YOUNG'S INEQUALITY

Theorem 1 (Young's inequality)**.** *For any four real numbers a, b, p, q such that $a \geq 0$, $b \geq 0$, $p > 0$, $q > 0$, and $1/p + 1/q = 1$, we have $p > 1$, $q > 1$, and*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \tag{1}$$

with equality iff $a^p = b^q$.

Proof. First, note that since p and q are positive, their reciprocals $1/p$ and $1/q$ are also positive. Since $1/p + 1/q = 1$, it follows that $1/p$ and $1/q$ are both strictly less than 1 which means $p > 1$ and $q > 1$.

As for (1): first note that if $b = 0$, then this inequality reduces to $0 \leq a^p/p$, which certainly holds since a and p are non-negative, and non-negativity is preserved by powers and quotients. Moreover we have equality iff $a^p/p = 0$, i.e. $a^p = 0$, and since $b^q = 0$ this is equivalent to $a^p = b^q$.

Otherwise, we have $b^q > 0$, so we can divide both sides of (1) by b^q to see that it is equivalent to

$$\frac{a}{b^{q-1}} \leq \frac{a^p}{pb^q} + \frac{1}{q},$$

i.e.

$$\frac{a}{b^{q-1}} \leq \frac{1}{p}\left(\frac{a}{b^{\frac{q}{p}}}\right)^p + \frac{1}{q}. \tag{2}$$

Now since $1/p + 1/q = 1$, and hence $1/p = 1 - 1/q$, we have

$$\frac{q}{p} = q \cdot \frac{1}{p} = q\left(1 - \frac{1}{q}\right) = q - 1.$$

Hence if we let $\alpha = a/b^{q-1} = a/b^{q/p}$, we can rewrite (2) as

$$\alpha \leq \frac{\alpha^p}{p} + \frac{1}{q}. \tag{3}$$

We have

$$\frac{\alpha^p}{p} + \frac{1}{q} = \frac{\alpha^p}{p} + \left(1 - \frac{1}{p}\right) = \frac{\alpha^p - 1}{p} + 1,$$

so (3) is in turn equivalent to

$$\alpha - 1 \leq \frac{\alpha^p - 1}{p},$$

i.e.

$$p(\alpha - 1) \leq \alpha^p - 1,$$

i.e.

$$p(\alpha - 1) + 1 \leq \alpha^p.$$

This holds by Bernoulli's inequality with $\alpha - 1$ substituted in place of x and p substituted in place of r (these substitutions are allowed since $\alpha = a/b^{q-1} \geq 0$, so that $\alpha - 1 \geq -1$, and $p > 1$). Moreover we have equality iff $\alpha - 1 = 0$, i.e. $\alpha = 1$, i.e. $a/b^{q/p} = 1$, i.e. $a = b^{q/p}$, i.e. $a^p = b^q$. $\square$