Understand Math
Reasons for the Rules

by

Andrew Kelley, PhD

Book cover designed by Michael Cole.

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Last updated on February 1, 2025.

Dedicated to Dr. Scott.
Of all my teachers in 1st through 12th grade, you were the best.

Endorsements

"Looks very thoughtfully written for easy understanding."
- Mike Summerhays, Engineering Fellow

"That makes so much sense."
- Mary Ann Brubaker, 85 years old (commenting on the fractions chapter)

"I thought the fractions chapter was quite clear and easy to understand and follow."
- Elizabeth Mayner, former homeschool mom

"An impressive book."
- Marilyn Frydrych, math teacher and math lab coordinator at Pikes Peak State College

"I completed chapter 1 and really enjoyed the process of relearning and better understanding fractions! Your examples are easy to identify with, your explanations are clear, and your conversational style made me feel like I had the help of a tutor via the text."
- Gayle Meredith, senior library associate

"Most math teachers (and students!) are deeply familiar with the common impulse in mathematics education to "just learn the formula," apply it, and move on. After all, that's the goal, right? Well, actually no, not if you really want to learn mathematics. What I so appreciate about Andrew Kelley's work is his relentless emphasis on understanding real mathematics – the 'why' behind the formula – and his skillful explanation that makes those beautiful concepts accessible to the student. I believe Andrew's work can turn what many students experience as dull and dry manipulation of indecipherable symbols into a delightful exercise in finally understanding the beauty, usefulness, and design of mathematics. You won't be disappointed."
- Jim DeKorne, retired math teacher and former principal of Colorado Springs Christian Schools

"So far, I'm loving it!...I really like your descriptions and how you develop ideas/concepts, especially in discussing the inherent properties of seemingly simple ideas (which, of course, are not usually simple)...I also very much like how you explain the logic and thought processes behind each concept. You present so much more than a set of rules (most math textbooks don't do much more than presenting a set of rules); you really give the reader a sense of how these ideas and concepts were developed and why...You also do a great job of connecting the concepts...Your Motivation questions are excellent, really drawing readers to consider the concepts more deeply...You've combined so many deeply connected ideas and provided understanding of why these concepts are important, and how they might be used and applied."
- Erica Hastert, college math teacher

"The book will make a great companion to math textbooks throughout subjects like algebra, advanced algebra, geometry, trigonometry, and calculus. The conversational style the book is written in makes it easy to understand and follow. It would make a welcome addition to texts used by homeschool parents to help explain new concepts, new math principles, and new ways of thinking that are typically stumbling blocks for many being introduced to math throughout the Jr. High–High school years."
- Jim Crowder, PhD, Mathematician and Engineering Fellow

Preface

Understanding is a fountain of life to those who have it...
–Solomon, King of Israel, (ca. 950 B.C.) Proverbs 16:22 (NASB)

Do you want to understand math? This book is for you. Does math seem like a bunch of arbitrary rules? This book is here to explain clearly why most of the rules aren't arbitrary at all. There are good reasons for them!

The purpose of this book is to *explain the reasons* for the main rules learned in middle school and high school mathematics (i.e., for 6th through 12th grade). My goal throughout the book is to explain *why* the rules work the way they do. I *don't* assume you already know all the rules, and so I review each rule before explaining the reason behind it. For most rules, I do not give rigorous proofs. Rather, I instead focus on the essence for why the rules work. I focus on the heart of why they are true. My intent has been to be informative.

? **Question.** *Why should you learn reasons for the rules?*

If you understand why a rule works and know why it is true, then you are more likely to remember the rule, use it correctly, and know when it applies and when it doesn't. Having real insight can also set you apart. If you can only mechanically follow rules without understanding them, what sets you apart from a computer? Machines can do math, but they still lack real understanding. People who don't want to be replaceable by a computer, should seek to *understand*. Further, when math makes sense, it is much more enjoyable.

This book isn't meant to be a standard textbook used for just one year of school. Rather, I have tried to make it the best **supplement** to use while studying math in middle school and high school. Homeschoolers and those teaching themselves should find this book incredibly helpful. Also, any adult who has not done math in many years but wants to learn more should find this book helpful. Although my target audience is people who are teaching themselves math, teachers may also benefit from this book by learning new ways to explain concepts. Nevertheless, this book is geared to students.

Note to the reader

Is learning how to walk difficult? If you've ever watched a very small child learn to walk, then perhaps you agree with me that *learning* to walk is indeed difficult. Of course, *after* someone gets the hang of it, walking eventually becomes easy. All this is the same with math:

Math is *hard*... until you get it, and then it's easy.

This book is written so that people who aren't strong at math can understand it. I've tried to explain things simply and tried to get at the heart of what's going on.

If you have access to a very patient and skilled teacher who has a deep knowledge of mathematics, then that's the easiest way to learn. I have attempted to make this book be as close to that as I can. A great math teacher will ask many questions, and so this book also contains many questions in it, and to get the most from it, you should try to answer most of them. Even with a less-than-ideal background in math, you probably can answer most questions labeled **Motivation** or **Exercise**. Questions labeled **Motivation** are usually meant to lead you to a point I am trying to make. These are extremely important questions to attempt before reading the answers.

! **Key Fact 1.** *In math, there is no substitute for thinking and attempting problems yourself.*

When learning to walk, it is necessary to get on your feet and try. Falling sometimes is inevitable and is not at all a failure. Failing to try at all is what will hinder growth. So don't be afraid that you might make a mistake! Mistakes are normal. After an honest attempt at a **Motivation** or **Exercise**, you can immediately catch any errors by then reading the solution.

The questions labeled **Exercise** are meant to have answers that are very similar to what was discussed prior to them. They are there simply to practice and cement what you just learned. I recommend that you solve each **Exercise**. Also, if you are tempted to skip the **Exercise** problems, please don't give in to that temptation, not even if they seem straightforward. Only skip them if you have already mastered the material in that section. There is a real danger in math in fooling yourself into thinking you understand something when you haven't quite made it there. Doing some practice helps a lot. Further, even if you do understand it, practicing will help you remember. There is *not* an overabundance of questions labeled **Exercise** because this book is not meant to drill the rules into your head by endless repetition. Hence, you shouldn't be bored at all if you do all of the exercises. For questions labeled **Motivation** or **Exercise**, be sure to answer them before reading the answers. Read with pencil and paper at hand!

? **Question.** *Do you have to answer the main questions (which are in a box like this) before reading the answer?*

No, each **Question** written this way (in a gray box, with a big question mark to the left) is *not* intended to be immediately answerable by the reader. Rather, they are the main questions which I, the author, will always answer myself. They often are the questions that the whole section is devoted to answering.

Finally, each **Challenge** is intended to cover material a little more advanced than the main flow of text. Readers who want to stretch themselves should attempt to answer them before reading the answer that follows. *All* readers should at least read each **Challenge** and their answers.

Just as math builds on itself, so does this book. If you consider yourself ready for any particular chapter however, then feel free to read that first. For advanced readers who want to skip, I do highly recommend at least reading the main **Questions** and **Key Facts** in each chapter you intend to skip. They are highlighted in gray boxes to set them apart.

? **Question.** *How else can you get the most from this book?*

You will need something to write with as you read, since you will have to answer questions as you go along. One practical suggestion is to have an additional piece of paper handy. Whenever you come to an exercise or motivation, you are encouraged to cover up the answer as you read the problem so that you don't accidentally see the answer before you work on figuring it out yourself. You will need self-control. If you skip the work, then you're also skipping the possible learning. There is no magic formula that makes learning effortless, and so you should commit to thinking through what you read and attempt exercises and motivations before reading the answer.

What you get out of this book is determined by what you put into it.

This is no different from learning anything else, which brings us to a central truth:

> **!** **Key Fact 2.** *Learning **anything** well takes a lot of effort.*

Do you want to learn how to play the violin or cello or French horn? Then you will need to do a lot of practice. Do you want to learn a new language? Then you'll need to put a lot of time into it. Do you want to become a doctor, nurse, lawyer, or engineer? Then you'll need to study a lot. The same is true of math. In this book, I try to make things as easy as possible, but there is still some amount of effort and work required to master it, just like anything else. Also, note that reading a math book takes much more effort than reading a novel. You have to think! And it often helps to reread a sentence, paragraph, or section. If something doesn't make perfect sense, then stop to think about it for a little bit, or go back and reread that section a second or third time. This is hard work, but "all hard work brings a profit" as it says in the book of Proverbs (NIV). So work *hard*, because it will pay off.

A note to parents

This book is intended as the best *supplement* to traditional curriculum. Without writing a 2,000 page book, it simply isn't possible to write a single book that thoroughly covers all of 6th grade through 12th grade math. While I try to focus on what is important, if I leave something out, that doesn't guarantee that it is unimportant. Rather, I have attempted to include anything important for which I have something insightful to say.

Please require your son or daughter to attempt each question labeled **Motivation** or **Exercise** before reading the solution that follows. Readers will get much more out of it if they at least try these problems before reading the answers.

If you want to work through this book with your son or daughter, then let me encourage you that you *can* learn as much of this material as you're motivated to learn. This is true even if you haven't done math in many years and even if you struggled with math when you were in school. If you do want to learn this material yourself, then allow me to encourage you to *not* make this book your only resource. Whether or not you read this book with your kids, one way to see if they understand this material is if they can explain the topics to others. If your kids can explain what they learn, then that means they have digested the material and understand it well. Another way to see if they understand the material is if they start doing better in whatever curriculum the present book is a supplement to. Understanding the fundamentals well is helpful no matter what curriculum you use.

The background I assume of readers is that they have at least seen fractions, variables, and the equals symbol, but I don't assume they have a good grasp of any of those topics. Someone completely new to variables might be able to understand the first two chapters as they study their main curriculum.

The chapters on geometry, trigonometry, logic, probability, and calculus can serve well as a first introduction to those subjects, but I recommend using them only as a *supplement* to your main curriculum. In every chapter except Chapters 6 and 12 (and maybe Chapter 9), readers will need to do some practice problems from a traditional textbook in order to make the most of the present book.

Like math in general, this book builds on itself. For instance, I intentionally put the arithmetic chapter after the pre-algebra chapter to better cover arithmetic. While arithmetic is normally covered before 6th grade, I include a chapter on it after pre-algebra so that readers can see how a little bit of algebra explains what they have already learned in arithmetic. I recommend that if you have your student read the pre-algebra chapter, then also assign most of the chapter on arithmetic (skipping perhaps Section 3.6). Also, the topic of fractions is usually introduced before the 6th grade, but too many people in high school struggle with fractions. That is one reason I start with fractions; it is because it should help many students in middle school and high school.

As for other prerequisites, note that the calculus chapter does assume readers understand the core material in the algebra and geometry chapters. Most of the precalculus chapter will also be helpful for calculus. The final chapter is probably best left as optional, bonus material for those who are interested. It consists of extra material that does not normally fit in a typical curriculum. Much of it can be read after the logic chapter (but some parts can be read earlier). Section 12.9 uses probability, and Section 12.10 uses calculus (as do most sections that follow it).

About the author

Andrew is a mathematician with over 12 years of experience in math education. He has been a tutor, teaching assistant, and professor and has had the pleasure of working with students in elementary school, middle school, high school, and college. Some of his students have even been adult learners in their 50s. Andrew first knew he wanted to be a mathematician when he began tutoring math at Pikes Peak Community College. He found that helping people learn math is very satisfying. After having been a professor for four years, he now works as a software engineer and has enjoyed programming computers using various programming languages. In Andrew's free time, he enjoys spending time with his wife and with family and friends. He also likes to read, listen to music, learn more math, and do research in computer science.

Acknowledgements

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Finally, I'd like to thank my wife for carefully reviewing a majority of the chapters and providing many helpful suggestions for improvement. She reviewed more of the book and gave more feedback than any other reviewer. For anyone else I missed, I apologize but want to say thank you.

Contact the author

Did you find a mistake or have any feedback on this book? Let me encourage you to email me at my personal email. Don't forget the second "e" in my last name.

akelley2500@gmail.com

If you found an error, I offer the following payments for the **first person** to contact me:

- \$5 for the first to report a mathematical error
- \$3 for the first to report a non-mathematical typo or an unintentional grammatical error (except for commas and also except for apostrophes in expressions such as 2's or x's, which I intentionally write that way)

For people reading early drafts, I also often paid \$1 per "sufficiently" helpful suggestion, but you should not count on me still paying this. The website for this book is the following:

www.UnderstandMathRules.com

If you find an error, please first check if it is listed there.

Contents of this book

Chapter 1: Fractions – Do you wish that fractions made sense? Have you ever been confused by what you get if you have a fraction *inside* another fraction? This chapter makes fractions accessible by starting with the fundamentals and making progress step by step.
Chapter 2: Pre-algebra – Does math seem more difficult when you have to use variables? This chapter helps you understand variables so that they are not intimidating. This chapter also lays the foundation of equations and how to think about them. Further, why is $2^0 = 1$ and $2^{-1} = 1/2$? This chapter explains why, and in the process, it makes exponents approachable. Practically all of math is built on the foundation laid in this chapter.
Chapter 3: Arithmetic – How does the decimal system work? Why does multiplication of multi-digit numbers work the way it does? Why can't we divide by 0? How do computers represent whole numbers? This chapter answers these questions and more. It is a continuation of the pre-algebra chapter and builds off of it.
Chapter 4: Algebra – Do square roots or logarithms seem confusing? Would you like to master fundamental topics such as factoring? Do you want a solid foundation in the math that is used everywhere in science and engineering? This chapter will help you gain that understanding. Exactly what is classified as pre-algebra or algebra is somewhat arbitrary. (Similarly, splitting algebra into two parts also is arbitrary.) If you don't already understand variables or the concept of distributing, then the pre-algebra chapter would be very helpful to read first.
Chapter 5: Geometry – Would you like to learn why the area of a circle is πr^2? Do you want to understand why the Pythagorean theorem is true? This chapter explains both of these. Altogether, this chapter includes only a small sample of what can be covered in a high school geometry class. One reason this chapter is so short is that if geometry is taught in a proof-based way, then students taking such a course are already learning good reasons for the results that they learn. Most of the proofs that this book contains are found in the geometry and logic chapters. If you read this chapter, you may also want to read the logic chapter.

Chapter 6: Logic – Can you recognize logical fallacies in flawed arguments? The last section of this chapter can help you do so. It is written from a Christian perspective and covers both mathematical and non-mathematical fallacies. On a more mathematical note, did you know that there are infinitely many prime numbers? And would you like to know why? Do you want to know why $\sqrt{2}$ is an irrational number? This chapter gives understandable proofs for these results. Indeed, the first two sections of the chapter cover (a) basics of logic and (b) proofs. All of math is built on the foundation of logic. This chapter helps you understand that foundation. Some readers might skip the first two (more mathematical) sections and first read either of the last two. Section 6.3 covers fundamentals that equip readers with basic concepts for a grounded worldview.
Chapter 7: Trigonometry – Do you want to understand trigonometric identities or have an intuitive grasp of the trig functions? This chapter explains what is most fundamental about trigonometry. In addition to this, the precalculus and complex numbers chapters also talk about trigonometry.
Chapter 8: Complex numbers $i = \sqrt{-1}$ – Complex numbers are also called imaginary numbers. Does $\sqrt{-1}$ seem mysterious, weird, or useless to you? This chapter tries to demystify complex numbers and explain some ways in which they are useful. As a nice bonus, this chapter also gives a way to help remember two of the most fundamental trigonometric identities.
Chapter 9: Probability – This chapter explains what is most important in a first course on probability. It introduces set theory in order to help clear up a common source of confusion–when to multiply probabilities and when to add them. A reader who understands this material will be well prepared to excel in a college level course in probability.
Chapter 10: Precalculus – Do you want a deeper understanding of one of the most important concepts in all of math? Then you will want to understand what this chapter says about functions. This includes a section giving a graphical reason why $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$. This chapter focuses on preparing the reader for calculus and so leaves out some material that is not essential to master (assuming you just want to be prepared for calculus). However, a few useful, additional topics are also included.
Chapter 11: Calculus – Many students take calculus in college, and many take it in high school. Hence, calculus is both college level *and* high school level. Rather than give rigorous proofs for all the results, this chapter gives intuitive explanations instead. This includes a section explaining the essence of the Fundamental Theorem of Calculus. Also, because some students find infinite series so challenging, this chapter gives insight into what is otherwise considered a difficult topic. The material in this chapter is called "single variable calculus" and includes topics from what high schools in the United States call "Calculus AB" and "Calculus BC," which at a college might be called "Calculus I" and "Calculus II."
Chapter 12: What is mathematics? – Would you like to learn some fascinating math that is not normally taught in middle school or high school? Would you like to learn about what else there is to math beyond traditional curriculum? Then this chapter is for you. Note that other than perhaps Section 12.3, the first nine sections do not require any knowledge of calculus, but four of the remaining ones do use calculus. This final chapter is unique in that unlike all the other chapters of this book, some of the results are mentioned without giving adequate justification for them. After all, some theorems in math are understandable to a high schooler while simultaneously having proofs that take hundreds of pages long! This chapter includes some beautiful pieces of math to give a broader perspective on what math has been done that is beyond traditional curriculum. A little bit of the history of math is also included.

Contents

Chapter 1

Fractions

By applying yourself to the task of becoming a little better each and every day over a period of time, you will become a ***lot*** *better.*

–John Wooden, greatest basketball coach of all time

Let's begin with an important question:

? **Question 1.1.** *What does a fraction even mean?*

The meaning of the fraction one half, written $\frac{1}{2}$, is to take 1 of something, split it into 2 equal sized pieces (that together make up the whole), and then each of those 2 pieces is called $\frac{1}{2}$. Similarly, the meaning of one third, written $\frac{1}{3}$, is to split 1 of something into 3 equal sized pieces (that together make up the whole), and then each of those 3 pieces is called $\frac{1}{3}$.

Exercise 1.2. *What does the fraction one fourth mean? (A fourth is written as* $\frac{1}{4}$*.)*

Recall from the preface that everything labeled as **Motivation** or **Exercise** is something that you should do your best to answer first before reading the answer. If you are stuck on an exercise, one good thing to try is to reread the text that precedes the exercise. After a solid effort, you may then read the answer that follows.

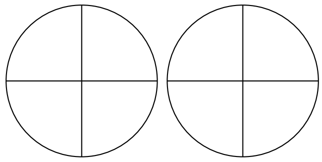
With that said, to get $\frac{1}{4}$ is to take 1 of something, split it into 4 equal sized pieces (that together make up the whole), and then each of those 4 pieces is called $\frac{1}{4}$.

Sometimes, the number on top of a fraction, which is called the **numerator**, is greater than 1. In this case, there are actually two common interpretations of such a fraction. We will spend time covering both interpretations so that you can become more comfortable with fractions. This will take some effort but is worth it.

For example, let's consider what $\frac{2}{5}$ means. One interpretation is that it is just 2 of whatever $\frac{1}{5}$ is, that is 2 times $\frac{1}{5}$. This is why $\frac{2}{5}$ can be read as "two fifths." The other meaning of $\frac{2}{5}$ is to take 2 of something, split *that* into 5 equal sized pieces (that together make the whole 2), and then each of those pieces is called $\frac{2}{5}$. This is why $\frac{2}{5}$ can be read as "2 divided by 5" and written as $2 \div 5$.

Note that the number on the bottom of a fraction is called the **denominator**. Also, note that the fraction $\frac{2}{5}$ can also be written as 2/5, and similarly for other fractions.

Let's do an example where we interpret a fraction as division. In particular, let's interpret the fraction $\frac{2}{8}$ as $2 \div 8$. To do this, let's pretend that we have 2 pizzas that we are dividing evenly among 8 people. Then we must divide the 2 pizzas evenly into 8 total parts. To do that, as can be seen in the picture below, each pizza needs to be cut into 4 pieces. Hence, 2 pizzas divided into 8 parts is one fourth of a pizza (per part).

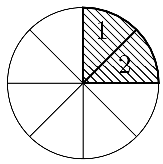


Exercise 1.3. *Suppose we have 2 pizzas still. How can we interpret the fraction $\frac{2}{4}$ in terms of the 2 pizzas? (This is similar to the $\frac{2}{8}$ we just looked at.)*

Since it is so incredibly important, please allow me to repeat that it is far better to attempt an exercise first and *then* read the answer carefully instead of just skipping to the answer before thinking through it. Let me encourage you that some amount of struggling in math problems is perfectly normal. Also, it's no problem at all to make mistakes when you first try on your own. Mistakes are perfectly normal and are opportunities to learn more.

We may think of the $\frac{2}{4}$ as dividing our 2 pizzas into 4 equal parts. To get 4 parts out of 2 pizzas, we must split both pizzas into 2 pieces each, or in other words, into halves. So then, 2 pizzas divided into 4 equal parts would make each part one half of a pizza. So the amount of pizza that $\frac{2}{4}$ here represents is one half of a pizza. What we have done in this exercise is divide 2 pizzas into 4 parts. We are thinking of $\frac{2}{4}$ as $2 \div 4$.

Before the next exercise, let's go back to the fraction $\frac{2}{8}$ from before Exercise 1.3. This time, let's interpret $\frac{2}{8}$ *not* as $2 \div 8$ but instead as 2 times $\frac{1}{8}$. To do this, let's pretend that we have only 1 pizza. Then $\frac{1}{8}$ of a pizza is an eighth of one pizza, and $\frac{2}{8}$ is then two eighths, but two eighths of a pizza make up one fourth of that pizza because four groups of two eighths make up eight eighths. So then, we find that $\frac{2}{8}$ pizzas is one fourth of a pizza, the same thing that we found just before Exercise 1.3.



Exercise 1.4. *Suppose we have only 1 pizza. How can we interpret the fraction $\frac{2}{4}$ in terms of the 1 pizza?*

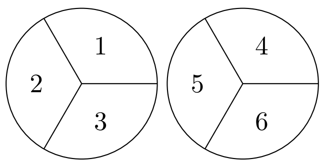
Here, instead of thinking of $\frac{2}{4}$ as 2 divided by 4, we can think of it as two fourths of a single pizza. That is, we are thinking of $\frac{2}{4}$ as 2 times whatever $\frac{1}{4}$ is. So we first must interpret $\frac{1}{4}$ as dividing up a *single* pizza into 4 equal parts. Each part is called a fourth. Next, to have 2 times $\frac{1}{4}$ is to have two fourths. Together, the two fourths make up half of the one pizza.

Notice that in Exercises 1.3 & 1.4, we end up with the same amount of pizza, whichever way we interpret the fraction $\frac{2}{4}$. We found that $\frac{2}{4}$ is $\frac{1}{2}$.

If you struggled with the previous two exercises, that's no problem. The following are more opportunities to practice how to think about fractions.

Exercise 1.5. *Suppose we have 2 small pies. How can we interpret the fraction $\frac{2}{6}$ in terms of the 2 pies?*

We may think of the $\frac{2}{6}$ as dividing 2 pies into 6 equal total parts. When we split each pie into three pieces, here are the six parts:



Each piece then represents $\frac{1}{6}$ of 2 pies, which is written as $\frac{2}{6}$. Hence, $\frac{2}{6}$ pies is one third of a pie. We are thinking of $2/6$ as $2 \div 6$.

Exercise 1.6. *Suppose we have only 1 pie. How can we interpret the fraction $\frac{2}{6}$ in terms of the 1 pie?*

Here, instead of thinking of $\frac{2}{6}$ as 2 divided by 6, we can think of it as two sixths of a single pie. Hence, we split the *one* pie into 6 equal parts and then take 2 of those parts. Together, the two sixths make up one third of the pie.

Notice that in Exercises 1.5 & 1.6, we end up with the same amount of pie, whichever way we interpret the fraction $\frac{2}{6}$. Either way, we get $\frac{2}{6}$ is $\frac{1}{3}$.

Before moving on, let me point out one more way people read a fraction out loud. The fraction $\frac{2}{7}$ can be read as "2 over 7." So then, the fraction $\frac{2}{7}$ can be read as "two sevenths," "two divided by seven," or "two over seven."

1.1 Adding fractions

Let's begin this section with how to add fractions that already have the same denominator. For instance $\dfrac{2}{7}+\dfrac{3}{7}=\dfrac{5}{7}$. As another example, $\dfrac{1}{9}+\dfrac{4}{9}=\dfrac{5}{9}$. The rule for adding fractions with the same denominator is that you only add their numerators and leave the denominator the same. If you already learned how to *multiply* fractions, then you may remember that when you multiply them, you multiply straight across in both the numerator and denominator.

?

Question 1.7. *When we add $\dfrac{2}{7}$ and $\dfrac{3}{7}$, why is it that we get*

$$\frac{2}{7}+\frac{3}{7}=\frac{5}{7},$$

and why is $\dfrac{2}{7}+\dfrac{3}{7}$ not equal to $\dfrac{5}{14}$? In other words, why does adding fractions work the way it does? Why don't we add straight across for both the numerators and denominators?

To understand this, let's answer the following questions:

$$\begin{aligned} \text{2 apples } + \text{3 apples} &= \text{what?} \\ \text{2 bricks } + \text{3 bricks} &= \text{what?} \\ \text{2 thing-a-ma-jigs } + \text{3 thing-a-ma-jigs} &= \text{what?} \end{aligned}$$

Yes, the answers are 5 apples, 5 bricks, and 5 thing-a-ma-jigs. What's a thing-a-ma-jig? Well, it actually doesn't matter what it is, because regardless of what it is, the answer to the last question is still 5 of them.

So, answer the following question next: (You don't have to know what a "seventh" is to answer the question, just as you don't need to know what a "thing-a-ma-jig" is.)

$$\text{2 sevenths } + \text{3 sevenths} = \text{what?}$$

Yes, we get 5 sevenths. Going back to the previous questions, notice that 2 apples plus 3 apples is **not** 5 *apple pies*. Similarly, 2 bricks plus 3 bricks **isn't** 5 *brick walls*. And so similarly, 2 sevenths plus 3 sevenths isn't 5 *fourteenths*. By the way, though it isn't needed to answer the above question, let's review:

Exercise 1.8. *What does it mean to have a seventh?*

Chapter 2

Pre-algebra

Progress comes slowly but steadily if you are patient and prepare diligently.
–John Wooden, greatest basketball coach of all time

2.1 What is a variable?

We often work with variables such as x or y, but what is a variable in the first place? It turns out that a variable is very similar to a pronoun: he, she, it, they. How can you tell what the word "she" refers to? Only by context. Similarly, you can only understand what the word "it" refers to by context, and the same goes for all the pronouns. Variables are similar. You can only know what a variable refers to by its context. Also, just as a pronoun is a name for a person (he/she) or several people (they/them), a variable also can be a name for a single number or for several numbers.

One difference between variables and pronouns though is that whenever you hear the word "she" or "they" in a sentence, you usually know who it is referring to (or at least a rough idea). With a variable, it is often more difficult to figure out what it is referring to. The key to remember is that when you don't know what a variable stands for, it is simply a name for a number (or multiple numbers) that you don't yet know.

A consequence of the fact that a variable is just a name for a number you don't know is that the things you can do with numbers are precisely what you can do with a variable, and how numbers are manipulated is how variables are manipulated.

! **Key Fact 2.1.** *A variable is just a name for a number (or multiple numbers).*

So what does $2x$ mean? It's just 2 times x. So if x happens to be 10, then $2x$ is 20. If instead x is 7, then $2x$ is 14.

Exercise 2.2. *If x is 9, then what is $2x$?*

Remember that you should attempt every problem labeled **Exercise** or **Motivation** before reading the answer that follows. To figure this out, all you have to do is multiply x (which is 9) by 2, and so $2 \cdot 9 = 18$.

Let's get back to what $2x$ means in general. If there is not enough information to know what x stands for, then all we can say is $2x$ is twice whatever x is. We could also write it as $x + x$, if we felt like doing so.

? **Question 2.3.** *Why is it the case that $3x + 2 \neq 5x$?*

To answer this question, let's first ask what $3x+2$ means. Well, $3x+2$ means to take whatever x is, multiply it by 3, and then add 2 to that product. To understand what $3x + 2$ means, we have to recognize that x can be *any* number. Let's see what happens if x happened to be 100. Then $3x + 2 = 300 + 2 = 302$, but notice that $5x = 500$. So in this example, $3x + 2 \neq 5x$. Let's try another example. What if $x = 50$, then what is $3x + 2$? It would be 152, unlike the 250 that $5x$ would be. So again, $3x + 2 \neq 5x$.

What $3x$ is, is three x's, and what 2 is, is two ones. The variable x very well might not be 1; in this case, three x's and two 1's doesn't give you five x's, and so that's the essential reason why in general $3x + 2 \neq 5x$. Now, *if* we had a particular number called x for which it is the case that $3x + 2 = 5x$, then x would *have* to be 1. But unless you know that x is 1, then x might be a hundred (or fifty or a thousand).

Let me repeat this. The expression $3x + 2$ is three x's plus two ones. If x were a hundred, then $3x + 2$ means three hundreds plus two ones (i.e., 302), which is not five hundreds (i.e., $5x$). Similarly, in $3x + 2$, if x happened to be a thousand, then $3x + 2$ means three thousands plus two ones (i.e., 3002), which is not five thousands (i.e., $5x$). So whether x is 100 or 1000, we have that three x's plus two ones isn't five x's.

2.2 Order of operations

To begin, let's state what the order of operations is:

1. First, do what is inside parentheses.
2. Then handle exponents.
3. Then perform all multiplications and divisions (left to right).
4. Then perform all additions and subtractions (left to right).

As stated, for addition and subtraction, if you are adding and subtracting multiple values in a row, the order is to do them left to right. So given

$$10 - 2 + 4,$$

we first compute $10-2$ as 8 and then add 4 to get 12. It is a mistake to first add 2 and 4 and then write $10 - 6$, which is 4 (a different answer).

If you have no subtractions and only have additions instead, then a very nice property of addition is that the order in which you add does not affect the final result. So starting with

$$1 + 2 + 3 + 4,$$

we can first add the 1 and 2 and then write $3+3+4$ and then write $6+4$ and end with 10. However, we could also add right to left instead to get $1 + 2 + 7$ and then $1 + 9$ and end up with 10 again. Also, to add $1+2+3+4$, we could instead add the 1 and 4 first to get 5 and also add the 2 and 3 to get 5. Then, we compute $5 + 5$ to get 10 again.

? **Question 2.4.** *Why are the order of operations what they are? Why do we multiply before adding? Why do we do exponents before multiplying?*

The vast majority of the rules covered in this book are not arbitrary. Those rules couldn't really by anything other than what they are if we want them to still make sense. It turns out though that there are some rules in math that actually are somewhat arbitrary, and so it is helpful to know when they are arbitrary. The order of operations turns out to be like the arbitrariness of language. Why is the English word for a book

"book" rather than "libro"? Well, there are historical reasons for it, but words are somewhat arbitrary symbols that could easily be something else. Everyone who speaks English though should agree that a "book" is a book and not say, a "dog." In order to communicate at all, we have to agree on what our symbols mean, and this is also definitely true in math. The order of operations is just one of those things that everyone has to agree on so that we can communicate. So, in the following expression, we do the exponent first, and then the multiplication, and then the addition:

$$5 \cdot 2^4 + 10.$$

The order of operations *could* have been different, *if everyone agreed to different rules*, but people haven't done that.

In $2x+4$, we are supposed to multiply the 2 and x first, and then add 4, but what if what you really wanted to convey is that you should add x and 4 first and then multiply by 2? Because that is one thing you might want to convey, there must be a way of specifying to do things in a different order than multiply first and then add. That's what parentheses are for. The symbols $2(x+4)$ mean to add x and 4 first, and then multiply that sum by 2.

Similarly, recall again that the expression $5 \cdot 2^4 + 10$ means by the order of operations to do the exponent 4 first, (and get $2^4 = 16$), and then you multiply 5 and 16 to get 80. After that, you add 80 and 10 to get 90. However, what if you really did want to multiply the 5 and 2 first? There has to be some way of conveying that, and there is, by using parentheses: The expression $(5 \cdot 2)^4 + 10$ means to first multiply 5 and 2 to get 10, and then you raise that to the 4th power to get 10,000, and then you add 10 to get 10,010, which is very different from the 90 we got earlier.

To communicate effectively, two people have to understand each other's language (or speak through an interpreter). The order of operations is just a way of ensuring that everyone agrees on what our language of mathematics means.

2.2.1 An advanced but important note

Given an expression, if you modify the order of operations to something *equivalent*, and if the modified order is **guaranteed** to yield the same answer, then the exact order of operations may be thought of as more of a suggestion than a requirement.

Let me do an example explaining this claim. Recall that when we computed $5 \cdot 2^4 + 10$ and when we followed the order of operations, we found it to be equal to 90. However, if we really did not want to compute all of 2^4 first (because multiplying 5 and 16 is harder to do in your head), then we can note that the number 2^4 is the same as $2 \cdot 2^3$. To evaluate $5 \cdot 2^4 + 10$, if we really wanted to, we could actually first change it to

$$5 \cdot 2 \cdot 2^3 + 10.$$

Then we can compute $5 \cdot 2$ and 2^3 *in the same step* and write

$$10 \cdot 8 + 10.$$

Then we get $80 + 10$ and finally 90. This is the same answer as before. Note that when we multiplied 5 and 2 in the same step as when we computed 2^3, we were technically not following the order of operations, but this modified order was equivalent.

Having said all this, if you struggle with the order of operations, then I recommend you be more strict in following it.

2.3 Equations and expressions

Let's begin with an important question.

> ? **Question 2.5.** *What are the things you can do with a variable?*

The answer depends on whether your variable, say x, appears in an equation, such as $2x - 5 = 11$, or whether there is no equals symbol. If there is no equals symbol, then all you have is something called an expression, such as $2x - 5$. Other examples of expressions are 118 or $87 \cdot 91 + 10$ or $5x$ or $12x^2$ or $\frac{x}{2}$ or $5x^3 + \pi x$ or even $(x-1)(x+5)$. In general, an expression is just something that represents a number (or several numbers) and *that does not have an equals symbol in it.* This is important enough to highlight:

> ! **Key Fact 2.6.** *An expression is something that represents a number (and so is not an equation).*

Occasionally, an expression will represent more than one number, such as ± 5, which means positive or negative 5, but usually, an expression is just one number.

An *equation*, on the other hand, is formed by putting an equals symbol between two expressions. So if we put an "=" between the expression $2x - 5$ and the expression 11, then we get the equation $2x - 5 = 11$.

If you have an expression all by itself (and so not as part of an equation), then what can you do with the expression? Well, the only things you can do are the things that don't change the value of the number(s) that the expression represents. For instance, if you had the expression $4x + 10$, then if you wanted, you could rewrite it as $10 + 4x$, or as $2x + 2x + 10$, or as $x + 3x + 8 + 2$, or as $4x + 10 + 0$, or $4x + 12 - 2$, etc.

Let's say we have an expression, such as 3, which for the sake of concreteness represents how many siblings I (the author) have. You could write that as $3 \cdot 1$, or $3 + 0$, or $0 + 3$, or $1 + 2$, or $2 + 1$, or $1 + 1 + 1$. What you *cannot* do is arbitrarily multiply that value by 2, to get 6. This is because I don't have six siblings. I have 3 of them (not counting brothers-in-law and sisters-in-law, which maybe should count, since they're great). But at the end of the day, you can't just change a 3 into a 6, since they mean different things.

Of course, your parents may or may not be able to double the number of siblings you have, but you yourself have no control over it (besides getting married and gaining in-laws). Suppose you have an expression only (and no equation). For example, let's say you have a 3, then you *are not permitted* to multiply it by 2, because you'd then be changing what it represents. You also can't add 10 to it (because 13 isn't 3), nor could you square 3, because 9 isn't 3. When you have an expression, the name of the game is to leave its value unchanged. You may change the way it *looks* (such as writing 3 as 2+1 or as $10 - 7$), but you cannot change its value. Of course, if you *really* wanted to change the value of an expression by say adding 1 to it, then you are free to do so, but just recognize that the new expression simply represents something different than the earlier expression. For example, if an expression represents how much money you have, then that is a value that changes as you make money, spend money, and donate money.

> ? **Question 2.7.** *What can you do with an equation?*

Let me begin with the punch line: You can do pretty much anything to an equation as long as you make sure that after each step, you still have an equality. Let's break

Chapter 3

Arithmetic

Technologies come and technologies go, but insight is forever.
–Michael Nielsen, quantum physicist

This chapter uses some of the concepts explained in the pre-algebra chapter. It turns out that knowing a little bit of algebra is useful in understanding arithmetic better.

3.1 How numbers are represented

This section lays the foundation needed to understand how we use numbers. Later in this chapter, we will discuss how multi-digit numbers are multiplied, namely, why the method we learn to multiply actually works. To do that though, we first must understand how we represent numbers.

Humans in the 21st century represent numbers in what is called the *decimal* system. We use 10 different digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.

> **!** **Key Fact 3.1.** *In the decimal system, if a number has multiple digits in it, the* ***position*** *of a digit affects how much that digit represents.*

For example, consider the number 323, or spelled out: three hundred twenty-three. The first "3" represents three *hundreds*, and the last "3" represents three *ones*. Having three hundreds is very different from having three ones. So in this number, the two 3's are counting different things: hundreds or ones. Also, the 2 in 323 is counting 2 *tens*.

In the decimal system, a digit might be counting ones, tens, hundreds, thousands, ten thousands, hundred thousands, millions, or even larger powers of 10. This chapter assumes you can already read numbers up to a few billion, but as a reminder, a million is a thousand thousands, namely $1{,}000 \cdot 1{,}000$, which is 1,000,000 or 10^6. Also, a billion is a thousand millions, or 1,000,000,000 or 10^9.

As another example, consider the number 7,654,321. Here, the 7 is counting millions (a million being 10^6); the 6 is counting hundred thousands (or $100 \cdot 1000 = 10^5$); the 5 is counting ten thousands (or $10 \cdot 1000 = 10^4$); the 4 is counting thousands (or 10^3); the 3 is counting hundreds (or 10^2); the 2 is counting tens (or 10^1); and the 1 is counting ones (or 10^0). On that note, if you aren't sure why we have the rule that $10^0 = 1$, then you are highly encouraged to read Section 2.5 from the pre-algebra chapter.

Exercise 3.2. *For each digit in the following eight-digit number, say what power of 10 the digit is counting:*

12,345,678

Chapter 4

Algebra

Wise people store up knowledge...

–King Solomon, Proverbs 10:14 (NASB)

Is it possible to thrive in this world without being able to walk? This question might appear to have nothing to do with algebra, but I will explain the connection. Let me start out by saying that yes, there are adults who cannot walk and yet are still thriving. Though not being able to walk is considered a disability, some people without legs (or with legs that don't work) have made the most of their situation. It must take incredible fortitude to be joyful without the ability to walk, but there are people who do exactly that.

Have you heard of Joni Eareckson Tada? She is a woman who, at the age of 17, became a quadriplegic. Though she is paralyzed from the shoulders down, she learned how to paint by holding a paintbrush in her mouth. She has also authored numerous books and is very active. Though her life is difficult (and includes chronic pain), she presses on.

Leaving seriousness for a moment, what if there was a sassy infant who could talk but hadn't yet learned to walk. He might argue with his parents that learning to walk is difficult, and he'd be right about that. He then might go on to say that at least in the United States, where accessibility is valued, people have worked hard to make things accessible to those in wheelchairs. There are elevators and ramps. Why learn to walk when you can still get around on wheels? Further, there are lots of helpful people who can push you, and some strong enough to carry you. Is learning to walk all that useful when you can still get around without walking? Learning to walk is just too difficult to be worth the hassle.

The hypothetical situation in the previous paragraph is absurd, but to a mathematician, algebra is like walking. Actually, for most people who have worked in any applied field of science or engineering, algebra might seem to be like knowing how to walk. (One difference, though, is that people usually learn to walk by the age of two, whereas people often learn algebra in middle school or high school.) To someone who has gone far in math, the two skills of (a) knowing how to walk and (b) knowing algebra are analogous skills. Just as walking is normally useful everywhere, so is algebra. Some people object to the usefulness of algebra by arguing that many people can get by just fine without it. Such people are correct that some professions use very little math. As for those who do need math, often a computer can do all the math for you. So what's the point in learning it? After all, it's difficult to learn, just as learning to walk is difficult!

My claim, then, is that you should learn algebra because it really is useful and because many professions require you to learn a reasonable amount of math in college.

Do you want to severely limit your options of what career you work in? I hope not. Even if you might know now what you want to do for an occupation, life happens and things change.

Sometimes, I, the author, hear people tell me that they gave up on their dream profession because the math required in college in their degree program was too difficult for them. This makes me sad. True, math is difficult to learn, but with hard work and guidance, you can make it through. Math doesn't have to be an unsurmountable obstacle. My hope is that this book provides some of that needed guidance, but you will still have to work hard. Okay, let's get back to doing some math.

As with almost the entirety of this book, I am focusing on *why* the rules of math have to be what they are. I mostly ignore applications. Someday, I might write a math book whose purpose is to show how useful math is by showing how it is used everywhere. God willing, I'd love to write such a book, but this is not that book.

4.1 Background

This chapter is not the first chapter in this book about algebra. Some of the fundamentals of algebra are included in the chapter called "Pre-algebra," and you should feel welcome to spend some time in that chapter. Are you completely comfortable with what a variable is? I explain variables in the pre-algebra chapter. Do you understand negative exponents and fractional exponents and why $2^0 = 1$? That also is in the pre-algebra chapter. In it, I also introduce the coordinate plane and how to graph lines. You are encouraged to spend some time in the pre-algebra chapter if you haven't yet.

4.2 Square roots, cube roots, and exponents being fractions like $1/2$ or $1/3$

Let's first cover square roots.

Motivation 4.1. *What positive number times itself equals 100? And what positive number times itself equals the number 36?*

Be sure to attempt each **Motivation** and **Exercise** before reading the answer that follows. Allow me to encourage you to read the preface if you haven't yet. The answers to this question are ten and six. What we have done is compute the square root of 100 and the square root of 36, which we write as $\sqrt{100}$ and $\sqrt{36}$ respectively.

The heart of the meaning of $\sqrt{100} = 10$ is that 10 times itself, i.e., 10^2, equals what's on the inside of the square root: 100. Similarly, the heart of the meaning of $\sqrt{36} = 6$ is that 6 times itself, i.e., 6^2, equals what's on the inside of the square root: 36.

As a couple other examples, we have that $\sqrt{16} = 4$ because $4^2 = 16$. Also, $\sqrt{49} = 7$ because $7^2 = 49$.

Exercise 4.2. *Compute the following: $\sqrt{4}$, $\sqrt{9}$, and $\sqrt{81}$.*

Hopefully you did the above exercise before continuing. We get that the answers are two, three, and nine respectively.

Recall that Chapter 3 explained why a negative number times a negative number is positive, and now, we will use that rule.

Motivation 4.3. *What is -10 times itself? What is -6 times itself?*

What we get are these numbers: positive one hundred and positive thirty-six respectively. So if we really *really* wanted to, we could say that -10 is **a** square root of 100 because -10 times itself is 100, and similarly, if we really wanted to, we could say that -6 is **a** square root of 36.

In other words, when computing $\sqrt{36}$, and so when we try to find *some* number that when multiplied by itself gives 36, then we perhaps have a choice in picking either positive or negative 6. In order to simplify our life and to have a common language to communicate with, whenever someone writes $\sqrt{36}$, we agree that it means *positive* 6 rather than maybe -6. If we really want -6, then we can simply write $-\sqrt{36}$, and if we want to speak of both -6 and positive 6, then we can write $\pm\sqrt{36}$, or just ± 6.

That brings us to the meaning of $\sqrt{x}$. The square root of x, i.e., the number $\sqrt{x}$, is a number that isn't negative and that when multiplied by itself equals x.

So $\sqrt{9} = 3$ because $3^2 = 9$ and also $3 \geq 0$. Similarly, $\sqrt{64} = 8$ because $8^2 = 64$ and $8 \geq 0$.

We have that $\sqrt{x} = \text{blah}$ means $\text{blah}^2 = x$ and that $\text{blah} \geq 0$.

? **Question 4.4.** *What does* $(\sqrt{x})^2$ *equal?*

You very well might be able to answer that question before reading on. Recall that $\sqrt{x}$ is a number (that isn't negative) such that if you multiply it by itself, you get x. So if we square $\sqrt{x}$, i.e., if we multiply $\sqrt{x}$ by itself, what should we get? We should get the number x. So the answer to the previous question is x.

If you were able to follow this section on square roots, then you're very close to also understanding cube roots. I will be much briefer on explaining cube roots.

The cube root of 8 is written $\sqrt[3]{8}$ (and is read as "the cube root of 8"). We have that $\sqrt[3]{8}$ is 2 because 2 cubed is 8. Similarly, $\sqrt[3]{125} = 5$ because $5^3 = 125$.

The meaning of $\sqrt[3]{x}$ is that it equals a number which, when multiplied by itself 3 times, gives x. In other words, if we cube the number $\sqrt[3]{x}$, then we get the number on the inside of the cube root: the number x.

Exercise 4.5. *Compute the following numbers:* $\sqrt[3]{27}$, $\sqrt[3]{1000}$, $\sqrt[3]{-125}$, *and* $\sqrt[3]{-1}$.

What we get are the numbers three, ten, negative five, and negative one respectively, because when you cube the numbers three, ten, negative five, and negative one, you get the numbers on the inside of the cube root: $3^3 = 27$, $10^3 = 1000$, $(-5)^3 = -125$, and $(-1)^3 = -1$.

So far in this section, I have only explained *how* to take square roots and cube roots (at least for examples that work out nicely). There is a rule I want to review briefly before I give a reason in the next subsection for this rule. The rule is that raising a number x to the $1/2$ power is the same as taking the square root of x. In other words, we say that $x^{1/2} = \sqrt{x}$. Similarly, the rule for cube roots is that $x^{1/3} = \sqrt[3]{x}$. For instance, $100^{1/2} = 10$ and $125^{1/3} = 5$.

It turns out that we can also take the square root (or cube root) of numbers and get a number with a decimal that goes on forever. For instance, using a computer or calculator, we can see that $\sqrt{2} = 1.41421356\ldots$, a number whose decimal expansion goes on forever (and with no repeating pattern). Here, I wrote down the first eight digits after the decimal point. If we stop writing digits after eight digits (or fifty digits or a million digits), then what we have is an approximation of $\sqrt{2}$. If we square the number $\sqrt{2}$ itself, what we get is 2 exactly, but if we square the approximation 1.414, then we get a number that is about 1.9994, which is pretty close to 2. If we instead find $(1.41421356)^2$, then we would get a number even closer to 2. As another example of a square root that has an infinite decimal expansion, we have $\sqrt{10} = 3.1622776\ldots$

The explanation for *why* certain rules with square roots work the way they do is omitted from the sample. Indeed, most pages are left out. However, since the algebra chapter is so long, and since it is rare to see clear explanations of logarithms, I have included the first part of my explanation of logarithms.

You are encouraged to skip the top part of the next page (which picks up in the middle of a thought).

The right-hand side is equal to $\pm\sqrt{b^2-4ac}/\sqrt{4a^2}$, which is $\pm\sqrt{b^2-4ac}/(2a)$, and so we may use this and subtract $b/(2a)$ from both sides to get

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a},$$

which may be rewritten as

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

We have thus solved for x in $ax^2+bx+c=0$ and so have derived the quadratic formula.

4.13 Logarithms

Apply your heart to instruction and your ear to words of knowledge.

–Solomon, King of Israel, (ca. 950 B.C.) Proverbs 23:12 (ESV)

This section answers the following:

Question 4.137. *What is a logarithm?*

One key to understanding logarithms is to understand exponents well, because logarithms are all about exponents. In fact, I will state something even stronger:

Key Fact 4.138. *The logarithm of a number* ***is*** *its exponent.*

This reality can be misunderstood, but I will spend some time here and explain what I mean. With proper guidance, logarithms do *not* have to be confusing.

Motivation 4.139. *What are the exponents of the following numbers? (a)* 10^3*, (b)* 10^6*, (c)* 10^4*, (d)* 10*, (e)* $\frac{1}{10}$*, (f)* $\sqrt{10}$*.*

One tricky thing about this question is that to answer it, you have to add an assumption to it. Indeed, I intended that you simply *assume* that I meant each of these numbers is 10 raised to some exponent, and it is *that* exponent that I was referring to. We say that 10 is the base. So then, with 10 as the base, the exponents are (a) three, (b) six, (c) four, (d) one. Next, (e) is slightly harder, but a tenth is ten raised to the negative one, and so negative one is the exponent (if 10 is the base). If you are unsure why raising 10 to the -1 yields $1/10$, then I highly recommend that you read Section 2.5 from the pre-algebra chapter. For (f), recall from when we covered fractional exponents that $\sqrt{10}$ equals 10 raised to the one half power, and so the answer to (f) is $1/2$ (where again, we assume that we are writing the numbers as 10 raised to some power).

What you have just done is calculate the logarithms of those six numbers. We may write $\log(10^3) = 3$ because the exponent of 10^3 is 3. We may also write that $\log(10^6) = 6$ because the exponent of 10^6 is 6. Similarly, we have $\log(10^2) = 2$, $\log(10) = 1$, $\log(\frac{1}{10}) = -1$, and $\log(\sqrt{10}) = 1/2$. Also, let me note here that the expression $\log(10^3)$ is *not* log "times" 10^3. Rather, $\log(10^3)$ means that we are plugging in the value of 10^3 into a function called log.

Motivation 4.140. *If we have an exponent in a number, is 10 always the base? Or is it the case that we can raise other numbers to some exponent?*

Of course, we are free to write 2^4 or 3^2 or 100^2, and so the answer is that when we have an exponent, the base very well might be something other than 10. Let's continue this thought in the next motivation:

Motivation 4.141. *If the logarithm of a number is just its exponent, then is the logarithm of 100 the number 1? (After all,* $100 = 100^1$*.) Or is it the case that the logarithm of 100 is 2? (After all,* $100 = 10^2$*.)*

It turns out that the answer to this question is that it depends on what base you're using for your logarithm! If you are using base 100, then the logarithm of 100 is 1 since $100 = 100^1$, but if you are using base 10, then the logarithm of 100 is 2, since $100 = 10^2$. In that same sense, we may say that the exponent of 100 is 1, since $100 = 100^1$, or we may say that the exponent of 100 is 2 when we are using 10 as the base.

The way we specify which base a logarithm is relative to is to write it as a subscript of the log. So we write $\log_{10} 100 = 2$ and say "log base 10 of 100 equals 2." Also, we write $\log_{100} 100 = 1$ and say "log base 100 of 100 equals 1."

Exercise 4.142. *Compute the following logarithms: (a)* $\log_{10}(10^7)$*, (b)* $\log_{10}(1{,}000)$*, (c)* $\log_{10}(1{,}000{,}000)$*, (d)* $\log_{10}(1/1000)$ *(e)* $\log_{100}(100^2)$*, (f)* $\log_{100}(100^5)$*, (g)* $\log_{100} 10$ *(h)* $\log_{100}(1{,}000{,}000)$*.* ***Note*** *that half-way through, I switched the base from 10 to 100.*

To answer this, remember that a logarithm is just telling us what the exponent is, where the assumed base is written as the subscript b in $\log_b(\ldots)$. The answers are (a) seven, (b) three, since $1{,}000 = 10^3$, (c) six, (d) negative three, since $1/1000 = 10^{-3}$ (e) two, (f) five, (g) 1/2, since $100^{1/2} = \sqrt{100} = 10$, (h) three, since $1{,}000{,}000 = 100^3$.

Besides logarithms with 10 as the base, there really are only two other bases that are very commonly used. The number 2 is often used as a base of a logarithm, especially in computer science. Also, there is an irrational number denoted e which is often the base of a logarithm. The number e is approximately 2.71828, but to remember it, you can just think of it as a number that is approximately 2.7 (or simplifying further, some number between 2 and 3). It turns out that e is a very important number, but its importance is only evident if you study calculus. When e is used as the base of a logarithm, instead of writing $\log_e(\ldots)$, people often write $\ln(\ldots)$, which stands for the "**n**atural **l**ogarithm" (with the order of the letters reversed). Again, exactly why the logarithm base e is somehow "natural" is not evident unless you take calculus.

Unfortunately, when $\log(\ldots)$ is written without any subscript b to the log, it is actually ambiguous. In advanced math, some mathematicians will sometimes write the natural logarithm as $\log(\ldots)$ (and so mean base e). Computer scientists will instead sometimes mean that $\log(\ldots)$ is $\log_2(\ldots)$. It appears to be the case that for most of the rest of the world, when people write $\log(\ldots)$, they mean $\log_{10}(\ldots)$.

Before the next exercise, let me do a quick example and state that $\log_2(8) = 3$, since $8 = 2^3$. In other words, the exponent of 8 is 3 when the base is 2.

Exercise 4.143. *Compute the following logarithms: (a)* $\log_2(2^8)$*, (b)* $\log_2(2^{10})$*, (c)* $\log_2(4)$*, (d)* $\log_2(2)$*, (e)* $\log_2(\sqrt[3]{2})$*, (f)* $\log_2(1)$*, (g)* $\log_2(1/8)$*, (h)* $\log_2(32)$*.*

To figure out what $\log_2(\text{blah})$ is, what we need to do is just figure out what the exponent of blah is when we write it as 2 to the something. For (a) and (b), the answer is already written for us. We have (a) eight, (b) ten, (c) two, since $4 = 2^2$, (d) 1, since $2 = 2^1$, (e) a third, since $\sqrt[3]{2} = 2^{1/3}$, (f) 0, since $1 = 2^0$ (and if you are not sure why $2^0 = 1$, then I highly recommend reading Section 2.5 from the pre-algebra chapter). Next, (g) negative three, since $1/8 = 2^{-3}$, and finally (h) five, since $32 = 2^5$.

So far, all the numbers we have taken a logarithm of have been numbers we can figure out without using a calculator. However, it is also possible to take the logarithm of numbers that don't work out so nicely. For example, we may compute $\log_{10}(98)$, but the easiest way of doing so is to use a calculator. It turns out that $\log_{10}(98) \approx 1.99$, but the exact decimal goes on forever without repeating (and so $\log_{10}(98)$ is an irrational number). The meaning of the statement that "$\log_{10}(98)$ approximately equals 1.99" is that $98 \approx 10^{1.99}$. It should make sense that the logarithm base 10 of 98 is a little less than 2, since 98 is a little less than 10^2.

Technically speaking, we may do logarithms for any base b such that $b > 0$ and $b \neq 1$.

Motivation 4.144. *Why can't we have 1 as the base of a logarithm? Why can't we compute* $\log_1(3)$*?*

The reason why 1 cannot be the base of a logarithm is that no matter what exponent we raise 1 to, we get 1. For instance, $1^2 = 1$, and $1^{200} = 1$. So if we want to ask what exponent we can raise 1 to so that we get 3, the answer is that there is no such exponent.

It is a little less obvious why we cannot use a negative number as the base of a logarithm. On that note, we cannot have -2 as a base of a logarithm. For instance, $\log_{-2}(8)$ makes no sense because there is no real number k such that $(-2)^k = 8$.

We are ready to state a definition of the logarithm with base b:

> **!** **Key Fact 4.145.** *Let $b > 0$ with $b \neq 1$. The function $\log_b(x)$ just tells us what exponent of b gives x. In other symbols, $\log_b(x)$ equals the number y such that $b^y = x$.*

So far, we have been in the process of answering the following question:

> **?** **Question 4.146.** *What is the meaning of* $\log_b(x)$*? (It is read as "log base b of x.")*

Our main answer from earlier in this section is that a logarithm is just the exponent of the number that is being plugged into the logarithm. It turns out though that there is one other perspective that is important for understanding logarithms. This key is to know that a logarithm is just the opposite (i.e., inverse) of an exponential function. My assumption in the rest of this section is that you have read the entirety of the section up to this point.

Before moving on to our second view of what logarithms are, let's briefly do one more exercise.

Exercise 4.147. *Without a calculator, compute the following logarithms: (a)* $\log_3(1/9)$*, (b)* $\log_{10}(1{,}000{,}000{,}000)$*, (c)* $\log_7(49)$*, (d)* $\log_8(1)$*, (e)* $\log_3(81)$*.*

We get (a) negative two, since $1/9 = 3^{-2}$, (b) nine, since there are nine zeros, (c) two, since $7^2 = 49$, (d) zero, since $8^0 = 1$, (e) four, since $81 = 9 \cdot 9 = 3^2 \cdot 3^2 = 3^4$.

4.13.1 The logarithm as the inverse of the exponential function

My aim until now is to help you be thoroughly comfortable with thinking of logarithms as exponents. The logarithm of a number is just the exponent of that number (relative to some base). Now, we are ready to cover the next primary interpretation of logarithms as functions. For the moment, we will restrict ourselves to using 2 as a base. The following is the fact we will focus on next.

> **!** **Key Fact 4.148.** *The function $f(x) = \log_2(x)$ is the inverse of the exponential function $g(x) = 2^x$. (The term "inverse" is defined below.)*

The **inverse** of a function is a function that swaps the input and output. In other words, the inverse of a function f with $f(x) = y$ is a function g such that $g(y) = x$. See also Section 10.2.1 from the precalculus chapter to read more on inverse functions.

Key Fact 4.148 is one possibility of how we can *define* the logarithm for base 2 (assuming we already had defined the exponential function $g(x) = 2^x$). It would be good to understand the meaning of the fact that the log base 2 function is the inverse of the exponential function with base 2, and so consider the following table of exponents and powers of 2:

exponents	-3	-2	-1	0	1	2	3	4	5
powers	1/8	1/4	1/2	1	2	4	8	16	32

The exponential function $g(x) = 2^x$ takes as input any exponent x. The output of this exponential function (that I am calling g), when given an input x, is the power of 2 written 2^x:

$$x \longrightarrow \boxed{\text{the function } g} \longrightarrow 2^x$$

The logarithm (with base 2) does the exact opposite thing as g. In fact, we can write this:

$$2^x \longrightarrow \boxed{\text{the function } \log_2} \longrightarrow x$$

which can also be written as $\log_2(2^x) = x$, which says that the exponent of 2^x (relative to base 2) is x. Going back to the above table, we may replace 2^x with y. The logarithm takes as input any y that is a power of 2, and then the output is the exponent x for which $2^x = y$.

One thing that the above table does not illustrate is that the allowed exponents x in 2^x include *any* real number (not just integers).

So then, the exponential function $g(x) = 2^x$ begins with any exponent x and transforms it into a power of 2. Complementary to this is the logarithm function $f(y) = \log_2(y)$ which begins with any power of 2 (written as y) and transforms it into the exponent x of 2 for which $2^x = y$. The logarithm and exponential functions do the exact opposite thing. Let's summarize where we've been:

> **!** **Key Fact 4.149.** *The exponential function transforms exponents into powers, and the logarithm transforms powers into exponents.*

So then, the exponential and logarithm functions are *inverses* of each other. This means that if you start with an input and apply one of the functions and then apply the other function, you end up where you started (i.e., with the input that you started with). Let's see what this looks like. Start with the input x. We will first apply $g(x) = 2^x$ and then we will take the output of g and put it into the $\log_2$ function. That looks like this:

$$x \longrightarrow \boxed{\text{the function } g} \longrightarrow 2^x \longrightarrow \boxed{\text{the function } \log_2} \longrightarrow x$$

Notice that we end up with x, which is what we started from. This can be written out algebraically as $\log_2(g(x)) = x$. In other symbols, since $g(x) = 2^x$, the figure shows this:

$$\log_2(2^x) = x$$

Chapter 5

Geometry

There is no royal road to geometry.

–Euclid (ca. 300 B.C.)

5.1 The Pythagorean Theorem

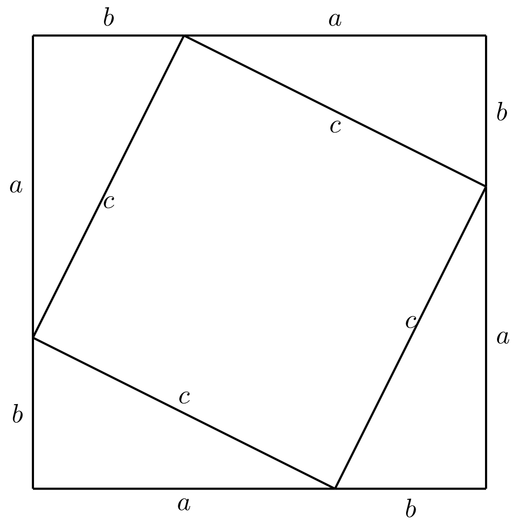
Let's begin with the Pythagorean Theorem, which we state in Theorem 5.1. Recall that a right triangle is a triangle with a 90° angle in it (which is called a right angle), and the hypotenuse is the longest side of a right triangle, the side opposite the right angle.

Theorem 5.1 (The Pythagorean Theorem)**.** *Suppose a right triangle has a hypotenuse of length c and the two other sides of length a and b. Then*

$$a^2 + b^2 = c^2.$$

? **Question 5.2.** *How can we know with absolute certainty that the Pythagorean Theorem is true?*

Consider the figure below, which is a square with sides of length $a + b$ containing a smaller, tilted square with sides of length c inside of it.



Challenge 5.3. *Use the figure above to come up with a proof of Theorem 5.1. Hint: Compute the area of the large square in two ways.*

Let A be the area of the large square. Then $A = (a+b)^2$. Distributing shows

$$A = (a+b)^2 = a^2 + 2ab + b^2.$$

But A is also the sum of the areas of the four triangles and the tilted square. (Recall that the area of a triangle is 1/2 base times height.) Hence,

$$A = 4 \cdot \frac{1}{2}ab + c^2 = 2ab + c^2.$$

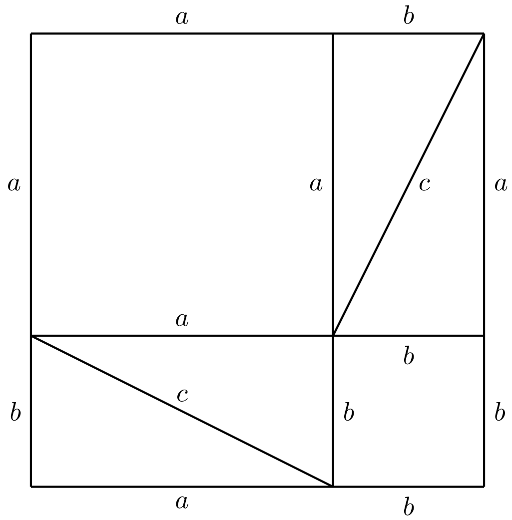
We may then equate the two different expressions for A to get

$$a^2 + 2ab + b^2 = 2ab + c^2,$$

and then we are done by subtracting $2ab$ from both sides.

? **Question 5.4.** *Does the above proof explain the Pythagorean Theorem fully?*

I hope readers like the proof just given, but sometimes, one proof might give more insight than another proof. Also, why should we only give one proof when we can give two? This next proof is only a slight variation of the previous one, but you might like it even more.



Notice that this large square has the same shape as the large square on the previous page, since they both are squares with sides of length $a + b$. Notice what is left if you subtract the areas of the four triangles: In the figure on this page, you're left with the areas of the two smaller squares: $a^2 + b^2$, but on the previous page, when you take away the four triangles, you're left with c^2. As a result, we have

$$a^2 + b^2 = c^2.$$

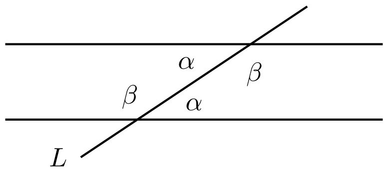
One thing I really like about this second proof is that it shows how you can rearrange geometrical shapes to show that the theorem is true. Note, however, the similarity with the first proof, where we also had to subtract the areas of four triangles $4 \cdot \frac{1}{2}ab$, which equals $2ab$, from two sides of an equation.

5.2 Why do the angles in a triangle add to $180°$?

This section focuses on the question in the section title, which is restated here:

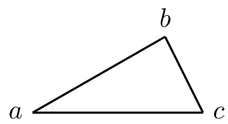
> **Question 5.5.** *Why is the sum of the angles in a triangle* $180°$*?*

To explain this, we will take one result from geometry for granted.

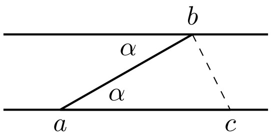


This figure consists of two parallel lines, with another line (line L) passing through them. The two angles labeled α (the Greek letter "alpha") are equal, and the two angles labeled β (the Greek letter "beta") are equal. The corresponding angles are equal because of the two parallel lines. We will take this result for granted in part because it looks intuitive, but in a geometry class, you would actually prove this result from simpler ones.

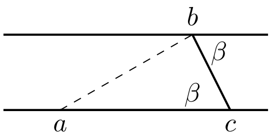
With that said, consider the following triangle:



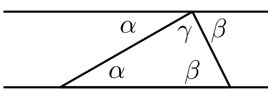
We will show that the angles of this triangle add to $180°$. What we will do is extend the line segment ac into a full horizontal line. We will also draw a line through b that is parallel to ac:



Because we have parallel lines, the two angles labeled α are equal.



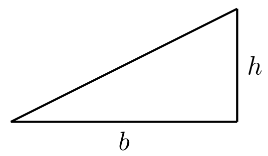
Because we have parallel lines, the two angles labeled β are equal. Combined, we have the following figure (with also the labeled angle γ, the Greek letter "gamma"):



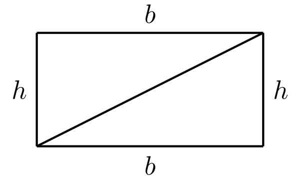
Notice that the angles α, β, and γ together form a straight line. Hence, they sum to $180°$. That concludes our proof on the sum of the angles in a triangle.

5.3 A universally applicable formula about area

For some triangles, it is not difficult to see why their area is $\frac{1}{2}bh$, where b is the length of the base of the triangle and h is its height. This section goes beyond that, but we will begin at the beginning.

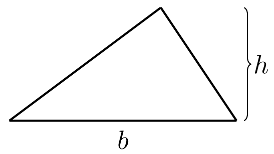


To see that this *right* triangle has area $\frac{1}{2}bh$, we need only see that it is half of a rectangle with width b and height h:

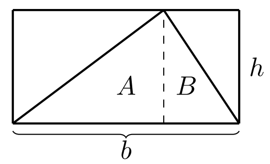


(Note that a reason why the rectangle has area bh is given in the section "Multiplication as repeated addition" in the arithmetic chapter.)

A little bit harder is why the following triangle also has area $\frac{1}{2}bh$:



To see why the same formula works for this triangle too, we can again embed it in a rectangle with width b and height h:

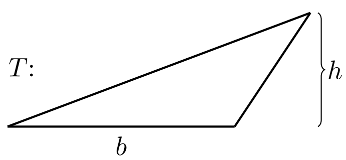


What we can notice then is that the triangle with base b and height h can be split into two right triangles, A and B, that share the dashed side that has a height of h. Both A and B are half of the rectangle on their side of the dashed line, and so their total area is half of bh.

> ? **Question 5.6.** *What if the top of a triangle is not directly above a part of its base? Why is the area of the triangle still $\frac{1}{2}bh$?*

Consider the following triangle:

Triangle T:



Chapter 6

Logic

Mathematics is the music of reason.

–James Joseph Sylvester, mathematician

To understand math well, it is necessary to gain a good grasp of logic. In particular, to write or understand proofs, you must know basic logic. The following question might enter your mind:

> ? **Question 6.1.** *What is a proof?*

This chapter will answer this question, but to give a short answer, a mathematical proof is simply a rigorous argument that will convince someone that a certain mathematical statement is true. Here, the "someone" referred to in the previous sentence is anyone who has the necessary background. Examples of proofs will be given in this chapter. You might ask the following question though:

Motivation 6.2. *Doesn't the rest of this book contain many proofs?*

Thinking so would be quite reasonable, but in most cases, it hasn't been my intention to write out full proofs of the rules/statements I've been showing to be true. My goal has been to be convincing and not to worry about making the arguments completely rigorous. (Also, some of the rules I've explained are actually *definitions*, for which it doesn't make sense to prove. When I try to convince readers that a certain rule/definition is "correct," I mean to simply show why any such rule couldn't be anything other than what it is, assuming it is sensible.) Getting back to the topic...

> ? **Question 6.3.** *Why do proofs even matter? Aren't convincing, non-rigorous arguments good enough? Why do mathematicians value proof?*

Those are fair questions to ask. First, note that what may convince one reader might not convince someone else who is more skeptical. So for some people, an argument that isn't rigorous isn't convincing. This is partly due to the following phenomenon: Even if a mathematical statement appears to be true, that doesn't guarantee it to be true.

Let me give an example. First, let me introduce some notation. The symbols $\lceil x \rceil$ mean to round x up to the next closest integer larger than x (if x isn't an integer) and to leave x alone if x is already an integer. For instance $\lceil 7.2 \rceil = 8$, $\lceil 19.9997 \rceil = 20$, and $\lceil 45 \rceil = 45$. Similarly, the symbols $\lfloor x \rfloor$ mean to round x down to the next closest integer smaller than x (if x isn't an integer) and to leave x alone if x is already an integer.

The functions $g(x) = \lceil x \rceil$ and $f(x) = \lfloor x \rfloor$ are called respectively the ceiling and floor functions.

$$\text{Does} \quad \left\lceil \frac{2}{2^{1/n}-1} \right\rceil = \left\lfloor \frac{2n}{\ln 2} \right\rfloor \quad \text{for all positive integers } n?$$

You might guess, "Well of course not! Those two formulas look so different!" But if you plug in all the positive integers less than a million, you'd find that the formula is always true![1] The formula is also accurate for all integers up to a billion, or a trillion, or even seven hundred trillion! It turns out that the smallest integer n for which the formula isn't true is $n = 777,451,915,729,368$. It's quite incredible that the formula works for so long but finally fails.[2]

As a result, one reason why mathematicians like proofs is to steer clear of error. We want to be absolutely certain that a result is true.

So does "rigor" in a proof mean you must write out a very long proof? Not necessarily. For instance, you are allowed to use the statements of theorems already proved without reproving them.

There is another very important reason why mathematicians value proofs; it is in order to *understand* something. An air-tight (i.e., rigorous) argument of why a mathematical statement is true should give insight as to *why* it is true. That's one good reason why mathematicians often like to have multiple proofs for the same statement.

One final reason I'll mention why mathematicians value proofs is that it is often the case that a proof includes some result, technique, or key insight that is useful to help do other mathematics.

6.1 The basics of logic

6.1.1 Conditional statements

Logic is used to deduce true statements that are consequences of other true statements. Consequently, it is important to understand the act of one statement implying another. Central to logic is something called a conditional statement. Suppose A and B are statements. We call a statement of the following form a conditional statement:

If A, then B.

Conditional statements can also be written as follows:

$$A \implies B.$$

The previous statement is read as "A implies B."

Here are some examples of conditional statements (which may or may not all be true):

1. If it is raining hard outside, then the ground is wet.

2. If you are reading this chapter, then you must be at least 14 years old.

3. If n is an even integer, then n^2 is even.

[1] If you use a computer to do this, then note that you need to make sure that the computer is using enough precision to represent the numbers accurately enough. Otherwise, it may incorrectly say that the two formulas aren't equal, such as for $n = 156{,}339$.

[2] For a reference, see what the eminent mathematician Richard Stanley wrote in an answer to a question here: `https://mathoverflow.net/questions/15444/examples-of-eventual-counterexamples`

I wager that the first of those statements is true. The second statement may or may not be true. The third one definitely is true.

There are two important questions to ask about the concept of conditional statements:

> **Question 6.4.** *Under what conditions is a conditional statement true? And how might you modify a conditional statement to get a related statement?*

Let me address the first question first. Suppose I am talking with my nephew Jeremiah and I tell him, "Jeremiah, if you clean your room today, then tomorrow, I'll buy you ice cream." If I weren't telling him the truth when I said this, what is the only way for Jeremiah to prove that my offer isn't true?

First consider that he didn't clean his room. I probably wouldn't be buying him ice cream then. So in that case, could he show that I was lying? No, of course not. The statement I made to Jeremiah would in this case be "Jeremiah, if you clean your room today [False], then tomorrow, I'll buy you ice cream [False]." So then, we *cannot* say a statement of the following form is false:

"If ⟨ some false statement ⟩, then ⟨ some false statement ⟩."

A conditional statement of this form is thus taken to be true.

Next, perhaps the reason he didn't clean his room was that he was busy cleaning the bathroom and kitchen (extra chores, let's say), and then his parents decided the family would watch a movie together. And what if then, I decided to buy him ice cream, even though he didn't clean his room? Would that show that I was lying? Again, no it wouldn't. "Jeremiah, if you clean your room today [False], then tomorrow, I'll buy you ice cream [True]." So then, we *cannot* say a statement of the following form is false:

"If ⟨ some false statement ⟩, then ⟨ some true statement ⟩."

A conditional statement of this form is thus also taken to be true.

To summarize the previous two paragraphs, under the condition that Jeremiah did *not* clean his room, he couldn't show that I was lying to him, whether or not I buy him ice cream. In general, when the condition in a conditional statement (the "if" part, i.e., the A in $A \implies B$) happens to be false, the whole conditional statement is considered true regardless of whether or not B is true.

Of course, if Jeremiah did clean his room, and then I bought him ice cream, then the statement I made would have been true. We have then that conditional statements of the following three forms are true:

- If ⟨ some false statement ⟩, then ⟨ some true statement ⟩.
- If ⟨ some false statement ⟩, then ⟨ some false statement ⟩.
- If ⟨ some true statement ⟩, then ⟨ some true statement ⟩.

The first two on the above list might be confusing, and so let me add that we say that the conditional statement is **vacuously** true whenever the condition is false (as in the first two on that list). In that case, you of course cannot conclude anything about whether or not the conclusion is true, and so the conditional statement would in that case do you no good in an attempt to try to deduce other conclusions.

What is the only way for Jeremiah to claim that my statement was a lie? Suppose Jeremiah really did clean his room today. Then of course he could say my offer isn't true precisely if in that case, I don't buy him ice cream the next day. A conditional statement is false when it is of the following form:

Chapter 7

Trigonometry

Do not worry about your difficulties in mathematics; I can assure you that mine are still greater.

– Albert Einstein

Some people say that trigonometry has a lot to do with triangles and angles. While I don't necessarily disagree, I would want to add that trigonometry also has a lot to do with circles. Trigonometry is very relevant when trying to describe a repeating wave, such as a musical note or a radio wave or visible light.

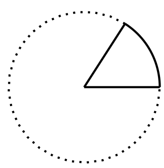
Let's begin with how angles are measured. You might be most familiar with measuring angles with degrees, where a degree is defined as the angle that takes up 1/360 of a circle. Hence, a right angle is $90°$. A degree turns out to be a very arbitrary unit of measurement for angles. Why should our unit of measurement of angles be 1/360 of a circle rather than 1/200 of a circle or 1/100 of a circle? There really is no good inherent reason for this, but it turns out that there is a unit of measure of angles that is less arbitrary. This unit of measure of angles, called the radian, appears foreign and unnatural at first, but it makes things simpler (especially when doing calculus).

7.1 Radians

Let's first answer the question of what a radian is.

> ? **Question 7.1.** *How big is a radian?*

Suppose we take a circle with a given radius r, and suppose we also have a string that has length one radius: r. Let's pick up that string and place it along the edge of the circle so that we have an arc of the circle of length r. Measuring the angle at the center of the circle corresponding to that arc, the resulting angle would take up exactly one radian. This is our definition of what a **radian** is.



To better understand radians, let's figure out approximately how many degrees equal one radian. To do that, first try answering the following question:

Motivation 7.2. *How many radiuses (or radii) does it take to measure the entire circumference of the circle?*

To reword the question in a more familiar way, consider what times the radius equals the circumference. What we have is that the circumference of a circle C with radius r is

$$C = 2\pi r.$$

If you don't recall the reason for this, then it would be a good idea to review Section 5.4.1 from the geometry chapter (which is brief).

So then, since $C = 2\pi r$, what that means is that it takes the length of 2π radiuses to measure the circumference of a circle. Recall from above that a radian corresponds to how much of the circle is formed when the arc length is one radius. With that said, we are ready for the next question.

Motivation 7.3. *How many radians does it take to go all the way around the whole circle (the whole* $360°$*)?*

What we get is that it takes 2π radians to go all the way around. Each radian corresponds to an arc length of one radius. We have then that 2π radians is $360°$.

If we want to know what one radian is in degrees, all we have to do is divide both sides of the equation 2π radians $= 360°$ by 2π. We conclude that 1 radian $= 360°/(2\pi) \approx 57.3°$. So basically, a radian is a tad less than $60°$, which should make sense since 60 is $1/6$ of 360 and 2π is a bit more than 6, which implies that a radius r has length a little less than $1/6$ of $2\pi r$.

Motivation 7.4. *Suppose an angle is two radians. Then what is the corresponding arc length of that part of a circle?*

What we get is that two radians make up an arc of the circle of length $2r$, where the circle has radius r.

Exercise 7.5. *If an arc of the circle has length* $3.4r$*, then what angle does that arc form?*

That arc forms 3.4 radians. Notice that we get the number 3.4 by simply dividing the arc length, namely $3.4r$, by r.

Motivation 7.6. *Suppose a circle has a radius of 10 inches. If an arc of that circle has length 51 inches, then what angle does that arc form?*

What we get is that the angle is 51 in/10 in $= 5.1$ radians. All we had to do was divide the arc length by the length of the radius.

This next point I want to make is a little tricky, but it is something that we shouldn't skip.

? **Question 7.7.** *Although I called the radian a unit of measure of angles, some people legitimately call a radian "unitless." What does that mean? And why is it "unitless"?*

A radian is in fact a *ratio* of two lengths: It is an arc length divided by the length of the radius. Hence, if we are measuring lengths in inches, the radian "unit" has units of inches/inches (i.e., inches divided by inches). But since anything (nonzero) divided by itself equals one, that means that the units cancel. If we were measuring the radius and arc length in meters, then $m/m = 1$ too, and so regardless of how we measure lengths, when we divide two lengths, the units cancel. *Any* ratio of two numbers having the same unit is in fact unitless. So then, a radian can actually be called unitless. Consequently, instead of writing "2π radians $= 360°$," people sometimes write the following equation:

$$2\pi = 360°.$$

Exercise 7.8. *Since 2π radians make up the whole circle, then how many radians equals $180°$?*

Because $180°$ is half of a circle, the answer is half of 2π, which gives π radians.

Exercise 7.9. *How many radians equal a right angle?*

A right angle is half of $180°$, and so the answer is half of π, which is $\pi/2$.

Exercise 7.10. *How many radians equal... (a) $45°$? (b) $30°$?*

One way of figuring this out is to see that $45°$ is half of a right angle, and so $45° = (1/2) \cdot \pi/2 = \pi/4$ (radians). Also, $30°$ is $1/6$ of $180°$, and since $180° = \pi$, we get that $30° = \pi/6$.

The previous two sentences are legitimate ways of answering the previous exercise, but we would also like to be able to convert other angles into radians that aren't as nice. For instance, how many radians is $37°$? Because, $2\pi = 360°$, dividing both sides by 2 gives $\pi = 180°$. Next, dividing by either π or $180°$ gives the equations

$$\frac{180°}{\pi} = 1, \qquad \text{and} \qquad \frac{\pi}{180°} = 1.$$

These fractions can be used to do unit conversions. So then to convert $37°$ into radians, just multiply by $\frac{\pi}{180°}$, and then the degrees unit would cancel. To get a decimal as an answer, we find that $37° \approx 0.646$ (radians).

Exercise 7.11. *(a) How many radians is $190°$? (b) How many degrees is 10 radians?*

For the first, we can just multiply by $\frac{\pi}{180°}$ to get 3.32, and to illustrate that the degrees really cancel, I'll write

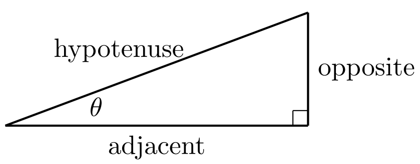
$$190 \text{ degrees} = 190 \cancel{\text{ degrees}} \times \frac{\pi \text{ radians}}{180 \cancel{\text{ degrees}}} \approx 3.32 \text{ radians}.$$

For the second problem, we can just multiply 10 by $\frac{180°}{\pi}$ to get $\approx 573°$, and just to illustrate that the radians cancel, I'll write

$$10 \text{ radians} = 10 \cancel{\text{ radians}} \times \frac{180 \text{ degrees}}{\pi \cancel{\text{ radians}}} \approx 573 \text{ degrees}.$$

7.2 Sine, cosine, tangent – triangle definition

It is common to use the Greek letter theta, written θ, to represent an angle. When we define the trigonometric functions using a triangle, we must use a *right* triangle:



Chapter 8

Complex numbers: $i = \sqrt{-1}$

The discerning heart seeks knowledge...

–Solomon, King of Israel, (ca. 950 B.C.) Proverbs 15:14 (NIV)

8.1 Numbers and the complex plane

We speak of something called "complex numbers," also known as "imaginary numbers." It is an unfortunate reality that some people abuse the notion of "imaginary" numbers by saying that in math, you can make up whatever you want. This claim, however, is misguided.

In order to understand complex numbers, we must talk about what we mean in the first place by "number." We will spend some time to answer this question:

? **Question 8.1.** *What does the word "number" mean?*

When teachers ask questions, they sometimes say that there are no wrong answers. If we were talking about something that was purely a matter of opinion, then assuming sincerity, we could say that in that context, there are no wrong answers, just different opinions, but Question 8.1 is not like that. It does have wrong answers. Even so, while there are plenty of wrong answers to this question, there does happen to be more than one valid answer to what constitutes a "number."

Let me next say that I dislike the term "complex number," and I also dislike the term "imaginary number." In my opinion, these terms are misleading. Granted, to the uninitiated, the "number" $\sqrt{-1}$ seems extremely complex and totally made up, but I hope to convince you that it can be a completely reasonable thing to think about and work with. Also, it turns out that in real life, so-called "imaginary" numbers turn out to be really useful (at least for some people, such as electrical engineers).

Allow me to communicate that plenty of numbers you are already comfortable with are in a sense, actually imaginary. For instance, how long is an (American) football field? You might answer 100 yards, and that is the "correct" answer, but an actual football field may really be 100.0001 yards or 99.998 yards (or it actually can differ depending on which part of the field you measure). Here, the number 100 is an idealization of reality and so in some sense is imaginary. Whenever you measure any quantity on a continuum (such as length, weight, or temperature), any number we use has only a certain amount of precision in it. The number used is an approximation of reality (that can even change slightly with time).

There are also many numbers that definitely are complex. For instance, the number 8125826056540367283927 is a plain old whole number, but it looks pretty complex to me. The value $i = \sqrt{-1}$ is *much* simpler than that complicated whole number.

One possible answer to Question 8.1 is that a number represents a count of how many of something we have. In this sense, a number is an answer to a question such as, "How many sisters do you have?" or, "How many marbles are in the sealed jar?" Answers to questions like these can be a whole number.

Motivation 8.2. *Is the concept of "number" restricted only to whole numbers, or instead, are there also numbers that don't count how many of something we have?*

I claim that the answer is that there are lots of numbers that aren't counting anything. Take for instance 13.4, which is thirteen and four tenths. Is there anyone who has 13.4 sisters (i.e., 13 and four tenths sisters)? No, that's just absurd. It just isn't possible to have 13 and four tenths sisters. So then, there are numbers that say something *other* than how many of something we have.

While 1 and 2 and 32 are numbers, of course 13.4 is a number too. It's just that having a fractional part means that it is much more appropriate describing how much, how big, or how long something is. In other words, 13.4 represents a quantity. It doesn't count something, but it measures something. You might have 13.4 ounces of chocolate, or you might have a model rocket that is 13.4 inches tall. So then, our second answer to Question 8.1 is that a number can measure the quantity of something, such as length, weight, height, or speed.

Motivation 8.3. *How about negative numbers? Are they numbers too?*

Yes, they are, and just as the number 13.4 is absurd in some contexts, so also negative numbers are absurd in some contexts. For instance, if someone told you that their TV screen is -18 inches wide (*negative* eighteen inches wide), how would you respond? Saying your TV is negative eighteen inches wide just doesn't make any sense at all.

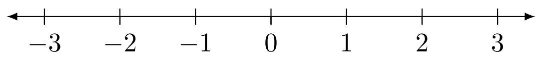
As stated in a previous chapter, negative numbers do make sense in some contexts. Suppose my nephew has no money saved at all but wants to buy a toy that costs \$18. Suppose that his parents do not give him the toy out of generosity. What they could do is buy the toy for Jeremiah under the condition that he works to pay it off later. Jeremiah could then go in debt \$18. Hence, if he were asked how much money he had, he could say that he's in debt \$18, and so has "negative 18 dollars," where the negative represents the "direction" of a debt rather than a savings. After he makes \$18, he would pay it to his parents and be at \$0.

A negative number accurately represents a debt, but it would be absurd to talk about a TV having a width of some negative amount. So just as 13.4 is absurd in some contexts and makes sense in other contexts, so also negative numbers are absurd in some contexts and make sense in other contexts. Unsurprisingly, the same is true of "complex numbers." Again, there are lots of situations in which fractional numbers are absurd, and there are lots of situations in which negative numbers are absurd. Similarly, there are lots of situations in which complex numbers are absurd. We should also recognize that fractional numbers, negative numbers, and complex numbers all have situations in which they *do* make sense. This chapter will focus on what complex numbers mean and when they do make sense.

Let's then state right at the outset that complex numbers do *not* represent a quantity. They represent something different entirely.

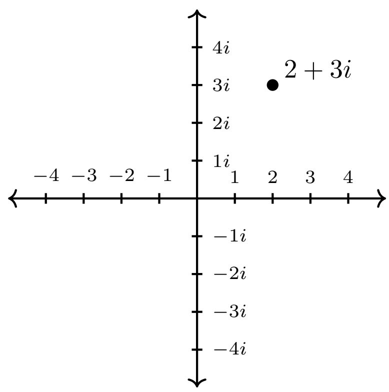
As I stated earlier in this section, in my opinion, the terms "complex numbers" and "imaginary numbers" both are misleading. The former makes it sound like they're complicated, and the latter makes it sound like they are somehow illegitimate. We can even say something like "i is not a real number," and we'd be using the technical terms correctly. This is somewhat unfortunate. In my opinion, a better name for the numbers called complex numbers would be "**2-dimensional numbers**" which we could abbreviate by saying "2D numbers." My hope is that as I explain these numbers, my preference for the term "2D numbers" will become clear. Although I hope to convince you that i is not overly complicated, I must admit that when you first work with it, the value i is *foreign*, and so just like learning a foreign language, it takes time to get used to it.

Recall that the set of "**real**" numbers can be represented by a number *line*:



This number line extends in both directions without any end, and each position on the number line corresponds to one real number. One interpretation then of a real number is that it represents a position on a line. Let's note here that a line is a *one* dimensional object.

What if we wanted to describe a position in *two* dimensions instead of one? One option is to use Cartesian coordinates, and here each point has both an x-coordinate and a y-coordinate, such as the point $(2, 3)$. The point $(2, 3)$ has an x-coordinate of 2 and a y-coordinate of 3.



Astute readers may think that because I have called complex numbers "2-dimensional numbers," then perhaps a point on the Cartesian plane can be represented by a single 2D number. If you're one of those readers, then you are absolutely correct! In fact, the point $(2, 3)$ in the plane can be written as the complex number $2+3i$. This sum might look like more than one number, but it's just a single number as much as 5 is a single number. So the number $2+3$ is a single number that is written as a sum of two numbers. Similarly, $2 + 3i$ is a single 2D number that is written as a sum of two numbers. (But $2+3i$ is *not* 5.) The number $2+3i$ is the position that can be found by starting at the origin (zero), and then the 2 says to go 2 to the right, and the $3i$ says to go 3 up; the "up" in "3 up" stands for the i in $3i$.

The point on this plane labeled $2+3i$ is a single, 2-dimensional number. Note that to specify this point, we have to use two real numbers (2 and 3), and using both numbers, we can write the single complex number $2 + 3i$.

As another example, the point in the plane $(-2.5, 7.7)$ can be written as the complex number $-2.5+7.7i$. Another example is that the point in the plane $(1, -4)$ can be written as the complex number $1 - 4i$.

Exercise 8.4. *The point* $(10, 8)$ *in the plane can be represented by what complex number?*

As usual, with all questions labeled "Exercise" or "Motivation," be sure to answer the question yourself before reading on to check your answer. You get a lot more out of reading if you think through the question and come up with an answer yourself before reading the answer. For this exercise, the complex number is $10 + 8i$.

Motivation 8.5. *The complex number i represents what point in the plane?*

If you are stuck, a hint is that i can also be written as $0 + 1i$. What we get is that the complex number i represents the point in the plane with the coordinates $(0, 1)$.

Chapter 9

Probability

Give portions to seven, yes to eight, for you do not know what disaster may come upon the land.
–King Solomon, Ecclesiastes 11:2, (NIV, 1984)

When math is abstract, it can be difficult to understand, but sometimes, a small amount of abstraction can actually simplify things. In probability, one particular thing students sometimes struggle with is distinguishing between when numbers should be added together and when they should be multiplied. It turns out that here, knowing a small amount of set theory can be quite beneficial and clarify matters. Hence, we will begin in the next section with a brief discussion of what are called *sets*, which will be explained.

Probability itself is a way of measuring the size of something just like counting is a way of measuring how many of something you have. Many questions that can be asked in probability simply boil down to being able to count how many of something you have.

The probability of something describes in some way "how big" it is, just like length, area, and volume are measures of how big something is. In probability, we are "measuring" (or calculating) the likelihood of certain events to occur. An event that is certain to occur has probability 1 (and so is as big as possible), and an event that cannot occur would have probability 0 (and so is as small as possible). If you flip a fair coin once, then the probability of heads is $1/2$. If you shoot a basketball from the half-court line, you may or may not get the basketball into the hoop (and so have two options), but since you'd be quite a ways away from the hoop, I'd guess the probability of you getting the ball into the hoop is quite a bit less than $1/2$. If you threw the ball from the half-court line a thousand times, and if you made it in only eight times out of a thousand, then that would suggest the probability of you making it into the hoop from there perhaps is approximately $8/1000$. This is only an approximate, since if you flip a fair coin 1000 times, you most likely won't get *exactly* 500 heads. Besides, unlike coin flips, you might actually have gotten better at shooting hoops as you went along.

In this chapter, we'll cover how to calculate probabilities. We will have no need here to shoot any basketballs a thousand times. Instead, we'll look at situations where exact probabilities can be calculated. By the end, you should be able to calculate the probability of getting exactly 500 heads if you flip a fair coin 1000 times. Finally, note that this chapter will only deal with finite sets. Some sets are infinite, such as the set of positive integers: $\{1, 2, 3, 4, \ldots\}$, but in this chapter, we'll only deal with finite sets.

9.1 Basic set theory

? **Question 9.1.** *What is a set?*

A set is just an unordered collection of elements. For example, as I am writing this sentence, my wife and I are the only people physically present in our condominium unit. So then, the set of all people physically present in our condo has exactly two elements in it: my wife and me. We could denote it as follows:

$$\{\text{Andrew, Andrew's wife}\}.$$

Since elements of sets have no order to them within the set, we could alternatively write the same set as follows:

$$\{\text{Andrew's wife, Andrew}\}.$$

Another set is the set of positive integers less than 5:

$$\{1, 2, 3, 4\},$$

which is the same as $\{2, 4, 1, 3\}$, but writing $\{1, 2, 3, 4\}$ just looks a lot nicer.

What kinds of elements can be in a set? Pretty much anything. Elements can be numbers, letters, words, people, countries, computers, transistors, or even other sets.

The set that has literally nothing in it at all comes up reasonably often in mathematics and is called the **empty set** and is denoted $\varnothing$. The number of elements in the empty set is 0.

The only data associated with a set is what is in it (and perhaps what is not in it). An element cannot be in a set multiple times. Otherwise, you'd be talking about a different mathematical concept, called a *multi*set, which we won't have need of in this book.

Motivation 9.2. *What is common to the two sets A and B below?*

$$A = \{4, 5, 6\} \quad \textit{and} \quad B = \{Peter, James, John\}$$

The thing in common to them is that they both have exactly three elements. Although their contents differ, the *structure* of the sets is identical. In fact, how many elements a set contains is the only structural property a set has. Though I don't plan to use this term later in this chapter, the number of elements in a set is called its **cardinality**. So A has cardinality 3, and so does the set B. The set $\varnothing$ has cardinality 0, and the set $\{a, b, c, d, e\}$ has cardinality 5. To denote the cardinality of a set A, we will write $|A|$. Hence, $|\{a, b, c, d, e\}| = 5$ and $|\varnothing| = 0$.

In probability, we often have a set that specifies all possible outcomes of some experiment. For instance, let's roll a regular, single die; all regular dice have six sides. Then the set of possible outcomes is the set $\{1, 2, 3, 4, 5, 6\}$. Consider the set consisting of rolling a 5 or 6, and call it A. So $A = \{5, 6\}$. Given A and given the set of all possibilities, it makes sense to say what's *not* in the set A. The set of all elements *not* in A is called the **complement** of A. This would be the set $\{1, 2, 3, 4\}$. Different people denote the complement of a set using different notation. Some write A^c, with "c" as a superscript, and others write $\overline{A}$, with a bar written over A. I personally prefer the notation $\overline{A}$, but you're welcome to write it either way (unless your teacher requests you to write it a certain way). Whichever you choose, you can read both symbols A^c and $\overline{A}$ as "the complement of A."

Exercise 9.3. *Suppose you draw a single card from a deck of regular cards (that consists of 52 cards). Let F denote the set of all face cards (i.e., the jacks, queens, and kings of the deck). What is $\overline{F}$, the complement of F? Also, find $|\overline{F}|$, the number of elements in $\overline{F}$.*

The cards not in F are the aces as well as the 2's through 10's. Since there are four suits, that makes $4 \cdot 10$ or 40 cards in $\overline{F}$. We can also arrive at $|\overline{F}| = 40$ by calculating $52 - 4 \cdot 3$.

Whenever we take the complement of a set, it is required that we understand the totality of all elements we are considering. In the deck of cards example, the set of all options of what to draw is the set of all 52 cards. This set of all options can be called the *universe of discourse*, and in probability, this set of all options is called the **sample space**.

Exercise 9.4. *Suppose we have dice with various numbers of sides. We have a blue die that is 12-sided, a red die that is 8-sided, and a green die that is 6-sided. All our dice have their sides numbered one through the total number of sides on that die. Depending on which die we pick, the set of all elements we are considering is one of the following:*

(a) $\{1, 2, 3, \ldots, 6\}$ *(for the green die)*

(b) $\{1, 2, 3, \ldots, 8\}$ *(for the red die)*

(c) $\{1, 2, 3, \ldots, 12\}$ *(for the blue die)*

Here, the ellipses mean to include all the integers in between. Let A be the set $\{1, 2, 3, 4, 5\}$. *For each of the above three dice, find $\overline{A}$.*

For (a), the green die, we would get that $\overline{A} = \{6\}$. For (b), the red die, we get $\overline{A} = \{6, 7, 8\}$. For (c), the blue die, we get $\overline{A} = \{6, 7, 8, 9, 10, 11, 12\}$. Let the reader be encouraged that whenever we want to find the complement of the set, it will be clear from context what the larger set is that specifies the universe of discourse (or sample space, which here, is determined by which die we are using).

? **Question 9.5.** *Why do we care about set complements?*

So far, we have covered what the complement of a set means but haven't yet included any motivation on why we care. Let me do that here.

Suppose Thomas is making an eight-character password, and the characters he uses for his password are lowercase letters and digits. So each of the eight characters has 36 options. Suppose further that Thomas has decided to make his password so that it contains at least one digit. What is the simplest way of figuring out how many options Thomas has? It turns out that it's easiest to actually count the passwords that have no digits in them, of which there are 26^8, and so the number of options Thomas has is $36^8 - 26^8$, but this is kind of jumping ahead of ourselves. All this should be more clear when we cover Cartesian products. The only point I want to make here is that if you want to count how many elements are in a set, sometimes it's a lot easier to count how many elements are *not* in the set (i.e., count how many elements are in the complement of the set).

Let me highlight a statement that is important enough to repeat:

! **Key Fact 9.6.** *The only data associated with a set is what is in it (and perhaps what is not in it).*

Chapter 10

Precalculus

Undoubtedly the most important concept in all of mathematics is that of a function...

– Michael Spivak, mathematician

What is the purpose of a course on precalculus? One goal is to help prepare students for calculus. Some people have a different purpose, however. Indeed, some seem to intend that precalculus be a time to introduce a variety of topics, not all of which are essential for calculus.

In my opinion, the most important thing to gain in a precalculus class is better preparation for calculus. Having taught calculus a number of times, I believe that the most difficult part of calculus is actually just algebra, and I think the second most common topic that some calculus students struggle with is trigonometry. If you have a solid understanding of algebra and trigonometry, and if you know the fundamentals of geometry (such as being able to work with similar triangles), then that puts you in a good position to succeed in calculus and higher mathematics.

In this chapter, I try to focus more on the topics that matter most for successfully learning calculus. However, I do include a little bit of extra material. In addition to the present chapter, you can consider the previous material on algebra, trigonometry, and geometry as being a part of "precalculus." Indeed, my favorite textbook on precalculus is a little book that has just three chapters, one on each of geometry, algebra, and trigonometry. (The book is *Precalculus Mathematics in a Nutshell* by George F. Simmons.)

There is one essential topic that students do not always learn thoroughly in algebra and trigonometry and which is indispensable to understanding calculus. That topic is the topic of functions.

10.1 Functions

If you want to understand mathematics at the precalculus level or beyond, it is paramount that you have a good grasp of the concept of functions. The algebra chapter includes a section on functions, which would be very good to read if you aren't already comfortable with them. Let's begin here with our first definition of what a function is. Later on in this section, we will cover a couple of alternate definitions that you might see in some traditional math books.

> **!** **Key Fact 10.1.** *A function is just a rule that, given an input, produces an output. The output is completely determined by what the input is.*

I chose this definition intentionally because I think it is pedagogically sound. However, because you might see other definitions elsewhere on what a function is, I am compelled to discuss other ways people define functions.

Some people word their definition slightly differently. In particular, some people write that a function is a rule that, given an input, produces a *unique* output. What do they mean by the output being unique? To answer this, consider the function $f(x) = \sqrt{x}$, and consider for instance $f(81)$. The value of $f(81)$ is 9. The value is *not* sometimes 9 and sometimes -9. Rather, as we have defined the meaning of the square root of non-negative numbers, $\sqrt{81}$ is *always* 9. If you really wanted to refer to both positive *and* negative 9, then you would need to write "$\pm$" in front of the square root like so: $\pm\sqrt{81}$. You could also write it in front of 9 itself: ± 9. There is no difference between ± 9 and $\pm\sqrt{81}$. In sum, $\sqrt{81}$ by itself is *always* 9.

If you have a function called f, and if a is anything we're allowed to plug into f (i.e., if a is in the domain of f), then $f(a)$ is *not* one thing on Mondays, a different thing on Wednesdays, and a different thing every other Friday. Rather, $f(a)$ is completely determined by a (assuming that f is given/defined/specified). In that sense, we could say that $f(a)$ is unique.

One thing easy to confuse in this discussion is that the "unique output" part does not rule out the possibility that a function may have two separate inputs that both produce the same output. Consider $g(x) = x^2$. Notice that $g(7) = 49$, but also $g(-7) = 49$. The two separate inputs of 7 and -7 both yield the output of 49. There is nothing at all wrong with this.

Exercise 10.2. *Let $h(\theta) = \sin(\theta)$. Find two different values of θ that produce the same output when plugged into h.*

Remember that you should attempt every problem labeled **Exercise** or **Motivation** before reading the answer that follows. In this case, there happen to be infinitely many different correct answers to this question, and you are welcome to check your work with a calculator, if you want to. One way of coming up with two angles for θ is to pick them as co-terminal angles (i.e., angles that differ by an integral multiple of 2π radians, which is $360°$). So for instance, you could have picked 0 and 2π (in radians, or $0°$ and $360°$ if using degrees), but another example is $\pi/4$ and $2\pi + \pi/4$.

> **!** **Key Fact 10.3.** *That a function produces a unique output for a given input simply means that the input determines the output. It does **not** mean that it is impossible to find two separate inputs that produce the same output.*

We have thus covered one variation on Key Fact 10.1 in defining what a function is.[1] Next, let's work toward another definition that appears to be very different.

Before we get a little abstract, I want to give more examples of some functions. All of the following arrows represent a function. I've intentionally made most of them non-mathematical in nature because the concept of a function is not exclusive to mathematics.

The following arrows signify that an input of a particular kind is associated with an output. Each arrow is specifying a function. While most functions used in textbooks

[1] As stated earlier, this alternate definition is that a function is a rule that, given an input, produces a unique output.

are named (such as "f" or "g"), the implicitly described functions below will remain nameless (except that I do use the name "f" later on for any one of them). In what follows, inputs are described on the left side, and outputs associated to the inputs are described on the right side. Consider the following associations:

a student at Cedarville University $\longrightarrow$ his/her student ID
any license plate $\longrightarrow$ the place that the vehicle is registered
any city $\longrightarrow$ the population of that city in 2025
any gasoline car $\longrightarrow$ the average miles per gallon of that car

Here are more examples:

any pet $\longrightarrow$ the weight (in pounds) of that pet
any human being $\longrightarrow$ his/her biological mother
a stretch of DNA encoding a protein $\longrightarrow$ the sequence of amino acids of that protein
the sequence of amino acids of a protein $\longrightarrow$ the 3D shape of the folded protein

If any particular sequence of amino acids that comprise a protein do not uniquely determine its 3-dimensional shape, then the last example above would not be a function. Here are more examples:

any year between 1789 and 2024 $\longrightarrow$ the president of the USA on May 1 that year
any year since Gutenberg's printing press $\longrightarrow$ the number of Bibles printed that year
any day since July 31, 1871 $\longrightarrow$ the rainfall in Colorado Springs that day

Allow me to give just a few more examples. For the following, only God knows the precise associations in full:

any human (born or unborn) $\longrightarrow$ the number of hairs on his/her head
any island in the ocean $\longrightarrow$ the number of grains of sand on its beaches
any goal or plan you have $\longrightarrow$ whether or not it will succeed

Some of the above examples are actually slightly fuzzy. Many people have a pet dog that they've owned since the dog was a puppy. You also might have owned a pet kitten. Of course, the weight of a puppy or kitten changes as time goes on. If we were using "function" in a more exact way, we would need to admit that a function doesn't change as time goes on. However, it is perfectly legitimate to have a function whose input is a moment in time, and whose output is determined by that moment in time.

Now that we've seen a number of examples, we are ready to try to unify them all. The following gets a little abstract, but I hope the above examples make what follows easier to understand. The following is not a definition, but it is a helpful way of thinking about functions:

> **!** **Key Fact 10.4.** *At its heart, a function is a way of associating inputs with outputs.*

Given a function f, when we write $f(a) = b$, we mean that a is an input that f associates with the output $f(a)$, which here happens to be called b. This can also be stated as "a yields b," or that "a produces b," or even that "f transforms a into b." We could also say that "a maps to b," but this latter usage of "maps" as a verb can be confusing to some. To explain the word "maps," notice that for any point a on a

Chapter 11

Calculus

> *When you improve a little each day, eventually big things occur... Not tomorrow, not the next day, but eventually a big gain is made. Don't look for the big quick improvement. Seek the small improvement one day at a time. That's the only way it happens—and when it happens, it lasts.*
>
> –John Wooden, greatest basketball coach of all time

Calculus is a beautiful diamond in the land of mathematics. The heart of calculus really boils down to three or four interconnected ideas, which when understood, are quite simple. But isn't calculus a difficult subject? What is the hardest part of calculus? In my opinion, the most difficult part of calculus actually happens to be... algebra. Yes, algebra. The second hardest part might be trigonometry. If you have a good handle on both algebra and trigonometry, then you are well prepared to learn calculus.

Let me mention here that I am assuming readers of this chapter have a decent background in algebra and in particular are comfortable with the concept of a function. However, we will keep the algebra here to a minimum.

? **Question 11.1.** *What are the three or four most important concepts in calculus?*

To really answer this question is the goal of the next four sections. It isn't a problem if you haven't heard of these concepts before, and so let me briefly just mention the names of those four concepts: (1) limits, (2) derivatives, (3) integrals, and also (4) the Fundamental Theorem of Calculus. These four ideas are closely related. In fact, derivatives and integrals are built on the foundation of limits, and the Fundamental Theorem of Calculus is precisely about the relationship between derivatives and integrals.

11.1 Limits

The concept of a limit is foundational to calculus. The present section answers the following question.

? **Question 11.2.** *What is a limit?*

Limits are all about numbers getting closer and closer to some specific value. For instance, suppose you are on a desert island with no water except for a one liter water bottle full of water that never gets refilled. You decide that starting today, each day you will drink half of the water left. (That way, you always have some water left.) Today, you drink 1/2 of the liter. Tomorrow, you drink 1/2 of the 1/2 left (so 1/4 of a liter of water).

Motivation 11.3. *By the end of tomorrow, how much water will be left?*

To get the answer, notice that if you always drink half of what's left, then the amount you drink that day is how much is left. So there will be 1/4 of a liter left.

The next day (the day after tomorrow), you'll drink 1/2 of what's left (so 1/8 of a liter of water).

Motivation 11.4. *By the end of the day after tomorrow, how much water will be left?*

Since the third day we drink $(1/2)^3$ of the water, or 1/8, then that is how much will be left.

Motivation 11.5. *The amount of water you have left in your water bottle is approaching what as the days go by?*

In our hypothetical scenario, the amount is getting closer and closer to zero. What we have computed then is a limit. The amount of water left in your water bottle approaches zero, as time goes on. That can be thought of as follows. Consider the following sequence of numbers:

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$$

Then the limit of that sequence is 0. In symbols, we write it in the following way, where we use the variable n to represent the day number:

$$\lim_{n\to\infty} \frac{1}{2^n} = 0.$$

This is read as follows: "The limit as n approaches infinity of 1 over 2 to the n is 0."

We will next turn to a question that has confused many people:

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Question 11.6. *Consider 0.99999... (repeating 9's without end):*

$$\textit{Is } 0.99999\ldots = 1?$$

As stated, by 0.99999..., I mean that the 9's go on and on forever, whatever that means. Now that we mention it, that leads us to the following question which will help us answer the previous one:

?

Question 11.7. *What do we really mean by an infinite decimal (by a decimal that goes on forever)?*

Since we are considering the number $0.99999\ldots$, let's give it a name, say L.

Motivation 11.8. *Does L equal 0.9?*

The answer is no since L is bigger because it has more 9's in its decimal expansion.

Motivation 11.9. *Does L equal 0.999?*

Well, the same reasoning says that no, $L \neq 0.999$, since L is bigger due to having more 9's in its decimal expansion.

Motivation 11.10. *Does L equal 0.99999?*

Just as before, the answer is that no, L is bigger due to having more 9's in its decimal expansion.

We almost have the answer on what L is:

Motivation 11.11. *Is there a number that the following sequence is getting closer and closer to?*

$$0.9, 0.99, 0.999, 0.9999, 0.99999, \ldots \tag{11.1}$$

If you think about it, you could probably say that yes, those numbers are getting closer and closer to 1. Is the 10^{th} number on that list equal to 1? No. How about the 50^{th}? Nope, but the 50^{th} number (a number with fifty 9's after the decimal point) sure is super close to 1.

What we mean then by the repeating decimal is that it equals the limit. This is actually a *definition* of what we mean by the infinite decimal. As you write down more and more decimals in its decimal expansion, you get closer and closer to the limit even if you *never*, no matter how many 9's you write down, ever reach the number 1. Of course, I gave the number $0.99999\ldots$ the name L because that letter suggests that the number is a limit, which it is.

So the meaning of L is that it is the *limit* of the sequence (11.1) above, which is in fact 1. In other words, the answer to Question 11.6 is yes. Of course, there is a much simpler "proof" that $0.9999\ldots$ equals 1, but it takes infinite decimals for granted:

Simplified "proof". Note that $1/3 + 1/3 + 1/3 = 1$, but we have the following:

$$\frac{1}{3} = 0.33333\ldots$$

So just add three infinite decimal expansions (of repeating 3's) together to get repeating 9's. Therefore, since $1/3 + 1/3 + 1/3$ equals 1, and since it also equals $0.99999\ldots$, then $0.99999\ldots = 1$. □

The reason why I put the word "proof" in quotes (and also why I say it is simplified) is that this argument glosses over the important details of what infinite decimal expansions even mean. While the argument is sound, it just jumps over subtleties that are explained more in the discussion prior to it. There is nothing wrong with sometimes skipping subtleties, but I brought up infinite decimals for the very purpose of talking about limits, which the above simplified proof ignores.

Going back to limits, just as 1 is the limit of the sequence 0.9, 0.99, 0.999, 0.9999, ..., we have that π is the limit of the following sequence:

$$3, 3.1, 3.14, 3.141, 3.1415, 3.14159, \ldots$$

But of course, there is no pattern discernible here to the digits of π. So to answer Question 11.7, the meaning of an infinite decimal expansion is that it equals the limit of what you get by using more and more of the digits in the expansion.

11.1.1 Two tips for limits

When dealing with limits and infinity, there are two important principles to keep in mind. Let the phrase "a small number" denote some tiny positive number, and let "a big number" denote some huge positive number. Here are the two important but simple principles to remember for limits:

$$\frac{1}{\text{a small number}} = \text{a big number} \qquad \text{and}$$

$$\frac{1}{\text{a big number}} = \text{a small number}$$

To use this most effectively, you should note that the "1" in the above fractions can be replaced with any positive number.

Before applying the above principles in some examples, let me address the inquisitive readers who might ask the following:

> ? **Question 11.12.** *What do you mean by "small" or "large"? In other words, what really is a small number, and what is a big number?*

After all, is 1/100000 a small number? Is a billion a big number? Believe it or not, the answer to Question 11.12 is quite important and relevant for scientists, software engineers, economists, and many other people.

Motivation 11.13. *Can you think of a context where the number 1 is enormously large?*

As always for "Motivation" questions, please do think about this for a minute. Here is an example I thought of: Imagine eating 1 buffalo for dinner. It would be impossible for a human to, in a single setting, eat all the meat in 1 entire buffalo (assuming the buffalo is a healthy adult buffalo).

Motivation 11.14. *Can you think of a context where the number* 10^{12} *(one trillion) is tiny?*

Actually, what I will do instead is show that the number 10^{22} is sometimes incredibly small. Indeed, to drink only 10^{22} molecules of water in a day is a minuscule amount of water to drink, being less than a gram of water. Hence, in the context of counting how many molecules of water you drink in a day, the number 10^{22} is tiny.

Motivation 11.15. *If the number 1 is sometimes enormously big, and if the number* 10^{22} *is sometimes tiny, then what does that say on what a small number or a big number is?*

What it says is that what is small and what is big simply depends on context. The number 1 might be small or big or just a medium sized number; it all depends on the context.

This is quite important because sometimes people try to intimidate others with a number such as a billion or a trillion, saying how huge it is, when in fact you need to know a context to determine how big it is.

Next, let's go over examples of using the two principles mentioned at the beginning of Section 11.1.1 by finding the following limits.

We will find...

$$A = \lim_{x\to\infty} \frac{2}{x}, \qquad \text{and} \qquad B = \lim_{x\to 0} \frac{5}{x^2}.$$

For A, if you aren't sure, you can begin by just plugging in some positive number for x in the expression $2/x$, such as $x = 10$. Then $\frac{2}{10} = 0.2$ is what $2/x$ simplifies to. But don't stop there, we can plug in 1000 for x. Then $\frac{2}{1000} = 0.002$ is what $2/x$ simplifies to. What happens if x is even bigger? Then $2/x$ gets even smaller.

Motivation 11.16. *What do you think the limit A is? Recall* $A = \lim_{x\to\infty} \frac{2}{x}$.

If you aren't sure, you can plug in an even bigger number for x. What does $\frac{2}{x}$ approach as x gets larger and larger? What it approaches is zero, and so A equals zero.

Chapter 12

What is Mathematics?

Finally, brothers, whatever is true, whatever is honorable, whatever is just, whatever is pure, whatever is lovely, whatever is commendable, if there is any excellence, if there is anything worthy of praise, think about these things.

–Paul, Philippians 4:8 (ESV)

Is math ever interesting? Does it have anything to say beyond the typical stuff we learned in school? Yes!–to both of those questions, and if you give me the chance, I'd like to show that to you in this chapter.

True, I will soon give a definition of mathematics. However, to know what math is about, it is necessary to have *examples* of beautiful mathematics. So a general definition as you might find in a dictionary won't by itself give the whole picture.

To understand what math is about, you must see good examples of mathematics. Of course, if you have read the previous chapters of this book, then you will have seen many examples of mathematics. Some academics high in their towers might look down on the math we've done, but despite what they say, what we've gone through in this book so far really is *real* mathematics.

? **Question 12.1.** *So... what is mathematics?*

A boring person might say that it's what we've been doing throughout this whole book! But that's pretty obvious. Although the present chapter attempts to answer this question by giving beautiful examples of math you have *not* seen before, allow me to give a definition here.

Mathematics is the art and rigorous study of patterns.

Math involves abstract ideas and reasoning logically about them. The patterns might have to do with shape or with quantities that can be counted or measured. Central to math are ideas. Thankfully, a rigorous study of patterns turns out to be very helpful when trying to understand the world around us. Some mathematicians (called "pure mathematicians") largely ignore the world around them and focus only on pure ideas and abstract concepts. Others (called "applied mathematicians") try to find applications of their ideas in the outside world. Applied mathematicians create and analyze models of some phenomena in the real world. There is much variety in math, just as there is much variety among mathematicians. This chapter (and this book as a whole) focuses more on math itself, mostly ignoring applications to the real world. This bias or focus simply reflects my own background.

If math really has enormous variety in it, wouldn't it be sad if a book that includes a broad spectrum of math didn't also include some really cool stuff that you've never seen before? Someone might object to this question and say that the *explanations* given in this book are cool and new to many readers. However, wouldn't it be nice to do more than just restrict ourselves to the rules you learned in school?

Don't get me wrong. I think the earlier chapters include some really cool stuff. However, there is a lot of fascinating mathematics that most people don't learn in middle school or high school.

When talking about interesting mathematics, I have a choice to make: Should I only include results that I can give full justification for why they are true, or should I permit myself to share other extremely cool math too? Are there true mathematical statements which are much easier to state than to prove? Yes, there are many such truths in math. My decision then is to include some of the beautiful diamonds of math, even if including a proof or justification is too much to ask.

So let me take you on a tour in the mathematical landscape and show you some of the beautiful views. You can appreciate the scenery even if you never read another math book. Of course, a few readers just might treat this as only the start of an adventure. Whether or not you want to read another math book, let me say that it's perfectly okay (good even) if you have other pursuits besides mathematics. I'll admit it, math isn't the *only* important thing in life, but it is important to many people. Regardless, I hope this book helps many readers to want to learn more math.

12.1 How this book differs from traditional math

In practice, mathematicians often work out examples in order to gain insight. However, when it is time to write a proof, the examples are usually left out entirely. In this book, rather than intend to always give proofs, I have often instead given examples. Also, proofs are supposed to be rigorous, but to get a proof, you sometimes have to give a lot more detail than just the main ideas. I have chosen here to focus on the essence of what is going on rather than include all the gory details. There is a time to include such detail, but I don't think the present book is the best place for such rigor.

Astute readers might notice that some of the rules I have explained in this book are listed in other books as *definitions*. Can there be reasons for definitions, or are they arbitrary? When something is given as a definition, people often think they must accept it blindly. In fact, one popular math book[1] at one point specifically states this when talking about the fact that $\sqrt{2} = 2^{1/2}$. That author says, "There is nothing to understand, for this is a definition of what we mean..." But unfortunately, that author is misguided here. As a quick recap, the key equation for exponents is that (for $b \neq 0$) we want that for all x and y,

$$b^x b^y = b^{x+y}. \qquad \text{(the key equation for exponents)}$$

This rule works perfectly well when x and y are positive integers. It is precisely *because* we want this rule to be true for *all* x and y that we are forced into agreeing that $2^{1/2}$ is a number whose square is 2. (It is also due to the above key equation that negative exponents work the way they do. If we want the key equation for exponents to hold for all real values of x and y, then we are forced into the fact that $b^{-x} = 1/b^x$, which some people state as a definition.) So then, even though some people declare $2^{1/2}$ to be *defined* as $\sqrt{2}$, that doesn't mean that there is no reason why such a definition is made in the first place. Saying otherwise is quite misguided.

[1] It is the book *Algebra 2* by John H. Saxon.

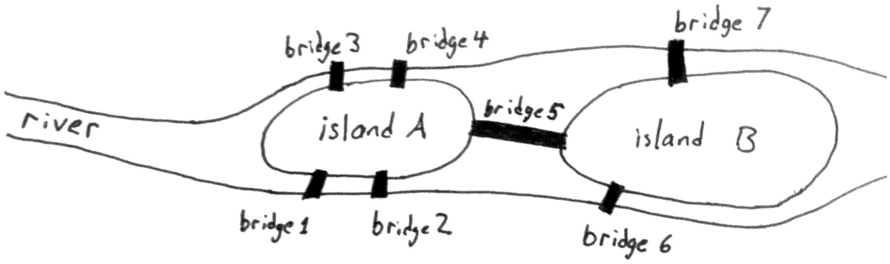
There is one more facet of this discussion that is worth mentioning. The issue is that in mathematics, it is often the case that there are multiple, equivalent ways of defining something. Precisely which definition is used to describe a given concept is sometimes a matter of opinion, taste, and expediency. Although math is objective, and although people cannot make up anything at all (if they want it to be useful and true), it is still the case that there often is much choice in how concepts are described.

For example, consider the functions $f(x) = 2^x$ and $g(x) = \log_2(x)$. These two functions are inverses of each other. Should you first define 2^x and then define $\log_2(x)$ as the inverse of 2^x? Or should you first define $\log_2(x)$ and then define 2^x as the inverse of $\log_2(x)$? As a third option, should you instead define both functions 2^x *and* $\log_2(x)$ separately (without reference to the other one), and then prove a theorem that they are inverses of each other? Already, we have three options of how to arrive at 2^x and $\log_2(x)$, and it turns out that there are yet more options than what I listed (such as starting with the more convenient number e as the base and then build off of that). If you really want to define 2^x and $\log_2(x)$ properly (for all appropriate values of x), it actually requires a bit of calculus, but I think people should be introduced to these functions before they take calculus.

It probably is the case that when most people see a definition, they think that it must be accepted blindly. That is one reason why I have not chosen to list all such rules as definitions. Rather, I have described the rules as what they need to be in order for some pattern to hold true. Math is all about patterns, and so I believe my approach is justified.

12.2 The Königsberg bridge problem

There was a city named Königsberg that was on the Pregel river in Prussia. Part of the city was on two islands in the river, and part of the city was on either bank. There were seven bridges connecting the different land masses as follows (not drawn to scale):



The city was thus separated into four land masses: two islands and the two sides of the river. The citizens of Königsberg wondered whether it was possible, starting on one of the four land masses, to cross every bridge *exactly once* and end up in the land mass you started at.

Of course, you are not allowed to cheat by crossing from one land mass to another by any means other than a bridge. (For example, you are not allowed to swim or use a boat.)

A solution to this problem (either in the affirmative or a proof that it was impossible) was given by the mathematician Leonhard Euler in the 1700s.

Exercise 12.2. *Spend a few minutes trying to solve this problem yourself. What is your opinion on whether or not it is possible, starting from one land mass, to cross each*

Index