BERNOULLI'S INEQUALITY

Definition 1. A function $f : X \to \mathbb{R}$ on a real vector space is *strictly concave* iff for every $t \in (0, 1)$ and any two distinct elements x and y of X, we have

$$f(tx + (1 - t)y) > tf(x) + (1 - t)f(y).$$

Lemma 1. *The function* $\ln : (0, \infty) \to \mathbb{R} : x \mapsto \ln x$ *is strictly concave.*

Proof. For every $x \in (0, \infty)$, we have $\ln'(x) = 1/x$, and hence $\ln''(x) = -1/x^2$. Since $x \neq 0$, we have $x^2 > 0$ and hence $-1/x^2 < 0$; so $\ln''$ is negative-valued; hence $\ln'$ is strictly antitone; hence $\ln$ is strictly concave. □

Theorem 1 (Bernoulli's inequality)**.** *For any two real numbers x and r such that $x \geq -1$ and $r \geq 0$, the following statements hold:*

(1) *If $x = 0$, $r = 0$ or $r = 1$, we have $(1 + x)^r = 1 + rx$.*
(2) *If $x \neq 0$ and $0 < r < 1$, we have $(1 + x)^r < 1 + rx$.*
(3) *If $x \neq 0$ and $r > 1$, we have $(1 + x)^r > 1 + rx$.*

Proof. For (1), observe that:

- If $x = 0$, then $(1 + x)^r = 1^r = 1 = 1 + r \cdot 0 = 1 + rx$.
- If $r = 0$, then $(1 + x)^r = (1 + x)^0 = 1 + 0 \cdot x = 1 + rx$.
- If $r = 1$, then $(1 + x)^r = 1 + x = 1 + 1 \cdot x = 1 + rx$.

Henceforth we may assume $x \neq 0$ and $r \notin \{0, 1\}$.

For (2) and (3) we will proceed by taking logarithms of both sides of the inequalities, but first we need to deal with the fact that these logarithms may not exist.

The logarithm $\ln(1 + x)$ exists iff $1 + x > 0$, i.e. $x > -1$. We know that $x \geq -1$. If $x = -1$, then $(1 + x)^r = 0^r = 0$ and $1 + rx = 1 - r$, so we have

$$\begin{aligned} (1 + x)^r < 1 + rx &\iff 0 < 1 - r, \quad \text{i.e.} \quad r < 1, \\ (1 + x)^r > 1 + rx &\iff 0 > 1 - r, \quad \text{i.e.} \quad r > 1, \end{aligned}$$

which is consistent with (2) and (3). So henceforth we may assume $x > -1$, so that $\ln(1 + x)$ exists. This also means that $(1 + x)^r > 0$ (since $(1 + x)^r$ can be 0 only when $1 + x = 0$).

The logarithm $\ln(1 + rx)$ exists iff $1 + rx > 0$, i.e. $rx > -1$. If $rx \leq -1$, then $1 + rx \leq 0 < (1 + x)^r$. This is what we expect in (3), and indeed, in this case we also have $rx \geq -r$ (since $x \geq -1$ and $r \geq 0$) and hence (by transitivity) $-1 \geq -r$, i.e. $r \leq 1$; so there is no contradiction with (2). Henceforth we may assume $rx > -1$, so that $\ln(1 + rx)$ exists.

To complete the proof of (2), apply $\ln$ to both sides of the inequality

$$(1 + x)^r < 1 + rx$$

to see that it is equivalent to

$$r \ln(1 + x) < \ln(1 + rx), \quad \text{i.e.} \quad \ln(1 + x) < \frac{\ln(1 + rx)}{r}.$$

Now if $0 < r < 1$, then since ln is strictly concave, and $1+rx = r(1+x)+(1-r)\cdot 1$, and $1+x \neq 1$ (since $x \neq 0$), we have

$$\ln(1+rx) > r\ln(1+x) + (1-r)\ln 1 = r\ln(1+x)$$

and hence $(1+x)^r < 1+rx$.

To complete the proof of (3), apply ln to both sides of the inequality

$$(1+x)^r > 1+rx$$

to see that it is equivalent to

$$r\ln(1+x) > \ln(1+rx), \quad \text{i.e.} \quad \ln(1+x) > \frac{\ln(1+x)}{r}.$$

Now if $r > 1$, then since ln is strictly concave, and $1+x = 1/r\cdot(1+rx)+(1-1/r)\cdot 1$, and $1+rx \neq 1$ (since x and r are nonzero, so $xr \neq 0$), we have

$$\ln(1+x) > \frac{1}{r}\ln(1+rx) + \left(1-\frac{1}{r}\right)\ln 1 = \frac{\ln(1+rx)}{r}$$

and hence $(1+x)^r > 1+rx$. $\square$