all $k \geq 0$.
- 2) If $m \mid n$, then $F_m \subseteq F_n$.
- 3) $F_m \cap F_n = F_{\gcd(m,n)}$.
- 4) $F_m \subseteq F_n$ if and only if $m \mid n$.

Proposition 13.3. Every finite extension of a finite field is cyclic.

Proof. Let $K = \mathbb{Z}/p$. It is enough to show that F_n/F_m is cyclic if m divides n .

We first prove that F_n/K is cyclic. Let $\sigma \in \text{Gal}(F_n/K)$ be given by $\sigma(x) = x^p$. Note that $|\text{Gal}(F_n/K)| = [F_n : K] = n$. Since $\sigma^i(x) = x^{p^i}$ for all $i \geq 0$, in particular, $\sigma^n(x) = x^{p^n} = x$. Thus $\sigma^n = \text{id}$ and hence $|\sigma|$ divides n . Let $s = |\sigma|$. We know that $F_n^\times = F_n \setminus \{0\}$ is cyclic, say $F_n^\times = \langle g \rangle$. Since $|g| = p^n - 1$,

$$g = \sigma^s(g) = g^{p^s}$$

and hence $p^s \equiv 1 \pmod{p^n - 1}$. Thus $p^n - 1$ divides $p^s - 1$ and hence n divides s . Therefore $n = s$ and $\text{Gal}(F_n/K) = \langle \sigma \rangle$.

For the general case note that if m divides n , then $\text{Gal}(F_n/F_m)$ is a subgroup of $\text{Gal}(F_n/K)$. Since $\text{Gal}(F_n/K)$ is cyclic, the claim follows. \square

If $K = \mathbb{Z}/p$ and m divides n , the subextension F_m corresponds to the unique subgroup of index m of $\text{Gal}(F_n/K) = \langle \sigma \rangle$. This subgroup is $\langle \sigma^m \rangle$, where

$$\sigma^m(x) = x^{p^m} = x^{|F_m|}.$$

Note that $\text{Gal}(F_n/F_m) = \langle \sigma^m \rangle$. The map σ^m is known as the **Frobenius automorphism**.

Exercise 13.4. Let E/K be an extension of finite fields. Then E/K is cyclic and $\text{Gal}(E/K) = \langle \tau \rangle$, where $\tau(x) = x^{|K|}$.

Lecture 10

§14. Cyclotomic extensions

For $n \geq 1$ let $G_n(K) = \{x \in K : x^n = 1\}$ be the set of n -roots of one in K . Note that $G_n(K)$ is a cyclic subgroup of K^\times and that $|G_n(K)|$ divides n .

Example 14.1. $G_n(\mathbb{R}) = \{-1, 1\}$ if n is odd and $G_n = \{1\}$ if n is even.

Exercise 14.2. Let K be a field of characteristic $p > 0$. Let $n = p^s m$ for some m not divisible by p . Then $G_n(K) = G_m(K)$.

Exercise 14.3. Let q be a prime number. Then $G_n(\mathbb{Z}/q) \simeq \mathbb{Z}/\gcd(n, q-1)$.

Similarly, one can prove that if K is a finite field, then $G_n(K)$ is a cyclic group of order $\gcd(n, |K^\times|)$.

Example 14.4. If C is algebraically closed of characteristic coprime with n , then $G_n(C)$ is cyclic of order n , as $X^n - 1$ has all his roots in C and does not contain multiple roots.

Let K be an algebraically closed field and n be such that n is coprime with the characteristic of K . The set of **primitive n -roots** is defined as

$$H_n(K) = \{x \in G_n(K) : |x| = n\}.$$

Definition 14.5. Let K be an algebraically closed field and n be such that n is coprime with the characteristic of K . The **n -th cyclotomic polynomial** is defined as

$$\Phi_n = \prod_{x \in H_n(K)} (X - x) \in K[X].$$

For $n \geq 1$ the Euler's function is defined as

$$\varphi(n) = |\{k : 1 \leq k \leq n, \gcd(k, n) = 1\}|.$$

For example, $\varphi(4) = 2$, $\varphi(8) = \varphi(10) = 4$ and $\varphi(p) = p - 1$ for every prime p .

Proposition 14.6. *Let K be an algebraically closed field and n be such that n is coprime with the characteristic of K . Let A be the ring of integers of K .*

- 1) $\deg \Phi_n = \varphi(n)$.
 2) $\Phi_n \in A[X]$.

Proof. The first statement is clear. Let us prove 2) by induction on n . The case $n = 1$ is trivial, as $\Phi_1 = X - 1$. Assume that $\Phi_d \in A[X]$ for all d such that $d < n$. In particular,

$$\gamma = \prod_{\substack{d|n \\ d \neq n}} \Phi_d \in A[X].$$

Since γ is monic, it follows that $\frac{X^n - 1}{\gamma} \in A[X]$. Now the claim follows from

$$X^n - 1 = \prod_{d|n} \Phi_d = \Phi_n \prod_{\substack{d|n \\ d \neq n}} \Phi_d = \Phi_n \gamma. \quad \square$$

By taking degree in the equality $X^n - 1 = \prod_{d|n} \Phi_d$ one gets

$$n = \sum_{d|n} \varphi(d).$$

Definition 14.7. Let $n \geq 2$ and K be a field of characteristic coprime with n . A **cyclotomic extension** of K of index n is a decomposition field of $X^n - 1$ over K .

Let C be an algebraic closure of K and $n \geq 2$ be coprime with the characteristic of K . It follows from Definition 14.7 that a cyclotomic extension of index n is of the form $K(\omega)/K$ for some $\omega \in H_n(C)$.

Proposition 14.8. *A cyclotomic extension of index n is abelian and of degree a divisor of $\varphi(n)$.*

Proof. Let C be an algebraic closure of K and $n \geq 2$ be coprime with the characteristic of K . Let $\omega \in H_n(C)$ and $K(\omega)/K$ be a cyclotomic extension. Then $K(\omega)/K$ is a Galois extension, as it is a decomposition field of a separable polynomial. Let $U = \mathcal{U}(\mathbb{Z}/n)$ be the group of units of \mathbb{Z}/n and

$$\lambda: \text{Gal}(K(\omega)/K) \rightarrow U, \quad \sigma \mapsto m_\sigma,$$

where m_σ is such that $\sigma(\omega) = \omega^{m_\sigma}$. The map λ is well-defined and it is a group homomorphism, as if $\sigma, \tau \in \text{Gal}(K(\omega)/K)$, then, since

$$(\tau\sigma)(\omega) = \tau(\sigma(\omega)) = \tau(\omega^{m_\sigma}) = (\omega^{m_\sigma})^{m_\tau} = \omega^{m_\sigma m_\tau},$$

it follows that $\lambda(\sigma)\lambda(\tau) = \lambda(\sigma\tau)$. Since λ is injective, $\text{Gal}(K(\omega)/K)$ is isomorphic to a subgroup of the abelian group U . Hence $\text{Gal}(K(\omega)/K)$ is abelian. Moreover, $[K(\omega) : K] = |\text{Gal}(K(\omega)/K)|$ is a divisor of $|U| = \varphi(n)$. \square

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Exercise 14.9. Prove that a cyclotomic extension $K(\omega)/K$ has degree $\varphi(n)$ if and only if Φ_n is irreducible over K .

Note that Φ_n is irreducible over \mathbb{Q} . Some concrete examples:

$$\Phi_1 = X - 1, \quad \Phi_2 = X + 1, \quad \Phi_3 = X^2 + X + 1, \quad \Phi_6 = X^2 - X + 1.$$

If p is a prime number, then $\Phi_p = X^{p-1} + \cdots + X + 1$.

Example 14.10. Φ_5 is irreducible over $\mathbb{Z}/2$. First note that $\Phi_5 = X^4 + \cdots + X + 1$ does not have roots in $\mathbb{Z}/2$. If Φ_5 is reducible, then, since $X^2 + X + 1$ is the unique monic irreducible polynomial over $\mathbb{Z}/2$, it follows that

$$\Phi_5 = (X^2 + X + 1)(X^2 + X + 1) = (X^2 + X + 1)^2 = X^4 + X^2 + 1,$$

a contradiction.

Exercise 14.11. Prove that $\Phi_{12} = X^4 - X^2 + 1$ is not irreducible over $\mathbb{Z}/5$.

§15. Hilbert's theorem

Let G be a group and A be a (left) G -module. This means that A is an abelian group and there exists a map

$$G \times A \rightarrow A, \quad (g, a) \mapsto g \cdot a$$

such that $1 \cdot a = a$ for all $a \in A$, $(gh) \cdot a = g \cdot (h \cdot a)$ for all $g, h \in G$ and $a \in A$ and $g \cdot (a + b) = g \cdot a + g \cdot b$ for all $g \in G$ and $a, b \in A$.

Definition 15.1. Let A be a G -module. A **derivation** of A is a map $d: G \rightarrow A$ such that $d(gh) = g \cdot d(h) + d(g)$ for all $g, h \in G$.

Let A be a G -module and $a \in A$. The map $d(g) = g \cdot a - a$ is a derivation of A . Such derivations are known as **inner derivations**.

Exercise 15.2. Let E/K be a finite Galois extension and $G = \text{Gal}(E/K)$. Then the (multiplicative) group E^\times is a G -module with $\sigma \cdot x = \sigma(x)$.

In the context of the previous exercise, let $d: G \rightarrow E^\times$ be a derivation. For $\tau \in G$ let $x_\tau = d(\tau)$. Then

$$x_{\sigma\tau} = d(\sigma\tau) = (\sigma \cdot d(\tau))d(\sigma) = \sigma(x_\tau)x_\sigma.$$

Is this true that $x_\sigma = \frac{\sigma(c)}{c} = (\sigma \cdot c)c^{-1}$ for some $c \in E^\times$?

Proposition 15.3. Let E/K be a finite Galois extension and $G = \text{Gal}(E/K)$.

1) Let $\{x_\tau : \tau \in G\} \subseteq E^\times$ be such that $x_{\sigma\tau} = \sigma(x_\tau)x_\sigma$. Then there exists $c \in E^\times$ such that $x_\sigma = \frac{\sigma(c)}{c}$ for all $\sigma \in G$.

2) Let $\{x_\tau : \tau \in G\} \subseteq E$ be such that $x_{\sigma\tau} = \sigma(x_\tau) + x_\sigma$. Then there exists $c \in E$ such that $x_\sigma = \sigma(c) - c$ for all $\sigma \in G$.

Proof. We prove 1). By Dedekind's theorem, the homomorphism $\sum_{\tau \in G} x_\tau^{-1} \tau$ is non-zero. Thus there exists $a \in E^\times$ such that $\sum_{\tau \in G} x_\tau^{-1} \tau(a) = c \in E^\times$. If $\sigma \in G$, then

$$\sigma(c) = \sum_{\tau \in G} \sigma(x_\tau^{-1})(\sigma\tau)(a) = \sum_{\tau \in G} x_{\sigma\tau}^{-1} x_\tau \sigma\tau(a) = x_\sigma \sum_{\tau \in G} x_{\sigma\tau}^{-1} \sigma\tau(a) = x_\sigma c.$$

We now prove 2). By Dedekind's theorem, $\sum_{\tau \in G} \tau \neq 0$. Since it is a linear form in E , it is surjective. In particular, there exists $a \in E$ such that $\sum_{\tau \in G} \tau(a) = 1$. Let $c = -\sum_{\tau \in G} x_\tau \tau(a)$. If $\sigma \in G$, then

$$\begin{aligned} \sigma(c) &= - \sum_{\tau \in G} \sigma(x_\tau) \sigma\tau(a) \\ &= - \sum_{\tau \in G} (x_{\sigma\tau} - x_\sigma) \sigma\tau(a) \\ &= - \sum_{\tau \in G} x_{\sigma\tau} \sigma\tau(a) + \sum_{\tau \in G} x_\sigma \sigma\tau(a) = c + x_\sigma. \end{aligned}$$

Hence $x_\sigma = \sigma(c) - c$. □

The following result is known as Hilbert's 90 theorem.

Theorem 15.4 (Hilbert). *Let E/K be a cyclic extension of degree n and let $G = \text{Gal}(E/K) = \langle \sigma \rangle$.*

- 1) *Let $x \in E^\times$ be such that $\text{norm}_{E/K}(x) = 1$. There exists $b \in E^\times$ such that $x = \sigma(b)/b$.*
- 2) *Let $x \in E$ be such that $\text{trace}_{E/K}(x) = 0$. There exists $b \in E$ such that $x = \sigma(b) - b$.*

Proof. Let us prove 1). Note that $G = \{\sigma^i : 0 \leq i < n\}$. For $i \in \{0, \dots, n-1\}$ let $x_{\sigma^i} = \prod_{k=0}^{i-1} \sigma^k(x)$. In particular, $x_\sigma = x$. We now check that $\{x_{\sigma^i}\}$ satisfy the assumptions of the previous proposition:

$$\begin{aligned} \sigma^j(x_{\sigma^i}) &= \prod_{k=0}^{i-1} \sigma^{k+j}(x) = \prod_{k=j}^{i+j-1} \sigma^k(x) \\ &= \prod_{k=0}^{i+j-1} \sigma^k(x) \left(\prod_{k=0}^{j-1} \sigma^k(x) \right)^{-1} = \prod_{k=0}^{i+j-1} \sigma^k(x) x_{\sigma^j}^{-1}. \end{aligned}$$

If $i+j < n$, then

$$\sigma^j \sigma^i = \sigma^{i+j} \implies \sigma^j(x_{\sigma^i}) = x_{\sigma^i \sigma^j} x_{\sigma^j}^{-1} \implies x_{\sigma^j \sigma^i} = \sigma^j(x_{\sigma^i}) x_{\sigma^j}.$$

If $i+j = n+r$ for some $r \in \{0, \dots, n-1\}$, then, since $i+j < 2n$ and $\sigma^i \sigma^j = \sigma^r$,

$$\sigma^j(x_{\sigma^i}) = \prod_{k=0}^{n-1} \sigma^k(x) \prod_{k=n}^{n+r-1} \sigma^k(x) x_{\sigma^j}^{-1} = \prod_{k=0}^{r-1} \sigma^k(x) x_{\sigma^j}^{-1} = x_{\sigma^r} x_{\sigma^j}^{-1} = x_{\sigma^j \sigma^i} x_{\sigma^j}^{-1}$$

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and hence $x_{\sigma^j \sigma^i} = \sigma^j(x_{\sigma^i})x_{\sigma^j}$. By the previous lemma, there exists $c \in E^\times$ such that $x_{\sigma^i} = \frac{\sigma^i(c)}{c}$. In particular, $x = x_\sigma = \sigma(c)/c$.

The second statement is similar and it is left as an exercise. \square

If A , B and C are groups (written multiplicatively) and $f: A \rightarrow B$ and $g: B \rightarrow C$ are group homomorphism, the sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

of groups and homomorphisms is said to be **exact** if $f(A) = \ker g$. For example, the sequence

$$1 \longrightarrow B \xrightarrow{g} C$$

is exact if and only if g is injective, as the first map represents the trivial homomorphism. Similarly, the sequence

$$A \xrightarrow{f} B \longrightarrow 1$$

is exact if and only if f is surjective.

Corollary 15.5. *Let E/K be finite and cyclic with $\text{Gal}(E/K) = \langle \sigma \rangle$.*

1) *The sequence*

$$1 \longrightarrow K^\times \hookrightarrow E^\times \xrightarrow{\rho} E^\times \xrightarrow{\text{norm}_{E/K}} K^\times$$

is exact, where $\rho(z) = \sigma(z)/z$.

2) *The sequence*

$$0 \longrightarrow K \hookrightarrow E \xrightarrow{\lambda} E \xrightarrow{\text{trace}_{E/K}} K$$

is exact, where $\lambda(z) = \sigma(z) - z$.

Proof. We only prove 1). Note that $K^\times \hookrightarrow E^\times$ is the inclusion map. If $z \in \ker \rho$, then $\sigma(z) = z$ and hence $z \in K$. The sequence

$$E^\times \xrightarrow{\rho} E^\times \xrightarrow{\text{norm}_{E/K}} K^\times$$

is exact by Hilbert's theorem. \square

Lecture 11

Proposition 15.6. *Let $n \geq 2$ and K be a field containing a primitive n -root of one. If $a \in K^\times$ and E/K is a decomposition field of $f = X^n - a$, then E/K is cyclic of degree d , where d divides n . Moreover,*

$$d = \min\{k : a^k \in K^\times\},$$

where $K^\times = \{x \in K : x = y^n \text{ for some } y \in K\}$. Conversely, if E/K is cyclic of degree n , then E/K is a decomposition field of an irreducible polynomial of the form $X^n - a$ for some $a \in K^\times$.

Proof. A decomposition field of f over K is of the form $K(\alpha)$, where $\alpha^n = a$. Thus $K(\alpha)/K$ is a Galois extension. If $\sigma \in \text{Gal}(K(\alpha)/K)$, then $\sigma(\alpha)$ is a root of f , so $\sigma(\alpha) = \omega_\sigma \alpha$, where $\omega_\sigma \in G_n(K)$. This means that there exists an injective map

$$\lambda: \text{Gal}(K(\alpha)/K) \rightarrow G_n(K), \quad \sigma \mapsto \omega_\sigma.$$

Moreover, λ is a group homomorphism, as

$$\sigma\tau(\alpha) = \sigma(\tau(\alpha)) = \sigma(\omega_\tau \alpha) = \omega_\tau \sigma(\alpha) = \omega_\tau \omega_\sigma \alpha.$$

Therefore $\text{Gal}(K(\alpha)/K)$ is isomorphic to a subgroup of $G_n(K)$. In particular, $\text{Gal}(K(\alpha)/K)$ is cyclic and $|\text{Gal}(K(\alpha)/K)|$ divides n .

Let $d = |\text{Gal}(K(\alpha)/K)|$. Since $a = \alpha^n$,

$$\text{norm}(\alpha)^n = \text{norm}(a) = a^d.$$

Thus $a^d \in K^\times$, as $\text{norm}(\alpha) \in K$. If $a^k \in K^\times$, say $a^k = c^n$ for some $c \in K$, then

$$a^k = c^n = (\alpha^n)^k = (\alpha^k)^n \implies \alpha^k = c\omega \in K$$

for some $\omega \in G_n(K)$. Thus α is a root of $X^k - \alpha^k \in K[X]$ and hence $k \geq d$.

Note that $f(\alpha, K) = X^d - \alpha^d$.

Let E/K be cyclic of degree n . Assume that $\text{Gal}(E/K) = \langle \sigma \rangle$. If ω is a primitive n -root of one,

$$\text{norm}_{E/K}(\omega) = 1 = \omega^n.$$

By Hilbert's theorem, there exists $b \in E^\times$ such that $\omega = \sigma(b)/b$. Thus $\sigma(b) = \omega b$ and hence $\sigma^i(b) = \omega^i b$ for all $i \geq 0$. Since $|\{b, \sigma(b), \dots, \sigma^{n-1}(b)\}| = n$, it follows that $E = K(b)$. Moreover,

$$\sigma(b^n) = \sigma(b)^n = (\omega b)^n = b^n$$

and hence $b^n \in K$. This means that E/K is a decomposition field of $X^n - b^n$. Note that $X^n - b^n$ is irreducible, as $[E : K] = [K(b) : K] = n$. \square

Proposition 15.7. *Let K be a field of characteristic $p > 0$.*

- 1) *Let $a \in K$ and $f = X^p - X - a$. Then f is irreducible over K or all the roots of f belong to K . In the first case, if b is a root of f , then $K(b)/K$ is a cyclic extension of degree p .*
- 2) *Every cyclic extension of degree p is a decomposition field of an irreducible polynomial of the form $X^p - X - a$.*

Proof. We first prove 1). Let K_0 be the prime field of K . Note that $K_0 \simeq \mathbb{Z}/p$. Let b be a root of f and let $x \in K_0$. Then

$$f(b+x) = (b+x)^p - (b+x) - a = (b^p - b - a) + (x^p - x) = 0$$

and thus $\{b+x : x \in K_0\}$ is the set of roots of f . Note that $f' = -1$, so f has no multiple roots.

We claim that if $b \notin K$, then f is irreducible. If f is not irreducible, then $f = gh$ for some $g, h \in K[X]$ such that $0 < \deg g < p$. There exists a subset S of K_0 such that $g = \prod_{x \in S} (X - (b+x))$ and hence

$$|S|b + \sum_{x \in S} x = \sum_{x \in S} (b+x) \in K.$$

This implies that $|S|b \in K$ and hence, since $|S| \in K^\times$, it follows that $b \in K$.

Since $K(b)/K$ is a decomposition field of a separable polynomial, $K(b)/K$ is a Galois extension. Moreover, $|\text{Gal}(K(b)/K)| = [K(b) : K] = p$ and hence $\text{Gal}(K(b)/K)$ is cyclic.

We now prove 2). Let E/K be cyclic of degree p . Assume that $\text{Gal}(E/K) = \langle \sigma \rangle$. Since $\text{trace}_{E/K}(1) = p = 0$, Hilbert's theorem implies that there exists $b \in E$ such that $\sigma(b) = b+1$. In particular, $b \notin K$ and thus $E = K(b)$. Moreover, since

$$\sigma(b^p - b) = \sigma(b)^p - \sigma(b) = (b+1)^p - (b+1) = b^p - b,$$

it follows that $b^p - b \in K$. Thus $f(b, K) = X^p - X - (b^p - b) \in K[X]$. \square

§16. Symmetric polynomials

Let K be a field and $\{t_1, \dots, t_n\}$ be an algebraic independent set over K . Let $E = K(t_1, \dots, t_n)$ and $f = \prod_{i=1}^n (X - t_i) \in E[X]$. Then

$$f = X^n + \sum_{i=1}^n (-1)^i s_i X^{n-i},$$

where

$$\begin{aligned} s_1 &= t_1 + t_2 + \cdots + t_n, \\ s_2 &= \sum_{1 \leq i < j \leq n} t_i t_j, \\ &\vdots \\ s_n &= t_1 t_2 \cdots t_n. \end{aligned}$$

For example,

$$(X - t_1)(X - t_2)(X - t_3) = X^3 - (t_1 + t_2 + t_3)X^2 + (t_1 t_2 + t_2 t_3 + t_1 t_3)X - t_1 t_2 t_3.$$

The polynomials s_1, s_2, \dots, s_n are known as the **elementary symmetric polynomials** in the variables t_1, \dots, t_n . Note that $\deg s_i = i$.

Let $\sigma \in \mathbb{S}_n$ and

$$\alpha_\sigma: K[t_1, \dots, t_n] \rightarrow K[t_1, \dots, t_n], \quad t_i \mapsto t_{\sigma(i)} \quad \text{for all } i.$$

Then α_σ is a bijective homomorphism of K -algebras. In fact, $\alpha_\sigma^{-1} = \alpha_{\sigma^{-1}}$. Note that

$$\alpha_\sigma(h(t_1, \dots, t_n)) = h(t_{\sigma(1)}, \dots, t_{\sigma(n)}).$$

Since α_σ is injective, it induces an element $\widehat{\sigma} \in \text{Gal}(E/K)$ given by

$$\widehat{\sigma} \left(\frac{h}{g} \right) = \frac{\alpha_\sigma(h)}{\alpha_\sigma(g)}.$$

The map $\mathbb{S}_n \rightarrow \text{Gal}(E/K)$, $\sigma \mapsto \widehat{\sigma}$, is an injective group homomorphism. Thus $\{\widehat{\sigma} : \sigma \in \mathbb{S}_n\} \simeq \mathbb{S}_n$.

Definition 16.1. Let $g \in K[t_1, \dots, t_n]$. Then g is **symmetric** if $\widehat{\sigma}(g) = g$ for all $\sigma \in \mathbb{S}_n$.

We write P to denote the set of symmetric polynomials in $K[t_1, \dots, t_n]$. Clearly, P is a subalgebra of $K[t_1, \dots, t_n]$. The following statements hold:

- 1) $K \subseteq P$.
- 2) $\sum_{i=1}^n t_i^r \in P$ for all $r \geq 1$.

- 3) $s_i \in P$ for all i .
 4) $K(P) \subseteq {}^G E$, where $G = \{\widehat{\sigma} : \sigma \in \mathbb{S}_n\}$.

Let $F = K(s_1, s_2, \dots, s_n)$. Then E/F is a Galois extension, as it is a decomposition field of f .

Proposition 16.2. $[E : F] \leq n!$.

Proof. We proceed by induction on n . The case $n = 1$ is clear, as $E = F$. Assume that $n > 1$. Let u_1, \dots, u_{n-1} be the elementary symmetric polynomials in t_1, \dots, t_{n-1} . Then

$$s_i = u_i + t_n u_{i-1}$$

for all $i \in \{1, \dots, n\}$, where $u_0 = 1$ and $u_n = 0$. Note that $u_1 = s_1 - t_n$ and $u_i = s_i - t_n u_{i-1}$ for all i . Since $K(s_1, \dots, s_n, t_n) = K(u_1, \dots, u_{n-1}, t_n)$,

$$F(t_n) = K(u_1, \dots, u_{n-1}, t_n) = K(t_n)(u_1, \dots, u_{n-1})$$

and

$$[E : F] = [E : F(t_n)][F(t_n) : F] \leq n[E : F(t_n)].$$

Note that $E = K(t_1, \dots, t_n) = K(t_n)(t_1, \dots, t_{n-1})$. By the inductive hypothesis, $[E : F(t_n)] \leq (n-1)!$ and hence $[E : F] \leq n!$, as desired. \square

Theorem 16.3. ${}^G E = F$.

Proof. By Artin's theorem,

$$[{}^G E : F] = \frac{[E : F]}{[E : {}^G E]} \leq \frac{n!}{[E : {}^G E]} = 1$$

and hence ${}^G E = F$. \square

Exercise 16.4. Prove that $\text{Gal}(E/F) \simeq \mathbb{S}_n$.

Exercise 16.5. Prove that $\{s_1, \dots, s_n\}$ is algebraically independent over K .

Exercise 16.6. Prove that every symmetric polynomial in t_1, \dots, t_n can be written as a rational fraction in s_1, \dots, s_n .

§17. Solvable groups

Let G be a group. If $x, y \in G$ we define the **commutator** of x and y as

$$[x, y] = xyx^{-1}y^{-1}.$$

Note that $[x, y] = 1$ if and only if $xy = yx$. Moreover, $[x, y]^{-1} = [y, x]$. The **commutator subgroup** $[G, G]$ of G is defined as the subgroup of G generated by all commutators, i.e.

$$[G, G] = \langle [x, y] : x, y \in G \rangle.$$

This means that every element of $[G, G]$ is a finite product of commutators, so every element of $[G, G]$ is of the form $\prod_{i=1}^m [x_i, y_i]$. In general, the commutator subgroup is not equal to the set of commutators!

Exercise 17.1. Let G be a group. Prove the following facts:

- 1) G is abelian if and only if $[G, G] = \{1\}$.
- 2) $[G, G]$ is a normal subgroup of G .
- 3) $G/[G, G]$ is abelian.
- 4) If H is a subgroup of G and $[G, G] \subseteq H$, then H is normal in G .
- 5) If H is a normal subgroup of G , then G/H is abelian if and only if $[G, G] \subseteq H$.

Definition 17.2. Let G be a group. The **derived series** of G is defined as $G^{(0)} = G$ and $G^{(k+1)} = [G^{(k)}, G^{(k)}]$ for $k \geq 0$.

Exercise 17.3. Prove that $G^{(k)}$ is normal in G for all k .

Why derived series? We cannot explain this here, but let us use the following notation. We write $G' = [G, G]$, $G'' = [G', G']$... Note that

$$G \supseteq G' \supseteq G'' \supseteq \dots$$

Exercise 17.4. Let $n \geq 3$. Prove that $[\mathbb{S}_n, \mathbb{S}_n] = \mathbb{A}_n$.

Example 17.5. Let $K = \{\text{id}, (12)(34), (13)(24), (14)(23)\}$. Then K is a normal subgroup of \mathbb{A}_4 . One proves that $[\mathbb{A}_4, \mathbb{A}_4] = K$.

Example 17.6. Let $n \geq 5$. Since \mathbb{A}_n is a non-abelian simple group, it follows that $[\mathbb{A}_n, \mathbb{A}_n] = \mathbb{A}_n$.

Let us show that \mathbb{A}_5 is a non-abelian simple group. Hence it is not solvable:

```
julia> A5 = alternating_group(5)
Alt( [ 1 .. 5 ] )
julia> is_abelian(A5)
false
julia> is_simple(A5)
true
julia> is_solvable(A5)
false
```

Definition 17.7. A group G is **solvable** if and only if $G^{(m)} = \{1\}$ for some m .

Clearly, every abelian group is solvable.

Exercise 17.8. Prove that \mathbb{S}_n is solvable if and only if $n \leq 4$.

Let us compute (with the computer software Oscar) the derived series of the symmetric group \mathbb{S}_4 . The calculation shows that \mathbb{S}_4 is solvable:

```
julia> G = symmetric_group(4);
julia> derived_series(G)
4-element Vector{PermGroup}:
 Sym( [ 1 .. 4 ] )
 Alt( [ 1 .. 4 ] )
 Group([ (1,4) (2,3), (1,2) (3,4) ])
 Group(())
julia> [order(x) for x in derived_series(G)]
4-element Vector{fmpz}:
 24
 12
  4
  1
julia> is_solvable(G)
true
```

Proposition 17.9. *Let G be a group and H be a subgroup of G . The following statements hold:*

- 1) *If G is solvable, then H is solvable.*
- 2) *If H is normal in G and G is solvable, then G/H is solvable.*
- 3) *If H is normal in G and H and G/H are solvable, then G is solvable.*

Proof. The first statement follows from the fact that $H^{(i)} \subseteq G^{(i)}$ holds for all i .

Assume now that H is normal in G . Let $Q = G/H$ and $\pi: G \rightarrow Q$ be the canonical map. By induction one proves that $\pi(G^{(i)}) = Q^{(i)}$ for all $i \geq 0$. The case where $i = 0$ is trivial, as π is surjective. If the result holds for some $i \geq 0$, then

$$\pi(G^{(i+1)}) = \pi([G^{(i)}, G^{(i)}]) = [\pi(G^{(i)}), \pi(G^{(i)})] = [Q^{(i)}, Q^{(i)}] = Q^{(i+1)}.$$

We now prove 2). Since G is solvable, $G^{(n)} = \{1\}$ for some n . Thus Q is solvable, as $Q^n = \pi(G^{(n)}) = \pi(\{1\}) = \{1\}$.

We finally prove 3). Since Q is solvable, $Q^{(n)} = \{1\}$ for some n . Since $\pi(G^{(n)}) = Q^{(n)} = \{1\}$, it follows that $G^{(n)} \subseteq H$. Since H is solvable,

$$G^{(n+m)} \subseteq (G^{(n)})^{(m)} \subseteq H^{(m)} = \{1\}$$

for some m . Thus G is solvable. □

An application:

Proposition 17.10. *Let G be a finite p -group. Then G is solvable.*

Proof. Assume the result is not true. Let G be a finite p -group of minimal order that is not solvable. Since G is a p -group, $Z(G) \neq \{1\}$. Since $|G|$ is minimal, $G/Z(G)$ is

a solvable p -group. Since $Z(G)$ is abelian, $Z(G)$ is solvable. Now G is solvable by Proposition 17.9. \square

We finish this discussion with two important theorems (without proof) about finite solvable groups.

Theorem 17.11 (Burnside). *Let p and q be prime numbers. If G is a group of order $p^a q^b$, then G is solvable.*

The proof appears in courses on representation theory of finite groups.

Theorem 17.12 (Feit–Thompson). *Every finite group of odd order is solvable.*

The proof of the theorem is extremely hard. It occupies a full volume of *Pacific Journal of Mathematics*, see [1].

Lecture 12

§18. Radical extensions

Definition 18.1. An extension E/K is **radical** if $E = K(x_1, \dots, x_m)$ such that for each $i \in \{1, \dots, m\}$ there exists $a_i \in \mathbb{Z}$ such that $x_i^{a_i} \in K(x_1, \dots, x_{i-1})$.

Note that radical extensions are finite.

Example 18.2. Let E be a decomposition field of $X^4 - 2$ over \mathbb{Q} . Then E/\mathbb{Q} is radical, as $E = \mathbb{Q}(\sqrt[4]{2}, i)$.

Example 18.3. Let $\alpha, \beta \in \mathbb{C}$ be such that $\alpha^2 = 2$ and $\beta^5 = 1 + \alpha$. The number $\sqrt[5]{1 + \sqrt{2}}$ belongs to the radical extension $\mathbb{Q}(\alpha, \beta)/\mathbb{Q}$.

Theorem 18.4. Let K be of characteristic zero and R/K be a radical extension. If E/K is a subextension of R/K , then $\text{Gal}(E/K)$ is solvable.

Proof. Without loss of generality, we may assume that E/K is a Galois extension. In fact, let $G = \text{Gal}(E/K)$ and $K_0 = {}^G E$. Then E/K_0 is a Galois extension and $\text{Gal}(E/K_0) = G$ by Artin's theorem. Thus, replacing K by K_0 if needed, we may assume that E/K is Galois.

Let L be the normal closure of R in some algebraic closure C that contains R . Note that if $R = K(x_1, \dots, x_m)$, then

$$L = K(\{\sigma_i(x_j) : 1 \leq i \leq s, 1 \leq j \leq m\}),$$

where $\text{Hom}(R/K, C/K) = \{\sigma_1, \dots, \sigma_s\}$.

Claim. L/K is radical.

Since $x_j^{a_j} \in K(x_1, \dots, x_{j-1})$ for some integer a_j ,

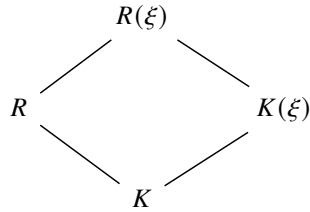
$$\sigma_i(x_j)^{a_j} = \sigma_i(x_j^{a_j}) \in \sigma_i(K(x_1, \dots, x_{j-1})) = K(\sigma_i(x_1), \dots, \sigma_i(x_{j-1}))$$

Thus L/K is radical and Galois.

We may assume then that E/K and R/K are both Galois.

Since $\text{Gal}(E/K) \simeq \text{Gal}(R/K)/\text{Gal}(R/E)$, we only need to prove that $\text{Gal}(R/K)$ is solvable.

Let ξ be a primitive n -th root of one (in some algebraic closure of K that contains R). Consider the diagram



Then

- 1) $K(\xi)/K$ and $R(\xi)/R$ are abelian.
- 2) $R(\xi)/K$ is Galois.
- 3) $\text{Gal}(R/K) \simeq \text{Gal}(R(\xi)/K)/\text{Gal}(R(\xi)/R)$.
- 4) $\text{Gal}(K(\xi)/K) \simeq \text{Gal}(R(\xi)/K)/\text{Gal}(R(\xi)/K(\xi))$.

The third item implies that we need to show that $\text{Gal}(R(\xi)/K)$ is solvable. By the fourth item it suffices to show that $\text{Gal}(R(\xi)/K(\xi))$ is solvable.

Since $R = K(x_1, \dots, x_m)$,

$$R(\xi) = K(x_1, \dots, x_m, \xi) = K(\xi)(x_1, \dots, x_m)$$

and hence $R(\xi)/K(\xi)$ is radical. This means that without loss of generality we may assume that K contains primitive n -roots of one. For example, if $R = K(x_1, \dots, x_m)$ and $x_i^{a_i} \in K(x_1, \dots, x_{i-1})$, then we may assume that K contains a primitive a_i -root of one. We proceed by induction on m . The case $m = 0$ is trivial. Assume that the claim holds for some $m \geq 0$. Let $L = K(x_1)$. Then L/K is a decomposition field of $X^{a_1} - x_1^{a_1}$, and hence L/K is a cyclic extension. Thus $\text{Gal}(L/K)$ is cyclic (and hence, in particular, solvable). Let H be the subgroup that corresponds to L , that is $H = \text{Gal}(R/L)$ (here we use Galois' correspondence). Then H is normal in $\text{Gal}(R/K)$. Since $R = K(x_1, \dots, x_m) = L(x_2, \dots, x_m)$, R/L is radical and Galois. By the inductive hypothesis, $\text{Gal}(R/L)$ is solvable. Since

$$\text{Gal}(L/K) \simeq \text{Gal}(R/K)/\text{Gal}(R/L),$$

it follows that $\text{Gal}(R/K)$ is solvable. □

Definition 18.5. Let $f \in K[X]$ and E be a decomposition field of f over K . We say that f is **solvable by radicals** if there is a radical extension R/K such that $E \subseteq R$.

The general polynomial of degree two is solvable by radicals, as its Galois group is solvable (in fact, it is isomorphic to \mathbb{S}_2).

Exercise 18.6. Prove that $f = X^2 - s_1X + s_2 \in \mathbb{Q}[X]$ is solvable by radicals.

Theorem 18.4 translates into the following result:

Exercise 18.7. If $f \in K[X]$ is solvable by radicals, then $\text{Gal}(f, K)$ is solvable.

As a consequence, the general polynomial of degree $n \geq 5$ is not solvable by radicals, as its Galois group is isomorphic to \mathbb{S}_5 .

Example 18.8. Let p be a prime number and $f = X^5 - 2pX + p \in \mathbb{Q}[X]$. We claim that f is not solvable by radicals.

By Gauss' theorem one proves that f has no rational roots.

Since $f(-1)f(1) < 0$ and $\deg f$ is odd, one proves that f has at least three real roots. Moreover, f has exactly three real roots, as $f' = 5X^4 - 2p$. Let $x_1, x_2 \in \mathbb{C} \setminus \mathbb{R}$ and $x_3, x_4, x_5 \in \mathbb{R}$ be the roots of f .

By Eisenstein's theorem, f is irreducible.

Let E/\mathbb{Q} be a decomposition field of f . Then $\text{Gal}(f, \mathbb{Q}) = \text{Gal}(E/\mathbb{Q})$ is isomorphic to a subgroup G of \mathbb{S}_5 . Since f is irreducible, 5 divides $[E : \mathbb{Q}] = |G|$. In particular, by Cauchy's theorem, G contains an element σ of order five. This element is a 5-cycle, so without loss of generality we may assume that $\sigma = (x_1x_2x_3x_4x_5)$. Note that $(x_1x_2) \in G$. Thus $G \simeq \mathbb{S}_5$ and hence G is not solvable.

Exercise 18.9. Let $f = X^6 + 2X^5 - 5X^4 + 9X^3 - 5X^2 + 2X + 1 \in \mathbb{Q}[X]$ is solvable by radicals.

Some solutions

6.1 Let $\{v_i : i \in I\}$ be a basis of V over K . For each $i \in I$ let $f_i : V \rightarrow F$, $f_i(v_j) = \delta_{ij}$. Then $\{f_i : i \in I\}$ is linearly independent over F . In fact, let $\sum a_i f_i = 0$, where each $a_i \in F$. Then $a_i = 0$ for almost all i . If $j \in I$, then

$$0 = \left(\sum a_i f_i \right) (v_j) = \sum a_i f_i(v_j) = a_j.$$

Now assume that $\dim_K V = n$. Let $\{v_1, \dots, v_n\}$ be a basis of V over K . We claim that $\{f_1, \dots, f_n\}$ is a basis of $\text{Hom}_K(V, F)$ over F . If $g \in \text{Hom}_K(V, F)$, then $g = \sum g(v_i) f_i$. If $1 \leq k \leq n$, then

$$\left(\sum g(v_i) f_i \right) (v_k) = \sum g(v_i) f_i(v_k) = g(v_k).$$

6.4 We need to find a bijective map

$$\text{Hom}(E/K, C/K) \rightarrow \text{Hom}(E/K, C_1/K).$$

If $\sigma \in \text{Hom}(E/K, C/K)$, then $\theta^{-1}\sigma \in \text{Hom}(E/K, C_1/K)$. If $\varphi \in \text{Hom}(E/K, C_1/K)$, then $\theta\varphi \in \text{Hom}(E/K, C/K)$. The maps $\sigma \mapsto \theta^{-1}\sigma$ and $\varphi \mapsto \theta\varphi$ are inverse to each other.

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