Galois theory

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Contents

Introduction

The notes correspond to the bachelor course *Galois theory* of the Vrije Universiteit Brussel, Faculty of Sciences, Department of Mathematics and Data Sciences. The course is divided into twelve two-hour lectures.

The material is somewhat standard. Basic texts on fields and Galois theory are for example [**3**] and [**4**].

As usual, we also mention a set of great expository papers by Keith Conrad, the notes are extremely well-written and useful at every stage of a mathematical career.

Several chapters contain optional paragraphs that give examples of how to apply OSCAR Computer Algebra System to concrete problems in Galois theory.

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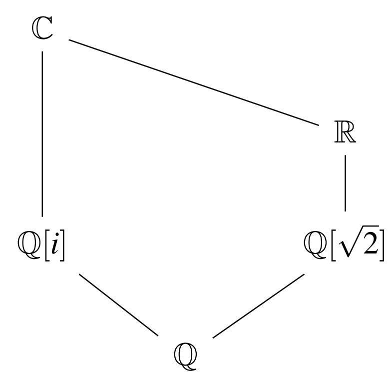
Lecture 1. 12/02/2024

§ 1.1. Fields. Recall that a **field** is a commutative ring such that $1 \neq 0$ and every non-zero element is invertible. Examples of (infinite) fields are $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$. If p is a prime number, then $\mathbb{Z}/p$ is a field.

Example 1.1. The abelian group $\mathbb{Z}/2 \times \mathbb{Z}/2$ is a field with multiplication

$$(a,b)(c,d) = (ac+bd, ad+bc+bd).$$

Example 1.2. $\mathbb{Q}[i] = \{a+bi : a,b \in \mathbb{Q}\}$ and $\mathbb{Q}[\sqrt{2}]$ are fields.



Exercise 1.3. Prove that $\mathbb{Q}[i]$ and $\mathbb{Q}[\sqrt{2}]$ are not isomorphic as fields.

If R is a ring, there exists a unique ring homomorphism $\mathbb{Z} \to R$, $m \mapsto m1$. The image

$$\{m1 : m \in \mathbb{Z}\}$$

of this homomorphism is a subring of R and it is known as the **ring of integers** of R. The kernel is a subgroup of $\mathbb{Z}$ generated by some $t \geq 0$. The integer t is the **characteristic** of the ring R.

Exercise 1.4. The characteristic of a field is either zero or a prime number.

Example 1.5. The characteristic of the field of Example 1.1 is two. Why?

Recall that a commutative ring R is an **integral domain** if $xy = 0 \implies x = 0$ or $y = 0$. Fields are integral domains.

Exercise 1.6. Let K be a field. Prove that the following statements are equivalent:

1) K is of characteristic zero.
2) The additive order of 1 is infinite.
3) The additive order of each $x \neq 0$ is infinite.
4) The ring of integers of K is isomorphic to $\mathbb{Z}$.

Exercise 1.7. Let K be a field. Prove that the following statements are equivalent:

1) K is of characteristic p.
2) The additive order of 1 is p.
3) The additive order of each $x \neq 0$ is p.
4) The ring of integers of K is isomorphic to $\mathbb{Z}/p$.

Definition 1.8. A **subfield** of a ring R is a subring of R that is also a field.

Note that if K is a subfield of E, then the characteristic of K coincides with the characteristic of E. Moreover, if $K \to L$ is a field homomorphism, then K and L have the same characteristic.

Exercise 1.9. Let K be a field of characteristic p. Prove that $K \to K$, $x \mapsto x^{p^n}$, is a field homomorphism for all $n \in \mathbb{Z}_{\geq 0}$.

Note that finite fields are of characteristic p.

Let K be a subfield of a field E. Then E is a K-vector space with the usual scalar multiplication $K \times E \to E$, $(\lambda, x) \mapsto \lambda x$.

Definition 1.10. A field K is **prime** if there are no proper subfields of K.

Examples of prime fields are $\mathbb{Q}$ and $\mathbb{Z}/p$ for a prime number p.

Proposition 1.11. *Let K be a field. The following statements hold:*

1) *K contains a unique prime field, it is known as the **prime subfield** of K.*
2) *The prime subfield of K is either isomorphic to $\mathbb{Q}$ if the characteristic of K is zero, or it is isomorphic to $\mathbb{Z}/p$ for some prime number p if the characteristic of K is p.*

Proof. To prove the first claim let L be the intersection of all the subfields of K. Then L is a subfield of K. If F is a subfield of L, then F is a subfield of K. Thus $L \subseteq F$ and hence $F = L$, which proves that L is prime. If L_1 is a subfield of K and L_1 is prime, then $L \subseteq L_1$ and hence $L = L_1$.

Let K_0 be the prime field of K. Suppose that K is of characteristic $p > 0$. Then the ring $K_{\mathbb{Z}}$ of integers of K is a field isomorphic to $\mathbb{Z}/p$ and hence $K_0 \simeq K_{\mathbb{Z}}$. Suppose now that the characteristic of K is zero. Let $E = \{m1/n1 : m, n \in \mathbb{Z}, n \neq 0\}$. We claim that $K_0 = E$. Since $K_{\mathbb{Z}} \subseteq K_0$, it follows that $E \subseteq K_0$. Hence $E = K_0$, as E is a subfield of K. □

Definition 1.12. Let E be a field and K be a subfield of E. Then E is a **field extension** of K. We will use the notation E/K.

If E is an extension of K, then E is a K-vector space.

Definition 1.13. The **degree** of an extension E of K is the integer $\dim_K E$. It will be denoted by $[E : K]$.

We say that E is a **finite extension** of K if $[E : K]$ is finite.

Example 1.14. Let K be a field. Then $[K : K] = 1$. Conversely, if E is an extension of K and $[E : K] = 1$, then $K = E$. If not, let $x \in E \setminus K$. We claim that $\{1, x\}$ is linearly independent over K. Indeed, if $a1 + bx = 0$ for some $a, b \in K$, then $bx = -a$. If $b \neq 0$, then $x = -a/b \in K$, a contradiction. If $b = 0$, then $a = 0$.

We know that $[\mathbb{C} : \mathbb{R}] = 2$.

Example 1.15. A basis of $\mathbb{Q}[\sqrt{2}]$ over $\mathbb{Q}$ is given by $\{1,\sqrt{2}\}$. Then $[\mathbb{Q}[\sqrt{2}]:\mathbb{Q}]=2$. The calculations can be easily done by computer:

```
julia> E, a = quadratic_field(2)
(Real quadratic field defined by x^2 - 2, sqrt(2))

julia> characteristic(E)
0

julia> K = prime_field(E)
Rational Field

julia> degree(E)
2

julia> basis(E)
2-element Vector{nf_elem}:
 1
 sqrt(2)

julia> one(K)==one(E)
true

julia> zero(K)==zero(E)
true
```

Example 1.16. Since $\mathbb{Q}$ is numerable and $\mathbb{R}$ is not, $[\mathbb{R}:\mathbb{Q}]>\aleph_0$. If $\{x_i : i\in\mathbb{Z}_{>0}\}$ is a numerable basis of $\mathbb{R}$ over $\mathbb{Q}$, for each n consider the $\mathbb{Q}$-vector space V_n generated by $\{x_1,\dots,x_n\}$. Then

$$\mathbb{R}=\bigcup_{n\geq 1} V_n,$$

is numerable, as each V_n is numerable, a contradiction.

If E is an extension of K and E is finite, then $[E:K]$ is finite.

Proposition 1.17. *Let K be a finite field. Then $|K|=p^m$ for some prime number p and some $m\geq 1$.*

Proof. We know the prime subfield K_0 of K is isomorphic to $\mathbb{Z}/p$. In particular, $|K_0|=p$. Since K is finite, $[K:K_0]=m$ for some m. If $\{x_1,\dots,x_m\}$ is a basis of K over K_0, then each element of K can be written uniquely as $\sum_{i=1}^m a_ix_i$ for some $a_1,\dots,a_m\in K_0$. Then there is a bijection between K and K_0^m and hence $|K|=|K_0^m|=p^m$. □

We now perform some basic calculations with a finite field of eight elements:

```
julia> E, x = FiniteField(2, 3, "x")
(Finite field of degree 3 over F_2, x)

julia> characteristic(E)
2

julia> prime_field(E)
Galois field with characteristic 2

julia> degree(E)
```

```
3

julia> size(E)
8

julia> [z for z in E]
8-element Vector{fq_nmod}:
 0
 1
 x
 x + 1
 x^2
 x^2 + 1
 x^2 + x
 x^2 + x + 1
```

Definition 1.18. Let E be an extension of K. A **subextension** F/K of E/K is a subfield F of E that contains K, that is $K \subseteq F \subseteq E$.

Definition 1.19. Let E and E_1 be extensions over K. An **extension homomorphism**

$$E/K \to E_1/K$$

is a field homomorphism $\sigma\colon E \to E_1$ such that $\sigma(x) = x$ for all $x \in K$.

To describe the homomorphism $\sigma\colon E/K \to E_1/K$ of the extensions over K one typically writes the commutative diagram

$$\begin{tikzcd} K \arrow[r, equal] \arrow[d, hook] & K \arrow[d, hook] \\ E \arrow[r, "\sigma"] & E_1 \end{tikzcd}$$

We write $\mathrm{Hom}(E/K, E_1/K)$ to denote the set of homomorphism $E/K \to E_1/K$ of extensions of K. Note that if $\sigma \in \mathrm{Hom}(E/K, E_1/K)$, then σ is a K-linear map, as

$$\sigma(\lambda x) = \sigma(\lambda)\sigma(x) = \lambda\sigma(x)$$

for all $\lambda \in K$ and $x \in E$.

Example 1.20. The conjugation map $\mathbb{C} \to \mathbb{C}$, $z \mapsto \bar{z}$, is an endomorphism of $\mathbb{C}$ as an extension over $\mathbb{R}$. Let $\varphi \in \mathrm{Hom}(\mathbb{C}/\mathbb{R}, \mathbb{C}/\mathbb{R})$. Then

$$\varphi(x+iy) = \varphi(x) + \varphi(i)\varphi(y) = x + \varphi(i)y$$

for all $x, y \in \mathbb{R}$. Since $\varphi(i)^2 = \varphi(i^2) = \varphi(-1) = -1$, it follows that $\varphi(i) \in \{-i, i\}$. Thus either $\varphi(x+iy) = x+iy$ or $\varphi(x+iy) = x-iy$.

Exercise 1.21. Let K be a field, K_0 be its prime field and $\sigma\colon K \to K$ be a field homomorphism. Prove that $\sigma \in \mathrm{Hom}(K/K_0, K/K_0)$.

If E/K is an extension, then

$$\begin{aligned} \mathrm{Aut}(E/K) &= \{\sigma\colon E/K \to E/K \text{ is a bijective extension homomorphism}\} \\ &= \{\sigma\colon E \to E : \sigma \text{ is a bijective field homomorphism with } \sigma|_K = \mathrm{id}\} \end{aligned}$$

is a group with composition.

Definition 1.22. Let E/K be an extension. The **Galois group** of E/K is the group $\operatorname{Aut}(E/K)$ and it will be denoted by $\operatorname{Gal}(E/K)$.

A typical example: $\operatorname{Gal}(\mathbb{C}/\mathbb{R}) \simeq \mathbb{Z}/2$.
As an example, we show with the computer that $\operatorname{Gal}(\mathbb{Q}[\sqrt{2}]/\mathbb{Q}) \simeq \mathbb{Z}/2$:

```
julia> E, x = quadratic_field(2)
(Real quadratic field defined by x^2 - 2, sqrt(2))
julia> characteristic(E)
0
julia> G, C = galois_group(E);
julia> describe(G)
"C2"
julia> order(G)
2
```

Example 1.23. Let $\theta = \sqrt[3]{2}$ and let $E = \{a + b\theta + c\theta^2 : a, b, c \in \mathbb{Q}\}$. Note that

$$a + b\theta + c\theta^2 = 0 \iff a = b = c = 0.$$

Then E is an extension of $\mathbb{Q}$ such that $[E : \mathbb{Q}] = 3$. We claim that $\operatorname{Gal}(E/\mathbb{Q})$ is trivial. If $\sigma \in \operatorname{Gal}(E/\mathbb{Q})$ and $z = a + b\theta + c\theta^2$, then $\sigma(z) = a + b\sigma(\theta) + c\sigma^2(\theta)$. Since

$$\sigma(\theta)^3 = \sigma(\theta^3) = \sigma(2) = 2,$$

it follows that $\sigma(\theta) = \theta$ and therefore $\sigma = \operatorname{id}$.

Exercise 1.24. Prove that the polynomial $X^3 - 2$ is irreducible in $\mathbb{Q}[X]$.

The previous exercise can easily be solved using computers:

```
julia> R, x = PolynomialRing(QQ, "x");
julia> is_irreducible(x^3-2)
true
```

The following exercise is known as the *Eisenstein's irreducibility criterion*:

Exercise 1.25. Let A be a unique factorization domain and K be its fraction field. Let $f = \sum_{i=0}^{n} a_i X^i \in A[X]$ be a polynomial of degree $n > 0$. Assume that there exists a prime element $p \in A$ such that $p \mid a_i$ for all $i \in \{0, 1, \dots, n-1\}$, $p \nmid a_n$ and $p^2 \nmid a_0$. Then f is irreducible in $K[X]$.

Exercise 1.26. Prove that the polynomials

$$f = X^{10} + 60X^7 + 82X^6 - 36X^3 + 2,$$
$$g = 3X^{10} + 15X^2 - 45,$$

are irreducible in $\mathbb{Z}[X]$.

Exercise 1.27. Is the polynomial $f = 3(X^{10} + 5X^2 - 15)$ irreducible in $\mathbb{Z}[X]$?

If E/K is an extension and S is a subset of E, then there exists a unique smallest subextension F/K of E/K such that $S \subseteq F$. In fact,

$$F = \bigcap \{T : T \text{ is a subfield of } E \text{ that contains } K \cup S\}$$

If L/K is a subextension of E/K such that $S \subseteq L$, then $F \subseteq L$ by definition. The extension F is known as the **subextension generated by** S and it will be denoted by $K(S)$. If $S = \{x_1, \dots, x_n\}$ is finite, then $K(S) = K(x_1, \dots, x_n)$ is said to be of **finite type**.

Example 1.28. If $\{e_1, \dots, e_n\}$ is a basis of E over K, then $E = K(e_1, \dots, e_n)$.

Example 1.29. The field $\mathbb{Q}(\sqrt{2})$ is precisely the extension of $\mathbb{R}/\mathbb{Q}$ generated by $\sqrt{2}$.

Let E/K be an extension and S and T be subsets of E. Then

$$K(S \cup T) = K(S)(T) = K(T)(S).$$

If, moreover, $S \subseteq T$, then $K(S) \subseteq K(T)$.

§ 1.2. Algebraic extensions.

Definition 1.30. Let E/K be an extension. An element $x \in E$ is **algebraic** over K if there exists a non-zero polynomial $f(X) \in K[X]$ such that $f(x) = 0$. If x is not algebraic over K, then it is called **transcendental** over K.

Definition 1.31. An extension E/K is **algebraic** if every $x \in E$ is algebraic over K.

If K is a field, every $x \in K$ is algebraic over K, as x is a root of $X - x \in K[X]$. In particular, K/K is an algebraic extension.

Example 1.32. $\mathbb{C}/\mathbb{R}$ is an algebraic extension. If $z \in \mathbb{C} \setminus \mathbb{R}$, then z is a root of the polynomial $X^2 - (z + \bar{z})X + |z|^2 \in \mathbb{R}[X]$.

If F/K is an extension $x \in E$ is algebraic over K for some field $E \supseteq F$, then x is algebraic over F.

Example 1.33. $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is algebraic, as the number $a + b\sqrt{2}$ is a root of the polynomial $X^2 - 2aX + (a^2 - 2b^2) \in \mathbb{Q}[X]$.

The extension $\mathbb{C}/\mathbb{Q}$ is not algebraic. For example, Hermite proved that e is transcendental over $\mathbb{Q}$; see [**4**, Theorem 24.4]. Lindemann's theorem states that π is not algebraic over $\mathbb{Q}$; see [**4**, Theorem 24.5].

Example 1.34. Let $a = \sqrt{2}$ and $b = \sqrt[3]{3}$. Both a and b are algebraic numbers over $\mathbb{Q}$. Let us show that $a + b$ is also algebraic. Let $f(X) = X^3 - 3 \in \mathbb{Q}[X]$. Then $f(b) = 0$. Note that the polynomial

$$g(X) = f(X - a) = X^3 - 3aX^2 + 3aX - a^3 - 3 \in \mathbb{Q}(a)[X]$$

is such that $g(a + b) = 0$. How can we find a polynomial with coefficients in $\mathbb{Q}$ that vanishes on $a + b$? We do the "conjugation" trick:

$$h(X) = f(X - a)f(X + a) = X^6 - 6X^4 - 6X^3 + 12X^2 - 36X + 1 \in \mathbb{Q}[X].$$

Note that $h(a + b) = 0$. How can you prove that ab is also algebraic over $\mathbb{Q}$?

Lecture 2. 19/02/2024

If E/K is an extension and $x \in E$ is algebraic over K, then the evaluation homomorphism $K[X] \to E$, $p \mapsto p(x)$, is not injective. In particular, its kernel is a non-zero ideal. Hence it is generated by a monic polynomial f.

DEFINITION 2.1. Let E/K be an extension and $x \in E$ be an algebraic element. The monic polynomial that generates the kernel of $K[X] \to E$, $f \mapsto f(x)$, is known as the **minimal polynomial** of x over K and it will be denoted by $f(x,K)$. The **degree** of x over K is then $\deg f(x,K)$.

Some basic properties of the minimal polynomial of an algebraic element:

PROPOSITION 2.2. *Let E/K be an extension and $x \in E$. Assume that x is algebraic over K.*

1) *If $g \in K[X] \setminus \{0\}$ is such that $g(x) = 0$, then $f(x,K)$ divides g and $\deg f(x,K) \leq \deg g$.*
2) *$f(x,K)$ is irreducible in $K[X]$.*
3) *If F/K is a subextension of E/K, then $f(x,F)$ divides $f(x,K)$.*

PROOF. Write $f = f(x,K)$ to denote the minimal polynomial of x. To prove 1) note that $g(x) = 0$ implies that g belongs to the kernel of the evaluation map, so g is a multiple of f. To prove 2) note that if $f = pq$ for some $p,q \in K[X]$ such that $0 < \deg p, \deg q < \deg f$, then $f(x) = 0$ implies that either $p(x) = 0$ or $q(x) = 0$, a contradiction. Finally, we prove 3). Since $f \in K[X] \subseteq F[X]$ and $f(x) = 0$, it follows from 1) that $f(x,F)$ divides f. □

Some easy examples: $f(i,\mathbb{R}) = X^2+1$, $f(i,\mathbb{C}) = X - i$ and $f(\sqrt[3]{2},\mathbb{Q}) = X^3 - 2$:

```
julia> E, x = radical_extension(3, QQ(2), "x");

julia> minpoly(x)
x^3 - 2

julia> F, y = quadratic_field(-1);

julia> minpoly(y)
x^2 + 1
```

EXAMPLE 2.3. Let us compute $f(\sqrt{2}+\sqrt{3},\mathbb{Q})$. Let $\alpha = \sqrt{2}+\sqrt{3}$. Then

$$\alpha - \sqrt{2} = \sqrt{3} \implies (\alpha-\sqrt{2})^2 = 3 \implies \alpha^2 - 2\sqrt{2}\alpha + 2 = 3$$
$$\implies \alpha^2 - 1 = 2\sqrt{2}\alpha \implies (\alpha^2-1)^2 = 8\alpha^2 \implies \alpha^4 - 10\alpha^2 + 1 = 0.$$

Thus α is a root of $g = X^4 - 10X^2 + 1$. To prove that $g = f(\alpha,\mathbb{Q})$ it is enough to prove that g is irreducible in $\mathbb{Q}[X]$. First note that the roots of g are $\sqrt{2}+\sqrt{3}$, $\sqrt{2}-\sqrt{3}$, $-\sqrt{2}+\sqrt{3}$ and $-\sqrt{2}-\sqrt{3}$. This means that if g is not irreducible, then $g = hh_1$ for some polynomials $h, h_1 \in \mathbb{Q}[X]$ such that $\deg h = \deg h_1 = 2$. This is not possible, as $(\sqrt{2}+\sqrt{3}) + (\sqrt{2}-\sqrt{3}) = 2\sqrt{2} \notin \mathbb{Q}$, $(\sqrt{2}+\sqrt{3}) + (-\sqrt{2}+\sqrt{3}) = 2\sqrt{3} \notin \mathbb{Q}$ and $(\sqrt{2}+\sqrt{3})(-\sqrt{2}-\sqrt{3}) = -5-2\sqrt{6} \notin \mathbb{Q}$.

PROPOSITION 2.4. *Let F/K be a subextension and E/K. Then*

$$[E:K] = [E:F][F:K].$$

PROOF. Let $\{e_i : i \in I\}$ be a basis of E over F and $\{f_j : j \in J\}$ be a basis of F over K. If $x \in E$, then $x = \sum_i \lambda_i e_i$ (finite sum) for some $\lambda_i \in F$. For each i, $\lambda_i = \sum_j a_{ij} f_j$ (finite sum) for some $a_{ij} \in K$. Then $x = \sum_i \sum_j a_{ij}(f_j e_i)$. This means that $\{f_j e_i : i \in I, j \in J\}$ generates E as a K-vector space. Let us prove that $\{f_je_i : i \in I, j \in J\}$ is linearly independent. If $\sum_i \sum_j a_{ij}(f_je_i) = 0$ (finite sum) for some $a_{ij} \in K$, then

$$0 = \sum_i \left(\sum_j a_{ij} f_j \right) e_i \implies \sum_j a_{ij} f_j = 0 \text{ for all } i \in I$$
$$\implies a_{ij} = 0 \text{ for all } i \in I \text{ and } j \in J. \qquad \square$$

We state a lemma:

Lemma 2.5. *If A is a finite-dimensional commutative algebra over K and A is an integral domain, then A is a field.*

Proof. Let $a \in A \setminus \{0\}$. We need to prove that there exists $b \in A$ such that $ab = 1$. Let $\theta \colon A \to A$, $x \mapsto ax$. Note that θ is K-linear transformation, as

$$\theta(x+y) = a(x+y) = ax + ay = \theta(x) + \theta(y), \quad \theta(\lambda x) = a(\lambda x) = \lambda(ax) = \lambda\theta(x),$$

for all $x, y \in A$ and $\lambda \in K$. It is injective since A is an integral domain. Since $\dim_K A < \infty$, it follows that θ is an isomorphism. In particular, $\theta(A) = A$, which implies that there exists $b \in A$ such that $1 = ab$. $\square$

Let E/K be an extension and $x \in E$. Then

$$K[x] = \{f(x) : f \in K[X]\}$$

is a subring of E that contains K. Note that $K[x]$ is a K-vector space.

More generally, if $x_1, \dots, x_n \in E$, then

$$K[x_1, \dots, x_n] = \{f(x_1, \dots, x_n) : f \in K[X_1, \dots, X_n]\}$$

is a subring of E. Note that $K[x_1, \dots, x_n]$ is a K-vector space. Clearly, $K[x_1, \dots, x_n]$ is a domain and

$$K(x_1, \dots, x_n) = \left\{ \frac{f(x_1, \dots, x_n)}{g(x_1, \dots, x_n)} : f, g \in K[X_1, \dots, X_n] \text{ with } g(x_1, \dots, x_n) \neq 0 \right\}$$

is the extension of K generated by $x_1, \dots, x_n$. Note that

$$K(x_1, \dots, x_n) = (K(x_1, \dots, x_{n-1}))(x_n).$$

The previous construction can be generalized. Let I be a non-empty set. For each $i \in I$, let X_i be a variable. Consider the polynomial ring $K[\{X_i : i \in I\}]$ and let $S = \{x_i : i \in I\}$ be a subset of E. There exists a unique algebra homomorphism

$$K[\{X_i : i \in I\}] \to E$$

such that $X_i \mapsto x_i$ for all $i \in I$. The image is denoted by $K[S]$. In particular, an element $z \in K[S]$ is of the form

$$z = h(x_1, \dots, x_n)$$

for a polynomial $h \in K[X_1, \dots, X_n]$ in finitely many variables $X_1, \dots, X_n$ and $x_1, \dots, x_n \in S$.

Exercise 2.6. Prove that $\mathbb{Q}[\sqrt{2}] = \mathbb{Q}(\sqrt{2})$.

The exercise is not an accident.

Theorem 2.7. *Let E/K be an extension and $x \in E \setminus K$. The following statements are equivalent:*

1) *x is algebraic over K.*

2) $\dim_K K[x] < \infty$.
3) *$K[x]$ is a field.*
4) $K[x] = K(x)$.

Proof. We first prove 1) $\implies$ 2). Let $z \in K[x]$, say $z = h(x)$ for some $h \in K[X]$. There exists $g \in K[X]$ such that $g \neq 0$ and $g(x) = 0$. Divide h by g to obtain polynomials $q, r \in K[X]$ such that $h = gq + r$, where $r = 0$ or $\deg r < \deg g$. This implies that

$$z = h(x) = g(x)q(x) + r(x) = r(x).$$

If $\deg g = m$, then $r = \sum_{i=0}^{m-1} a_i X^i$ for some $a_0, \dots, a_{m-1} \in K$. Thus

$$z = \sum_{i=0}^{m-1} a_i x^i$$

and hence $K[x] \subseteq \langle 1, x, \dots, x^{m-1} \rangle$.

The previous lemma proves that 2) $\implies$ 3).

It is trivial that 3) $\implies$ 4).

It remains to prove that 4) $\implies$ 1). Since $x \neq 0$, $1/x \in K(x) = K[x]$. There exists $a_0, \dots, a_n \in K$ such that $1/x = a_0 + a_1 x + \cdots + a_n x^n$. Thus

$$a_n x^{n+1} + \cdots + a_1 x^2 + a_0 x - 1 = 0,$$

and hence x is a root of $a_n X^{n+1} + \cdots + a_0 X - 1 \in K[X] \setminus \{0\}$. □

Note that if x is algebraic over K, then $K[x] \simeq K[X]/(f(x,K))$.

Exercise 2.8. Let E/K be an extension and $x \in E$ be an algebraic element over K. Prove that the degree of x over K is equal to $[K(x) : K]$.

Corollary 2.9. *If E/K is finite, then E/K is algebraic.*

Proof. Let $n = [E : K]$ and $x \in E \setminus K$. The set $\{1, x, \dots, x^n\}$ has $n+1$ elements, so it is linearly dependent. There exist $a_0, \dots, a_n \in K$, not all zero, such that

$$a_0 + a_1 x + \cdots + a_n x^n = 0.$$

Thus x is a root of the non-zero polynomial $a_0 + a_1 X + \cdots + a_n X^n \in K[X]$. □

In Example 1.34 we proved that $\sqrt{2} + \sqrt[3]{3}$ and $\sqrt{2}\sqrt[3]{3}$ are algebraic over $\mathbb{Q}$. This can be easily proved now with Corollary 2.9.

Exercise 2.10. Let E/K be an extension and a and b be algebraic over K. Prove that $a + b$ and ab are algebraic over K.

We note that the converse of Corollary 2.9 result does not hold.

Corollary 2.11. *If E/K is an extension and $x_1, \dots, x_n \in E$ are algebraic over K, then $K(x_1, \dots, x_n)/K$ is finite and $K(x_1, \dots, x_n) = K[x_1, \dots, x_n]$.*

PROOF. We proceed by induction on n. The case $n=1$ follows immediately from the theorem. So assume the result holds for some $n \geq 1$. Since the extensions $K(x_1,\dots,x_n)/K(x_1,\dots,x_{n-1})$ and $K(x_1,\dots,x_{n-1})/K$ are both finite, it follows that $K(x_1,\dots,x_n)/K$ is finite. Moreover,

$$\begin{aligned} K(x_1,\dots,x_n) &= K(x_1,\dots,x_{n-1})(x_n) \\ &= K(x_1,\dots,x_{n-1})[x_n] = K[x_1,\dots,x_{n-1}][x_n] = K[x_1,\dots,x_n]. \end{aligned}$$

□

COROLLARY 2.12. *Let $E=K(S)$ for some set S. Then E/K is algebraic if and only if x is algebraic over K for all $x \in S$.*

PROOF. Let us prove the non-trivial implication. Let $z \in K(S)$. In particular, there exists a finite subset $T \subseteq S$ such that $z \in K(T)$. The previous result implies that $K(T)/K$ is algebraic, and hence z is algebraic. □

If E/K is an extension, let

$$\overline{K}_E = \{x \in E : x \text{ is algebraic over } K\}.$$

COROLLARY 2.13. *If E/K is an extension, then $\overline{K}_E$ is a subfield of E that contains K. Moreover, $K(\overline{K}_E)=\overline{K}_E$ and $K(\overline{K}_E)/K$ is algebraic.*

PROOF. By definition, $K(\overline{K}_E)/K$ is algebraic. Thus $K(\overline{K}_E) \subseteq \overline{K}_E$. From this, it follows that $K(\overline{K}_E)=\overline{K}_E$. □

The following exercise is now almost trivial:

EXERCISE 2.14. Let E/K be an extension of finite type; this means that $E=K(S)$ for some finite set S. Prove that E/K is algebraic if and only if E/K is finite.

Let $\overline{\mathbb{Q}} = \{\alpha \in \mathbb{C} : \alpha \text{ is algebraic over } \mathbb{Q}\}$. Then $\overline{\mathbb{Q}}$ is the field of algebraic numbers. Can you compute $[\overline{\mathbb{Q}}:\mathbb{Q}]$?

EXERCISE 2.15. Prove that $[\mathbb{Q}[\sqrt[3]{2}]:\mathbb{Q}]=3$.

For the previous exercise, you may use Eisenstein's criterion.

EXERCISE 2.16. Let $E=\mathbb{Q}[i,\sqrt{2}]=\mathbb{Q}[\sqrt{2}][i]$. Prove that $[E:\mathbb{Q}]=4$.

EXERCISE 2.17. Let $E=\mathbb{Q}[\sqrt{2},\sqrt[3]{5}]$.

1) Compute $[E:\mathbb{Q}]$.
2) Prove that $E=\mathbb{Q}[\sqrt{2}+\sqrt[3]{5}]$.
3) Find the minimal polynomial of $\sqrt{2}+\sqrt[3]{5}$ over $\mathbb{Q}$.

EXERCISE 2.18. Find the minimal polynomials of $\sqrt[4]{3}i$ over $\mathbb{Q}[i]$ and over $\mathbb{Q}[\sqrt{3}]$.

EXERCISE 2.19. Find the minimal polynomial of $\sqrt{2}+\sqrt[3]{5}i$ over $\mathbb{Q}[i]$.

Algebraic field extensions form a nice class of extensions. The same happens with finite field extensions.

Proposition 2.20. *Let F/K be a subextension of E/K. Then E/K is algebraic if and only if E/F and F/K are algebraic.*

Proof. If E/K is algebraic, then E/F and F/K are both algebraic, as $K \subseteq F \subseteq E$. Let us assume that E/F and F/K are both algebraic. Let $x \in E$ and let L be the subextension over K generated by the coefficients of $f(x,F)$, the minimal polynomial of x over F. Then L/K is finite, since it is generated by finitely many algebraic elements. Moreover, x is algebraic over L. Since

$$[L(x):K] = [L(x):L][L:K] < \infty,$$

$L(x)/K$ is algebraic. In particular, x is algebraic over K. □

Exercise 2.21. Let F/K be a subextension of E/K. Prove that E/K is finite if and only if E/F and F/K are finite.

Let K be a field and $K \subseteq F \subseteq L$ and $K \subseteq F \subseteq L$ be fields. The **composite** of E and F is defined as

$$EF = K(E \cup F) = F(E) = E(F)$$

and it is equal to the smallest field that contains E and F. Here is the picture:

$$\begin{tikzcd}[column sep=small]
 & L \arrow[d, dash] & \\
 & EF \arrow[dl, dash] \arrow[dr, dash] & \\
E \arrow[dr, dash] & & F \arrow[dl, dash] \\
 & K &
\end{tikzcd}$$

Exercise 2.22. Let E and F be fields. Prove that

$$EF = \left\{ \sum_{i=1}^{m} e_i f_i : m \in \mathbb{Z}_{>0}, e_i \in E, f_i \in F \text{ for all } i \in \{1,\dots,m\} \right\}.$$

Exercise 2.23. If $E = \mathbb{Q}(\sqrt{2})$ and $F = \mathbb{Q}(\sqrt{3})$, then $EF = \mathbb{Q}(\sqrt{2},\sqrt{3})$. Compute $[\mathbb{Q}(\sqrt{2},\sqrt{3}):\mathbb{Q}]$ and $\mathbb{Q}(\sqrt{2}) \cap \mathbb{Q}(\sqrt{3})$.

Exercise 2.24. Let $\xi \in \mathbb{C}$ be a primitive cubic root of one. If $E = \mathbb{Q}(\sqrt[3]{2})$ and $F = \mathbb{Q}(\xi)$, then $EF = \mathbb{Q}(\sqrt[3]{2},\xi)$. Compute $[\mathbb{Q}(\sqrt[3]{2},\xi):\mathbb{Q}]$ and $\mathbb{Q}(\sqrt[3]{2}) \cap \mathbb{Q}(\xi)$.

Exercise 2.25. Let E/K and F/K be extensions, where both E and F are subfields of a field L. If F/K is algebraic, then EF/E is algebraic.

Exercise 2.26. Let E/K and F/K be extensions, where both E and F are subfields of a field L. If F/K is finite, then EF/E is finite.

The solution to the previous exercise shows, in particular, that $[EF : E] \leq [F : K]$.

Lecture 3. 26/02/2024

LEMMA 3.1. *Let $\sigma\colon K\to L$ be a field homomorphism. Then there exists an extension E/K and a field isomorphism $\varphi\colon E\to L$ such that $\varphi|_K=\sigma$.*

PROOF. Note that $\sigma\colon K\to\sigma(K)$ is bijective. Let A be a set in bijection with $L\setminus\sigma(K)$ and disjoint with K. Let $E=K\cup A$. If $\theta\colon A\to L\setminus\sigma(K)$ is bijective, then let

$$\varphi\colon E\to L,\quad \varphi(x)=\begin{cases}\sigma(x) & \text{if } x\in K,\\ \theta(x) & \text{if } x\in A.\end{cases}$$

Then φ is a bijective map such that $\varphi|_K=\sigma$. Transport the operations of L onto E, that is to define binary operations on E as follows:

$$(x,y)\mapsto x\oplus y=\varphi^{-1}(\varphi(x)+\varphi(y)),\qquad (x,y)\mapsto x\odot y=\varphi^{-1}(\varphi(x)\varphi(y)).$$

Then, for example,

$$x\oplus y=\varphi^{-1}(\varphi(x)+\varphi(y))=\varphi^{-1}(\sigma(x)+\sigma(y))=\varphi^{-1}(\sigma(x+y))=\varphi^{-1}(\varphi(x+y))=x+y$$

for all $x,y\in K$. □

If $\sigma\colon A\to B$ is a ring homomorphism, then σ induces a ring homomorphism $\overline{\sigma}\colon A[X]\to B[X]$, $\sum_i a_iX^i\mapsto\sum_i\sigma(a_i)X^i$.

THEOREM 3.2. *Let K be a field and $f\in K[X]$ be such that $\deg f>0$. Then there exists an extension E/K such that f admits a root in E.*

PROOF. We may assume that f is irreducible over K. Let $L=K[X]/(f)$ and $\pi\colon K[X]\to L$ be the canonical map. Then L is a field (the reader should explain why). Let $\sigma\colon K\to L$, $a\mapsto\pi(aX^0)$, and $g=\overline{\sigma}(f)\in L[X]$.

We claim that $\pi(X)$ is a root of g in L. Suppose that $f=\sum_i a_iX^i$. Then

$$\begin{aligned}g(\pi(X))&=\overline{\sigma}(f)(\pi(X))\\&=\sum_i\sigma(a_i)\pi(X)^i=\sum_i\pi(a_iX^0)\pi(X^i)=\pi(\sum a_iX^i)=\pi(f)=0.\end{aligned}$$

The previous lemma states that there exists an extension E/K and an isomorphism $\varphi\colon E\to L$ such that $\varphi|_K=\sigma$. Note that $\varphi(x)=0$ if and only if $x=0$. If $u=\pi(X)$, then $\varphi^{-1}(u)$ is a root of f in E, as

$$\begin{aligned}\varphi(f(\varphi^{-1}(u)))&=\varphi\left(\sum_i a_i\varphi^{-1}(u)^i\right)=\varphi\left(\sum_i a_i\varphi^{-1}(u^i)\right)\\&=\sum_i\varphi(a_i)u^i=\sum_i\sigma(a_i)u^i=g(u)=0.\end{aligned}$$ □

As a corollary, if K is a field and $f_1,\dots,f_n\in K[X]$ are polynomials of positive degree, then there exists an extension E/K such that each f_i admits a root in E. This is proved by induction on n.

DEFINITION 3.3. A field K is **algebraically closed** if each $f\in K[X]$ of positive degree admits a root in K.

The *fundamental theorem of algebra* states that $\mathbb{C}$ is algebraically closed. A typical proof uses complex analysis. Later we will give a proof of this result using Galois theory.

PROPOSITION 3.4. *The following statements are equivalent:*

1) *K is algebraically closed.*
2) *If $f \in K[X]$ is irreducible, then* $\deg f = 1$.
3) *If $f \in K[X]$ is non-zero, then f decomposes linearly in $K[X]$, that is*

$$f = a\prod_{i=1}^{n}(X-\alpha_i)^{m_i}$$

for some $a \in K$ and $\alpha_1,\dots,\alpha_n \in K$.
4) *If E/K is algebraic, then $E = K$.*

Proof. 1) $\implies$ 2 $\implies$ 3) are exercises.
Let us prove that 3) $\implies$ 4). Let $x \in E$. Decompose $f(x,K)$ linearly in $K[X]$ as

$$f(x,K) = a\prod_{i=1}^{n}(X-\alpha_i)^{m_i}$$

and evaluate on x to obtain that $x = \alpha_j$ for some j.

To prove that 4) $\implies$ 1) let $f \in K[X]$ be such that $\deg f > 0$. There exists an extension E/K such that f has a root x in E. The extension $K(x)/K$ is algebraic and hence $K(x) = K$, so $x \in K$. □

§ 3.1. Artin's theorem.

Definition 3.5. An **algebraic closure** of a field K is an algebraic extension C/K such that C is algebraically closed.

For example, $\mathbb{C}/\mathbb{R}$ is an algebraic closure but $\mathbb{C}/\mathbb{Q}$ is not.

Proposition 3.6. *Let C be algebraically closed and $\sigma\colon K \to C$ be a field homomorphism. If E/K is algebraic, then there exists a field homomorphism $\varphi\colon E \to C$ such that $\varphi|_K = \sigma$.*

Proof. Suppose first that $E = K(x)$ and let $f = f(x,K)$. Let $\overline{\sigma}(f) \in C[X]$ and let $y \in C$ be a root of $\overline{\sigma}(f)$. If $z \in E$, then $z = g(x)$ for some $g \in K[X]$. Let $\varphi\colon E \to C$, $z \mapsto \overline{\sigma}(g)(y)$.

The map φ is well-defined. If $z = h(x)$ for some $h \in K[X]$, then

$$0 = g(x) - h(x) = (g-h)(x)$$

and thus f divides $g-h$. In particular, $\overline{\sigma}(f)$ divides $\overline{\sigma}(g-h) = \overline{\sigma}(g) - \overline{\sigma}(h)$ and hence

$$(\overline{\sigma}(g) - \overline{\sigma}(h))(y) = 0.$$

It is an exercise to show that the map φ is a ring homomorphism.
Let $a \in K$. It follows that $\varphi|_K = \sigma$, as

$$\varphi(a) = \overline{\sigma}(aX^0)(y) = \sigma(a)$$

Let us now prove the proposition in full generality. Let X be the set of pairs (F,τ), where F is a subfield of E that contains K and $\tau\colon F \to C$ is a field homomorphism such that $\tau|_K = \sigma$. Note that $(K,\sigma) \in X$, so X is non-empty. Moreover, X is partially ordered by

$$(F,\tau) \leq (F_1,\tau_1) \iff F \subseteq F_1 \text{ and } \tau_1|_F = \tau.$$

If $\{(F_i,\tau_i) : i \in I\}$ is a chain in X, then $F = \cup_{i\in I} F_i$ is a subfield of E that contains K. Moreover, if $z \in F$, then $z \in F_i$ for some $i \in I$ and then one defines $\tau(z) = \tau_i(z)$. It is an exercise to prove that τ is well-defined. Since $(F,\tau) \in X$ is an upper bound, Zorn's lemma implies that there exists a maximal element $(E_1,\theta) \in X$. We claim that $E = E_1$. If not, let $z \in E \setminus E_1$. Since we know the proposition is true for the extension $E_1(z)/E_1$, let $\rho\colon E_1(z) \to C$ be a field homomorphism such that $\rho|_{E_1} = \theta$.

Then, in particular, $\rho|_K = \sigma$. This implies that $(E_1(z), \rho) \in X$ and hence $(E_1, \theta) < (E_1(z), \rho)$, a contradiction to the maximality of (E_1, θ). $\square$

Lecture 4. 04/03/2024

The previous proposition will be used to prove that the algebraic closure always exists.

THEOREM 4.1 (Artin). *Let K be a field. Then K admits an algebraic closure C/K. If C_1/K is an algebraic closure, then the extensions C/K and C_1/K are isomorphic.*

PROOF. Let us first prove the uniqueness. The previous proposition implies the existence of an extension homomorphism $\varphi\colon C \to C_1$. Let $y \in C_1$ and $f = f(y,K)$ be the minimal polynomial of y in K. Since f admits a factorization

$$f = \lambda \prod (X - \alpha_i)^{m_i}$$

in $C[X]$, it follows that

$$f = \overline{\varphi}(f) = \varphi(\lambda) \prod (X - \varphi(\alpha_i))^{m_i}$$

Since $0 = f(y)$, we conclude that $y = \varphi(\alpha_j)$ for some j. In particular, φ is surjective and hence φ is bijective.

We now prove the existence. Let us assume that K admits an extension E/K with E algebraically closed. We will prove later that this extension indeed exists; at the moment, we only want to get an algebraic extension from this setting. Let

$$F = \{x \in E : x \text{ is algebraic over } K\}.$$

Then F/K is algebraic. Let $g \in F[X]$ be such that $\deg g > 0$. Since E is algebraically closed, g admits a root α in E. In particular, α is algebraic over F and hence α is algebraic over K. This implies that $\alpha \in F$, thus F is algebraically closed. This proves that F/K is an algebraic closure.

Let us prove that there exists an extension E_1/K such that every polynomial $f \in K[X]$ with $\deg f > 0$ has a root in E_1. Let $\{f_i : i \in I\}$ be the family of monic irreducible polynomials with coefficients in K. We may think that $f_i = f_i(X_i)$. Let $R = K[\{X_i : i \in I\}]$ and let J be the ideal of R generated by the $f_i(X_i)$. We claim that $J \neq R$. If not, $1 \in J$, so

$$1 = \sum_{j=1}^{m} g_j f_{i_j}(X_{i_j})$$

for some $g_1, \dots, g_m \in R$. There exists an extension F/K such that f_{i_j} has a root α_j in F for all j. Let

$$\tau\colon R \to F, \quad \tau(X_k) = \begin{cases} \alpha_j & \text{if } k = i_j, \\ 0 & \text{if } k \notin \{i_1, \dots, i_m\}. \end{cases}$$

Then τ is a ring homomorphism and

$$1 = \tau(1) = \sum_{j=1}^{m} \tau(g_j) f_{i_j}(\alpha_j) = 0,$$

a contradiction.

Since J is a proper ideal, it is contained in a maximal ideal M. Let $L = R/M$ and let $\sigma\colon K \to L$ be the composition $K \hookrightarrow R \to R/M = L$, where $\pi\colon R \to R/M$ is the canonical map. As we did before, $\pi(X_i)$ is a root of $\overline{\sigma}(f_i)$ for all i. And there exists an extension E_1/K such that every f_i has a root in E_1. Proceeding in this way, we construct a sequence

$$E_1 \subseteq E_2 \subseteq \cdots$$

of fields such that every polynomial of positive degree and coefficients in E_k admits a root in E_{k+1}. Let $E = \cup E_k$. We claim that E is algebraically closed. In fact, let $g \in E[X]$ be such that $\deg g > 0$. Then, since $g \in E_r[X]$ for some r, it follows that g has a root in $E_{r+1} \subseteq E$. □

§ 4.1. Decomposition fields.

Definition 4.2. Let K be a field and $f \in K[X]$ be such that $\deg f > 0$. A **decomposition field** of f over K is field E that contains K and that satisfies the following properties:

1) f factorizes linearly in $E[X]$.
2) If F is a field such that $K \subseteq F \subseteq E$ and f factorizes linearly in $F[X]$, then $F = E$.

Easy examples:

Example 4.3. $\mathbb{C}$ is a decomposition field of $X^2+1 \in \mathbb{R}[X]$.

Example 4.4. $\mathbb{Q}[\sqrt{2}]$ is a decomposition field of $X^2-2 \in \mathbb{Q}[X]$.

Example 4.5. The decomposition field of $f = X^2-2$ over $\mathbb{Z}/7$ is precisely $\mathbb{Z}/7$, as 3 and 4 are the roots of f in $\mathbb{Z}/7$.

Example 4.6. $\mathbb{Q}(\sqrt[3]{2})$ is not a decomposition field of $X^3-2 \in \mathbb{Q}[X]$. However, if ω is a primitive cubic root of one, then $\mathbb{Q}(\sqrt[3]{2}, \omega)$ is a decomposition field of the polynomial $X^3-2 \in \mathbb{Q}[X]$.

Proposition 4.7. *E is a decomposition field of $f \in K[X]$ if and only if f factorizes linearly in $E[X]$ and $E = K(x_1, \dots, x_n)$, where $x_1, \dots, x_n$ are the roots of f.*

Proof. Let $f = a\prod_{i=1}^{r}(X-x_i)^{n_i}$ and $F = K(x_1, \dots, x_r)$ with $x_1, \dots, x_r \in E$. Since f factorizes linearly in $F[X]$, it follows that $F = E$. Conversely, let $E = K(x_1, \dots, x_r)$ and assume that f factorizes linearly in $F[X]$. Then, in particular, $x_1, \dots, x_r \in F$. Hence $E \subseteq F$ and $F = E$. □

One immediately obtains the following consequence: If E is a decomposition field of $f \in K[X]$, then E/K is finite.

Theorem 4.8. *Let $f \in K[X]$ be such that* $\deg f > 0$. *There exists a (unique up to extension isomorphism) decomposition field of f over K.*

Proof. Let C/K be an algebraic closure. Write

$$f = a\prod_{i=1}^{r}(X-x_i)^{n_i}$$

in $C[X]$. Then $E = K(x_1, \dots, x_r)$ is a decomposition field of f over K.

Let us prove the uniqueness: if E_1/K is a decomposition field of f over K, then E_1/K is algebraic and thus Proposition 3.6 implies that there exists $\varphi \in \operatorname{Hom}(E_1/K, C/K)$, that is $\varphi\colon E_1 \to C$ is a field homomorphism such that $\varphi|_K$ is the identity. Factorize f linearly in $E_1[X]$ and apply $\overline{\varphi}$:

$$f = a\prod_{j=1}^{s}(X-y_j)^{m_j} \implies f = \overline{\varphi}(f) = \varphi(a)\prod_{j=1}^{s}(X-\varphi(y_j))^{m_j}$$

so f factorizes linearly in $\varphi(E_1)[X]$. Moreover, $E_1 = K(y_1, \dots, y_s)$ and

$$\varphi(E_1) = K(\varphi(y_1), \dots, \varphi(y_s)).$$

Thus $\varphi(E_1)$ is a decomposition field of f. Since $\varphi(E_1) \subseteq C$, it follows that $\varphi(E_1) = E$. □

EXERCISE 4.9. If C is an algebraic closure of K and $\varphi \in \mathrm{Hom}(C/K, C/K)$, then φ is an isomorphism.

Let C be an algebraic closure of K and $G = \mathrm{Gal}(C/K)$. The group G acts on C

$$\sigma \cdot x = \sigma(x), \quad \sigma \in G, x \in C.$$

The orbits are of the form

$$O_G(x) = \{\sigma(x) : \sigma \in G\} = \{y \in C : y = \sigma(x) \text{ for some } \sigma \in G\}$$

The elements $x, y \in C$ are **conjugate** if $y = \sigma(x)$ for some $\sigma \in G$.

PROPOSITION 4.10. *Let C be an algebraic closure of K and $x, y \in C$. Then x and y are conjugate if and only if $f(x,K) = f(y,K)$. In particular, $O_G(x)$ is finite.*

PROOF. Let $G = \mathrm{Gal}(C/K)$. If x and y are conjugate, say $y = \sigma(x)$ for some $\sigma \in G$, let us write $g = f(x,K)$ as

$$g = X^n + \sum_{i=0}^{n-1} a_i X^i.$$

Then $0 = g(x) = x^n + \sum_{i=0}^{n-1} a_i x^i$ and hence y is a root of g, as

$$\begin{aligned} 0 = \sigma\left(x^n + \sum_{i=0}^{n-1} a_i x^i\right) &= \sigma(x)^n + \sum_{i=0}^{n-1} \sigma(a_i)\sigma(x)^i \\ &= \sigma(x)^n + \sum_{i=0}^{n-1} a_i \sigma(x)^i = y^n + \sum_{i=0}^{n-1} a_i y^i. \end{aligned}$$

Thus $f(y,K) = g$.

Conversely, assume that $f(x,K) = f(y,K)$. Let $g = f(x,K) = f(y,K)$ and let

$$\varphi\colon K[x] \to K[y], \quad h(x) \mapsto h(y).$$

Let us show that the map φ is well-defined: we need to show that if $h_1(x) = h_2(x)$, then

$$h_1(y) = \varphi(h_1(x)) = \varphi(h_2(x)) = h_2(y).$$

If $h_1(x) = h_2(x)$, then

$$(h_1 - h_2)(x) = h_1(x) - h_2(x) = 0.$$

This implies that g divides $h_1 - h_2$. In particular, $h_1(y) = h_2(y)$.

A straightforward calculation shows that φ is a field homomorphism such that $\varphi|_K = \mathrm{id}$, this means that φ is an extension homomorphism such that $\varphi(x) = y$. There exists $\sigma \in \mathrm{Hom}(C/K, C/K)$ such that $\sigma|_{K[x]} = \varphi$. Since σ is bijective (this is left as an exercise, you did something similar before), $\sigma(x) = \varphi(x) = y$ and hence $O_G(x) = O_G(y)$. □

PROPOSITION 4.11. *Let C be an algebraic closure of K and $x \in C$. Then*

$$f(x,K) = \prod_{y \in O_G(x)} (X - y)^m$$

for some m.

Proof. For each $y \in O_G(x)$ let m_y be the multiplicity of y in $f(x,K)$. Then, for example, $f(x,K) = (X-x)^{m_x} g$ for some g. If $y \in O_G(x)$, then $y = \sigma(x)$ for some $\sigma \in \mathrm{Gal}(C/K)$. Since

$$\overline{\sigma}(f(x,K)) = f(x,K) = (X-y)^{m_x}\overline{\sigma}(g),$$

it follows that $m_y \geq m_x$. By symmetry, we conclude that $m_x = m_y$. $\square$

The previous proposition shows, in particular, that all the roots of an irreducible polynomial $f \in K[X]$ in an algebraic closure C of K have the same multiplicity. This is not true if f is not irreducible. Find an example.

Definition 4.12. Let K be a field and $\{f_i : i \in I\}$ be a non-empty family of polynomials of positive degree with coefficients in K. A **decomposition field** of $\{f_i : i \in I\}$ is an extension E/K such that every f_i factorizes linearly in $E[X]$ and if F/K is a sub extension of E/K such that every f_i factorizes linearly in $F[X]$, then $F = E$.

Exercise 4.13. Prove that E/K is a decomposition field of $\{f_i : i \in I\}$ if and only if every f_i factorizes linearly in $E[X]$ and $E = K(S)$ where $S = \{\text{roots of } f_i \text{ for all } i\}$.

Exercise 4.14. Prove that if E/K is a decomposition field of $\{f_i : i \in I\}$, then E/K is algebraic. If, moreover, I is finite, then E/K is a decomposition field of $\prod_{i \in I} f_i$.

Exercise 4.15. Prove that there exists a decomposition field of $\{f_i : i \in I\}$ and it is unique up to extension isomorphism.

Exercise 4.16. Let $f = X^3 - X - 1 \in (\mathbb{Z}/3)[X]$ and E be a decomposition field of f. Compute $[E : \mathbb{Z}/3]$.

What about the decomposition field of $f = X^3 - X - 1 \in \mathbb{Q}[X]$?

Exercise 4.17. Let $f = X^4 - 5x^2 + 5 \in \mathbb{Q}[X]$ and E be a decomposition field of f. Compute $[E : \mathbb{Q}]$ and $\mathrm{Gal}(E/K)$.

Lecture 5. 11/03/2024

§ 5.1. Normal extensions.

PROPOSITION 5.1. *Let E/K be an algebraic extension and $\sigma \in \mathrm{Hom}(E/K, E/K)$. Then σ is bijective.*

PROOF. Let $x \in E$ and C be an algebraic closure of K that contains E. There exists a field homomorphism $\varphi\colon C \to C$ such that $\varphi|_E = \sigma$. Thus $\varphi|_K = \sigma|_K = \mathrm{id}_K$. Let $G = \mathrm{Gal}(C/K)$. Then $\varphi \in G$. If $z \in O_G(x)$, then $z = \tau(x)$ for some $\tau \in G$ and hence

$$\varphi(z) = \varphi(\tau(x)) = (\varphi\tau)(x).$$

This implies that $\varphi(z) \in O_G(x)$ and $\varphi(O_G(x)) = O_G(x)$. The restriction $\sigma|_{E\cap O_G(x)}$ is injective. Then

$$\begin{aligned}\sigma(E \cap O_G(x)) &= \varphi(E \cap O_G(x))\\ &= \varphi(E) \cap \varphi(O_G(x)) = \sigma(E) \cap O_G(x) \subseteq E \cap O_G(x).\end{aligned}$$

Since $|E \cap O_G(x)| < \infty$, it follows that $E \cap O_G(x) = \sigma(E \cap O_G(x))$ and hence x belongs to the image of σ. □

DEFINITION 5.2. Let E/K be an algebraic extension and C be an algebraic closure of K containing E. Then E/K is **normal** if $\sigma(E) \subseteq E$ for all $\sigma \in \mathrm{Hom}(E/K, C/K)$.

Note that $\sigma(E) \subseteq E$ in the previous definition is equivalent to $\sigma(E) = E$.

EXAMPLE 5.3. The extension $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is not normal. Why?

Some trivial examples of normal extensions: K/K is normal and if C is an algebraic closure of K, then C/K is normal.

EXAMPLE 5.4. The extension $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is normal. Every extension generated by algebraic elements of degree two is normal.

EXERCISE 5.5. Let ξ be a primitive cubic root of one. Then $\mathbb{Q}(\sqrt[3]{2}, \xi)/\mathbb{Q}$ is normal.

The following result is practical but technical. That is why we leave the proof as an exercise.

EXERCISE 5.6. Prove that the previous definition depends only on E (and not on the algebraic closure C).

Some properties:

PROPOSITION 5.7. *Let E/K be a normal extension and $f \in K[X]$ be an irreducible polynomial that admits a root x in E. Then f factorizes linearly in E.*

PROOF. We may assume that f is monic. Let C/K be an algebraic closure of K containing E. Let y be a root of f in C. Since $f = f(x,K) = f(y,K)$, it follows that $y = \sigma(x)$ for some $\sigma \in \mathrm{Gal}(C/K)$. Since E/K is normal, $\sigma|_E\colon E \to C$ is an automorphism of E/K, that is $\sigma(E) \subseteq E$. In particular, $y \in E$. □

Let $K \subseteq F \subseteq E$ be a tower of fields. If E/K is normal, then E/F is normal. However, Note that E/K normal does not imply F/K normal, as this would imply that every extension is normal. Moreover, E/F normal and F/K normal do not imply E/K normal.

Example 5.8. The extensions $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ are both normal, but $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$ is not normal, as the roots of X^4-2 are $\sqrt[4]{2}$, $-\sqrt[4]{2}$, $\sqrt[4]{2}i$ and $-\sqrt[4]{2}i$.

Recall that if C is an algebraic closure of K and $x \in C$, then

$$f(x,K) = \prod (X-y)^m,$$

where the product is taken over all $y \in O_{\mathrm{Gal}(C/K)}(x)$. If E/K is normal and $x \in E$, then there exists m such that

$$f(x,K) = \prod (X-y)^m,$$

where the product is taken over all $y \in O_{\mathrm{Gal}(E/K)}(x)$.

Proposition 5.9. *Let E/K and F/K be extensions. If F/K is normal, then EF/E is normal.*

Proof. Let C be an algebraic closure of E containing EF (this exists because EF/E is algebraic). Let $\sigma \in \mathrm{Hom}(EF/E, C/E)$. We claim that $\sigma(EF) = EF$. Let

$$\overline{K} = \{x \in C : x \text{ is algebraic over } K\}.$$

Then $\overline{K}$ is an algebraic closure over K and $F \subseteq \overline{K}$. Since F/K is normal and $\sigma|_F \in \mathrm{Hom}(F/K, \overline{K}/K)$, it follows that $\sigma(F) = F$. If $z \in EF$, then $z = \sum_{i=1}^m e_i f_i$ for some $e_1,\dots,e_m \in E$ and $f_1,\dots,f_m \in F$. Since $\sigma(e_i) = e_i$ for all i,

$$\sigma(z) = \sum_{i=1}^m \sigma(e_i)\sigma(f_i) = \sum_{i=1}^m e_i \sigma(f_i) \in EF. \qquad \square$$

What is the relation between normal extensions and decomposition fields? The notions look deeply related. The following proposition serves as an explanation:

Proposition 5.10. *Let E/K be an algebraic extension. Then E/K is normal if and only if E/K is the decomposition field of a family of polynomials of $K[X]$ of positive degree.*

Proof. Assume first that E/K is a normal extension. Let $G = \mathrm{Gal}(E/K)$. If $x \in E$ and $f(x,K) = \prod_{y \in O_G(x)} (X-y)^m$, then $f(x,K)$ factorizes linearly in $E[X]$. Thus E/K is a decomposition field of the family $\{f(x,K) : x \in E\}$.

Conversely, assume that E/K is a decomposition field of the family $\{f_i : i \in I\}$. Then $E = K(S)$ where S is the set of roots of the polynomials f_i. Let C/K be an algebraic closure of K that contains E and let $\sigma \in \mathrm{Hom}(E/K, C/K)$. Let $x \in S$. Then x is a root of some $f_j = \sum a_k X^k$. Since $f_j(x) = 0$, it follows that $\sigma(x)$ is a root of f_j, as

$$f_j(\sigma(x)) = \sum a_k \sigma(x)^k = \sum \sigma(a_k)\sigma(x^k) = \sigma\left(\sum a_k x^k\right) = \sigma(0) = 0.$$

Hence $\sigma(E) \subseteq E$. $\square$

Exercise 5.11. Let $E = \mathbb{Q}[\sqrt[4]{7} + \sqrt{2}]$.

1) Prove that $E/\mathbb{Q}$ is not normal.
2) Compute $[E : \mathbb{Q}]$.
3) Compute $\mathrm{Gal}(E/\mathbb{Q})$.

§ 5.2. Dedekind's theorem. Note that every extension homomorphism $E/K \to F/K$ is, in particular, a K-linear map $E \to F$, that is

$$\operatorname{Hom}(E/K, F/K) \subseteq \operatorname{Hom}_K(E, F).$$

If F/K is an extension and V is a K-vector space, the set $\operatorname{Hom}_K(V,F)$ of K-linear maps is a vector space over F with $(a \cdot f)(v) = af(v)$ for $a \in F$, $f \in \operatorname{Hom}_K(V,F)$ and $v \in V$.

EXERCISE 5.12. Let V be a K-vector space. Prove that $\dim_F \operatorname{Hom}_K(V,F) \geq \dim_K V$. Moreover, if $\dim_K V < \infty$, then $\dim_F \operatorname{Hom}_K(V,F) = \dim_K V$.

If V is a vector space and S is a (possibly infinite) subset of V, then S is linearly independent if every finite subset of S is linearly independent.

THEOREM 5.13 (Dedekind). *Let E/K and F/K be extensions and let $\{\varphi_i : i \in I\}$ be a subset of $\operatorname{Hom}(E/K, F/K)$, i.e. a family of extension homomorphisms. Assume that $\varphi_i \neq \varphi_j$ if $i \neq j$. Then the subset $\{\varphi_i : i \in I\} \subseteq \operatorname{Hom}_K(E,F)$ is linearly independent over F.*

PROOF. Assume it is not. Let $\{\varphi_1, \dots, \varphi_n\}$ be linearly dependent over F with n minimal. Clearly, $n > 1$. We may assume that

$$\sum_{i=1}^{n} a_i \varphi_i = 0 \tag{5.1}$$

for some $a_1, \dots, a_n \in F$ all different from zero. Let $z \in E \setminus \{0\}$ be such that $\varphi_1(z) \neq \varphi_2(z)$. If $x \in E$, then

$$0 = \left(\sum_{i=1}^{n} a_i \varphi_i\right)(xz) = \sum_{i=1}^{n} a_i \varphi_i(xz) = \sum_{i=1}^{n} a_i \varphi_i(x)\varphi_i(z) = \left(\sum_{i=1}^{n} (a_i \varphi_i(z))\varphi_i\right)(x).$$

Thus

$$\sum_{i=1}^{n} (a_i \varphi_i(z))\varphi_i = 0.$$

Since $\sum_{i=1}^{n} a_i \varphi_i = 0$ and $\varphi_1(z) \neq 0$,

$$a_1 \varphi_1 + a_2 \frac{\varphi_2(z)}{\varphi_1(z)} \varphi_2 + \cdots + a_n \frac{\varphi_n(z)}{\varphi_1(z)} \varphi_n = 0. \tag{5.2}$$

Thus, subtracting (5.1) and (5.2),

$$\left(a_2 - a_2 \frac{\varphi_2(z)}{\varphi_1(z)}\right) \varphi_2 + \cdots + \left(a_n - a_n \frac{\varphi_n(z)}{\varphi_1(z)}\right) \varphi_n = 0.$$

Since $a_n \neq 0$ and $\varphi_2(z) \neq \varphi_1(z)$, the scalar $a_2 - a_2 \frac{\varphi_2(z)}{\varphi_1(z)} \neq 0$ and hence $\{\varphi_2, \dots, \varphi_n\}$ is linearly dependent, a contradiction. $\square$

If E/K and F/K are extensions, let $\gamma(E/K, F/K) = |\operatorname{Hom}(E/K, F/K)|$.

EXERCISE 5.14. Prove the following statements:

1) $\gamma(E/K, F/K) \leq \dim_F \operatorname{Hom}_K(E,F)$.
2) If $[E:K] < \infty$, then $\gamma(E/K, F/K) \leq [E:K]$.
3) If x is algebraic over K, then $\gamma(K(x)/K, F/K) \leq \deg f(x,K)$.

If C is an algebraic closure of K, then we define $\gamma(E/K) = \gamma(E/K, C/K)$. This definition does not depend on the algebraic closure.

EXERCISE 5.15. If C and C_1 are algebraic closures of K, then

$$|\operatorname{Hom}(E/K, C/K)| = |\operatorname{Hom}(E/K, C_1/K)|.$$

PROPOSITION 5.16. *Let C be an algebraic closure of K and $G = \operatorname{Gal}(C/K)$. If $x \in C$, then $\gamma(K(x)/K) = |O_G(x)|$.*

PROOF. If $\sigma \in \operatorname{Hom}(K(x)/K, C/K)$, then there exists $\phi \in G$ such that $\phi|_{K(x)} = \sigma$. Thus

$$\sigma(x) = \phi(x) \in O_G(x).$$

Conversely, if $y \in O_G(x)$, then there exists $\tau \in G$ such that $y = \tau(x)$. Hence

$$\tau|_{K(x)} \in \operatorname{Hom}(K(x)/K, C/K)$$

and $\tau|_{K(x)}(x) = y$. Since our sets are then in bijective correspondence, the claim follows. □

EXERCISE 5.17. If E/K is finite, then $|\operatorname{Gal}(E/K)| \leq [E : K]$. Moreover, E/K is normal if and only if $|\operatorname{Gal}(E/K)| = \gamma(E/K)$.

Lecture 6. 18/03/2024

If $t\colon A\to B$ is a surjective map, then $a\sim a_1 \iff t(a)=t(a_1)$ defines an equivalence relation on A. The set $\overline{A}$ of equivalence classes is in bijective correspondence with B, $\overline{A}\to B$, $\overline{a}\mapsto t(a)$. Moreover, if $|t^{-1}(\{b\})|=m$ for all $b\in B$, then $|A|=m|\overline{A}|=m|B|$.

Proposition 6.1. *Let E/K be algebraic and F/K be a subextension such that E/F is finite. Then $\gamma(E/K)=\gamma(E/F)\gamma(F/K)$.*

Proof. Assume that $E=F(x)$. Let C be an algebraic closure of K containing E and $G=\mathrm{Gal}(C/F)$. Let $f=f(x,F)=\sum b_iX^i$.

The map

$$\lambda\colon \mathrm{Hom}(E/K,C/K)\to\mathrm{Hom}(F/K,C/K),\quad \sigma\mapsto\sigma|_F,$$

is well-defined. It is surjective: if $\varphi\in\mathrm{Hom}(F/K,C/K)$, then $\varphi\colon F\to C$ is, in particular, a field homomorphism. Since E/F is algebraic, by Proposition 3.6 there exists a field homomorphism $\sigma\colon E\to C$ such that $\sigma|_F=\varphi$. Since $\sigma|_K=\varphi|_K=\mathrm{id}$, in particular $\sigma\in\mathrm{Hom}(E/K,C/K)$.

For $\varphi\in\mathrm{Hom}(F/K,C/K)$,

$$\lambda^{-1}(\{\varphi\})=\{\sigma\in\mathrm{Hom}(E/K,C/K):\sigma|_F=\varphi\}$$

and let R_φ be the set of roots (in C) of the polynomial $\overline{\varphi}(f)=\sum\varphi(b_i)X^i$.

Claim. The map $\alpha\colon\lambda^{-1}(\{\varphi\})\to R_\varphi$, $\sigma\mapsto\sigma(x)$, is well-defined.

We need to show that $\sigma(x)$ is a root of $\overline{\varphi}(f)$:

$$\begin{aligned}\overline{\varphi}(f)(\sigma(x))&=\sum\varphi(b_i)\sigma(x)^i=\sum\sigma(b_i)\sigma(x^i)\\&=\sum\sigma(b_ix^i)=\sigma\left(\sum b_ix^i\right)=\sigma(f(x))=\sigma(0)=0.\end{aligned}$$

Claim. The map $\beta\colon R_\varphi\to\lambda^{-1}(\{\varphi\})$, $y\mapsto\sigma_y$, where $\sigma_y(z)=\overline{\varphi}(h)(y)$ if $z=h(x)$, is well-defined.

We need to show that if $z=h(x)$ and $z=h_1(x)$ for some $h,h_1\in F[X]$, then $\overline{\varphi}(h)(y)=\overline{\varphi}(h_1)(y)$. The assumptions imply that $(h-h_1)(x)=0$ and hence f divides $h-h_1$. Since $\overline{\varphi}$ is a ring homomorphism, $\overline{\varphi}(f)$ divides $\overline{\varphi}(h)-\overline{\varphi}(h_1)$. This implies $(\overline{\varphi}(h)-\overline{\varphi}(h_1))(y)=0$. We also need to show that $\sigma_y|_F=\varphi$: if $a\in F$, then write $a=aX^0\in F[X]$. Thus $\sigma_y(a)=\overline{\varphi}(aX^0)(y)=\varphi(a)\in C$. It is now an exercise to prove that $\sigma_y\in\mathrm{Hom}(E/K,C/K)$.

Claim. $|\lambda^{-1}(\{\varphi\})|=|R_\varphi|$.

For this we need to show that β is the inverse of α, that is $\alpha\circ\beta=\mathrm{id}$ and $\beta\circ\alpha=\mathrm{id}$. To prove that $\beta\circ\alpha=\mathrm{id}$ let σ be such that $\sigma|_F=\varphi$. Then $y=\sigma(x)\in R_\varphi$. Let $z=h(x)=\sum a_ix^i\in F[x]=E$. Then

$$\overline{\varphi}(h)(y)=\sum\varphi(a_i)y^i=\sum\sigma(a_i)y^i=\sigma\left(\sum a_ix^i\right)=\sigma(z).$$

Conversely, if $y\in R_\varphi$, then

$$\alpha(\sigma_y)=\sigma_y(x)=y,$$

as $\sigma_y(x)=\overline{\varphi}(X)(y)=y$.

Claim. If $\phi\in\mathrm{Gal}(C/K)$ is such that $\phi|_F=\varphi$, then $|\phi^{-1}(R_\varphi)|=|R_\varphi|$ and

$$O_G(x)=\phi^{-1}(R_\varphi).$$

Let us first prove $O_G(x) \supseteq \phi^{-1}(R_\varphi)$. If $y \in R_\varphi$, then

$$\begin{aligned} f(\phi^{-1}(y)) &= \sum b_i \phi^{-1}(y^i) = \phi^{-1}\left(\sum \phi(b_i) y^i\right) \\ &= \phi^{-1}(\overline{\varphi}(f)(y)) = \phi^{-1}(0) = 0. \end{aligned}$$

Then $f(x,F) = f(\phi^{-1}(y),F)$. By Proposition 4.10, $\phi^{-1}(y) \in O_G(x)$. Now we prove $O_G(x) \subseteq \phi^{-1}(R_\varphi)$. Let $z \in O_G(x)$. Then $\overline{\varphi}(f)(\phi(z)) = 0$, as

$$\begin{aligned} \overline{\varphi}(f)(\phi(z)) &= \sum \varphi(b_i)\phi(z^i) \\ &= \sum \phi(b_i)\phi(z^i) = \phi\left(\sum b_i z^i\right) = \phi(f(z)) = \phi(0) = 0. \end{aligned}$$

Thus $\phi(z) \in R_\varphi$ and hence $z \in \phi^{-1}(R_\varphi)$. It follows that $|\lambda^{-1}(\varphi)| = |O_G(x)|$ for all φ. By using the argument before the proposition,

$$\begin{aligned} \gamma(E/K) &= |\operatorname{Hom}(E/K, C/K)| \\ &= |O_G(x)||\operatorname{Hom}(F/K, C/K)| \\ &= |O_G(x)|\gamma(F/K). \end{aligned}$$

Since $\gamma(E/K) = \gamma(F(x)/F) = |O_G(x)|$ by Proposition 5.16, the claim follows.

For the general case, we assume that $E = F(x_1, \dots, x_n)$. We proceed by induction on n. If $n = 0$, then $E = F$ and the result is trivial. If $n > 0$, let $L = F[x_1, \dots, x_{n-1}]$ and $E = L(x_n)$. The case proved implies that $\gamma(E/F) = \gamma(E/L)\gamma(L/F)$. By the inductive hypothesis, $\gamma(L/K) = \gamma(L/F)\gamma(F/K)$. Thus

$$\gamma(E/F)\gamma(F/K) = \gamma(E/L)\gamma(L/F)\gamma(F/K) = \gamma(E/L)\gamma(L/K) = \gamma(E/K),$$

again using the previous case. □

§ 6.1. Separable extensions.

Definition 6.2. Let E/K be an extension and $x \in E$ an algebraic element. Then x is **separable** over K if x is a simple root of $f(x,K)$.

An algebraic extension E/K is **separable** if every $x \in E$ is separable over K. Then K/K is separable.

Exercise 6.3. Prove that an element x is separable over K if and only if x is a simple root of a polynomial with coefficients in K.

If F/K is a subextension of E/K and $x \in E$ is separable over K, then x is separable over F.

Exercise 6.4. If C is an algebraic closure of K, $x \in C$ and $G = \operatorname{Gal}(C/K)$. Prove that the following statements are equivalent:

1) x is separable over K.
2) Every $y \in O_G(x)$ is separable over K.
3) $\gamma(K(x)/K) = [K(x):K] = \deg f(x,K)$.

Let K be any field and $g \in K[X]$. Let z be a root of g. Then z is a multiple root of g if and only if z is a root of g'.

EXERCISE 6.5. Prove that if K has characteristic zero or K is finite, then every algebraic extension of K is separable.

EXAMPLE 6.6. Let $E = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Then $[E : \mathbb{Q}] = 4$ and $\mathrm{Gal}(E/Q) \simeq C_2 \times C_2$. The extension E/Q is normal, as it is the decomposition field of $(X^2-2)(X^2-3)$ and it is separable as $\mathbb{Q}$ has characteristic zero.

EXAMPLE 6.7. Let E be a decomposition field of X^4-2 over $\mathbb{Q}$. Then $E/\mathbb{Q}$ is normal and separable. Note that $E = \mathbb{Q}(\sqrt[4]{2}, i)$, so

$$[E : \mathbb{Q}] = 8 = |\mathrm{Gal}(E/\mathbb{Q})|.$$

Let us compute $\mathrm{Gal}(E/\mathbb{Q})$. If $\sigma \in \mathrm{Gal}(E/\mathbb{Q})$, then $\sigma(\sqrt[4]{2}) \in \{\sqrt[4]{2}, -\sqrt[4]{2}, \sqrt[4]{2}i, -\sqrt[4]{2}i\}$ and $\sigma(i) \in \{-i, i\}$. Two examples are

$$\alpha\colon \begin{cases} \sqrt[4]{2} \mapsto \sqrt[4]{2}i, \\ i \mapsto i, \end{cases} \qquad \beta\colon \begin{cases} \sqrt[4]{2} \mapsto \sqrt[4]{2}, \\ i \mapsto -i. \end{cases}$$

It follows that $\mathrm{Gal}(E/\mathbb{Q})$ is isomorphic to the group $\langle \alpha, \beta \rangle$, which turns out to be isomorphic to the dihedral group of eight elements.

Another consequence: If $E = K(S)$, then E/K is separable if and only if every $x \in S$ is separable over K. One first does the case $E = K(x)$ and then proceeds by induction.

EXERCISE 6.8. Let $K \subseteq F \subseteq E$ be a tower of fields. Prove that E/K is separable if and only if F/K and E/F are separable.

EXERCISE 6.9. Let E/K and F/K be extensions. Prove that if F/K is separable, then EF/E is separable.

Lecture 7. 24/03/2024

If E/K is algebraic, then

$$F = \{x \in E : x \text{ is separable over } K\}$$

is a subfield of E that contains K. It is known as the **separable closure** of K with respect to E. Note that $F = K(F)$, as $K(F)$ is separable because it is generated by separable elements. Moreover, F/K is separable and E/F is a **purely inseparable** extension, meaning that for every $x \in E \setminus F$, the polynomial $f(x,F)$ is not separable.

Proposition 7.1. *If E/K is separable and finite, then $E = K(x)$ for some $x \in E$.*

Proof. Let us assume that K is finite. Then E is finite and hence the multiplicative group $E^\times = E \setminus \{0\}$ is cyclic, say $E^\times = \langle x \rangle$. It follows that $E = K(x)$.

Let us now assume that K is infinite. We first consider the case $E = K(x,y)$. The general case $E = K(x_1,\dots,x_n)$ is left as an exercise, one needs to proceed by induction. Let $n = [E:K]$ and C be an algebraic closure of K containing E. Write $\operatorname{Hom}(E/K,C/K) = \{\sigma_1,\dots,\sigma_n\}$. Let

$$f = \prod_{1 \leq i < j \leq n} \big((\sigma_i(y) - \sigma_j(y)) + X(\sigma_i(x) - \sigma_j(x))\big) \in C[X].$$

Then $f \neq 0$, as f is a product of non-zero polynomials. Since K is infinite, there exists a non-zero $c \in K$ such that $f(c) \neq 0$. For any $r,s \in \{1,\dots,n\}$ with $r \neq s$,

$$\sigma_r(y) - \sigma_s(y) + c(\sigma_r(x) - \sigma_s(x)) \neq 0,$$

as $f(c) \neq 0$. It follows that $\sigma_r(y+cx) \neq \sigma_s(y+cx)$. Thus $\gamma(K(y+cx)/K) \geq n$. Now

$$n \geq [K(y+cx):K] = \gamma(K(y+cx)/K) \geq n,$$

so $[K(y+cx):K] = n$ and hence $K(y+cx) = E$. $\square$

For example, $\mathbb{Q}(\sqrt{2},i) = \mathbb{Q}(\sqrt{2}+i)$.

Proposition 7.2. *Let E/K be a finite extension. Then $E = K(x)$ for some $x \in E$ if and only if E/K admits finitely many subextensions.*

Proof. We may assume that K is infinite; otherwise, the result is trivial. We first prove $\Longrightarrow$. Let us assume that $E = K(x)$ for some x. We claim that the map

$$\Psi\colon \{F : K \subseteq F \subseteq E\} \to \{g \in K[X] : g \text{ is a monic divisor of } f(x,K)\},\quad F \mapsto f(x,F),$$

is injective. Take F_0 such that $K \subseteq F_0 \subseteq F \subseteq E$ and $f(x,F) = f(x,F_0)$. Then

$$[E:F_0] = [F_0(x):F_0] = \deg f(x,F_0) = m = [F(x):F] = [E:F]$$

and hence $F = F_0$. It follows that Ψ is injective and therefore there are finitely many fields between K and E.

Let us prove $\Longleftarrow$. As before, let us assume that $E = K(x,y)$. For each $a \in K$ we consider the extension $K(ay+x)/K$. By assumption, there exist $a,b \in K$ such that $a \neq b$ and $K(x+ay) = K(x+by) = L$. We claim that $L = E$. Note that $x+ay \in L$ and $x+by \in L$, so $(a-b)y \in L$ and hence, since $K \subseteq L$, it follows that $y \in L$. Thus $x \in L$ and therefore $L = E$. $\square$

As a consequence, if E/K is finite and separable, then E/K admits finitely many subextensions.

§ 7.1. Galois extensions. Let E/K be an algebraic extension. Assume that $E = K(S)$ and let C be an algebraic closure of K containing E. Let

$$T = \{y \in C : y \text{ is a root of } f(x,K) \text{ for } x \in S\}$$

and let $L = K(T)$. Then $E \subseteq L$, as $S \subseteq T$. The extension L/K is normal, as L/K is a decomposition field of the family $\{f(x,K) : x \in S\}$. Moreover, L is the smallest normal extension of K containing E. The field L is the **normal closure** of E (with respect to C).

Exercise 7.3. If E/K is finite, then L/K is finite

Exercise 7.4. If E/K is separable, then L/K is separable.

Let E/K be an extension and $S \subseteq \mathrm{Gal}(E/K)$ be a subset. the set

$$^{S}E = \{x \in E : \sigma(x) = x \text{ for all } \sigma \in S\}$$

is a subfield of E that contains K. The subfield ^{S}E is known as the **fixed field** of S.

Definition 7.5. Let E/K be an algebraic extension and $G = \mathrm{Gal}(E/K)$. Then E/K is a **Galois extension** if $^{G}E = K$.

Clearly, K/K is a Galois extension. Note that $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is not a Galois extension. Why?

Exercise 7.6. Prove that $\mathbb{Q}(\sqrt{2},\sqrt{3})/\mathbb{Q}$ is a Galois extension.

Exercise 7.7. If the characteristic of K is different from two, then every quadratic extension of K is a Galois extension.

Exercise 7.8. Let E/K be an algebraic extension and $G = \mathrm{Gal}(E/K)$. Let $F = {}^{G}E$. Prove that $\mathrm{Gal}(E/F) = G$ and hence E/F is a Galois extension.

Proposition 7.9. *Let E/K be an algebraic extension. Then E/K is a Galois extension if and only if E/K is normal and separable.*

Proof. Let $G = \mathrm{Gal}(E/K)$. Let us first assume that E/K is Galois. For $x \in E$ let $f_x = \prod_{y \in O_G(x)}(X - y) = \sum a_i X^i \in E[X]$. If $\varphi \in G$, then

$$\overline{\varphi}(f_x) = \prod_{y \in O_G(x)} (X - \varphi(y)) = f_x,$$

as if $O_G(x) = \{\sigma_1(x), \dots, \sigma_r(x)\}$, then $\varphi(\sigma_i(x)) = (\varphi\sigma_i)(x) = \sigma_j(x)$ for some j. Since

$$\sum a_i X^i = f_x = \overline{\varphi}(f_x) = \sum \varphi(a_i) X^i,$$

it follows that $a_i \in {}^{G}E = K$ for all i. Thus $f_x \in K[X]$ and E/K is a decomposition field of the family $\{f_x : x \in E\}$. In particular, E/K is normal. Moreover, x is a simple root of $f_x \in K[X]$ and hence x is separable over K.

Conversely, let $x \in {}^{G}E$. Since E/K is normal, then $f(x,K) = \prod_{y \in O_G(x)}(X - y)^m$ for some m. Since E/K is separable, $m = 1$. Thus $f(x,K) = \prod_{y \in O_G(x)}(X - y) = X - x$ and $x \in K$. $\square$

Definition 7.10. Let K be a field and $f \in K[X]$. Then f is **separable** if all roots of f are simple (in some algebraic closure of K).

Proposition 7.11. *Let E/K be a finite extension. Then E/K is a Galois extension if and only if E is a decomposition field over K of a separable polynomial $f \in K[X]$.*

Proof. Let us assume first that E/K is a Galois extension. Since E/K is finite and separable, $E = K(x)$ by Proposition 7.1. Then E/K is a decomposition field of $f(x,K)$ since E/K is normal. Since E/K is separable, x is separable over K. Thus x is a simple root of $f(x,K)$ and hence $f(x,K)$ is separable.

Conversely, let $x_1,\dots,x_r$ be the roots of a separable polynomial $f \in K[X]$. Then $E = K(x_1,\dots,x_r)$ is separable and normal. $\square$

In the previous case, $\mathrm{Gal}(E/K)$ is known as the **Galois group** of the polynomial f. The notation is $\mathrm{Gal}(f,K)$. If $n = \deg f$ and $x_1,\dots,x_n$ are the roots of f, then any $\varphi \in \mathrm{Gal}(f,K)$ permutes the roots of f, that is φ permutes the set $\{x_1,\dots,x_n\}$. In particular, $\mathrm{Gal}(f,K)$ is isomorphic to a subgroup of $\mathbb{S}_n$ and hence $|\mathrm{Gal}(f,K)|$ divides $n!$.

Proposition 7.12. *Let E/K be a normal extension and F be the separable closure of K with respect to E. Then F/K is a Galois extension.*

Proof. Let C/K be an algebraic closure such that $E \subseteq C$. Let $\sigma \in \mathrm{Hom}(F/K, C/K)$. and let $\varphi \in \mathrm{Hom}(E/K, C/K)$ be such that $\varphi|_F = \sigma$. Since E/K is normal, $\varphi(E) = E$. Let $x \in F$. Then $\sigma(x) = \varphi(x) \in E$. Thus $f(\sigma(x),K) = f(x,K)$ and $\sigma(x)$ is separable over K, which implies that $\sigma(x) \in F$. Thus F/K is normal. Since F/K is separable, it follows that F/K is a Galois extension by Proposition 7.9. $\square$

Some easy facts.

Exercise 7.13. Let E/K be a separable extension and L/K be the normal closure of E in some algebraic closure C that contains E. Prove that L/K is a Galois extension.

Exercise 7.14. Let E/K be a finite extension. Prove that E/K is Galois if and only if $[E:K] = |\mathrm{Gal}(E/K)|$.

For the previous exercise, note that if E/K is a finite extension, then

$$|\mathrm{Gal}(E/K)| \leq \gamma(E/K) \leq [E:K].$$

The first inequality is equality if and only if E/K is normal. The second inequality is equality if and only if E/K is separable.

Exercise 7.15. Let E/K be a Galois extension and F/K be a subextension of E/K. Prove that E/F is a Galois extension.

Theorem 7.16 (Artin). *Let E be a field and G be a finite group of automorphisms of E. If $K = {}^{G}E$, then E/K is a Galois extension, $[E:K] = |G|$ and $\mathrm{Gal}(E/K) = G$.*

Before proving the theorem, we need a lemma.

Lemma 7.17. *Let E/K be a separable extension such that* $\deg f(x,K) \leq m$ *for all $x \in E$. Then E/K is finite and $[E:K] \leq m$.*

Proof. Let $z \in E$ be of maximal degree. If $x \in E$, then $K(x,z)/K$ is separable. Then $K(x,z) = K(y)$ for some y. It follows that
$$K(z) \subseteq K(x,z) = K(y).$$
Since $\deg f(z,K) \leq \deg f(y,K)$, $\deg f(z,K) = \deg f(y,K)$. Hence $K(y) = K(z)$. In particular, $x \in K(z)$ and therefore $E = K(z)$. □

Now we are ready to prove Artin's theorem:

Proof of Theorem 7.16. Note that $G \subseteq \mathrm{Gal}(E/K)$. Let $x \in E$ and
$$f_x = \prod_{y \in O_G(x)} (X - y).$$
Since $f_x \in K[X]$, the extension E/K is normal and separable (as it is a decomposition field of a family of separable polynomials), so E/K is a Galois extension. Moreover,
$$\deg f(x,K) \leq \deg f_x = |O_G(x)| \leq |G|.$$
By the previous lemma, E/K is finite and $[E:K] \leq |G|$. This implies that $|\mathrm{Gal}(E/K)| = [E:K] \leq |G|$ and hence $|\mathrm{Gal}(E/K)| = |G|$. □

Example 7.18. Let $E = K(X,Y)$ and $\sigma\colon K[X,Y] \to E$ be the ring homomorphism given by $\sigma(X) = Y$ and $\sigma(Y) = X$. Note that σ is bijective, as $\sigma^2 = \mathrm{id}$. The map σ induces a field homomorphism $\overline{\sigma}\colon E \to E$ such that $\overline{\sigma}^2 = \mathrm{id}$. Recall that such a homomorphism is given by $f/g \mapsto \sigma(f)/\sigma(g)$. Let $G = \langle \overline{\sigma} \rangle$. Then $|G| = 2$. We claim that ${}^G E = K(X+Y, XY)$. Let $F = K(X+Y, XY)$. We only prove that ${}^G E \subseteq F$, as the other inclusion is trivial. Artin's theorem implies that $[E : {}^G E] = 2$ and $E = F(X)$, as X is a root of the polynomial $Z^2 - (X+Y)Z + XY$. Then $[E:F] \leq 2$ and $[{}^G E : F] = 1$.

Lecture 8. 15/04/2024

§ 8.1. Galois' correspondence.

Theorem 8.1 (Galois). *Let E/K be a finite Galois extension and $G = \mathrm{Gal}(E/K)$. There exists a bijective correspondence*

$$\{F : K \subseteq F \subseteq E \text{ subfields}\} \leftrightarrow \{S : S \text{ is a subgroup of } G\}$$

The correspondence is given by $F \mapsto \mathrm{Gal}(E/F)$ and ${}^{S}E \leftarrow S$. Moreover, normal subextensions of E/K correspond to normal subgroups of G.

$$\begin{tikzcd}
 & & E \arrow[dl, dash] \arrow[dr, dash] & \\
 & F \arrow[dl, dash] \arrow[dr, dash] & & \{1\} \arrow[dl, dash] \\
K \arrow[dr, dash] & & S \arrow[dl, dash] & \\
 & G & &
\end{tikzcd}$$

Proof. Let α and β be the maps $\alpha(F) = \mathrm{Gal}(E/F)$ and $\beta(S) = {}^{S}E$. A routine exercise shows that α and β are well-defined. We first note that

$$\beta(\alpha(F)) = \beta(\mathrm{Gal}(E/F)) = {}^{\mathrm{Gal}(E/F)}E = F$$

since E/F is a Galois extension. Moreover,

$$\alpha(\beta(S)) = \alpha({}^{S}E) = \mathrm{Gal}(E/{}^{S}E) = S$$

by Artin's theorem, as S is finite.

Let F be a subfield of E containing K and $S = \alpha(F)$. Then

$$[F : K] = \frac{[E : K]}{[E : F]} = \frac{|G|}{|S|} = (G : S).$$

Let C be an algebraic closure of K that contains E. If $S = \mathrm{Gal}(E/F)$, then $F = {}^{S}E$.

We need to prove that F/K is normal if and only if S is normal in G. Let us first prove $\implies$. Let $\tau \in S$ and $\sigma \in G$. Since F/K is normal, $\sigma|_F \in \mathrm{Aut}(F)$. Thus $\sigma^{-1}(F) = F$. In particular, if $x \in F$, then $\sigma^{-1}(x) \in F$ and

$$\sigma\tau\sigma^{-1}(x) = \sigma\sigma^{-1}(x) = x.$$

Conversely, let $\varphi \in \mathrm{Hom}(F/K, C/K)$. There exists $\Phi \colon E \to C$ such that $\Phi|_F = \varphi$. Since E/K is normal, $\Phi(E) = E$ and hence $\Phi \in G$. We claim that $\varphi(x) \in F$ for all $x \in F$. Note that $F = {}^{S}E$, so

$$\tau\varphi(x) = \tau\Phi(x) = \Phi\Phi^{-1}\tau\Phi(x) = \Phi(x) = \varphi(x)$$

for all $\tau \in S$, as $\Phi^{-1}\tau\Phi \in S$. This means that $\varphi(x) \in {}^{S}E = F$.

Let us compute $\mathrm{Gal}(F/K)$. Since F/K is normal, the map $\lambda \colon G \to \mathrm{Gal}(F/K)$, $\sigma \mapsto \sigma|_F$, is a surjective group homomorphism such that $\ker\lambda = S$. The first isomorphism theorem implies that $\mathrm{Gal}(F/K) \simeq G/S$. □

Some easy consequences.

EXERCISE 8.2. If E/K is a Galois extension of degree n and p is a prime number dividing n, then E/K admits a subextension of degree n/p.

EXERCISE 8.3. If E/K is a Galois extension of degree $p^\alpha m$ with p a prime number coprime with m, then E/K admits a subextension of degree m.

DEFINITION 8.4. An extension E/K is **abelian** if E/K is a Galois extension with $\mathrm{Gal}(E/K)$ abelian.

EXERCISE 8.5. If E/K is an abelian extension of degree n and d divides n, then E/K admits a subextension of degree d.

DEFINITION 8.6. An extension E/K is **cyclic** if E/K is a Galois extension with $\mathrm{Gal}(E/K)$ cyclic.

EXAMPLE 8.7. The extension $\mathbb{Q}(\sqrt{2},\sqrt{3})/\mathbb{Q}$ admits exactly three non-trivial subextensions:

$$\mathbb{Q}(\sqrt{2})/\mathbb{Q}, \quad \mathbb{Q}(\sqrt{3})/\mathbb{Q}, \quad \mathbb{Q}(\sqrt{6})/\mathbb{Q},$$

as $\mathrm{Gal}(\mathbb{Q}(\sqrt{2},\sqrt{3})/Q) \simeq C_2 \times C_2$.

EXAMPLE 8.8. Let $\omega \in \mathbb{C} \setminus \{1\}$ be such that $\omega^5 = 1$. Then

$$f(\omega,\mathbb{Q}) = 1 + X + X^2 + X^3 + X^4$$

and $\mathbb{Q}(\omega)/\mathbb{Q}$ has degree four. Moreover, $\mathbb{Q}(\omega)/\mathbb{Q}$ is a Galois extension and $\mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) \simeq C_4$. If $\sigma \in \mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})$, then $\sigma(\omega) = \omega^i$ for some $i \in \{1,\dots,4\}$. Moreover, for every $i \in \{1,\dots,4\}$ the map $\omega \mapsto \omega^i$ induces an automorphism of $\mathbb{Q}(\omega)/\mathbb{Q}$. Thus $|\mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})| = 4$. Now

$$\sigma_i^k = \mathrm{id} \iff \omega^{i^k} = \sigma_i^k(\omega) = \omega \iff i^k \equiv 1 \bmod 5.$$

Thus the map σ_2 given by $\omega \mapsto \omega^2$ has order four.

Since $\mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) = \langle \sigma \rangle$, where $\sigma(\omega) = \omega^2$, is cyclic of order four, the extension $\mathbb{Q}(\omega)/\mathbb{Q}$ has a unique degree-two subtextension $F/\mathbb{Q}$. Note that $|\langle \sigma^2 \rangle| = 2$ and $\sigma^2(\omega) = \omega^4 = \omega^{-1}$. Thus $F = {}^{\langle \sigma^2 \rangle}\mathbb{Q}(\omega)$. Let $\theta = \omega + \omega^{-1}$. Then

$$\theta^2 = \omega^2 + \omega^3 + 2 = -(1 + \omega + \omega^{-1}) + 2 = 1 - \theta$$

and hence θ is a root of $X^2 + X - 1$. It follows that

$$\theta \in \{(-1+\sqrt{5})/2, (-1-\sqrt{5})/2\}.$$

Therefore $F = \mathbb{Q}(\sqrt{5})$.

Let us mention some other consequences.

EXERCISE 8.9. Let E/K be a finite Galois extension and $F_1,\dots,F_n$ fields such that $K \subseteq F_i \subseteq E$ for all $i \in \{1,\dots,n\}$. For every i let $S_i = \mathrm{Gal}(E/F_i)$. Then

$$\mathrm{Gal}\left(E/\bigcap_{i=1}^n F_i\right) = \left\langle \bigcup_{i=1}^n S_i \right\rangle, \quad \mathrm{Gal}\left(E/\prod_{i=1}^n F_i\right) = \bigcap_{i=1}^n S_i.$$

The following statement is a concrete application of the previous exercise.

Exercise 8.10. Let E/K be a finite Galois extension and $G = \mathrm{Gal}(E/K)$. Assume that G is the direct product $G = S \times T$ of the groups S and T. Let $F = {}^{S}E$ and $L = {}^{T}E$. Then $F \cap L = K$ and $FL = E$.

Proposition 8.11. *Let $E_1/K, \dots, E_r/K$ be Galois extensions. If $E = \prod_{i=1}^{r} E_i$, then E/K is a Galois extension. If, moreover, each E_i/K is finite, then*

$$\theta \colon \mathrm{Gal}(E/K) \to \mathrm{Gal}(E_1/K) \times \cdots \times \mathrm{Gal}(E_r/K), \quad \sigma \mapsto (\sigma|_{E_1}, \dots, \sigma|_{E_r}),$$

is an injective group homomorphism.

Proof. We only do the first part in the case $r = 2$, the general case is left as an exercise. Since E_1/K is algebraic, then E_1E_2/E_2 is algebraic. Since E_2/K is algebraic, E_1E_2/K is algebraic. Similarly, E_1E_2/K is separable.

Let C/K be an algebraic closure such that $E_1E_2 \subseteq C$. If $\sigma \in \mathrm{Hom}(E_1E_2/K, C/K)$, then $\sigma(E_1E_2) \subseteq \sigma(E_1)\sigma(E_2) = E_1E_2$ (do this calculation as an exercise using the fact that E_1/K and E_2/K are normal extensions). Thus E_1E_2/K is normal.

If both E_1/K and E_2/K are finite, then E_1E_2/K is finite.

Then θ is a group homomorphism. We claim that the map θ is injective. Let $\sigma \in \ker\theta$. Then $\sigma|_{E_i} = \mathrm{id}_{E_i}$ for all $i \in \{1, \dots, r\}$. Let $S = \langle\sigma\rangle \subseteq \mathrm{Gal}(E/K)$ and $F = {}^{S}E$. Then $E_i \subseteq F$ for all $i \in \{1, \dots, r\}$ and hence $E \subseteq F$. It follows that $F = E = {}^{\{\mathrm{id}\}}E$ and therefore $S = \{\mathrm{id}\}$, so $\sigma = \mathrm{id}$. $\square$

Exercise 8.12. Let $E_1/K, \dots, E_r/K$ be finite Galois extensions such that for each j one has $E_j \cap (E_1 \cdots E_{j-1}E_{j+1} \cdots E_r) = K$. Then

$$\mathrm{Gal}(E/K) \simeq \mathrm{Gal}(E_1/K) \times \cdots \times \mathrm{Gal}(E_r/K).$$

In this case, $[E : K] = \prod_{i=1}^{r} [E_i : K]$.

§ 8.2. The fundamental theorem of algebra. We now present an easy proof of the fundamental theorem of algebra based on the ideas of Galois Theory. We need the following well-known facts:

1) Every real polynomial of odd degree admits a real root. This means that $\mathbb{R}$ does not admit extension of odd degree > 1.
2) Every complex number admits a square root in $\mathbb{C}$. This means that $\mathbb{C}$ does not admit degree-two extensions.

Theorem 8.13. *The field $\mathbb{C}$ is algebraically closed.*

Proof. Let $E/\mathbb{C}$ be an algebraic finite extension. Then $E/\mathbb{R}$ is finite separable extension of even degree. There exists a Galois extension $L/\mathbb{R}$ such that $E \subseteq L$, so $[L : \mathbb{R}]$ is even. Let $G = \mathrm{Gal}(L/\mathbb{R})$. Then $|G| = 2^m s$ for some odd number s. If T is a 2-Sylow subgroup of G, then there exists a subextension $F/\mathbb{R}$ of degree s. Since $\mathbb{R}$ does not admit extensions of odd degree > 1, $s = 1$ and hence G is a 2-group. Since $L/\mathbb{R}$ is a Galois extension, $L/\mathbb{C}$ is a Galois extension. In particular, $|\mathrm{Gal}(L/\mathbb{C})| = 2^{m-1}$. If $m > 1$, let U be a subgroup of $\mathrm{Gal}(L/\mathbb{C})$ of order 2^{m-2}. Then U corresponds to a subextension $L_1/\mathbb{C}$ of degree two, a contradiction. Hence $m = 1$ and $[L : \mathbb{C}] = 1$, so $L = \mathbb{C}$ and $E = \mathbb{C}$. $\square$

§ 8.3. Purely inseparable extensions. Let E/K be an algebraic extension. In page 7 we defined the **separable closure** of K with respect to E as the field

$$F = \{x \in E : x \text{ is separable over } K\}.$$

Note that $K \subseteq F \subseteq E$ and $F = K(F)$. Moreover, F/K is separable and E/F is a **purely inseparable** extension, meaning that for every $x \in E \setminus F$, the polynomial $f(x,F)$ is not separable.

The number $[E : F]$ is known as the **degree of inseparability** of E/K. We write $[E : K]_{\text{ins}} = [E : F]$. Clearly, E/K is separable if and only if $[E : K]_{\text{ins}} = 1$ and E/K is purely inseparable if and only if $[E : K]_{\text{ins}} = [E : K]$.

Proposition 8.14. *Let K be a field of characteristic $p > 0$ and E/K be an algebraic extension. The following statements are equivalent:*

1) *E/K is purely inseparable.*
2) *If $x \in E$, then $x^{p^m} \in K$ for some $m \geq 0$.*
3) *If $x \in E$, then $f(x,K) = X^{p^m} - a$ for some $a \in K$ and $m \geq 0$.*
4) *$\gamma(E/K) = 1$.*

Proof. We first prove 1) $\implies$ 2). Let $x \in E$ and $f = f(x,K)$. Assume x is not separable. Then $f(x) = 0$ and $f'(x) = 0$, as x is not a simple root. Since $\deg f' < \deg f$ and f is the minimal polynomial of x, it follows that $f' = 0$. The coefficients of f' are of the form ka_k. Since E is a field, $a_k = 0$ if k is not divisible by p. If $a_k \neq 0$, then $k = pm$ for some $m \geq 0$. It follows that $f = g(X^p)$ for some $g \in K[X]$ with $\deg g < \deg f$. We now proceed by induction on the degree of x. The result is true for elements of degree one. So assume the result holds for the element of degree $\leq n$ for some $n \geq 1$. If $x \in E$ is such that $\deg f(x,K) = n+1$, then, since $f(x,K) = g(X^p)$, the element x^p has degree $\leq n$. By the inductive hypothesis, $x^{p^{m+1}} = (x^p)^{p^m} \in K$.

We now prove 2) $\implies$ 3). Let $x \in E$ and m be the minimal positive integer such that $x^{p^m} \in K$. Then x is a root of $X^{p^m} - x^{p^m} \in K[X]$. Since $X^{p^m} - x^{p^m} = (X - x)^{p^m}$, it follows that

$$f(x,K) = (X - x)^r = X^r + \cdots + (-1)^r x^r$$

for some $r \in \{1, \dots, p^m\}$. Write $r = p^s t$ for some integer t coprime with p and s such that $0 \leq s \leq m$. Let $a, b \in \mathbb{Z}$ be such that $ar + bp^m = p^s$. Then

$$x^{p^s} = x^{ar+bp^m} = (x^r)^a \left(x^{p^m}\right)^b \in K.$$

The minimality of m implies that $s \geq m$ and hence $s = m$. Now $p^m t = p^s t = r \leq p^m$, so $t = 1$. This means $f(x,K) = X^{p^m} - x^{p^m}$.

We now prove 3) $\implies$ 4). Let C/K be an algebraic closure that contains E and $\sigma \in \operatorname{Hom}(E/K, C/K)$. Let $x \in E$. We claim that $\sigma(x) = x$. Since $f(x,K) = X^{p^m} - a$,

$$(\sigma(x))^{p^m} = \sigma\left(x^{p^m}\right) = \sigma(a) = a = x^{p^m}.$$

It follows that $\sigma(x)$ is a root of $X^{p^m} - x^{p^m} = (X - x)^{p^m}$. Thus $\sigma(x) = x$.

Finally, we prove that 4) $\implies$ 1). Let C be an algebraic closure of K containing E. Then $\operatorname{Gal}(E/K) = \operatorname{Hom}(E/K, C/K) = \{\mathrm{id}\}$, as $\gamma(E/K) = 1$. If $x \in E$ is separable over K, then

$$f(x,K) = \prod_{y \in O_{\operatorname{Gal}(E/K)}(x)} (X - y) = X - x \in K[X].$$

Thus $x \in K$ and hence E/K is purely inseparable. $\square$

Some consequences:

Exercise 8.15. Let K be a field of characteristic $p>0$ and E/K be finite and purely inseparable. Then $[E:K]=p^s$ for some prime number p and some s. Moreover, $x^{[E:K]}\in K$.

For the first part of the previous exercise, write $E=K(x_1,\dots,x_n)$ and proceed by induction on n.

Exercise 8.16. Let K be of characteristic $p>0$ and E/K be a finite extension such that $[E:K]$ is not divisible by p. Then E/K is separable.

Let K be of characteristic $p>0$, E/K be finite and F be the separable closure of K in E. Since

$$\gamma(E/K)=\gamma(E/F)\gamma(F/K)=\gamma(F/K),$$

it follows that

$$[E:K]=[E:F]\gamma(E/K)=[E:K]_{\text{ins}}\gamma(E/K).$$

§ 8.4. Norm and trace.

Definition 8.17. Let E/K be a finite extension and C/K be an algebraic closure that contains E. Let $A=\operatorname{Hom}(E/K,C/K)$. For $x\in E$ we define the **trace** of x in E/K as

$$\operatorname{trace}_{E/K}(x)=[E:K]_{\text{ins}}\sum_{\varphi\in A}\varphi(x)$$

and the **norm** of x in E/K as

$$\operatorname{norm}_{E/K}(x)=\left(\prod_{\varphi\in A}\varphi(x)\right)^{[E:K]_{\text{ins}}}.$$

As an optional exercise, one can show that these definitions do not depend on the algebraic closure.

We collect some basic properties as an exercise:

Exercise 8.18. Let E/K be a finite extension. The following statements hold:

1) If E/K is not separable, then $\operatorname{trace}_{E/K}(x)=0$ for all $x\in E$.
2) If $x\in K$, then $\operatorname{trace}_{E/K}(x)=[E:K]x$.
3) $\operatorname{trace}_{E/K}(x)\in K$ for all $x\in E$.
4) $\operatorname{norm}_{E/K}(x)=0$ if and only if $x=0$.
5) If $x\in K$, then $\operatorname{norm}_{E/K}(x)=x^{[E:K]}$.
6) $\operatorname{norm}_{E/K}(x)\in K$ for all $x\in E$.

One proves, moreover, that $\operatorname{trace}_{E/K}\colon E\to K$ satisfies

$$\operatorname{trace}_{E/K}(x+\lambda y)=\operatorname{trace}_{E/K}(x)+\lambda\operatorname{trace}_{E/K}(y)$$

for all $x,y\in E$ and $\lambda\in K$, that is to say that $\operatorname{trace}_{E/K}\colon E\to K$ is a linear form in E The norm $\operatorname{norm}_{E/K}\colon E^\times\to K^\times$ is a group homomorphism.

EXERCISE 8.19. Let E/K be a finite extension and $x \in E$. If

$$f(x,K) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0,$$

then $\operatorname{norm}_{E/K}(x) = ((-1)^n a_0)^{[E:K(x)]}$ and $\operatorname{trace}_{E/K}(x) = -[E:K(x)]a_{n-1}$.

EXAMPLE 8.20. Let $E = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Then

$$\operatorname{trace}_{E/\mathbb{Q}}(\sqrt{2}) = 0, \qquad \operatorname{norm}_{E/\mathbb{Q}}(\sqrt{2}) = 4,$$
$$\operatorname{trace}_{E/\mathbb{Q}(\sqrt{2})}(\sqrt{2}) = 2\sqrt{2}, \qquad \operatorname{norm}_{E/\mathbb{Q}(\sqrt{2})}(\sqrt{2}) = 2.$$

EXAMPLE 8.21. If E/K is a finite Galois extension, then

$$\operatorname{trace}_{E/K}(x) = \sum_{\sigma \in \operatorname{Gal}(E/K)} \sigma(x) \quad \text{and} \quad \operatorname{trace}_{E/K}(x) = \prod_{\sigma \in \operatorname{Gal}(E/K)} \sigma(x)$$

for all $x \in E$. In particular, since $E = K(y)$ for some y by Proposition 7.1,

$$\operatorname{trace}_{E/K}(y) = -a_{n-1} \quad \text{and} \quad \operatorname{norm}_{E/K}(y) = (-1)^n a_0,$$

where $f(y,K) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$.

Lecture 9. 22/04/2024

§ 9.1. Finite fields. In this section, p will be a prime number.

PROPOSITION 9.1. *Let m be a positive integer. Up to isomorphism, there exists a unique field F_m of size p^m.*

PROOF. Let C be an algebraic closure of the field $\mathbb{Z}/p$ and let $F_m = \{x \in C : x^{p^m} = x\}$ be the set of roots of $X^{p^m} - X$. Since the polynomial $X^{p^m} - X$ has no multiple roots, $|F_m| = p^m$. Moreover, F_m is the unique subfield of C of size p^m.

To prove the uniqueness, it is enough to note that if K is a field of p^m elements, then K is the decomposition field of $X^{p^m} - X$ over $\mathbb{Z}/p$. □

Let $K = \mathbb{Z}/p$ and C be an algebraic closure of K. We claim that $C = \cup_k F_k$. If $x \in C$, then x is algebraic over K. Since $K(x)/K$ is finite, $K(x)$ is a finite field, say $|K| = p^r$ for some r. Then $x^{p^r} = x$ and hence $x \in F_r$.

EXERCISE 9.2. Prove the following statements:

1) If $x \in F_r$, then $x^{p^{rk}} = x$ for all $k \geq 0$.
2) If $m \mid n$, then $F_m \subseteq F_n$.
3) $F_m \cap F_n = F_{\gcd(m,n)}$.
4) $F_m \subseteq F_n$ if and only if $m \mid n$.

PROPOSITION 9.3. *Every finite extension of a finite field is cyclic.*

PROOF. Let $K = \mathbb{Z}/p$. It is enough to show that F_n/F_m is cyclic if m divides n.

We first prove that F_n/K is cyclic. Let

$$\sigma\colon F_n \to F_n, \quad x \mapsto x^p.$$

Then $\sigma \in \mathrm{Gal}(F_n/K)$ (it is bijective because all field homomorphisms are injective and F_n is finite).

Note that F_n/K is a Galois extension, as F_n is the splitting field over K of the separable polynomial $X^{p^n} - X \in K[X]$. Thus $|\mathrm{Gal}(F_n/K)| = [F_n : K] = n$.

We claim that σ generated $\mathrm{Gal}(F_n/K)$. Since $\sigma^i(x) = x^{p^i}$ for all $i \geq 0$, in particular, $\sigma^n(x) = x^{p^n} = x$. Thus $\sigma^n = \mathrm{id}$ and hence $|\sigma|$ divides n. Let $s = |\sigma|$. We know that $F_n^\times = F_n \setminus \{0\}$ is cyclic, say $F_n^\times = \langle g \rangle$. Since $|g| = p^n - 1$,

$$g = \sigma^s(g) = g^{p^s}$$

and hence $p^s \equiv 1 \bmod (p^n - 1)$. Thus $p^n - 1$ divides $p^s - 1$ and hence n divides s. Therefore $n = s$ and $\mathrm{Gal}(F_n/K) = \langle \sigma \rangle$.

For the general case note that if m divides n, then $\mathrm{Gal}(F_n/F_m)$ is a subgroup of $\mathrm{Gal}(F_n/K)$. Since $\mathrm{Gal}(F_n/K)$ is cyclic, the claim follows. □

If $K = \mathbb{Z}/p$ and m divides n, the subextension F_m corresponds to the unique subgroup of index m of $\mathrm{Gal}(F_n/K) = \langle \sigma \rangle$. This subgroup is $\langle \sigma^m \rangle$, where

$$\sigma^m(x) = x^{p^m} = x^{|F_m|}.$$

Note that $\mathrm{Gal}(F_n/F_m) = \langle \sigma^m \rangle$. The map σ^m is known as the **Frobenius automorphism**.

EXERCISE 9.4. Let E/K be an extension of finite fields. Then E/K is cyclic and $\mathrm{Gal}(E/K) = \langle \tau \rangle$, where $\tau(x) = x^{|K|}$.

§ 9.2. Cyclotomic extensions. For $n \geq 1$ let $G_n(K) = \{x \in K : x^n = 1\}$ be the set of n-roots of one in K. Note that $G_n(K)$ is a cyclic subgroup of $K^\times$ and that $|G_n(K)|$ divides n.

EXAMPLE 9.5. $G_n(\mathbb{R}) = \{-1, 1\}$ if n is odd and $G_n = \{1\}$ if n is even.

EXERCISE 9.6. Let K be a field of characteristic $p > 0$. Let $n = p^s m$ for some m not divisible by p. Then $G_n(K) = G_m(K)$.

EXERCISE 9.7. Let q be a prime number. Then $G_n(\mathbb{Z}/q) \simeq \mathbb{Z}/\gcd(n, q-1)$.

Similarly, one can prove that if K is a finite field, then $G_n(K)$ is a cyclic group of order $\gcd(n, |K^\times|)$.

EXAMPLE 9.8. If C is algebraically closed of characteristic coprime with n, then $G_n(C)$ is cyclic of order n, as $X^n - 1$ has all its roots in C and does not contain multiple roots.

Let K be an algebraically closed field and n be such that n is coprime with the characteristic of K. The set of **primitive n-roots** is defined as

$$H_n(K) = \{x \in G_n(K) : |x| = n\}.$$

DEFINITION 9.9. Let K be an algebraically closed field and n be such that n is coprime with the characteristic of K. The **n-th cyclotomic polynomial** is defined as

$$\Phi_n = \prod_{x \in H_n(K)} (X - x) \in K[X].$$

For $n \geq 1$ the Euler's function is defined as

$$\varphi(n) = |\{k : 1 \leq k \leq n,\ \gcd(k, n) = 1\}|.$$

For example, $\varphi(4) = 2$, $\varphi(8) = \varphi(10) = 4$ and $\varphi(p) = p - 1$ for every prime p.

PROPOSITION 9.10. *Let K be an algebraically closed field and n be such that n is coprime with the characteristic of K. Let A be the ring of integers of K.*

1) $\deg \Phi_n = \varphi(n)$.
2) $\Phi_n \in A[X]$.

PROOF. The first statement is clear. Let us prove 2) by induction on n. The case $n = 1$ is trivial, as $\Phi_1 = X - 1$. Assume that $\Phi_d \in A[X]$ for all d such that $d < n$. In particular,

$$\gamma = \prod_{\substack{d \mid n \\ d \neq n}} \Phi_d \in A[X].$$

Since γ is monic, it follows that $\frac{X^n - 1}{\gamma} \in A[X]$. Now the claim follows from

$$X^n - 1 = \prod_{d \mid n} \Phi_d = \Phi_n \prod_{\substack{d \mid n \\ d \neq n}} \Phi_d = \Phi_n \gamma. \qquad \square$$

By taking degree in the equality $X^n - 1 = \prod_{d|n} \Phi_d$ one gets

$$n = \sum_{d|n} \varphi(d).$$

Definition 9.11. Let $n \geq 2$ and K be a field of characteristic coprime with n. A **cyclotomic extension** of K of index n is a decomposition field of $X^n - 1$ over K.

Let C be an algebraic closure of K and $n \geq 2$ be coprime with the characteristic of K. If follows from Definition 9.11 that a cyclotomic extension of index n is of the form $K(\omega)/K$ for some $\omega \in H_n(K)$.

Proposition 9.12. *A cyclotomic extension of index n is abelian and of degree a divisor of* $\varphi(n)$.

Proof. Let C be an algebraic closure of K and $n \geq 2$ be coprime with the characteristic of K. Let $\omega \in H_n(C)$ and $K(\omega)/K$ be a cyclotomic extension. Then $K(\omega)/K$ is a Galois extension, as it is a decomposition field of a separable polynomial. Let $U = \mathscr{U}(\mathbb{Z}/n)$ be the group of units of $\mathbb{Z}/n$ and

$$\lambda \colon \operatorname{Gal}(K(\omega)/K) \to U, \quad \sigma \mapsto m_\sigma,$$

where m_σ is such that $\sigma(\omega) = \omega^{m_\sigma}$. The map λ is well-defined and it is a group homomorphism, as if $\sigma, \tau \in \operatorname{Gal}(K(\omega)/K)$, then, since

$$(\tau\sigma)(\omega) = \tau(\sigma(\omega)) = \tau(\omega^{m_\sigma}) = (\omega^{m_\sigma})^{m_\tau} = \omega^{m_\sigma m_\tau},$$

it follows that $\lambda(\sigma)\lambda(\tau) = \lambda(\sigma\tau)$. Since λ is injective, $\operatorname{Gal}(K(\omega)/K)$ is isomorphic to a subgroup of the abelian group U. Hence $\operatorname{Gal}(K(\omega)/K)$ is abelian. Moreover, $[K(\omega) : K] = |\operatorname{Gal}(K(\omega)/K)|$ is a divisor of $|U| = \varphi(n)$. □

Exercise 9.13. Prove that a cyclotomic extension $K(\omega)/K$ has degree $\varphi(n)$ if and only if Φ_n is irreducible over K.

Note that Φ_n is irreducible over $\mathbb{Q}$. Some concrete examples:

$$\Phi_1 = X - 1, \quad \Phi_2 = X + 1, \quad \Phi_3 = X^2 + X + 1, \quad \Phi_6 = X^2 - X + 1.$$

If p is a prime number, then $\Phi_p = X^{p-1} + \cdots + X + 1$.

Example 9.14. Φ_5 is irreducible over $\mathbb{Z}/2$. First note that $\Phi_5 = X^4 + \cdots + X + 1$ does not have roots in $\mathbb{Z}/2$. If Φ_5 is reducible, then, since $X^2 + X + 1$ is the unique degree-two monic irreducible polynomial over $\mathbb{Z}/2$, it follows that

$$\Phi_5 = (X^2 + X + 1)(X^2 + X + 1) = (X^2 + X + 1)^2 = X^4 + X^2 + 1,$$

a contradiction.

Exercise 9.15. Prove that $\Phi_{12} = X^4 - X^2 + 1$ is not irreducible over $\mathbb{Z}/5$.

§ 9.3. Hilbert's theorem 90.

Theorem 9.16 (Hilbert). *Let E/K be a cyclic extension. Assume that* $\operatorname{Gal}(E/K)$ *is generated by τ. For $a \in E$,* $\operatorname{norm}_{E/K}(a) = 1$ *if and only if $a = b/\tau(b)$ for some $b \in L \setminus \{0\}$.*

PROOF. Let $n = |G|$. We first prove $\Longleftarrow$. If $a = b/\tau(b)$ and $b \neq 0$, then

$$\mathrm{norm}_{E/K}(a) = a\tau(a)\tau^2(a)\cdots\tau^{n-1}(a) = \frac{b}{\tau(b)}\frac{\tau(b)}{\tau^2(b)}\cdots\frac{\tau^{n-1}(b)}{\tau^n(b)} = 1.$$

Now we prove $\Longrightarrow$. Let $a \in E$ be such that $\mathrm{norm}_{E/K}(a) = 1$. For $c \in E$ let

$$\begin{aligned}
d_0 &= ac,\\
d_1 &= a\tau(a)\tau(c),\\
d_2 &= a\tau(a)\tau^2(a)\tau^2(c),\\
&\vdots\\
d_{n-1} &= \underbrace{a\tau(a)\cdots\tau^{n-1}(a)}_{=\mathrm{norm}_{E/K}(a)}\tau^{n-1}(c) = \tau^{n-1}(c).
\end{aligned}$$

Then

$$a\tau(d_j) = a\tau(a)\cdots\tau^{j+1}(a)\tau^{j+1}(c) = d_{j+1}$$

for all $j \in \{0,\dots,n-2\}$. Let $b = d_0 + \cdots + d_{n-1}$. Then $b \neq 0$, otherwise, if $b = 0$, then, for every $c \in E$,

$$0 = ac + (a\tau(a))\tau(c) + \cdots + (a\tau(a)\cdots\tau^{n-1}(a))\tau^{n-1}(c)$$

implies that $a = 0$ by Dedekind's theorem, a contradiction. So let $c \in E$ be such that $b \neq 0$. Then

$$\begin{aligned}
\tau(b) &= \tau(d_0) + \cdots + \tau(d_{n-1})\\
&= \tau(ac) + \tau(a\tau(c)) + \cdots + \tau(\tau^{n-1}(c))\\
&= \frac{1}{a}(d_1 + \cdots + d_{n-1}) + \tau^n(c)\\
&= \frac{1}{a}(d_0 + \cdots + d_{n-1})\\
&= b/a.
\end{aligned}$$

□

EXERCISE 9.17. Let E/K be a cyclic extension. Assume that $\mathrm{Gal}(E/K)$ is generated by τ. Prove that for $a \in E$, $\mathrm{trace}_{E/K}(a) = 0$ if an only if $a = b - \tau(b)$ for some $b \in L \setminus \{0\}$.

COROLLARY 9.18. *Let $a,b,c \in \mathbb{Z}$ be such that $a^2 + b^2 = c^2$. Then*

$$(a,b,c) = \lambda(r^2 - s^2, -2rs, r^2 + s^2)$$

for some $r,s \in \mathbb{Z}$ and some $\lambda \in \mathbb{Z}$.

PROOF. We work with the extension $\mathbb{Q}(i)/\mathbb{Q}$. Note that $\mathrm{Gal}(\mathbb{Q}(i),\mathbb{Q}) = \{\mathrm{id},\gamma\}$ is cyclic, where $\gamma\colon \mathbb{Q}(i) \to \mathbb{Q}(i)$, $z \mapsto \bar{z}$, is the complex conjugation. We may assume that $c \neq 0$, otherwise $a = b = 0$ and the result is trivial. Write $(a/c)^2 + (b/c)^2 = 1$ and let $\alpha = (a/c) + (b/c)i \in \mathbb{Q}(i)$. Then $\mathrm{norm}_{\mathbb{Q}(i)/\mathbb{Q}}(\alpha) = 1$. By Hilbert's theorem, there exists $\beta \in \mathbb{Q}(i) \setminus \{0\}$ such that

$$\alpha = a + bi = \frac{\gamma(\beta)}{\beta}.$$

Note that if $m\in\mathbb{Z}\setminus\{0\}$, then $\frac{\gamma(m\beta)}{m\beta}=\frac{\gamma(\beta)}{\beta}$. There exists $m\in\mathbb{Z}\setminus\{0\}$ such that $m\beta\in\mathbb{Z}[i]$, say $m\beta=r+is$ with $r,s\in\mathbb{Z}$. Then

$$\alpha=\frac{\gamma(\beta)}{\beta}=\frac{\gamma(m\beta)}{m\beta}=\frac{r-is}{r+is}=\frac{r^2-s^2-2rsi}{r^2+s^2}.$$

From this the claim follows. □

Exercise 9.19. Let $A,B\in\mathbb{Z}$ be such that A^2-4B is not a square. Prove that a solution $(x,y,z)\in\mathbb{Z}^3$ of $x^2+Axy+By^2=z^2$ is proportional to

$$(r^2-Bs^2, 2rs+As^2, r^2+Ars+Bs^2).$$

§ 9.4. Group cohomology. Let G be a group and A be a **(left) G-module**. This means that A is an abelian group together with a map

$$G\times A\to A,\quad (g,a)\mapsto g\cdot a$$

such that $1\cdot a=a$ for all $a\in A$, $(gh)\cdot a=g\cdot(h\cdot a)$ for all $g,h\in G$ and $a\in A$ and $g\cdot(a+b)=g\cdot a+g\cdot b$ for all $g\in G$ and $a,b\in A$.

Example 9.20. The group $\operatorname{Gal}(\mathbb{C}/\mathbb{R})$ acts on $\mathbb{C}$ and $\mathbb{C}^\times$. Moreover, it acts trivially on $\mathbb{R}$ and $\mathbb{R}^\times$.

More generally, if E/K is a finite Galois extension, then the Galois group $\operatorname{Gal}(E/K)$ acts on E and $E^\times$.

Definition 9.21. Let G be a group and M and N be G-modules. A map $f\colon M\to N$ is a **homomorphism** of G-modules if $f(\sigma\cdot m)=\sigma\cdot f(m)$ for all $m\in M$ and $\sigma\in G$.

Definition 9.22. Let G be a group and M be a G-module. The submodule of **G-invariants** is defined as

$$M^G=\{m\in M:\sigma\cdot m=m\text{ for all }\sigma\in G\}.$$

Note that M^G is the largest submodule of the G-module M where G acts trivially. For example, if $G=\operatorname{Gal}(E/K)$, then $E^G=K$.

Proposition 9.23. *Let G be a group. If the sequence of G-modules and G-module homomorphism*

$$0\longrightarrow P\xrightarrow{\alpha}M\xrightarrow{\beta}N\longrightarrow 0$$

is exact, then

$$0\longrightarrow P^G\xrightarrow{\alpha^0}M^G\xrightarrow{\beta^0}N^G$$

is exact, where α^0 is the restriction $\alpha|_{P^G}$ of α to P^G and β^0 is the restriction $\beta|_{M^G}$ of β to M^G.

Proof. Since α is injective, the restriction α^0 is injective.
Note that $\ker\beta^0=\ker\beta\cap M^G\subseteq\ker\beta$.
We claim that $\alpha^0(P^G)=\alpha(P)\cap M^G$. If $m\in\alpha(P)\cap M^G$, then $\alpha(p)=m$ for some $p\in P$ and $\sigma\cdot m=m$. Since

$$\alpha(p)=m=\sigma\cdot m=\sigma\cdot\alpha(p)=\alpha(\sigma\cdot p),$$

$\sigma \cdot p - p \in \ker \alpha = \{0\}$. Hence $\sigma \cdot p = p$ and $p \in P^G$. Conversely, if $m \in \alpha^0(P^G)$, then $m = \alpha(p)$ for some $p \in P^G$. If $\sigma \in G$, then

$$\sigma \cdot m = \sigma \cdot \alpha(p) = \alpha(\sigma \cdot p) = \alpha(p) = m.$$

Hence $m \in M^G \cap \alpha(P)$.

Now

$$\alpha^0(P^G) = \alpha(P) \cap M^G = \ker \beta \cap M^G = \ker \beta^0. \qquad \square$$

Note that in the previous proposition, we did not prove that the map $\beta|_{M^G}$ is surjective.

EXAMPLE 9.24. Let $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Consider the following exact sequence of G-modules:

$$1 \longrightarrow \{-1,1\} \longrightarrow \mathbb{C}^\times \xrightarrow{\beta} \mathbb{C}^\times \longrightarrow 1$$

where $\beta(z) = z^2$. Note that β is surjective. Take invariants to obtain the sequence

$$0 \longrightarrow \{-1,1\} \longrightarrow \mathbb{R}^\times \xrightarrow{\beta^0} \mathbb{R}^\times$$

where $\beta^0(x) = x^2$. Note that β^0 is not surjective!

DEFINITION 9.25. Let G be a group and N be a G-module. We define

$$H^0(G,M) = M^G,$$
$$C^1(G,M) = \{\phi \colon G \to M : \phi \text{ is a map}\},$$
$$Z^1(G,M) = \{\phi \in C^1(G,M) : \phi(\sigma\tau) = \phi(\sigma) + \sigma \cdot \phi(\tau) \text{ for all } \sigma, \tau \in G\},$$

Note that $Z^1(G,M)$ is an abelian group with the operation

$$(\phi + \phi_1)(\sigma) = \phi(\sigma) + \phi_1(\sigma).$$

Moreover, if $\phi \in Z^1(G,M)$, then $\phi(1_G) = 0_M$. To prove this fact, note that

$$\phi(1_G) = \phi(1_G 1_G) = \phi(1_G) + 1_G \cdot \phi(1_G) = \phi(1_G) + \phi(1_G)$$

implies that $\phi(1_G) = 0_M$.

EXAMPLE 9.26. Let G be a group and M be a G-module. Fix $m \in M$. Then the map $\phi \colon G \to M$, $\phi(\sigma) = \sigma \cdot m - m$, is an element of $Z^1(G,M)$, because

$$\begin{aligned}\phi(\sigma\tau) &= (\sigma\tau) \cdot m - m \\ &= (\sigma\tau) \cdot m - \sigma \cdot m + \sigma \cdot m - m \\ &= \sigma \cdot (\tau \cdot m - m) + \sigma \cdot m - m \\ &= \sigma \cdot \phi(\tau) + \phi(\sigma)\end{aligned}$$

for all $\sigma, \tau \in G$.

DEFINITION 9.27. Let G be a group and M be a G-module. The set $B^1(G,M)$ of **coboundaries** is the set of elements $\phi \in C^1(G,M)$ such that there is a fixed $m \in M$ such that $\phi(\sigma) = \sigma \cdot m = m$ for all $\sigma \in G$.

We proved in Example 9.26 that $B^1(G,M) \subseteq Z^1(G,M)$. A direct calculation shows that, in fact, $B^1(G,M)$ is a subgroup of $Z^1(G,M)$.

Definition 9.28. Let G be a group and M be a G-module. The **first cohomology group** of G with coefficients in M is defined as the quotient

$$H^1(G,M) = Z^1(G,M)/B^1(G,M).$$

Example 9.29. If G acts trivially on M, then

$$H^0(G,M) = M^G = M, \quad B^1(G,M) = \{0\}, \quad Z^1(G,M) = \mathrm{Hom}(G,M).$$

Hence $H^1(G,M) \simeq \mathrm{Hom}(G,M)$.

Example 9.30. Let $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{\mathrm{id}, \gamma\}$, where $\gamma\colon \mathbb{C} \to \mathbb{C}$, $z \mapsto \bar{z}$, is the complex conjugation. Then

$$H^0(G,\mathbb{R}^\times) = \left(\mathbb{R}^\times\right)^G = \mathbb{R}^\times.$$

Since G acts trivially on $\mathbb{R}^\times$,

$$H^1(G,\mathbb{R}^\times) = \mathrm{Hom}(G,\mathbb{R}^\times) \simeq \mathrm{Hom}(G,\{-1,1\}) \simeq \mathbb{Z}/2.$$

The following lemma will be useful.

Lemma 9.31. *Let G be a group and $\alpha\colon M \to N$ be a homomorphism of G-modules. Then*

$$\alpha^1 \colon H^1(G,M) \to H^1(G,N), \quad \phi + B^1(G,M) \mapsto \alpha \circ \phi + B^1(G,N),$$

is a group homomorphism.

Proof. Let us prove that the map α^1 is well-defined. If $\phi - \phi' \in B^1(G,M)$, then there exists a fixed $m \in M$ such that $(\phi - \phi')(\sigma) = \sigma \cdot m - m$ for all $\sigma \in G$. Let $n = \alpha(m) \in N$. For $\sigma \in G$,

$$\alpha((\phi - \phi')(\sigma)) = \alpha(\sigma \cdot m - m) = \sigma \cdot \alpha(m) - \alpha(m) = \sigma \cdot n - n.$$

Thus $\alpha \circ \phi - \alpha \circ \phi' \in B^1(G,N)$.

We now prove that α^1 is a group homomorphism. If $\phi, \phi' \in Z^1(G,M)$, then

$$\begin{aligned}
\alpha^1(\phi + B^1(G,M) + \phi' + B^1(G,M)) &= \alpha^1(\phi + \phi' + B^1(G,M)) \\
&= \alpha \circ (\phi + \phi') + B^1(G,N) \\
&= \alpha \circ \phi + \alpha \circ \phi' + B^1(G,N) \\
&= \alpha \circ \phi + B^1(G,N) + \alpha \circ \phi' + B^1(G,N) \\
&= \alpha^1(\phi + B^1(G,M)) + \alpha^1(\phi' + B^1(G,M)).
\end{aligned}$$ $\square$

We will provide a detailed proof of the upcoming result. The theorem will be established by applying a **diagram chasing** technique, a widely used tool in homological algebra. The proof is tedious and may seem intricate, but the method essentially involves "chasing" elements around a (commutative) diagram until we attain the desired result.

Theorem 9.32. *Let G be a group and*

$$0 \longrightarrow P \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$$

be an exact sequence of G-modules and G-module homomorphism. Then there exists a ***connection homomorphism*** δ *such that the sequence*

$$\begin{array}{l} 0 \longrightarrow H^0(G,P) \xrightarrow{\alpha^0} H^0(G,M) \xrightarrow{\beta^0} H^0(G,N) \\ \xrightarrow{\delta} H^1(G,P) \xrightarrow{\alpha^1} H^1(G,M) \xrightarrow{\beta^1} H^1(G,N) \end{array} \tag{9.1}$$

of abelian groups and group homomorphisms is exact.

Proof. By Proposition 9.23, the sequence is exact at $H^0(G,P) = P^G$, $H^0(G,M) = M^G$ and $H^0(G,N) = N^G$. Note that, in particular, $\alpha\colon P \to \alpha(P)$ is a bijective group homomorphism.

Let us construct the connecting homomorphism $\delta\colon H^0(G,N) \to H^1(G,P)$. For $n \in N^G$, let $m \in M$ be such that $\beta(m) = n$. We define $\delta(n) = \phi + B^1(G,P)$, where

$$\phi(\sigma) = \alpha^{-1}(\sigma \cdot m - m).$$

Note that $\sigma \cdot m - m \in \operatorname{im}\alpha = \ker\beta$, as

$$\beta(\sigma \cdot m - m) = \sigma \cdot \beta(m) - \beta(m) = \sigma \cdot n - n = 0.$$

Let us prove that the map δ is well-defined: if $m, m' \in M$ are such that $\beta(m) = \beta(m') = n$, then $m - m' \in \ker\beta = \alpha(P)$. For $\sigma \in G$, write $\phi'(\sigma) = \sigma \cdot m' - m'$. Since $m - m' = \alpha(p)$ for some $p \in P$ and α^{-1} is a homomorphism of G-modules,

$$\begin{aligned} \phi(\sigma) - \phi'(\sigma) &= \alpha^{-1}(\sigma \cdot m - m) - \alpha^{-1}(\sigma \cdot m' - m') \\ &= \alpha^{-1}(\sigma \cdot (m - m')) - \alpha^{-1}(m - m') \\ &= \alpha^{-1}(\sigma \cdot \alpha(p) - \alpha(p)) \\ &= \sigma \cdot p - p. \end{aligned}$$

Thus $\phi - \phi' \in B^1(G,P)$.

Note that $\phi \in Z^1(G,P)$, because

$$\begin{aligned} \phi(\sigma\tau) &= \alpha^{-1}((\sigma\tau) \cdot m - m) \\ &= \alpha^{-1}((\sigma\tau) \cdot m - \sigma \cdot m + \sigma \cdot m - m) \\ &= \alpha^{-1}(\sigma \cdot (\tau \cdot m - m)) + \alpha^{-1}(\sigma \cdot m - m) \\ &= \sigma \cdot \phi(\tau) + \phi(\sigma) \end{aligned}$$

holds for all $\sigma, \tau \in G$.

We now prove that the sequence (9.1) is exact at $H^0(G,N) = N^G$. We need to prove that $\ker\delta = \operatorname{im}\beta^0$. To prove $\supseteq$ note that if $m \in M^G$ is such that $\delta(\beta(m)) = \phi + B^1(G,P)$, then

$$\phi(\sigma) = \alpha^{-1}(\sigma \cdot m - m) = 0.$$

Conversely, if $n \in \ker\delta$, then there exists $m \in M$ such that $\beta(m) = n$ and $\delta(\beta(m)) = \phi + B^1(G,P)$, where $\phi \in B^1(G,P)$ and $\phi(\sigma) = \alpha^{-1}(\sigma \cdot m - m)$ for all $\sigma \in G$. Since $\phi \in B^1(G,P)$, there exists $p \in P$ such that $\phi(\sigma) = \sigma \cdot p - p$ for all $\sigma \in G$. Note that

$$\beta(m - \alpha(p)) = \beta(m) - \beta(\alpha(p)) = \beta(m) = n.$$

Moreover, $m - \alpha(p) \in M^G$, as $\sigma \cdot (m - \alpha(p)) = m - \alpha(p)$. Hence $n \in \operatorname{im}\beta^0$.

We now prove that (9.1) is exact at $H^1(G,P)$, that is $\operatorname{im}\delta = \ker\alpha^1$. To prove $\subseteq$ note that for $n \in N^G$, $\delta(n) = \phi + B^1(G,P)$, where $\phi(\sigma) = \alpha^{-1}(\sigma\cdot m - m)$ for all $\sigma \in G$ and some $m \in M$ such that $\beta(m) = n$. In particular, $\alpha\circ\phi \in B^1(G,M)$. Then

$$\alpha^1(\phi + B^1(G,P)) = \alpha\circ\phi + B^1(G,M) = B^1(G,M).$$

To prove $\supseteq$, let $\phi + B^1(G,P) \in \ker\alpha^1$. Then $\alpha\circ\phi \in B^1(G,M)$, that is $\alpha(\phi(\sigma)) = \sigma\cdot m - m$ for all $\sigma \in G$ and some $m \in M$. Note that

$$\delta(\beta(m)) = \psi + B^1(G,P),$$

where $\psi(\sigma) = \alpha^{-1}(\sigma\cdot m - m)$. This implies that $\alpha(\psi(\sigma)) = \alpha(\phi(\sigma))$ for all $\sigma \in G$. Since α is injective, $\psi = \phi$. Therefore $\phi + B^1(G,P)$ belongs to the image of δ.

Finally, we prove that the sequence (9.1) is exact at $H^1(G,M)$, that is $\operatorname{im}\alpha^1 = \ker\beta^1$. To prove $\subseteq$ note that

$$\beta^1(\alpha^1(\phi + B^1(G,P))) = \beta^1(\alpha\circ\phi + B^1(G,M)) = (\beta\circ\alpha)\circ\phi + B^1(G,N) = B^1(G,N).$$

Conversely, let $\phi + B^1(G,M) \in \ker\beta_1$. Then $\beta\circ\phi \in B^1(G,N)$. Thus there exists $n \in N$ such that $\beta(\phi(\sigma)) = \sigma\cdot n - n$ for all $\sigma \in G$. Since β is surjective, $n = \beta(m)$ for some $m \in M$. Hence

$$\beta(\phi(\sigma)) = \sigma\cdot n - n = \sigma\cdot\beta(m) - \beta(m) = \beta(\sigma\cdot m - m).$$

For each $\sigma \in G$, $\phi(\sigma) - (\sigma\cdot m - m) \in \ker\beta = \operatorname{im}\alpha$. and therefore $\phi(\sigma) - (\sigma\cdot m - m) = \alpha(\rho_\sigma)$. This defines a map $\rho\colon G \to P$, $\sigma \mapsto \rho_\sigma$. We claim that $\rho \in Z^1(G,P)$. If $\sigma,\tau \in G$, then

$$\begin{aligned}
\alpha(\rho_{\sigma\tau}) &= \phi(\sigma\tau) - ((\sigma\tau)\cdot m - m)\\
&= \phi(\sigma) + \sigma\cdot\phi(\tau) - (\sigma\cdot(\tau\cdot m - m) + \sigma\cdot m - m)\\
&= \alpha(\rho_\sigma) + \sigma\cdot\alpha(\rho_\tau).
\end{aligned}$$

Since α is injective, it follows that $\rho \in Z^1(G,P)$. Now

$$\alpha_1(\rho + B^1(G,P)) = \alpha\circ\rho + B^1(G,M) = \phi + B^1(G,M).$$

□

THEOREM 9.33. *Let G be a finite group and M be a G-module. Then*

$$|G|H^1(G,M) = \{0\}.$$

PROOF. Let $n = |G|$. It is enough to prove that if $\phi \in Z^1(G,M)$, then $n\phi \in B^1(G,M)$. Let $\phi \in Z^1(G,M)$ and $\sigma \in G$. Then

$$\phi(\sigma\tau) = \phi(\sigma) + \sigma\cdot\phi(\tau)$$

for all $\tau \in G$. Summing over all possible $\tau \in G$, we obtain that

$$\sum_{\tau\in G}\phi(\tau) = \sum_{\tau\in G}\phi(\sigma\tau) = \sum_{\tau\in G}\sigma\cdot\phi(\tau) + \sum_{\tau\in G}\phi(\sigma) = n\phi(\sigma). \tag{9.2}$$

Let $m = -\sum_{\tau\in G}\phi(\tau) \in M$. Then (9.2) can be rewritten as

$$-m = \sum_{\tau\in G}\phi(\tau) = \sigma\cdot\sum_{\tau\in G}\phi(\tau) + n\phi(\sigma) = -\sigma\cdot m + n\phi(\sigma).$$

Thus $n\phi(\sigma) = \sigma\cdot m - m$ and hence $n\phi \in B^1(G,M)$. □

EXERCISE 9.34. Let G be a finite group and M be a finite G-module of size coprime to $|G|$. Prove that $H^1(G,M) = \{0\}$.

EXERCISE 9.35. Let G be a finite group and M be a finitely generated G-module. Prove that $H^1(G,M)$ is finite.

Lecture 10. 29/04/2024

Proposition 10.1. *Let $n \geq 2$ and K be a field containing a primitive n-root of one. If $a \in K^\times$ and E/K is a decomposition field of $f = X^n - a$, then E/K is cyclic of degree d, where d divides n. Moreover,*

$$d = \min\{k : a^k \in K^n\},$$

where $K^n = \{x \in K : x = y^n \text{ for some } y \in K\}$. Conversely, if E/K is cyclic of degree n, then E/K is a decomposition field of an irreducible polynomial of the form $X^n - a$ for some $a \in K^\times$.

Proof. A decomposition field of f over K is of the form $K(\alpha)$, where $\alpha^n = a$. Thus $K(\alpha)/K$ is a Galois extension. If $\sigma \in \mathrm{Gal}(K(\alpha)/K)$, then $\sigma(\alpha)$ is a root of f, so $\sigma(\alpha) = \omega_\sigma \alpha$, where $\omega_\sigma \in G_n(K)$. This means that there exists an injective map

$$\lambda \colon \mathrm{Gal}(K(\alpha)/K) \to G_n(K), \quad \sigma \mapsto \omega_\sigma.$$

Moreover, λ is a group homomorphism, as

$$\sigma\tau(\alpha) = \sigma(\tau(\alpha)) = \sigma(\omega_\tau \alpha) = \omega_\tau \sigma(\alpha) = \omega_\tau \omega_\sigma \alpha.$$

Therefore $\mathrm{Gal}(K(\alpha)/K)$ is isomorphic to a subgroup of $G_n(K)$. In particular, $\mathrm{Gal}(K(\alpha)/K)$ is cyclic and $|\mathrm{Gal}(K(\alpha)/K)|$ divides n.

Let $d = |\mathrm{Gal}(K(\alpha)/K)|$. Since $a = \alpha^n$,

$$\mathrm{norm}_{K(\alpha)/K}(\alpha)^n = \mathrm{norm}_{K(\alpha)/K}(a) = a^d.$$

Thus $a^d \in K^n$, as $\mathrm{norm}_{K(\alpha)/K}(\alpha) \in K$. If $a^k \in K^n$, say $a^k = c^n$ for some $c \in K$, then

$$c^n = a^k = (\alpha^n)^k = (\alpha^k)^n \implies \alpha^k = c\omega \in K$$

for some $\omega \in G_n(K)$. Thus α is a root of $X^k - \alpha^k \in K[X]$ and hence $k \geq d$.

Note that $f(\alpha, K) = X^d - \alpha^d$.

Let E/K be cyclic of degree n. Assume that $\mathrm{Gal}(E/K) = \langle \sigma \rangle$. If ω is a primitive n-root of one,

$$\mathrm{norm}_{E/K}(\omega) = \omega^n = 1.$$

By Hilbert's theorem 90, there exists $b \in E^\times$ such that $\omega = \sigma(b)/b$. Thus $\sigma(b) = \omega b$ and hence $\sigma^i(b) = \omega^i b$ for all $i \geq 0$. Since $|\{b, \sigma(b), \dots, \sigma^{n-1}(b)\}| = n$, it follows that $E = K(b)$. Moreover,

$$\sigma(b^n) = \sigma(b)^n = (\omega b)^n = b^n$$

and hence $b^n \in K$. This means that E/K is a decomposition field of $X^n - b^n$. Note that $X^n - b^n$ is irreducible, as $[E : K] = [K(b) : K] = n$. □

Proposition 10.2. *Let K be a field of characteristic $p > 0$.*

1) *Let $a \in K$ and $f = X^p - X - a$. Then f is irreducible over K or all the roots of f belong to K. In the first case, if b is a root of f, then $K(b)/K$ is a cyclic extension of degree p.*
2) *Every cyclic extension of degree p is a decomposition field of an irreducible polynomial of the form $X^p - X - a$.*

Proof. We first prove 1). Let K_0 be the prime field of K. Note that $K_0 \simeq \mathbb{Z}/p$. Let b be a root of f and let $x \in K_0$. Then

$$f(b+x) = (b+x)^p - (b+x) - a = (b^p - b - a) + (x^p - x) = 0$$

and thus $\{b + x : x \in K_0\}$ is the set of roots of f. Note that $f' = -1$, so f has no multiple roots.

We claim that if $b\notin K$, then f is irreducible. If f is not irreducible, then $f=gh$ for some $g,h\in K[X]$ such that $0<\deg g<p$. There exists a subset S of K_0 such that $g=\prod_{x\in S}(X-(b+x))$ and hence

$$|S|b+\sum_{x\in S}x=\sum_{x\in S}(b+x)\in K.$$

This implies that $|S|b\in K$ and hence, since $|S|\in K^{\times}$, it follows that $b\in K$.

Since $K(b)/K$ is a decomposition field of a separable polynomial, $K(b)/K$ is a Galois extension. Moreover, $|\operatorname{Gal}(K(b)/K)|=[K(b):K]=p$ and hence $\operatorname{Gal}(K(b)/K)$ is cyclic.

We now prove 2). Let E/K be cyclic of degree p. Assume that $\operatorname{Gal}(E/K)=\langle\sigma\rangle$. Since $\operatorname{trace}_{E/K}(1)=p=0$, Hilbert's theorem implies that there exists $b\in E$ such that $\sigma(b)=b+1$. In particular, $b\notin K$ and thus $E=K(b)$. Moreover, since

$$\sigma(b^p-b)=\sigma(b)^p-\sigma(b)=(b+1)^p-(b+1)=b^p-b,$$

it follows that $b^p-b\in K$. Thus $f(b,K)=X^p-X-(b^p-b)\in K[X]$. □

§ 10.1. Symmetric polynomials. Let K be a field and $\{t_1,\dots,t_n\}$ be a commuting set of independent variables. Let $E=K(t_1,\dots,t_n)$ and $f=\prod_{i=1}^{n}(X-t_i)\in E[X]$. Then

$$f=X^n+\sum_{i=1}^{n}(-1)^i s_iX^{n-i},$$

where

$$\begin{aligned}
s_1&=t_1+t_2+\cdots+t_n,\\
s_2&=\sum_{1\leq i<j\leq n}t_it_j,\\
&\vdots\\
s_n&=t_1t_2\cdots t_n.
\end{aligned}$$

For example,

$$(X-t_1)(X-t_2)(X-t_3)=X^3-(t_1+t_2+t_3)X^2+(t_1t_2+t_2t_3+t_1t_3)X-t_1t_2t_3.$$

The polynomials $s_1,s_2,\dots,s_n$ are known as the **elementary symmetric polynomials** in the variables $t_1,\dots,t_n$. Note that $\deg s_i=i$.

Let $\sigma\in\mathbb{S}_n$ and

$$\alpha_\sigma\colon K[t_1,\dots,t_n]\to K[t_1,\dots,t_n],\quad t_i\mapsto t_{\sigma(i)}\quad\text{for all } i.$$

Then α_σ is a bijective homomorphism of K-algebras. In fact, $\alpha_\sigma^{-1}=\alpha_{\sigma^{-1}}$. Note that

$$\alpha_\sigma(h(t_1,\dots,t_n))=h(t_{\sigma(1)},\dots,t_{\sigma(n)}).$$

Since α_σ is injective, it induces an element $\widehat{\sigma}\in\operatorname{Gal}(E/K)$ given by

$$\widehat{\sigma}\left(\frac{h}{g}\right)=\frac{\alpha_\sigma(h)}{\alpha_\sigma(h)}.$$

The map $\mathbb{S}_n\to\operatorname{Gal}(E/K)$, $\sigma\mapsto\widehat{\sigma}$, is an injective group homomorphism. Thus $\{\widehat{\sigma}:\sigma\in\mathbb{S}_n\}\simeq\mathbb{S}_n$.

Definition 10.3. Let $g\in K[t_1,\dots,t_n]$. Then g is **symmetric** if $\widehat{\sigma}(g)=g$ for all $\sigma\in\mathbb{S}_n$.

We write P to denote the set of symmetric polynomials in $K[t_1,\dots,t_n]$. Clearly, P is a subalgebra of $K[t_1,\dots,t_n]$. The following statements hold:

1) $K \subseteq P$.
2) $\sum_{i=1}^{n} t_i^r \in P$ for all $r \geq 1$.
3) $s_i \in P$ for all i.
4) $K(P) \subseteq {}^G E$, where $G = \{\widehat{\sigma} : \sigma \in \mathbb{S}_n\}$.

Let $F = K(s_1, s_2, \dots, s_n)$. Then E/F is a Galois extension, as it is a decomposition field of f.

Proposition 10.4. $[E : F] \leq n!$.

Proof. We proceed by induction on n. The case $n = 1$ is clear, as $E = F$. Assume that $n > 1$. Let $u_1, \dots, u_{n-1}$ be the elementary symmetric polynomials in $t_1, \dots, t_{n-1}$. Then

$$s_i = u_i + t_n u_{i-1}$$

for all $i \in \{1, \dots, n\}$, where $u_0 = 1$ and $u_n = 0$. Note that $u_1 = s_1 - t_n$ and $u_i = s_i - t_n u_{i-1}$ for all i. Since $K(s_1, \dots, s_n, t_n) = K(u_1, \dots, u_{n-1}, t_n)$,

$$F(t_n) = K(u_1, \dots, u_{n-1}, t_n) = K(t_n)(u_1, \dots, u_{n-1})$$

and

$$[E : F] = [E : F(t_n)][F(t_n) : F] \leq n[E : F(t_n)].$$

Note that $E = K(t_1, \dots, t_n) = K(t_n)(t_1, \dots, t_{n-1})$. By the inductive hypothesis, $[E : F(t_n)] \leq (n-1)!$ and hence $[E : F] \leq n!$, as desired. □

Theorem 10.5. ${}^G E = F$.

Proof. By Artin's theorem,

$$\left[{}^G E : F\right] = \frac{[E : F]}{\left[E : {}^G E\right]} \leq \frac{n!}{\left[E : {}^G E\right]} = 1$$

and hence ${}^G E = F$. □

Exercise 10.6. Prove that $\mathrm{Gal}(E/F) \simeq \mathbb{S}_n$.

Exercise 10.7. Prove that $\{s_1, \dots, s_n\}$ is algebraically independent over K.

Exercise 10.8. Prove that every symmetric polynomial in $t_1, \dots, t_n$ can be written as a rational fraction in $s_1, \dots, s_n$.

§ 10.2. Solvable groups. Let G be a group. If $x, y \in G$ we define the **commutator** of x and y as

$$[x, y] = xyx^{-1}y^{-1}.$$

Note that $[x, y] = 1$ if and only if $xy = yx$. Moreover, $[x, y]^{-1} = [y, x]$. The **commutator (or derived) subgroup** $[G, G]$ of G is defined as the subgroup of G generated by all commutators, i.e.

$$[G, G] = \langle [x, y] : x, y \in G \rangle.$$

This means that every element of $[G, G]$ is a finite product of commutators, so every element of $[G, G]$ is of the form $\prod_{i=1}^{m} [x_i, y_i]$. In general, the commutator subgroup is not equal to the set of commutators!

EXAMPLE 10.9. This example is taken from the book [**1**] of Carmichael. Let G be the subgroup of $\mathbb{S}_{16}$ generated by the permutations

$$a = (13)(24), \qquad b = (57)(68),$$
$$c = (9\,11)(10\,12), \qquad d = (13\,15)(14\,16),$$
$$e = (13)(57)(9\,11), \qquad f = (12)(34)(13\,15),$$
$$g = (56)(78)(13\,14)(15\,16), \qquad h = (9\,10)(11\,12).$$

Then $[G,G]$ has order 16. However, the set $\{[x,y] : x,y \in G\}$ of commutators has 15 elements:

```
julia> a = @perm (1,3)(2,4);

julia> b = @perm (5,7)(6,8);

julia> c = @perm (9,11)(10,12);

julia> d = @perm (13,15)(14,16);

julia> e = @perm (1,3)(5,7)(9,11);

julia> f = @perm (1,2)(3,4)(13,15);

julia> g = @perm (5,6)(7,8)(13,14)(15,16);

julia> h = @perm (9,10)(11,12);

julia> S16 = symmetric_group(16);

julia> G = sub(S16, [a,b,c,d,e,f,g,h])[1];

julia> commutators = G -> Set(comm(x,y) for x in G, y in G);

julia> length(commutators(G))
15

julia> order(derived_subgroup(G)[1])
16
```

EXERCISE 10.10. Let G be a group. Prove the following facts:

1) G is abelian if and only if $[G,G] = \{1\}$.
2) $[G,G]$ is a normal subgroup of G.
3) $G/[G,G]$ is abelian.
4) If H is a subgroup of G and $[G,G] \subseteq H$, then H is normal in G.
5) If H is a normal subgroup of G, then G/H is abelian if and only if $[G,G] \subseteq H$.

DEFINITION 10.11. Let G be a group. The **derived series** of G is defined as $G^{(0)} = G$ and $G^{(k+1)} = [G^{(k)}, G^{(k)}]$ for $k \geq 0$.

EXERCISE 10.12. Prove that $G^{(k)}$ is normal in G for all k.

Why derived series? We cannot explain this here, but let us use the following notation. We write $G' = [G,G]$, $G'' = [G',G']$... Note that

$$G \supseteq G' \supseteq G'' \supseteq \cdots$$

Exercise 10.13. Let $n \geq 3$. Prove that $[\mathbb{S}_n, \mathbb{S}_n] = \mathbb{A}_n$.

Example 10.14. Let $K = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\}$. Then K is a normal subgroup of $\mathbb{A}_4$. One proves that $[\mathbb{A}_4, \mathbb{A}_4] = K$.

A group G is said to be **simple** if there are no proper non-trivial subgroups of G. If p is a prime number, then the group $\mathbb{Z}/p$ of integers modulo p is a simple group. We will prove later that $\mathbb{A}_n$ is simple if $n \geq 5$.

Example 10.15. Let $n \geq 5$. Since $\mathbb{A}_n$ is a non-abelian simple group, $[\mathbb{A}_n, \mathbb{A}_n] = \mathbb{A}_n$.

Let us show that $\mathbb{A}_5$ is a non-abelian simple group. Hence it is not solvable:

```
julia> A5 = alternating_group(5)
Alt( [ 1 .. 5 ] )

julia> is_abelian(A5)
false

julia> is_simple(A5)
true

julia> is_solvable(A5)
false
```

Definition 10.16. A group G is **solvable** if and only if $G^{(m)} = \{1\}$ for some m.

Every abelian group is solvable.

Exercise 10.17. Prove that $\mathbb{S}_n$ is solvable if and only if $n \leq 4$.

Let us compute (with the computer software Oscar) the derived series of the symmetric group $\mathbb{S}_4$. The calculation shows that $\mathbb{S}_4$ is solvable:

```
julia> G = symmetric_group(4);

julia> derived_series(G)
4-element Vector{PermGroup}:
 Sym( [ 1 .. 4 ] )
 Alt( [ 1 .. 4 ] )
 Group([ (1,4)(2,3), (1,2)(3,4) ])
 Group(())

julia> [order(x) for x in derived_series(G)]
4-element Vector{fmpz}:
 24
 12
 4
 1
```

```
julia> is_solvable(G)
true
```

Proposition 10.18. *Let G be a group and H be a subgroup of G. The following statements hold:*

1) *If G is solvable, then H is solvable.*
2) *If H is normal in G and G is solvable, then G/H is solvable.*
3) *If H is normal in G and H and G/H are solvable, then G is solvable.*

Proof. The first statement follows from the fact that $H^{(i)} \subseteq G^{(i)}$ holds for all i.

Assume now that H is normal in G. Let $Q = G/H$ and $\pi\colon G \to Q$ be the canonical map. By induction one proves that $\pi(G^{(i)}) = Q^{(i)}$ for all $i \geq 0$. The case where $i = 0$ is trivial, as π is surjective. If the result holds for some $i \geq 0$, then

$$\pi(G^{(i+1)}) = \pi([G^{(i)}, G^{(i)}]) = [\pi(G^{(i)}), \pi(G^{(i)})] = [Q^{(i)}, Q^{(i)}] = Q^{(i+1)}.$$

We now prove 2). Since G is solvable, $G^{(n)} = \{1\}$ for some n. Thus Q is solvable, as $Q^n = \pi(G^{(n)}) = \pi(\{1\}) = \{1\}$.

We finally prove 3). Since Q is solvable, $Q^{(n)} = \{1\}$ for some n. Moreover, since $\pi(G^{(n)}) = Q^{(n)} = \{1\}$, it follows that $G^{(n)} \subseteq H$. Since H is solvable,

$$G^{(n+m)} \subseteq (G^{(n)})^{(m)} \subseteq H^{(m)} = \{1\}$$

for some m. Thus G is solvable. □

An application:

Proposition 10.19. *Let G be a finite p-group. Then G is solvable.*

Proof. Assume the result is not true. Let G be a finite p-group of minimal order that is not solvable. Since G is a p-group, $Z(G) \neq \{1\}$. Since $|G|$ is minimal, $G/Z(G)$ is a solvable p-group. Since $Z(G)$ is abelian, $Z(G)$ is solvable. Now G is solvable by Proposition 10.18. □

Let G be a group. A subgroup N of G is said to be **maximal normal** if N is a normal subgroup of G and there is no other normal subgroup of G containing N.

Exercise 10.20. If a subgroup N of G is maximal (for the inclusion) and normal, then it is maximal normal. Show that the converse does not hold.

The following result is a direct consequence of the correspondence theorem:

Exercise 10.21. Let G be a group and N be a normal subgroup of G. Prove that N is maximal normal if and only if G/N is simple.

Maximal normal subgroups always exist in finite groups (they could be trivial). We can compute maximal normal subgroups as follows:

```
julia> maximal_normal_subgroups(symmetric_group(3))
1-element Vector{PermGroup}:
 Group([ (1,2,3) ])

julia> maximal_normal_subgroups(quaternion_group(8))
```

```
3-element Vector{PcGroup}:
 Group([ y2, x ])
 Group([ y2, y ])
 Group([ y2, x*y ])

julia> maximal_normal_subgroups(alternating_group(4))
1-element Vector{PermGroup}:
 Group([ (1,4)(2,3), (1,2)(3,4) ])
```

Exercise 10.22. Let G be a finite solvable group. Prove that if G is simple, then G is cyclic of prime order.

The following result will be important later:

Proposition 10.23. *Every finite solvable group contains a normal subgroup of prime index.*

Proof. Let G be a finite solvable group. Let M be a maximal normal subgroup of G (there is at least one, as G is finite). Since G/M is simple and solvable (see Proposition 10.18), G/M is cylic of prime order by Exercise 10.22. □

We finish this discussion with two important theorems (without proof) about finite solvable groups.

Theorem 10.24 (Burnside). *Let p and q be prime numbers. If G is a group of order p^aq^b, then G is solvable.*

The proof appears in courses on the representation theory of finite groups.

Theorem 10.25 (Feit–Thompson). *Every finite group of odd order is solvable.*

The proof of the theorem is extremely hard. It occupies a full volume of *Pacific Journal of Mathematics*, see [**2**].

§ 10.3. Simplicity of the alternating simple group. We will present a family of non-abelian simple groups. We start with some exercises.

Exercise 10.26. Let G be a group. Prove that G is simple if and only if $\{(g,g) : g \in G\}$ is a maximal subgroup of $G \times G$.

Exercise 10.27. Prove that $\mathbb{A}_n$ is generated by 3-cycles.

Exercise 10.28. Compute the commutator subgroup of $\mathbb{A}_n$ for $n \geq 2$.

Note that $\mathbb{A}_2$ and $\mathbb{A}_3$ are abelian. For $\mathbb{A}_4$, one proves that

$$[\mathbb{A}_4, \mathbb{A}_4] = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\}.$$

Finally, $[\mathbb{A}_n, \mathbb{A}_n] = \mathbb{A}_n$ for $n \geq 5$.

Let us compute some commutator subgroups (and the inclusion group homomorphism) with the computer:

```
julia> derived_subgroup(symmetric_group(3))
(Alt( [ 1 .. 3 ] ), Group homomorphism from
Alt( [ 1 .. 3 ] )
to
Sym( [ 1 .. 3 ] ))
```

Exercise 10.29. Let $n \geq 3$. Prove that $[\mathbb{S}_n, \mathbb{S}_n] = \mathbb{A}_n$.

Recall that every normal subgroup is a union of conjugacy classes. The group $\mathbb{A}_5$ has conjugacy classes of sizes 1, 15, 20, 12 and 12. It follows that the only possible normal subgroups of $\mathbb{A}_5$ are $\{\mathrm{id}\}$ and $\mathbb{A}_5$.

```
julia> A5 = alternating_group(5);

julia> [length(c) for c in conjugacy_classes(A5)]
5-element Vector{ZZRingElem}:
 1
 15
 20
 12
 12
```

Theorem 10.30 (Jordan). *Let $n \geq 5$. Then $\mathbb{A}_n$ is simple.*

Before proving the theorem, we need some preliminary results.

Every permutation $\rho \in \mathbb{S}_n$ decomposes as a product of disjoint cycles, say

$$\rho = (a_1 \cdots a_r)(b_1 \cdots b_s) \cdots (c_1 \cdots c_t)$$

where, by convention, we do not write cycles of length one. The cyclic structure of ρ is, by definition, the ordered sequence of integers $r, s, \dots t$, where, again by convention, we omit fixed points. For example, the cyclic structure of the transposition (ab) is 2, of $(abc)(d)$ is 3 and of $(123)(45)(789a)(bcd)(d)$ is 2,3,3,4.

Lemma 10.31. *If ρ_1 and ρ_2 are permutations in $\mathbb{S}_n$ with the same cyclic structure, then $\rho_2 = \sigma\rho_1\sigma^{-1}$ for some $\sigma \in \mathbb{S}_n$.*

Proof. Assume that

$$\rho_1 = (a_1 \cdots a_r)(b_1 \cdots b_s) \cdots (c_1 \cdots c_t),$$
$$\rho_2 = (x_1 \cdots x_r)(y_1 \cdots y_s) \cdots (z_1 \cdots z_t).$$

Let

$$\mathrm{Fix}(\rho_1) = \{x \in \{1, \dots, n\} : \rho_1(x) = x\} = \{k_1, \dots, k_m\}, \qquad \mathrm{Fix}(\rho_2) = \{l_1, \dots, l_m\}$$

be the fixed points of the permutations ρ_1 and ρ_2, respectively. Then

$$\sigma(x) = \begin{cases} x_j & \text{if } x = a_j \text{ for some } j, \\ y_j & \text{if } x = b_j \text{ for some } j, \\ \vdots & \\ z_j & \text{if } x = c_j \text{ for some } j, \\ l_j & \text{if } x = k_j \text{ for some } j, \end{cases}$$

is such that $\sigma\rho_1\sigma^{-1} = \rho_2$. □

What happens with the alternating group?

LEMMA 10.32. *If $\rho_1, \rho_2 \in \mathbb{S}_n$ are conjugate in $\mathbb{S}_n$ and $|\mathrm{Fix}(\rho_1)| \geq 2$, then $\mu\rho_1\mu^{-1} = \rho_2$ for some $\mu \in \mathbb{A}_n$.*

PROOF. Assume that $\rho_2 = \sigma\rho_1\sigma^{-1}$ for some $\sigma \in \mathbb{S}_n$. There are $a, b \in \{1, \dots, n\}$ such that $\rho_1(a) = a$, $\rho_1(b) = b$ and $a \neq b$. Let

$$\mu = \begin{cases} \sigma & \text{if } \sigma \in \mathbb{A}_n, \\ \sigma(ab) & \text{otherwise.} \end{cases}$$

Then $\mu \in \mathbb{A}_n$ and $\mu\rho_1\mu^{-1} = \rho_2$, as (ab) commutes with ρ_1. □

Let us discuss some examples.

EXAMPLE 10.33. If $\rho_1 = (23)(156)$ and $\rho_2 = (45)(123)$, then $\rho_2 = \sigma\rho_1\sigma^{-1}$ for

$$\sigma = \begin{pmatrix} 123456 \\ 145623 \end{pmatrix}.$$

EXAMPLE 10.34. The permutations $\rho_1 = (123)$ and $\rho_2 = (132)$ are conjugate in $\mathbb{S}_3$, as $(123) = \sigma(132)\sigma^{-1}$ if $\sigma = (23)$. However, ρ_1 and ρ_2 are not conjugate in $\mathbb{A}_3$.

Now we are ready to prove the theorem.

PROOF OF THEOREM 10.30. Let $N \neq \{\mathrm{id}\}$ be a normal subgroup of $\mathbb{A}_n$. If $(abc) \in N$, then every 3-cycle belongs to N, because all 3-cycles are conjugate in $\mathbb{S}_n$, and the previous lemma states that $(ijk) = \mu(abc)\mu^{-1} \in N$ for some $\mu \in \mathbb{A}_n$. Thus $N = \mathbb{A}_n$.

We claim that N contains a 3-cycle. Since $N \neq \{id\}$, there exists $\sigma \in N \setminus \{\mathrm{id}\}$. Let $m = |\sigma|$ and let p be a prime number dividing m. Then $\tau = \sigma^{m/p}$ has order p and hence $\tau = \rho_1 \cdots \rho_s$, where the ρ_j's are disjoint p-cycles.

If $p = 2$, then $1 = \mathrm{sign}(\tau) = (-1)^s$. Thus s is even. Write

$$\tau = (ab)(cd)\rho_3 \cdots \rho_s.$$

Since $\rho_3 \cdots \rho_s$ commutes with (abc) and (acb),

$$\underbrace{(abc)\tau(abc)^{-1}\tau^{-1}}_{\in N} = (abc)(ab)(cd)(acb)(ab)(cd) = (ac)(bd).$$

Hence $(ac)(bd) \in N$. Let $e \in \{1, \dots, n\} \setminus \{a, b, c, d\}$. Then

$$(ae)(bd) = (aec)\underbrace{(ac)(bd)}_{\in N}(aec)^{-1} \in N$$

and therefore

$$(aec) = (ac)(ae) = (ac)(bd)(ae)(bd) \in N.$$

If $p = 3$, without loss of generality, we may assume that $s \geq 2$ (otherwise, τ would be a 3-cycle). Then $\tau = (abc)(def)\rho_3 \cdots \rho_s$. Since (bcd) commutes with $\rho_3 \cdots \rho_s$ and N is normal in $\mathbb{A}_n$,

$$\underbrace{(bcd)\tau(bcd)^{-1}\tau^{-1}}_{\in N} = (bcd)(abc)(def)(bdc)(acb)(dfe) = (adbce)$$

and therefore

$$(adc) = (adb)(adbce)(adb)^{-1}(adbce)^{-1} \in N.$$

If $p > 3$, then $\tau = (abcd \cdots z)\rho_2 \cdots \rho_s$. In particular, (abc) commutes with $\rho_2 \cdots \rho_s$. Then

$$(abd) = (abc)\tau(abc)^{-1}\tau^{-1} \in N.$$

□

As an application, we compute the normal subgroups of the symmetric group $\mathbb{S}_n$.

EXERCISE 10.35. Compute the list of normal subgroups of $\mathbb{S}_n$ for $n \geq 2$.

Lecture 11. 06/05/2024

§ 11.1. Radical extensions.

Definition 11.1. An extension E/K is said to be **pure** of type m if $E = K(x)$ for some x such that $x^m \in K$.

Note that if $E = K(x)$ is a pure extension of type m and K contains m-th roots of one, then E/K is a splitting field of $X^m - x^m$.

Definition 11.2. The sequence $K = R_0 \subseteq R_1 \subseteq \cdots \subseteq R_m$ of fields is said to be a **radical tower** if each R_{i+1}/R_i is pure. In this case, R_m/K is a **radical extension**.

Note that radical extensions are finite.

Example 11.3. Let E be a decomposition field of $X^4 - 2$ over $\mathbb{Q}$. Then $E/\mathbb{Q}$ is radical, as $E = \mathbb{Q}(\sqrt[4]{2}, i)$.

Example 11.4. Let $\alpha, \beta \in \mathbb{C}$ be such that $\alpha^2 = 2$ and $\beta^5 = 1 + \alpha$. The number $\sqrt[5]{1+\sqrt{2}}$ belongs to the radical extension $\mathbb{Q}(\alpha, \beta)/\mathbb{Q}$.

Theorem 11.5. *Let K be of characteristic zero and R/K be a radical extension. If E/K is a subextension of R/K, then* $\operatorname{Gal}(E/K)$ *is solvable.*

Proof. Without loss of generality, we may assume that E/K is a Galois extension. To prove this fact, let $G = \operatorname{Gal}(E/K)$ and $F = {}^{G}E$. Then E/F is a Galois extension and $\operatorname{Gal}(E/F) = G$ by Artin's theorem. Thus, replacing K by F if needed, we may assume that E/K is Galois.

Let L be the normal closure of R in some algebraic closure C that contains R. Note that if $R = K(x_1, \ldots, x_m)$, then

$$L = K(\{\sigma_i(x_j) : 1 \le i \le s,\ 1 \le j \le m\}),$$

where $\operatorname{Hom}(R/K, C/K) = \{\sigma_1, \ldots, \sigma_s\}$.

Claim. L/K is radical.

Since $x_j^{a_j} \in K(x_1, \ldots, x_{j-1})$ for some integer a_j,

$$\sigma_i(x_j)^{a_j} = \sigma_i\left(x_j^{a_j}\right) \in \sigma_i(K(x_1, \ldots, x_{j-1}) = K(\sigma_i(x_1), \ldots, \sigma_i(x_{j-1}))$$

Thus L/K is radical and Galois.

We may assume then that E/K and R/K are both Galois.

Since $\operatorname{Gal}(E/K) \simeq \operatorname{Gal}(R/K)/\operatorname{Gal}(R/E)$, we only need to prove that $\operatorname{Gal}(R/K)$ is solvable.

For a positive integer n, let ξ be a primitive n-th root of one (in some algebraic closure of K that contains R). Consider the diagram

$$\begin{tikzcd}
 & R(\xi) \arrow[dl, dash] \arrow[dr, dash] & \\
R \arrow[dr, dash] & & K(\xi) \arrow[dl, dash] \\
 & K &
\end{tikzcd}$$

Then

1) $K(\xi)/K$ and $R(\xi)/R$ are abelian.
2) $R(\xi)/K$ is Galois.
3) $\mathrm{Gal}(R/K) \simeq \mathrm{Gal}(R(\xi)/K)/\mathrm{Gal}(R(\xi)/R)$.
4) $\mathrm{Gal}(K(\xi)/K) \simeq \mathrm{Gal}(R(\xi)/K)/\mathrm{Gal}(R(\xi)/K(\xi))$.

The third item implies that we need to show that $\mathrm{Gal}(R(\xi)/K)$ is solvable. By the fourth item, it suffices to show that $\mathrm{Gal}(R(\xi)/K(\xi))$ is solvable (because $\mathrm{Gal}(K(\xi)/K)$ is abelian and hence solvable).

Since $R = K(x_1,\dots,x_m)$,

$$R(\xi) = K(x_1,\dots,x_m,\xi) = K(\xi)(x_1,\dots,x_m)$$

and hence $R(\xi)/K(\xi)$ is radical. This means that without loss of generality, we may assume that K contains primitive n-roots of one. For example, if $R = K(x_1,\dots,x_m)$ and $x_i^{a_i} \in K(x_1,\dots,x_{i-1})$, then we may assume that K contains a primitive a_i-root of one. We proceed by induction on m. The case $m = 0$ is trivial. Assume that the claim holds for some $m \geq 0$. Let $L = K(x_1)$. Then L/K is a decomposition field of $X^{a_1} - x_1^{a_1}$, and hence L/K is a cyclic extension. Thus $\mathrm{Gal}(L/K)$ is cyclic (and hence, in particular, solvable). Let H be the subgroup that corresponds to L, that is $H = \mathrm{Gal}(R/L)$ (here, we use Galois' correspondence). Then H is normal in $\mathrm{Gal}(R/K)$. Since $R = K(x_1,\dots,x_m) = L(x_2,\dots,x_m)$, R/L is radical and Galois. By the inductive hypothesis, $\mathrm{Gal}(R/L)$ is solvable. Since

$$\mathrm{Gal}(L/K) \simeq \mathrm{Gal}(R/K)/\mathrm{Gal}(R/L),$$

it follows that $\mathrm{Gal}(R/K)$ is solvable. $\square$

Definition 11.6. Let $f \in K[X]$ and E be a decomposition field of f over K. We say that f is **solvable by radicals** if there is a radical extension R/K such that $E \subseteq R$.

The general polynomial of degree two is solvable by radicals, as its Galois group is solvable (in fact, isomorphic to $\mathbb{S}_2$).

Exercise 11.7. Prove that $f = X^2 - s_1X + s_2 \in \mathbb{Q}[X]$ is solvable by radicals.

Theorem 11.5 translates into the following result:

Exercise 11.8. Let K be a field of characteristic zero. If $f \in K[X]$ is solvable by radicals, then $\mathrm{Gal}(f,K)$ is solvable.

As a consequence, the general polynomial of degree $n \geq 5$ is not solvable by radicals, as its Galois group is isomorphic to $\mathbb{S}_5$.

Example 11.9. Let p be a prime number and $f = X^5 - 2pX + p \in \mathbb{Q}[X]$. We claim that f is not solvable by radicals.

By Gauss' theorem, one proves that f has no rational roots.

Note that $f' = 5X^4 - 2p$. Then $\alpha = \sqrt[4]{2p/5}$ and $\beta = -\sqrt[4]{2p/5}$ are are critical points. Since $f(\alpha) < 0$ and $f(\beta) > 0$, it follows that f has exactly three real roots. Let $x_1, x_2 \in \mathbb{C} \setminus \mathbb{R}$ and $x_3, x_4, x_5 \in \mathbb{R}$ be the roots of f.

By Eisenstein's theorem, f is irreducible.

Let $E/\mathbb{Q}$ be a decomposition field of f. Then $\mathrm{Gal}(f,\mathbb{Q}) = \mathrm{Gal}(E/\mathbb{Q})$ is isomorphic to a subgroup G of $\mathbb{S}_5$. Since f is irreducible, 5 divides $[E:\mathbb{Q}] = |G|$. In particular, by Cauchy's theorem, G contains an element σ of order five. This element is a 5-cycle, so without loss of generality, we may assume that $\sigma = (x_1x_2x_3x_4x_5)$. Note that $(x_1x_2) \in G$. Thus $G \simeq \mathbb{S}_5$ and hence G is not solvable.

Exercise 11.10. Let $f = X^6 + 2X^5 - 5X^4 + 9X^3 - 5X^2 + 2X + 1 \in \mathbb{Q}[X]$. Prove that f is solvable by radicals.

Some solutions

5.12. Let $\{v_i : i \in I\}$ be a basis of V over K. For each $i \in I$ let $f_i \colon V \to F$, $f_i(v_j) = \delta_{ij}$. Then $\{f_i : i \in I\}$ is linearly independent over F. In fact, let $\sum a_i f_i = 0$, where each $a_i \in F$. Then $a_i = 0$ for almost all i. If $j \in I$, then

$$0 = \left(\sum a_i f_i\right)(v_j) = \sum a_i f_i(v_j) = a_j.$$

Now assume that $\dim_K V = n$. Let $\{v_1, \dots, v_n\}$ be a basis of V over K. We claim that $\{f_1, \dots, f_n\}$ is a basis of $\mathrm{Hom}_K(V,F)$ over F. If $g \in \mathrm{Hom}_K(V,F)$, then $g = \sum g(v_i) f_i$. If $1 \leq k \leq n$, then

$$\left(\sum g(v_i) f_i\right)(v_k) = \sum g(v_i) f_i(v_k) = g(v_k).$$

5.15. We need to find a bijective map

$$\mathrm{Hom}(E/K, C/K) \to \mathrm{Hom}(E/K, C_1/K).$$

If $\sigma \in \mathrm{Hom}(E/K, C/K)$, then $\theta^{-1}\sigma \in \mathrm{Hom}(E/K, C_1/K)$. If $\varphi \in \mathrm{Hom}(E/K, C_1/K)$, then $\theta\varphi \in \mathrm{Hom}(E/K, C/K)$. The maps $\sigma \mapsto \theta^{-1}\sigma$ and $\varphi \mapsto \theta\varphi$ are inverse to each other.

10.22. If G is solvable, then $[G,G]$ is a proper normal subgroup of G. Since G is simple, $[G,G] = \{1\}$ and G is abelian. Thus G is cyclic of prime order.

10.26. Assume that G is simple. Let $A = G \times \{1\}$, $B = \{1\} \times G$ and $D = \{(x,x) : x \in G\}$ the diagonal subgroup of $G \times G$. Since

$$(g,h) = (g,1)(1,h) = (gh^{-1},1)(h,h)$$

it follows that $G = AB = AD$. Let M be a subgroup of $G \times G$ that contains D. Note that

$$M = M \cap (G \times G) = M \cap AD = (M \cap A)D.$$

Similarly, $M = (M \cap B)D$. Since A is normal in $G \times G$, $M \cap A$ is normal in $G \times G$ and $(M \cap A)B$ is normal in $MB = G \times G$. Using the second isomorphism theorem, we see that

$$M \cap A \simeq \frac{(M \cap A)B}{B}$$

is a normal subgroup of $(G \times G)/B \simeq A$. Since $A \simeq G$ is simple, either $M \cap A = \{1\}$ or $M \cap A = A$. Thus either $M = D$ or $BD = G \times G$. Therefore D is maximal.

References

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