Problem solutions
EP "Quantum field theory, string theory and mathematical physics"

Conformal field theory – I
(A.V. Litvinov)

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Contents

Theoretical minimum

Lecture notes (A.V. Litvinov).

Stress-energy tensor in QFT, conformal Ward identities

We consider Euclidean, $SO(D)$ invariant field theory with symmetric stress-energy tensor $T_{\mu\nu} = T_{\nu\mu}$ defined by

$$\delta_{\boldsymbol{\epsilon}} S = \int_{\mathbb{R}^D} \partial_\mu \epsilon_\nu T_{\mu\nu} d^D \boldsymbol{x} \tag{1}$$

The quantization of the theory reduces to the consideration of functional integrals of the form

$$\langle X \rangle = \frac{1}{Z} \int X e^{-S[\Phi]} [\mathcal{D}\Phi], \tag{2}$$

where X is a composite field. Usually, we take it in the form

$$X = \mathcal{O}_1(\boldsymbol{x}_1) ... \mathcal{O}_N(\boldsymbol{x}_N), \tag{3}$$

where $\mathcal{O}_r(\boldsymbol{x}_r)$'s are some local fields

$$\mathcal{O}(\boldsymbol{x}) = \mathcal{F}(\Phi(\boldsymbol{x}), \partial_\mu \Phi(\boldsymbol{x}), ...) \tag{4}$$

In principle, function $\mathcal{F}(...)$ can be arbitrary. The only property which we require, is that the fields finitely separated in the space are not allowed. The collection of all local fields is usually thought of as a vector space. One can imagine it as

$$\mathcal{A} = \text{span}\{\Phi^N(\boldsymbol{x}), \Phi^N(\boldsymbol{x})\partial_\mu \Phi(\boldsymbol{x}), \Phi^N(\boldsymbol{x})\partial_\mu \partial_\nu \Phi(\boldsymbol{x}), ...\} \tag{5}$$

The fields in (5) can be regarded as symbols for the true quantum fields. We will usually denote them as $\mathcal{O}_j(\boldsymbol{x})$, $j = 1, ..., \infty$, meaning that they form a basis in infinitedimensional vector space $\mathcal{A}$. Since in (2) we integrate over all functions $\Phi(\boldsymbol{x})$ we assume that the measure of integration is invariant with respect to translations

$$\mathcal{D}(\Phi(\boldsymbol{x}) + \epsilon(\boldsymbol{x})) = \mathcal{D}(\Phi(\boldsymbol{x})) \tag{6}$$

where $\epsilon(\boldsymbol{x})$ is an arbitrary function. It leads to the following identity

$$\begin{aligned} \sum_{k=1}^{n} \langle \mathcal{O}_1(\boldsymbol{x}_1) ... \delta_\epsilon \mathcal{O}_k(\boldsymbol{x}_k) ... \mathcal{O}_N(\boldsymbol{x}_N) \rangle &= \frac{1}{Z} \int e^{-S[\Phi]} [\mathcal{D}\Phi] \sum_{k=1}^{n} \mathcal{O}_1(\boldsymbol{x}_1) ... \delta_\epsilon \mathcal{O}_k(\boldsymbol{x}_k) ... \mathcal{O}_N(\boldsymbol{x}_N) = \\ &= \frac{1}{Z} \int \delta_\epsilon(\mathcal{O}_1(\boldsymbol{x}_1) ... \mathcal{O}_N(\boldsymbol{x}_N)) e^{-S[\Phi]} [\mathcal{D}\Phi] = -\frac{1}{Z} \int \mathcal{O}_1(\boldsymbol{x}_1) ... \mathcal{O}_N(\boldsymbol{x}_N) \delta_\epsilon e^{-S[\Phi]} [\mathcal{D}\Phi] \end{aligned} \tag{7}$$

$$\delta_\epsilon e^{-S[\Phi]} = -e^{-S[\Phi]} \delta_\epsilon S[\Phi] = -e^{-S[\Phi]} \int_{\mathbb{R}^D} \epsilon(\boldsymbol{x}) \left(\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \right) \right) d^D \boldsymbol{x} \tag{8}$$

$$\text{EOM}(\boldsymbol{x}) = \frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \right) \tag{9}$$

is the composite field which vanishes on-shell in classical field theory.

$$\sum_{k=1}^{n} \langle \mathcal{O}_1(\boldsymbol{x}_1) ... \delta_\epsilon \mathcal{O}_k(\boldsymbol{x}_k) ... \mathcal{O}_N(\boldsymbol{x}_N) \rangle = \int_{\mathbb{R}^D} \epsilon(\boldsymbol{x}) \langle \text{EOM}(\boldsymbol{x}) \mathcal{O}_1(\boldsymbol{x}_1) ... \mathcal{O}_N(\boldsymbol{x}_N) \rangle d^D \boldsymbol{x} \tag{10}$$

Now, we note that the function $\epsilon(\boldsymbol{x})$ is arbitrary. In particular, it can be taken such that $\epsilon(\boldsymbol{x}_k) = 0$. Then the left hand side of (10) should vanish by assumption of locality. Thus we have

$$\langle \text{EOM}(\boldsymbol{x}) \mathcal{O}_1(\boldsymbol{x}_1) ... \mathcal{O}_N(\boldsymbol{x}_N) \rangle = 0, \quad \boldsymbol{x} \neq \boldsymbol{x}_k \tag{11}$$

Definition 1. A field with this property, that is any correlation function involving this field vanishes unless its position $\boldsymbol{x}$ coincides with one of the other insertion points, is called *the redundant field.*

There are, in principle, infinitely many redundant fields in QFT. Formally, their existence is related to the more general transformations of integration variable

$$\Phi(\boldsymbol{x}) \to \Phi(\boldsymbol{x}) + \epsilon(\boldsymbol{x}) F[\Phi(\boldsymbol{x})] \tag{12}$$

Generally, we do not known if the measure transforms covariantly under this change. If we would known a Jacobian of this transformation we would find a new redundant field similar to EOM(x). An important class of transformations (12) comes from the symmetries of the theory. Natural assumption would be that if the action has some symmetry, then the measure should share the same symmetry as well. For example, we expect that $\mathcal{D}(\Phi(\boldsymbol{x}+\boldsymbol{\epsilon}))$ as a manifestation of the fact that the change $\boldsymbol{x} \to \boldsymbol{x}+\boldsymbol{\epsilon}$ just relabels the coordinates in the functional integral. It is easy to justify the invariance for a constant $\boldsymbol{\epsilon}$. But what if $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}(\boldsymbol{x})$ is a function, as in 2D CFT? In general, this is the source of anomaly. Exactly, for the transformation $\Phi(\boldsymbol{x}) \to \Phi(\boldsymbol{x}+\boldsymbol{\epsilon}(\boldsymbol{x})) = \Phi(\boldsymbol{x}) + \epsilon_\mu(\boldsymbol{x})\partial_\mu\Phi(x) + ...$ we don't expect measure issues. Therefore we have an identity

$$\sum_{k=1}^{n} \langle \mathcal{O}_1(\boldsymbol{x}_1)...\delta_\epsilon\mathcal{O}_k(\boldsymbol{x}_k)...\mathcal{O}_N(\boldsymbol{x}_N)\rangle = \int_{\mathbb{R}^D} \partial_\mu\epsilon_\nu(\boldsymbol{x}) \langle T^{\mu\nu}(\boldsymbol{x})\mathcal{O}_1(\boldsymbol{x}_1)...\mathcal{O}_N(\boldsymbol{x}_N)\rangle \, d^D\boldsymbol{x} \tag{13}$$

In QFT we take (13) as a definition of the stress-energy tensor.
Variation $\delta_\epsilon\mathcal{O}_j(\boldsymbol{x})$ of some local field $\mathcal{O}_j(\boldsymbol{x})$ for infinite small transformation of coordinates due to locality is linear combination of the function $\epsilon(x)$ and a finite number of its derivatives taken at the point $\boldsymbol{x}$. This can be written as

$$\delta_{\boldsymbol{\epsilon}}\mathcal{O}_j(\boldsymbol{x}) = \sum_{k=0}^{\nu_j} B_j^{(k-1)}(\boldsymbol{x}) \frac{d^k}{dx^k}\epsilon(\boldsymbol{x}), \tag{14}$$

where $B_j^{(k-1)}$ are local fields from set $\mathcal{A}$, и ν_j – some integer number.
Now, let $\mathbb{B}_k$ be the small ball surrounding the point $\boldsymbol{x}_k$, such that $\mathbb{B}_i \cap \mathbb{B}_k = \varnothing$ if $i \neq k$. Then we split the integral in the r.h.s. in (13) as

$$\int_{\mathbb{R}^D} = \sum_{k=1}^{N} \int_{\mathbb{B}_k} + \int_{\bar{\mathbb{R}}^D}, \tag{15}$$

where $\bar{\mathbb{R}}^D \cup \mathbb{B}_1 \cup ... \cup \mathbb{B}_N = \mathbb{R}^D$. The last integral can be transformed by parts

$$\int_{\bar{\mathbb{R}}^D} \partial_\mu\epsilon_\nu(\boldsymbol{x}) \langle T^{\mu\nu}(\boldsymbol{x})\mathcal{O}_1(\boldsymbol{x}_1)...\mathcal{O}_N(\boldsymbol{x}_N)\rangle \, d^D\boldsymbol{x} = -\int_{\bar{\mathbb{R}}^D} \epsilon_\nu(\boldsymbol{x}) \langle \partial_\mu T^{\mu\nu}(\boldsymbol{x})\mathcal{O}_1(\boldsymbol{x}_1)...\mathcal{O}_N(\boldsymbol{x}_N)\rangle \, d^D\boldsymbol{x} + \\ + \int_{\bar{\mathbb{R}}^D} \partial_\mu(\epsilon_\nu(\boldsymbol{x}) \langle T^{\mu\nu}(\boldsymbol{x})\mathcal{O}_1(\boldsymbol{x}_1)...\mathcal{O}_N(\boldsymbol{x}_N)\rangle) d^D\boldsymbol{x} \tag{16}$$

We can take $\boldsymbol{\epsilon}(\boldsymbol{x})$ such that

$$\boldsymbol{\epsilon}(\boldsymbol{x}) \begin{cases} = 0, & \boldsymbol{x} \in \mathbb{B}_k, \quad \forall k \in \{1, ..., N\}, \\ \neq 0, & \boldsymbol{x} \in \bar{\mathbb{R}}^D. \end{cases} \tag{17}$$

Then

$$\int_{\mathbb{B}_k} \partial_\mu\epsilon_\nu(\boldsymbol{x}) \langle T^{\mu\nu}(\boldsymbol{x})\mathcal{O}_1(\boldsymbol{x}_1)...\mathcal{O}_N(\boldsymbol{x}_N)\rangle \, d^D\boldsymbol{x} = 0 \tag{18}$$

$$\int_{\bar{\mathbb{R}}^D} \partial_\mu(\epsilon_\nu(\boldsymbol{x}) \langle T^{\mu\nu}(\boldsymbol{x})\mathcal{O}_1(\boldsymbol{x}_1)...\mathcal{O}_N(\boldsymbol{x}_N)\rangle)d^D\boldsymbol{x} = 0 \tag{19}$$

$$\langle \mathcal{O}_1(\boldsymbol{x}_1)...\delta_\epsilon\mathcal{O}_k(\boldsymbol{x}_k)...\mathcal{O}_N(\boldsymbol{x}_N)\rangle = 0 \to \langle \partial_\mu T^{\mu\nu}(\boldsymbol{x})\mathcal{O}_1(\boldsymbol{x}_1)...\mathcal{O}_N(\boldsymbol{x}_N)\rangle = 0, \quad \boldsymbol{x} \in \bar{\mathbb{R}}^D \tag{20}$$

We note that we can take the balls $\mathbb{B}_k$ arbitrary small and hence (21) is valid for all $\boldsymbol{x} \neq \boldsymbol{x}_k$, i.e. the correlation function (21) vanishes everywhere except for some delta functions supported at the insertion points $\boldsymbol{x}_1, ..., \boldsymbol{x}_N$.

$$\langle \partial_\mu T^{\mu\nu}(\boldsymbol{x})\mathcal{O}_1(\boldsymbol{x}_1)...\mathcal{O}_N(\boldsymbol{x}_N)\rangle = 0, \quad \boldsymbol{x} \neq \boldsymbol{x}_k \tag{21}$$

That is $\partial_\mu T^{\mu\nu}$ is a redundant field. Having in mind (21), we conclude that

$$\begin{aligned}\sum_{k=1}^{n} \langle \mathcal{O}_1(\boldsymbol{x}_1)...\delta_\epsilon\mathcal{O}_k(\boldsymbol{x}_k)...\mathcal{O}_N(\boldsymbol{x}_N)\rangle &= \sum_{k=1}^{N} \int_{\mathbb{B}_k} \partial_\mu\epsilon_\nu(\boldsymbol{x}) \langle T^{\mu\nu}(\boldsymbol{x})\mathcal{O}_1(\boldsymbol{x}_1)...\mathcal{O}_N(\boldsymbol{x}_N)\rangle\, d^D\boldsymbol{x} + \\ &+ \int_{\bar{\mathbb{R}}^D} \partial_\mu(\epsilon_\nu(\boldsymbol{x}) \langle T^{\mu\nu}(\boldsymbol{x})\mathcal{O}_1(\boldsymbol{x}_1)...\mathcal{O}_N(\boldsymbol{x}_N)\rangle)d^D\boldsymbol{x}\end{aligned} \tag{22}$$

Now, we specify everything to the case of $D = 2$ and scale $T^{\mu\nu} \to \frac{1}{2\pi}T^{\mu\nu}$ for future convenience (with Belavin book). Using the Green theorem

$$\int_{\mathcal{D}} \partial_\mu A^\mu d^2\boldsymbol{x} = \oint_{\partial\mathcal{D}} \varepsilon_{\mu\nu} A^\mu dx^\nu \tag{23}$$

we find

$$\begin{aligned}\sum_{k=1}^{n} \langle \mathcal{O}_1(\boldsymbol{x}_1)...\delta_\epsilon\mathcal{O}_k(\boldsymbol{x}_k)...\mathcal{O}_N(\boldsymbol{x}_N)\rangle &= \frac{1}{2\pi}\sum_{k=1}^{N} \int_{\mathbb{B}_k} \partial_\mu\epsilon_\nu(\boldsymbol{x}) \langle T^{\mu\nu}(\boldsymbol{x})\mathcal{O}_1(\boldsymbol{x}_1)...\mathcal{O}_N(\boldsymbol{x}_N)\rangle\, d^2\boldsymbol{x} - \\ &- \frac{1}{2\pi}\sum_{k=1}^{N} \oint_{\partial\mathbb{B}_k} \epsilon_\nu(\boldsymbol{x})\varepsilon_{\mu\lambda} \langle T^{\lambda\nu}(\boldsymbol{x})\mathcal{O}_1(\boldsymbol{x}_1)...\mathcal{O}_N(\boldsymbol{x}_N)\rangle\, dx^\mu,\end{aligned} \tag{24}$$

where the contour integral goes in the counterclockwise direction. Since, $\boldsymbol{\epsilon}$ is arbitrary we can take it non-zero only in the vicinity of the point $\boldsymbol{x}_k$. In this case only one term of the sum contributes in (24). We can rewrite (24), formally erasing an average sign, as

$$\delta_{\boldsymbol{\epsilon}}\mathcal{O}(\boldsymbol{x}) = \frac{1}{2\pi}\int_{\mathcal{D}_{\boldsymbol{x}}} \partial_\mu\epsilon_\nu(\boldsymbol{y})T^{\mu\nu}(\boldsymbol{y})\mathcal{O}(\boldsymbol{x})d^2\boldsymbol{y} - \frac{1}{2\pi}\oint_{\mathcal{C}_{\boldsymbol{x}}} \epsilon_\nu(\boldsymbol{y})\varepsilon_{\mu\lambda}T^{\lambda\nu}(\boldsymbol{y})\mathcal{O}(\boldsymbol{x})dy^\mu, \tag{25}$$

where $\mathcal{D}_{\boldsymbol{x}}$ is a small disk surrounding the point $\boldsymbol{x}$ and $\mathcal{C}_{\boldsymbol{x}}$ is its boundary.
Now, suppose that our theory is conformally invariant, that is $T_{\mu\nu}$ is traceless. In this case the first term in (25) does not contribute for conformal transformations $\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu \sim \delta_{\mu\nu}$. Moreover, taking into account peculiarities of 2D geometry

$$\epsilon(z) = \epsilon_1(\boldsymbol{x}) + i\epsilon_2(\boldsymbol{x}), \quad \bar{\epsilon}(\bar{z}) = \epsilon_1(\boldsymbol{x}) - i\epsilon_2(\boldsymbol{x}) \tag{26}$$

$$T(z) = \frac{1}{4}(T_{11}(\boldsymbol{x}) - T_{22}(\boldsymbol{x}) - 2iT_{12}(\boldsymbol{x})), \quad \bar{T}(\bar{z}) = \frac{1}{4}(T_{11}(\boldsymbol{x}) - T_{22}(\boldsymbol{x}) + 2iT_{12}(\boldsymbol{x})) \tag{27}$$

we find that $\langle T(z)\mathcal{O}_1(z_1,\bar{z}_1)...\mathcal{O}_N(z_N,\bar{z}_N)\rangle$ and $\langle \bar{T}(\bar{z})\mathcal{O}_1(z_1,\bar{z}_1)...\mathcal{O}_N(z_N,\bar{z}_N)\rangle$. Moreover variation of the field $\mathcal{O}(z,\bar{z})$ under the conformal change of coordinates $\boldsymbol{\epsilon} = (\epsilon,\bar{\epsilon}) : z \to z + \epsilon(z), \bar{z} \to \bar{z} + \bar{\epsilon}(\bar{z})$ is

$$\delta_{\boldsymbol{\epsilon}}\mathcal{O}(z,\bar{z}) = \frac{1}{2\pi i}\oint_{\mathcal{C}_z} \epsilon(\zeta)T(\zeta)\mathcal{O}(z,\bar{z})d\zeta + \frac{1}{2\pi i}\oint_{\mathcal{C}_{\bar{z}}} \bar{\epsilon}(\bar{\zeta})\bar{T}(\bar{\zeta})\mathcal{O}(z,\bar{z})d\bar{\zeta} \tag{28}$$

where both contours $\mathcal{C}_z$ and $\mathcal{C}_{\bar{z}}$ go in the counterclockwise direction. It is important, that correlation functions (27) not only holomorphic (antiholomorphic), but also single valued. It allows us to define the holomorphic variation of local fields (assuming that $\epsilon(\zeta)$ is single-valued as well)

$$\delta_\epsilon \mathcal{O}(z,\bar{z}) = \frac{1}{2\pi i}\oint_{\mathcal{C}_z} \epsilon(\zeta)T(\zeta)\mathcal{O}(z,\bar{z})d\zeta \tag{29}$$

Consider infinitesimal transformation of the very special form:

$$\epsilon_n(\zeta) = \alpha(\zeta - z)^{n+1}, \quad \alpha \ll 1, \quad n \geq -1 \tag{30}$$

Variation of $\mathcal{O}(z,\bar{z})$ under this special conformal transformation we denote:

$$\delta_{\epsilon_n}\mathcal{O}(z,\bar{z}) = \alpha L_n \mathcal{O}(z,\bar{z}) \tag{31}$$

$$L_n\mathcal{O}(z,\bar{z}) = \frac{1}{2\pi i}\oint_{C_z}(\zeta - z)^{n+1}T(\zeta)\mathcal{O}(z,\bar{z})d\zeta \tag{32}$$

For generic ϵ:

$$\epsilon(\zeta) = \alpha + \alpha(\zeta - z) + \frac{\alpha(\zeta - z)^2}{2!} + ... = \epsilon_{-1}(\zeta) + \epsilon_0(\zeta) + \frac{\epsilon_1(\zeta)}{2!} + ... \tag{33}$$

$$\delta_\epsilon\mathcal{O}(z,\bar{z}) = \epsilon(z)L_{-1}\mathcal{O}(z,\bar{z}) + \epsilon'(z)L_0\mathcal{O}(z,\bar{z}) + \frac{\epsilon''(z)}{2}L_1\mathcal{O}(z,\bar{z}) + ... \tag{34}$$

At least two of these new fields $L_n\mathcal{O}(z,\bar{z})$ we can identify

$$L_{-1}\mathcal{O}(z,\bar{z}) = \partial\mathcal{O}(z,\bar{z}), \quad L_0\mathcal{O}(z,\bar{z}) = \Delta_{\mathcal{O}}\mathcal{O}(z,\bar{z}), \tag{35}$$

where $\Delta_{\mathcal{O}}$ – *conformal dimension* of the field $\mathcal{O}$. Similarly, one can define antiholomorphic conformal dimension $\bar{\Delta}$ as $\bar{L}_0\mathcal{O} = \bar{\Delta}\mathcal{O}$. Altogether it corresponds to the following transformation rules

$$\mathcal{O}(z,\bar{z}) \to \lambda^{\Delta}\bar{\lambda}^{\bar{\Delta}}\mathcal{O}(\lambda z, \bar{\lambda}\bar{z}) \tag{36}$$

So that $\Delta + \bar{\Delta}$ can be identified with the scaling dimension and $\Delta - \bar{\Delta}$ with the spin.
Other fields $L_n\mathcal{O}(z,\bar{z})$ are some new fields which are unrelated to the original one $\mathcal{O}(z,\bar{z})$. In general, we expect that (34) contains only finitely many derivative terms, that is there should $\exists N > 0$:

$$L_n\mathcal{O}(z,\bar{z}) = 0, \quad n > N \tag{37}$$

that conformal dimensions of the fields $L_k\mathcal{O}$ are given by

$$\Delta_{L_k\mathcal{O}} = \Delta_{\mathcal{O}} - k \tag{38}$$

We assume that the spectra of conformal dimensions $\{\Delta_j\}$ is bounded from below. Actually, we might require even more and forbid negative conformal dimensions at all. It guaranties for example that the two-point functions

$$\langle\mathcal{O}(z,\bar{z}), \mathcal{O}(z',\bar{z}')\rangle \sim \frac{1}{|z - z'|^{4\Delta_{\mathcal{O}}}} \tag{39}$$

will fall at infinity. In any case, this restriction implies existence of *primary fields*, which we denote as Φ, having the most simple variation

$$\delta_\epsilon\Phi(z) = \epsilon(z)\partial\Phi(z) + \Delta\epsilon'(z)\Phi(z) \Leftrightarrow L_n\Phi = 0 \quad \forall n > 0 \tag{40}$$

Under generic, not infinitesimal, holomorphic transformation primary fields behave as generalized tensor fields

$$\Phi(z) \to \left(\frac{dw}{dz}\right)^{\Delta} \Phi(w) \tag{41}$$

Consider the Ward identity:

$$\sum_{j=1}^{N} \langle \mathcal{O}_1(z_1)...\delta_\epsilon \mathcal{O}_j(z_j)...\mathcal{O}_N(z_N)\rangle = \frac{1}{2\pi i}\sum_{j=1}^{N}\oint_{\mathcal{C}_{z_j}} \epsilon(\zeta)\,\langle T(\zeta)\mathcal{O}_1(z_1)...\mathcal{O}_N(z_N)\rangle \tag{42}$$

We assume that the correlation function $\langle T(\zeta)\mathcal{O}_1(z_1)...\mathcal{O}_N(z_N)\rangle$ is a single-valued function of ζ falling at infinity ($T(\zeta) \to 0$, $\zeta \to \infty$) with only possible singularities, the poles at the insertion point z_j. Then, taking $\epsilon(z) = \frac{\alpha}{z-\zeta}$, $\alpha \ll 1$, where $z \neq z_j$ and we find

$$\langle T(z)\mathcal{O}_1(z_1)...\mathcal{O}_N(z_N)\rangle = \sum_{j=1}^{N}\sum_{k=0}^{\nu_j} \frac{1}{(z-z_j)^{k+1}}\,\langle \mathcal{O}_1(z_1)...L_{k-1}\mathcal{O}_j(z_j)...\mathcal{O}_N(z_N)\rangle \tag{43}$$

It is important to mention that the contours $\mathcal{C}_{z_j}$ are very small circles, so that z in lies outside of all $\mathcal{C}_{z_j}$'s. This formula is known under the name of *conformal Ward identity*. It has a particularly neat form for primary fields

$$\langle T(z)\Phi_1(z_1)...\Phi_N(z_N)\rangle = \sum_{k=1}^{N}\left(\frac{\Delta_k}{(z-z_k)^2} + \frac{\partial_k}{z-z_k}\right)\langle \Phi_1(z_1)...\Phi_N(z_N)\rangle \tag{44}$$

One can rewrite it in the form of *operator product expansion (OPE)*

$$T(\zeta)\Phi(z) = \frac{\Delta\Phi(z)}{(\zeta-z)^2} + \frac{\partial\Phi(z)}{\zeta-z} + ... \tag{45}$$

where by ... we denote terms regular at $\zeta \to z$. Similarly, from (43) we find that

$$T(\zeta)\mathcal{O}(z) = ... + \frac{L_2\mathcal{O}(z)}{(\zeta-z)^4} + \frac{L_1\mathcal{O}(z)}{(\zeta-z)^3} + \frac{\Delta_{\mathcal{O}}\mathcal{O}(z)}{(\zeta-z)^2} + \frac{\partial\mathcal{O}(z)}{\xi-z} + ... \tag{46}$$

with finitely many simgular terms.

Now, as we saw before, the conformal dimension of the field $\mathcal{O}$ differs from the conformal dimension of some primary field Φ by an integer positive number. It suggests that, may be, $\mathcal{O}$ can be obtained from Φ. To do so, we consider regular part of OPE

$$T(\zeta)\Phi(z) = \frac{\Delta\Phi(z)}{(\zeta-z)^2} + \frac{\partial\Phi(z)}{\zeta-z} + L_{-2}\Phi(z) + (\zeta-z)L_{-3}\Phi(z) + (\zeta-z)^2 L_{-4}\Phi(z) + ... \tag{47}$$

where $L_{-k}\Phi(z)$ are, by definition, some new local fields (note that $L_{-1}\Phi(z) = \partial\Phi(z)$). Their existence can be justified by functional integral arguments. For example,

$$L_{-2}\Phi(z) \approx T(z)\Phi(z), \quad L_{-3}\Phi(z) \approx T'(z)\Phi(z) \tag{48}$$

where the symbol $\approx$ means some kind of regularization. We will make it simpler and just postulate, that (47) defines the new fields $L_{-k}\Phi(z)$, which will be called descendant fields (but not only them). It can be expressed as follows

$$L_{-k}\Phi(z) = \frac{1}{2\pi i}\oint_{\mathcal{C}_z} (\zeta-z)^{1-k} T(\zeta)\Phi(z)d\zeta \tag{49}$$

Using (44), one finds that

$$\langle L_{-k}\Phi(z)\Phi_1(z_1)...\Phi_N(z_N)\rangle = \frac{1}{2\pi i}\oint_{\mathcal{C}_z}(\xi-z)^{1-k}\langle T(\xi)\Phi(z)\Phi_1(z_1)...\Phi_N(z_N)\rangle\, d\xi = \\ = \hat{\mathcal{L}}_{-k}\langle \Phi(z)\Phi_1(z_1)...\Phi_N(z_N)\rangle \tag{50}$$

where the differential operator $\hat{\mathcal{L}}(z, z_k)$ is given by

$$\hat{\mathcal{L}}_{-k} = \sum_{j=1}^{N}\left(\frac{(k-1)\Delta_j}{(z_j-z)^k} - \frac{\partial_j}{(z_j-z)^{k-1}}\right) \tag{51}$$

Hamiltonian formalism in CFT, representation theory of Virasoro algebra, null-vectors

We define the representation $\mathcal{V}_\Delta$, which is known as Verma module

$$L_{-\boldsymbol{\lambda}}|\Delta\rangle \equiv L_{-\lambda_1}...L_{-\lambda_n}|\Delta\rangle\,, \quad \lambda_1 \geq \lambda_2 \geq ... \tag{52}$$

$$L_0|\Delta\rangle = \Delta|\Delta\rangle\,, \quad L_n|\Delta\rangle = 0, \quad n>0 \tag{53}$$

is decomposed into the direct sum of finite dimensional subspaces (here $|\boldsymbol{\lambda}| = \lambda_1+\lambda_2+...$).

$$\mathcal{V}_\Delta = \oplus_N \mathcal{V}_{\Delta,N}, \quad \mathcal{V}_{\Delta,N} = \mathrm{span}\{L_{-\boldsymbol{\lambda}}|\Delta\rangle : |\boldsymbol{\lambda}| = N\}, \tag{54}$$

which are eigenspaces of the operator L_0:

$$L_0 L_{-\boldsymbol{\lambda}}|\Delta\rangle = (\Delta+|\boldsymbol{\lambda}|)L_{-\boldsymbol{\lambda}}|\Delta\rangle \tag{55}$$

On first few levels one has

$$N=0 : |\Delta\rangle \tag{56}$$

$$N=1 : L_{-1}|\Delta\rangle \tag{57}$$

$$N=2 : L_{-2}|\Delta\rangle\,, L_{-1}^2|\Delta\rangle \tag{58}$$

$$N=3 : L_{-3}|\Delta\rangle\,, L_{-2}L_{-1}|\Delta\rangle\,, L_{-1}^3|\Delta\rangle \tag{59}$$

$$N=4 : L_{-4}|\Delta\rangle\,, L_{-3}L_{-1}|\Delta\rangle\,, L_{-2}^2|\Delta\rangle\,, L_{-2}L_{-1}^2|\Delta\rangle\,, L_{-1}^4|\Delta\rangle \tag{60}$$

In general there are $p(N)$ states in $\mathcal{V}_{\Delta,N}$, where $p(N)$ is the number of partitions of N. Generating function for the number of partitions (Euler formula):

$$\sum_{N=0}^{\infty} p(N)q^N = \prod_{k=1}^{\infty}\frac{1}{1-q^k} \tag{61}$$

It is convenient to define the character (holomorphic block of the partition function)

$$\chi_\Delta(q) \equiv \mathrm{Tr}(q^{L_0-\frac{c}{24}})\Big|_{\mathcal{V}_\Delta} = q^{\Delta-\frac{c}{24}}\sum_{N=0}^{\infty}p(N)q^N = \frac{q^{\Delta-\frac{c}{24}}}{\prod_{k=1}^{\infty}(1-q^k)} \tag{62}$$

So far, we assumed that the values of the conformal dimension Δ and of the central charge c are generic. In this case the Verma module $\mathcal{V}_\Delta$ is irreducible. However, interesting things happen for quantized values of Δ. Remember, that we have postulated that $\Phi_{\Delta=0} = I$ is an identity operator and hence

$$\partial I = L_{-1} I = 0 \tag{63}$$

as it should be for coordinate independent field. But does that consistent with the conformal symmetry? We have to check that

$$L_n L_{-1} |\Delta\rangle = 0. \quad n > 0 \tag{64}$$

Well, in our case $\Delta = 0$, but we leave it arbitrary in order to see how does that happen. Actually, the condition (64) is satisfied for all $n > 1$ identically. We only have to demand it for $n = 1$.

$$L_1 L_{-1} |\Delta\rangle = 2\Delta |\Delta\rangle = 0 \tag{65}$$

We see that $\Delta = 0$ is necessary condition for the vector $L_{-1} |\Delta\rangle$ to vanish. But not sufficient of course. We can claim that for $\Delta = 0$ one can remove the state $L_{-1} |\Delta\rangle$, as well as all its descendants $L_{-\boldsymbol{k}} L_{-1} |\Delta\rangle$ from our Hilbert space without violating the conformal symmetry.

Definition 2. Such state we call *a null-vector*.

The fact that the null-vector vanishes leads us to the trivial conclusion that any correlation function involving the identity operator should satisfy

$$\partial_z \langle I(z) \Phi_1(z_1) ... \Phi_N(z_N) \rangle = 0 \tag{66}$$

Now we try to generalize this. On level 2 we have two states $L_{-1}^2 |\Delta\rangle$ and $L_{-2} |\Delta\rangle$. Probably, we can find their linear combination which vanishes, or, at least, can be safely removed from $\mathcal{V}_\Delta$. We have to require

$$|\chi\rangle = (L_{-1}^2 + \lambda L_{-2}) |\Delta\rangle \,, \quad L_n |\chi\rangle = 0, \quad n > 0 \tag{67}$$

We note that here we have to impose two conditions with $n = 1$ and $n = 2$. For $n = 1$ we have

$$L_1 L_{-1}^2 = (L_{-1}L_1 + 2L_0) L_{-1} = L_{-1}(L_{-1}L_1 + 2L_0) + 2(L_{-1}L_0 + L_{-1}) = L_{-1}^2 L_1 + 4L_{-1}L_0 + 2L_{-1} \tag{68}$$

$$L_1 L_{-2} = L_{-2} L_1 + 3L_{-1} \tag{69}$$

$$L_1(L_{-1}^2 + \lambda L_{-2}) |\Delta\rangle = (4\Delta + 2 + 3\lambda) L_{-1} |\Delta\rangle = 0 \tag{70}$$

For $n = 2$ we have

$$L_2 L_{-1}^2 = (L_{-1}L_2 + 3L_1) L_{-1} = L_{-1}(L_{-1}L_2 + 3L_1) + 3(L_{-1}L_1 + 2L_0) = L_{-1}^2 L_2 + 6L_{-1}L_1 + 6L_0 \tag{71}$$

$$L_2 L_{-2} = L_{-2} L_2 + 4L_0 + \frac{c}{2} \tag{72}$$

$$L_2(L_{-1}^2 + \lambda L_{-2}) |\Delta\rangle = \left(6\Delta + \lambda\left(4 + \frac{c}{2}\right)\right) |\Delta\rangle = 0 \tag{73}$$

$$\begin{cases} 4\Delta + 2 + 3\lambda = 0, \\ 6\Delta + \lambda\left(4 + \frac{c}{2}\right) = 0. \end{cases} \to \begin{cases} \Delta = \frac{1}{16}(5 - c \pm \sqrt{(c-1)(c-25)}), \\ \lambda = -\frac{2(2\Delta+1)}{3}. \end{cases} \tag{74}$$

Going further, we consider a descendant on third level

$$|\chi\rangle = (L_{-1}^3 + \lambda_1 L_{-2} L_{-1} + \lambda_2 L_{-3}) |\Delta\rangle \tag{75}$$

If it is a null-vector it has to obey $L_1 |\chi\rangle = L_2 |\chi\rangle = L_3 |\chi\rangle = 0$, but since $L_3 = [L_2, L_1]$ it is enough to impose only first two conditions. Simple algebra gives

$$L_1 |\chi\rangle = (3(2\Delta + 2 + \lambda_1) L_{-1}^2 + 2(\Delta\lambda_1 + 2\lambda_2) L_{-2}) |\Delta\rangle = 0 \tag{76}$$

$$L_2 |\chi\rangle = \left(18\Delta + 6 + \left(4 + \frac{c}{2} + 4\Delta\right)\lambda_1 + 5\lambda_2\right) L_{-1} |\Delta\rangle = 0 \tag{77}$$

$$\begin{cases} 2\Delta + 2 + \lambda_1 = 0, \\ \Delta\lambda_1 + 2\lambda_2 = 0, \\ 18\Delta + 6 + \left(4 + \frac{c}{2} + 4\Delta\right)\lambda_1 + 5\lambda_2 = 0; \end{cases} \rightarrow \begin{cases} \Delta = \frac{1}{6}(7 - c \pm \sqrt{(c-1)(c-25)}), \\ \lambda_1 = -2 - 2\Delta, \\ \lambda_2 = \Delta(1+\Delta). \end{cases} \tag{78}$$

We see that the expressions for null-vectors (74) and (78) look very similar. One can simplify them by introducing Liouville like parametrization of the central charge and of conformal dimension

$$c = 1 + 6Q^2, \quad Q = b + \frac{1}{b}, \quad \Delta = \Delta(\alpha) = \alpha(Q - \alpha) \tag{79}$$

Then the singular vectors appear at the values

$$N = 2 : \alpha = -\frac{b}{2}, \quad \alpha = -\frac{1}{2b} \tag{80}$$

$$N = 3 : \alpha = -b, \quad \alpha = -\frac{1}{b} \tag{81}$$

Corresponding null-vectors have the form

$$N = 2 : (L_{-1}^2 + b^2 L_{-2}) |\Delta\rangle \tag{82}$$

$$N = 3 : (L_{-1}^3 + 4b^2 L_{-2}L_{-1} + 2b^2(2b^2+1)L_{-3}) |\Delta\rangle \tag{83}$$

and similar expressions for $b \to b^{-1}$. One can compute null-vectors on higher levels in a similar manner.

Theorem 1 (Kac-Feigin-Fuks)**.** *At at level N, for any two positive integers m and n such that $N = mn$, there exist a null vector*

$$|\chi_{m,n}\rangle = D_{m,n} |\Delta_{m,n}\rangle \tag{84}$$

with

$$\Delta = \Delta_{m,n} = \Delta(\alpha_{m,n}), \quad \alpha_{m,n} = -\frac{(m-1)b}{2} - \frac{(n-1)b^{-1}}{2} \tag{85}$$

Definition 3. The operator $D_{m,n}$ is known as *null vector creation operator.*

It is convenient to adopt the following normalization

$$D_{m,n} = L_{-1}^{mn} + c_1(b)L_{-2}L_{-1}^{mn-2} + c_2(b)L_{-3}L_{-1}^{mn-3} + ... \tag{86}$$

The coefficients $c_k(b)$ in can be recursively found

$$c_1(b) = \frac{mn}{6}((m^2-1)b^2 + (n^2-1)b^{-2}) \tag{87}$$

$$c_2(b) = \frac{m^2n^2}{12}((m^2-1)b^2 + (n^2-1)b^{-2}) + \frac{mn}{30}((m^2-1)((m^2-4)b^4 - 5b^2 - 5) + \\ + (n^2-1)((n^2-4)b^{-4} - 5b^{-2} - 5)) + \frac{mn(m^2n^2-1)}{6} \tag{88}$$

For generic values of the central charge c, $|\chi_{m,n}\rangle$ is the only singular vector in the Verma module $\mathcal{V}_{\Delta_{m,n}}$ with $\Delta = \Delta_{m,n} + mn = \Delta_{m,-n}$. We can define the factor space $\mathcal{V}_{\Delta_{m,n}}/\mathcal{V}_{\Delta_{m,-n}}$ without violating the conformal symmetry. The character of the corresponding factor

$$\chi'_{m,n}(q) = \frac{q^{\Delta_{m,n}-\frac{c}{24}}(1-q^{mn})}{\prod_{k=1}^{\infty}(1-q^k)} \tag{89}$$

It is convenient to think about representation theory of Virasoro algebra with the help of Shapovalov form, that is Hermitian form defined by

$$\langle\Delta|\Delta\rangle = 1, \quad L_n^{\dagger} = L_{-n} \tag{90}$$

Definition 4. *Gram matrix*

$$G_{\boldsymbol{\lambda},\boldsymbol{\mu}} = \langle\Delta|L_{\boldsymbol{\mu}}L_{-\boldsymbol{\lambda}}|\Delta\rangle \tag{91}$$

It's block diagonal matrix $G = \{G_0, G_1, G_2, ...\}$ with block sizes $p(N)\times p(N)$. The degeneracies of this matrix are closely related to the reducibility of the corresponding Verma module. For example, one has

$$G_1 = \langle\Delta|L_1L_{-1}|\Delta\rangle = \langle\Delta|2L_0|\Delta\rangle = 2\Delta\,\langle\Delta|\Delta\rangle = 2\Delta \tag{92}$$

The the determinant $\det G_1$ vanishes for degenerate dimension $\Delta = \Delta_{1,1} = 0$. In general, it is clear that any descendant of a singular vector is orthogonal to everything else in Verma module

$$\langle\Delta_{m,n}|L_{\boldsymbol{\mu}}L_{-\boldsymbol{\lambda}}|\chi_{m,n}\rangle = 0, \quad \forall\boldsymbol{\lambda},\boldsymbol{\mu} \tag{93}$$

That is we have $p(N-mn)$ singular vectors on level N of the form $L_{-\boldsymbol{\lambda}}\,|\chi_{m,n}\rangle$ with $|\boldsymbol{\lambda}| = N$, which implies that the determinant of the Shapovalov form on level N vanishes as

$$\det G_N \sim (\Delta - \Delta_{m,n})^{p(N-mn)} \tag{94}$$

In fact it can be shown that taking the product of all factors with $mn \leq N$ exhausts Kac determinant completely

$$\det G_N = C_N \prod_{mn\leq N}(\Delta - \Delta_{m,n})^{p(N-mn)} \tag{95}$$

for some numerical coefficient C_N. This formula is equivalent to Kac-Feigin-Fuks theorem. We will not prove it in full generality.

It is easy to show that (95) provides correct degree in Δ. Indeed, it is easy to show that the degree of the matrix element of Gram matrix is non-greater than $l(\boldsymbol{\lambda})$ and $l(\boldsymbol{\mu})$, where $l(\boldsymbol{\lambda})$ is the length of partition $\boldsymbol{\lambda}$. Thus the degree of Kac determinant is non greater than

$$\sum_{|\boldsymbol{\lambda}|=N} l(\boldsymbol{\lambda}) \tag{96}$$

Now we will proof a simple combinatorial fact that

$$\sum_{|\boldsymbol{\lambda}|=N} l(\boldsymbol{\lambda}) = \sum_{mn\leq N} p(N-mn) \tag{97}$$

Let $l_n(\boldsymbol{\lambda})$ is the number of part equal to n in partition $\boldsymbol{\lambda}$

$$\sum_{N=0}^{\infty} q^N \sum_{mn\leq N} p(N-mn) = \sum_{m,n>0} q^{mn} \sum_{N=0}^{\infty} q^N p(N) = \sum_{n>0} \frac{q^n}{1-q^n} \prod_{k>0} \frac{1}{1-q^k} =$$
$$= \sum_{n>0} \frac{q^n}{(1-q^n)^2} \prod_{k>0,k\neq n} \frac{1}{1-q^k} = \sum_{n>0} \frac{1}{1-q} \cdots \frac{1}{1-q^{n-1}} \frac{\partial}{\partial q^n}\left(\frac{1}{1-q^n}\right) \frac{1}{1-q^{n+1}} \cdots =$$
$$= \sum_{n>0}\sum_{|\boldsymbol{\lambda}|} l_n(\boldsymbol{\lambda}) q^{|\boldsymbol{\lambda}|} = \sum_{|\boldsymbol{\lambda}|} l(\boldsymbol{\lambda}) q^{|\boldsymbol{\lambda}|} \tag{98}$$

Free bosonic CFT I: path integral approach

Let us start with the theory of free massless bosonic field

$$S[\varphi]=\frac{1}{8\pi}\int(\partial_\mu\varphi(\boldsymbol{x}))^2d^2\boldsymbol{x} \tag{99}$$

First of all, we notice that this is our "patient": the theory is conformally invariant (at least classically). This follows from the identity

$$\int\partial_\mu\varphi(\boldsymbol{x})\partial_\mu\varphi(\boldsymbol{x})d^2\boldsymbol{x}=-2\int\partial\varphi(z,\bar{z})\bar{\partial}\varphi(z,\bar{z})dzd\bar{z},\quad z=x_1+ix_2,\bar{z}=x_1-ix_2 \tag{100}$$

$$S[\varphi]=-\frac{1}{4\pi}\int\partial\varphi(z,\bar{z})\bar{\partial}\varphi(z,\bar{z}) \tag{101}$$

In this complex form it is obvious, that the action is invariant under conformal transformations

$$z=f(\zeta),\quad \bar{z}=f^*(\bar{\zeta}) \tag{102}$$

The stress-energy tensor

$$T_{\mu\nu}=\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\nu\varphi-\delta_{\mu\nu}\mathcal{L}=\frac{1}{4\pi}\left(\partial_\mu\varphi\partial_\nu\varphi-\frac{1}{2}\delta_{\mu\nu}(\partial_\mu\varphi)^2\right) \tag{103}$$

is indeed traceless $T_{\mu\mu}=0$ and hence the components

$$T=-\frac{\pi}{2}(T_{11}-T_{22}-2iT_{12})=-\frac{1}{2}(\partial\varphi)^2,\quad \bar{T}=-\frac{\pi}{2}(T_{11}-T_{22}+2iT_{12})=-\frac{1}{2}(\bar{\partial}\varphi)^2 \tag{104}$$

obey $\bar{\partial}T=\partial\bar{T}=0$ on-shell.
Now let us study the theory (99) quantum mechanically. It is easy, since the theory is Gaussian. There are however some subtleties. Consider the partition function

$$Z=\int[\mathcal{D}\varphi]e^{-S[\varphi]} \tag{105}$$

This integral diverges since the action does not contain the zero mode φ_0 of the field $\varphi : Z\sim\int d\varphi_0$.
We define the measure $[\mathcal{D}\varphi]'$ simply as an integral over all non-zero modes of the field φ.
Moreover, anticipating that we will have to deal with infrared divergencies, we will consider our theory in a finite volume. That is we impose the periodic conditions

$$\varphi(x_1,x_2+2\pi R)=\varphi(x_1,x_2) \tag{106}$$

This theory is gaussian. Let us compute the two-point function in this theory

$$G(\boldsymbol{x}-\boldsymbol{y})=\langle\varphi(\boldsymbol{x})\varphi(\boldsymbol{y})\rangle=\frac{1}{Z}\int[\mathcal{D}\varphi]\varphi(\boldsymbol{x})\varphi(\boldsymbol{y})e^{-S[\varphi]} \tag{107}$$

$$S[\varphi]=\frac{1}{8\pi}\int\partial_\mu\varphi\partial_\mu\varphi d^2\boldsymbol{x}=-\frac{1}{2}\int\varphi\left(-\frac{1}{4\pi}\Delta\right)\varphi d^2\boldsymbol{x} \tag{108}$$

As usual in Gaussian theory, one has to invert the quadratic form

$$-\Delta G(\boldsymbol{x})=4\pi\delta^2_R(\boldsymbol{x}),\quad \delta^2_R(\boldsymbol{x})=\delta(x_1)\sum_{n=-\infty}^{\infty}\delta(x_2+2\pi nR) \tag{109}$$

$$-\Delta K(\boldsymbol{x}) = 4\pi\delta^2(\boldsymbol{x}) \tag{110}$$

Using $\Delta = \frac{1}{r}\partial_r(r\partial_r) + \frac{1}{r^2}\partial_\varphi^2$ and integrating last equation over the disk of radius r, we obtain

$$-rK'(r) = 2 \to K(|\boldsymbol{x}|) = -\log|\boldsymbol{x}|^2 + \log|l|^2 \tag{111}$$

$$G(\boldsymbol{x}) = \sum_{n=-\infty}^{\infty} K(|x_1 + ix_2 + 2\pi inR|) = -\log|z + 2\pi inR|^2 + \log|l_n|^2 \tag{112}$$

$$\sinh x = x\prod_{n=1}^{\infty}\left(1 + \frac{x^2}{\pi^2 n^2}\right) \tag{113}$$

$$G(\boldsymbol{x}) = -\log\left(4\sinh\frac{z}{2R}\sinh\frac{\bar{z}}{2R}\right) = -\log\frac{|\boldsymbol{x}|^2}{R^2} + \mathcal{O}\left(\frac{1}{R^2}\right), \quad R \to \infty \tag{114}$$

We treat R as an infrared cut-off: it is assumed to be infinite, but we keep it large in the intermediate calculations and then send $R \to \infty$ in the final answer.
Multipoint correlation functions are computed by the Wick rules:

$$\begin{aligned}\langle\varphi(\boldsymbol{x}_1)\varphi(\boldsymbol{x}_2)\varphi(\boldsymbol{x}_3)\varphi(\boldsymbol{x}_4)\rangle &= \langle\varphi(\boldsymbol{x}_1)\varphi(\boldsymbol{x}_2)\rangle\langle\varphi(\boldsymbol{x}_3)\varphi(\boldsymbol{x}_4)\rangle + \langle\varphi(\boldsymbol{x}_1)\varphi(\boldsymbol{x}_3)\rangle\langle\varphi(\boldsymbol{x}_2)\varphi(\boldsymbol{x}_4)\rangle + \\ &+ \langle\varphi(\boldsymbol{x}_1)\varphi(\boldsymbol{x}_4)\rangle\langle\varphi(\boldsymbol{x}_2)\varphi(\boldsymbol{x}_3)\rangle = G(\boldsymbol{x}_1-\boldsymbol{x}_2)G(\boldsymbol{x}_3-\boldsymbol{x}_4) + G(\boldsymbol{x}_1-\boldsymbol{x}_3)G(\boldsymbol{x}_2-\boldsymbol{x}_4) + \\ &+ G(\boldsymbol{x}_1-\boldsymbol{x}_4)G(\boldsymbol{x}_2-\boldsymbol{x}_3)\end{aligned} \tag{115}$$

We note that the field φ does not look like a conformal field, its correlation functions behave logarithmically rather than power-like. Conformal (primary) fields in the theory (99) are represented by the exponential fields $e^{i\alpha\varphi(\boldsymbol{x})}$, $\alpha \in \mathbb{R}$.
We are interested in multipoint correlation functions $\langle e^{i\alpha_1\varphi(\boldsymbol{x}_1)}...e^{i\alpha_n\varphi(\boldsymbol{x}_n)}\rangle$. One can compute these correlation functions by expanding exponents in series, then using the Wick theorem and then resuming again. But it is better and much easier to use the following general fact, that for any Φ functional linear in fundamental field φ: $\Phi = \int J(\boldsymbol{x})\varphi(\boldsymbol{x})d^2\boldsymbol{x}$ we have

$$\langle e^{\Phi}\rangle = e^{\frac{1}{2}\langle\Phi^2\rangle} \tag{116}$$

In our case

$$\Phi = i\sum_{k=1}^{n}\alpha_k\varphi(\boldsymbol{x}_k) = \int J(\boldsymbol{x})\varphi(\boldsymbol{x})d^2\boldsymbol{x}, \quad J(\boldsymbol{x}) = i\sum_{k=1}^{n}\alpha_k\delta^{(2)}(\boldsymbol{x}-\boldsymbol{x}_k) \tag{117}$$

Then we have

$$\langle e^{i\alpha_1\varphi(\boldsymbol{x}_1)}...e^{\alpha_n\varphi(\boldsymbol{x}_n)}\rangle = \exp\left(-\frac{1}{2}\sum_{k=1}^{n}\alpha_k^2\langle\varphi(\boldsymbol{x}_k)\varphi(\boldsymbol{x}_k)\rangle - \sum_{i<j}\alpha_i\alpha_j\langle\varphi(\boldsymbol{x}_i)\varphi(\boldsymbol{x}_j)\rangle\right) \tag{118}$$

At this point we have a UV problem, since

$$\langle\varphi(\boldsymbol{x})\varphi(\boldsymbol{x})\rangle = G(0) = \infty \tag{119}$$

A standard way to deal with it is to introduce the UV cut-off. It s not universal. There many ways to do it, or as one says, there are many regularization schemes. In renormalizable QFT physically observable quantities must be independent on regularization scheme used for their computation. We define the scheme as follows

$$\langle\varphi(\boldsymbol{x})\varphi(\boldsymbol{x})\rangle = -\log\frac{r_0^2}{R^2}, \quad r_0 \ll 1 \tag{120}$$

The correlation function has the form

$$\langle e^{i\alpha_1\varphi(\boldsymbol{x}_1)}...e^{i\alpha_n\varphi(\boldsymbol{x}_n)}\rangle = \frac{r_0^{\sum_k \alpha_k^2}}{R^{(\sum_k \alpha_k)^2}} \prod_{i<j} |\boldsymbol{x}_i - \boldsymbol{x}_j|^{2\alpha_i\alpha_j} \tag{121}$$

Observables should be independent on the UV cut-off. We define the new field

$$V_\alpha \equiv r_0^{-\alpha^2} e^{i\alpha\varphi} = z_0^{-\frac{\alpha^2}{2}} \bar{z}_0^{-\frac{\alpha^2}{2}} e^{i\alpha\varphi} \tag{122}$$

We note that the operator V_α depends explicitly on a scale and hence has an *anomalous* conformal dimension $\Delta(\alpha) = \bar{\Delta}(\alpha) = \frac{\alpha^2}{2}$.

$$\langle V_{\alpha_1}(z_1, \bar{z}_1)...V_{\alpha_n}(z_n, \bar{z}_n)\rangle = \begin{cases} \prod_{i<j} |z_i - z_j|^{2\alpha_i\alpha_j}, & \sum_{k=1}^{n} \alpha_k = 0 \\ 0, & \text{otherwise} \end{cases} \tag{123}$$

1 Classical field theory, Noether theorem, $T_{\mu\nu}$, scaling and conformal invariances

Problem 1.1. Consider electrodynamics

$$S = \frac{1}{4}\int F_{\mu\nu}^2 d^D\mathbf{x}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{124}$$

- Find canonical stress-energy tensor (i.e. pretending that A_μ's are scalars).
- Find modified stress-energy tensor (i.e. treat $\mathbf{A}$ as a vector).
- Find gravitational stress-energy tensor.

Solution.

- Canonical stress-energy tensor is defined as

$$T^c_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial^\mu\Phi_\lambda)}\partial_\nu\Phi_\lambda - \delta_{\mu\nu}\mathcal{L} \tag{125}$$

In our case:

$$\mathcal{L} = \frac{1}{4}F_{\mu\nu}^2 = \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 = \frac{1}{2}((\partial_\mu A_\nu)^2 - 2\partial_\mu A_\nu \partial^\nu A^\mu), \quad \Phi_\lambda = A_\lambda \tag{126}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial^\mu A_\lambda)} = \partial_\mu A^\lambda - \partial^\lambda A_\mu = F_\mu{}^\lambda \tag{127}$$

$$\boxed{T^c_{\mu\nu} = F_\mu{}^\lambda \partial_\nu A_\lambda - \frac{1}{4}\delta_{\mu\nu}F_{\sigma\rho}^2} \tag{128}$$

- Derive the transformation of the vector field. Invariant scalar ($\epsilon \to 0$):

$$A_\mu(x)dx^\mu = A_\mu(x+\epsilon)d(x+\epsilon)^\mu = A_\mu(x)dx^\mu + \epsilon_\nu\partial^\nu A_\mu(x)dx^\mu + A^\nu(x)\partial_\mu\epsilon_\nu dx^\mu \tag{129}$$

$$A_\lambda(x) \to A_\lambda(x) + \epsilon_\nu\partial^\nu A_\lambda(x) + \partial_\lambda\epsilon_\nu A^\nu(x) \tag{130}$$

$$A_\lambda \to \tilde{A}_\lambda = A_\lambda + \epsilon_\nu\partial^\nu A_\lambda + \partial_\mu\epsilon_\nu \Sigma_\lambda^{\mu\nu}, \quad \Sigma^\mu_{\nu\lambda} = \delta^\mu_\lambda A_\nu \tag{131}$$

Derive stress-energy tensor corresponding (131):

$$\delta S = \int \partial_\mu\epsilon_\nu T^{\mu\nu} d^D\mathbf{x} \tag{132}$$

$$\mathcal{L}(\tilde{A}_\lambda, \partial_\mu\tilde{A}_\lambda) = \mathcal{L}(A_\lambda, \partial_\mu A_\lambda) + \frac{\partial\mathcal{L}}{\partial A_\lambda}\epsilon^\nu\partial_\nu A_\lambda + \frac{\partial\mathcal{L}}{\partial A_\lambda}\partial_\mu\epsilon_\nu\Sigma_\lambda^{\mu\nu} + \tag{133}$$

$$+\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\lambda)}\partial_\mu\epsilon_\nu\partial^\nu A_\lambda + \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\lambda)}\epsilon_\nu\partial_\mu\partial^\nu A_\lambda + \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\lambda)}\partial_\mu(\partial_\kappa\epsilon_\nu\Sigma_\lambda^{\kappa\nu}) \tag{134}$$

$$\delta_\epsilon S = \int d^D\mathbf{x}\left(\epsilon^\nu\left(\frac{\partial\mathcal{L}}{\partial A_\lambda}\partial_\nu A_\lambda + \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\lambda)}\partial_\mu\partial_\nu A_\lambda\right) + \partial_\mu\epsilon^\nu\frac{\partial\mathcal{L}}{\partial\partial_\mu A_\lambda}\partial_\nu A_\lambda + \right. \tag{135}$$

$$\left. + \frac{\partial\mathcal{L}}{\partial A_\lambda}\partial_\mu\epsilon_\nu\Sigma_\lambda^{\mu\nu} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\lambda)}\partial_\mu(\partial_\kappa\epsilon_\nu\Sigma_\lambda^{\kappa\nu})\right) \tag{136}$$

Очень жаль, что это конец демо-версии данного файла! Для получения полной версии перейдите по секретной ссылке.