Seminar “Nonperturbative methods in QFT”

The 40th Anniversary of BPZ paper

Cubic $O(N)$ model on a sphere

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Based on:
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Plan

- Introduction: c, a, F-theorems, $O(N)$ model
- Renormalization: beta-functions, critical points
- Sphere free energy:
  - Dimensional continuation ($6 - \epsilon$-expansion)
  - Long Range Approach
- GL description of non-unitary minimal models

Introduction

Constraints on RG flows: c and a-theorems

Consider RG flow between two CFTs.

- In $d = 2$, the c-theorem for unitary theories [Zamolodchikov'86]:
  - $c(g_i, \mu)$ decreases monotonically under the RG flow.
  - At fixed points $c(g_i^*, \mu) = c_*$ is a constant, independent of scale (*central charge* of CFT).
- For non-unitary theories, the c_{eff}-theorem [Ravanini et al'17], where $c_{eff} = c - 24h_{min}$.
- In $d = 4$, the a-theorem: after the works of [Cardy'88; Jack, Osborn'90] a non-perturbative proof in [Komargodski, Schwimmer'11; Komargodski'11].
- In $d = 6$, the a-theorem [Cordova, Dumitrescu, Yin'15; Cordova, Dumitrescu, Intriligator'15].

Introduction

Constraints on RG flows: c and a-theorems

- The Weyl anomaly equation in even d:

$$\langle T^{\mu}_{\mu} \rangle \sim -(-1)^{d/2} a E_d + \sum_i c_i I_i,$$

where E_d is the Euler density term and c_i are the coefficients of other Weyl invariant curvature terms.

- In even d, the RG inequalities: $a_{UV} > a_{IR}$. In $d = 2$, there is only one Weyl anomaly coefficient $c = 3a$. From sphere free energy: $F = -\log Z_{S^d} = (-1)^{d/2} a \log R$.

- In odd d, there are no Weyl anomalies, and sphere free energy is independent of R.

Introduction

Constraints on RG flows: F-theorem

- In $d = 3$, the F-theorem: $F_{UV} > F_{IR}$ [Jafferis, Klebanov, Pufu, Safdi'11].
- It can equivalently formulated in terms of the entanglement entropy across a circle [Myers, Sinha'10; Casini, Huerta, Myers'11]. A proof of $3d$ F-theorem [Casini, Huerta'12; Liu, Mezei'13].
- In odd d, the RG inequality: $\tilde{F}_{UV} > \tilde{F}_{IR}$, where $\tilde{F} = (-1)^{(d+1)/2} F$ [Klebanov, Pufu, Safdi'11]. In $d = 1$, this coincides with the g-theorem for BCFT, where $g = \log Z_{S^1}$ [Affleck, Ludwig'91].
- Generalized F-theorem: $\tilde{F}_{UV} > \tilde{F}_{IR}$, where *generalized free energy* $\tilde{F} = -\sin(\pi d/2) F$ [Giombi, Klebanov'15]. In even d, the factor $\sin(\pi d/2)$ cancels the pole in F: $\tilde{F} = \pi a/2$. Some holographic evidence in [Kawano, Nakaguchi, Nishioka'15].

Introduction

F in 3D theories

- The calculation of F in some SUSY 3D CFTs may be reduced to finite dimensional integrals using the methods of localization [Pufu'17].

- For non-supersymmetric CFTs ($O(N)$ model, GNY model, Conformal QED, etc.), methods for calculation F:

  - Dimensional continuation of $\tilde{F}$ [Giombi, Klebanov'15; Fei, Giombi, Klebanov, Tarnopolsky'15-16; Tarnopolsky'16; Giombi, Himwich, Katsevich, Klebanov, Sun'24].

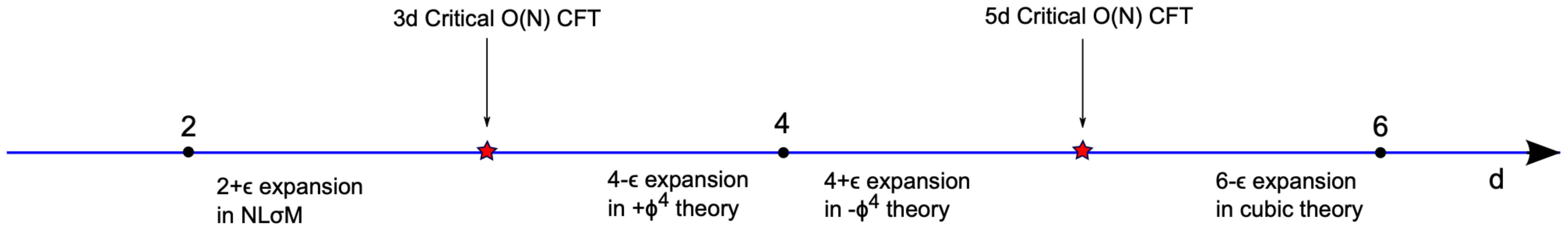
  - $1/N$-expansion [Klebanov, Pufu, Safdi'11; Klebanov, Pufu, Sachdev, Safdi'12; Tarnopolsky'16].

  - Fuzzy sphere [Zhu, Han, Huffman, Hofmann, He'23; Hu, Zhu, He'24].

  - Long Range Approach [Giombi, Himwich, Katsevich, Klebanov, Sun'24].

Introduction

$O(N)$ model



- Quartic $O(N)$ model in $d = 4 - \epsilon$ [Wilson, Fisher'72; Wilson, Kogut'74]:

$$S = \int d^d x \left(\frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{\lambda}{4}(\phi^i \phi^i)^2 \right).$$

- Weakly coupled IR fixed point $\lambda_* = \dfrac{8\pi^2}{N+8}\epsilon$. In $d = 4 + \epsilon$ the interaction is irrelevant, IR fixed point is a free theory, UV is $\lambda_* = -\dfrac{8\pi^2}{N+8}\epsilon$. Unstable fixed point?

- Large N, after the Hubbard-Stratonovich transformation:

$$S = \int d^d x \left(\frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{1}{2}\sigma \phi_i \phi_i - \frac{\sigma^2}{4\lambda} \right).$$

- UV completion is cubic $O(N)$ theory of $N+1$ fields ϕ_i and σ [Fei, Giombi, Klebanov'14]:

$$S = \int d^d x \left(\frac{1}{2}(\partial_\mu \phi_i)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{g_1}{2}\sigma \phi_i \phi_i + \frac{g_2}{6}\sigma^3 \right).$$

Dimensions Δ_σ and Δ_ϕ in $1/N$-expansion for cubic theory at fixed point g_1^* and g_2^* match with large N critical $O(N)$ theory [A. Vasiliev, Pismak, Khonkonen'81] expanded at $d = 6 - \epsilon$.

Renormalization of cubic $O(N)$ model

Flat-space warm-up

- Cubic $O(N)$ model in flat space $\mathbb{R}^d$:

$$S = \int d^d x \left(\frac{1}{2}(\partial_\mu \phi_0^i)^2 + \frac{1}{2}(\partial_\mu \sigma_0)^2 + \frac{1}{2} g_{1,0} \sigma_0 \phi_0^i \phi_0^i + \frac{1}{3!} g_{2,0} \sigma_0^3 \right).$$

- Dimensional regularization in $d = 6 - \epsilon$ ['t Hooft, Veltman'72] and the minimal subtraction (MS) scheme ['t Hooft'73]:

$$\phi_0^i = Z_\phi^{\frac{1}{2}} \phi^i, \quad \sigma_0 = Z_\sigma^{\frac{1}{2}} \sigma, \quad g_{1,0} = \mu^{\frac{\epsilon}{2}} Z_\phi^{-1} Z_\sigma^{-\frac{1}{2}} Z_{g_1} g_1, \quad g_{2,0} = \mu^{\frac{\epsilon}{2}} Z_\sigma^{-\frac{3}{2}} Z_{g_2} g_2,$$

$$Z_\phi = 1 + \delta_\phi, \quad Z_\sigma = 1 + \delta_\sigma, \quad Z_{g_1} = 1 + \frac{\delta_{g_1}}{g_1}, \quad Z_{g_2} = 1 + \frac{\delta_{g_2}}{g_2}.$$

- The beta-functions [Fei, Giombi, Klebanov'14; Fei, Giombi, Klebanov, Tarnopolsky'15; Gracey'15]:

$$\beta_{g_1} = -\frac{\epsilon}{2} g_1 + \frac{g_1((N-8)g_1^2 - 12 g_1 g_2 + g_2^2)}{12(4\pi)^3} + \mathcal{O}(g^5), \quad \beta_{g_2} = -\frac{\epsilon}{2} g_2 - \frac{4N g_1^3 - N g_1^2 g_2 + 3 g_2^3}{4(4\pi)^3} + \mathcal{O}(g^5).$$

Renormalization of cubic $O(N)$ model

Flat-space warm-up

- For $N > N_{crit} = 1038.26605 - 609.83890\epsilon - 306.17333\epsilon^2 + \mathcal{O}(\epsilon^3)$, real stable fixed points [Fei, Giombi, Klebanov, Tarnopolsky'15]. For all $N > 0$, real unstable points: for $N = 1$ there is a point $g_1^* = -g_2^*$ (the action $(\sigma + i\phi)^3 + (\sigma - i\phi)^3$ with Z_3 symmetry). This action appears in GL description of 3-state Potts model [Amit, Roginsky'79] (in $d = 2$ it's D_4-series version of $M(5,6)$). But $6-\epsilon$-expansion has grown coefficients, also $d_u \sim 2.5$.

- For $N < N'_{crit} = 1.02145 + 0.03253\epsilon - 0.00163\epsilon^2 + \mathcal{O}(\epsilon^3)$, there are stable non-unitary points [Fei, Giombi, Klebanov, Tarnopolsky'15]:

$$N = 0: \quad g_2^* = i\sqrt{\frac{2(4\pi)^3\epsilon}{3}}\left(1 + \frac{125}{324}\epsilon + \mathcal{O}(\epsilon^2)\right), \quad g_1^* = 0,$$

$$N = -2: \quad g_2^* = 2g_1^* = i\sqrt{\frac{4(4\pi)^3\epsilon}{5}}\left(1 + \frac{67}{180}\epsilon + \mathcal{O}(\epsilon^2)\right),$$

$$N = 1: \quad \begin{cases} g_1^* = 40i\sqrt{\frac{6\pi^3\epsilon}{499}}\left(1 + \frac{2633149}{7470030}\epsilon + \mathcal{O}(\epsilon^2)\right), \\ g_2^* = 48i\sqrt{\frac{6\pi^3\epsilon}{499}}\left(1 + \frac{227905}{498002}\epsilon + \mathcal{O}(\epsilon^2)\right). \end{cases}$$

- In $d = 2$, $N = 0$ theory corresponds to $M(2,5)$ [Fisher'78; Cardy'85], $N = -2$ - to $OSp(1|2)$ [Fei, Giombi, Klebanov, Tarnopolsky'15; Klebanov'22], $N = 1$ to a D_5-series version of $M(3,8)$ [Fei, Giombi, Klebanov, Tarnopolsky'14; Klebanov, Narovlansky, Sun, Tarnopolsky'22; Katsevich, Klebanov, Sun'24]. Also, for $N = 1$ there is unstable point $g_1^* = g_2^*$, corresponding to a D_6-series version of $M(3,10)$.

Renormalization of cubic $O(N)$ model

Flat-space warm-up

- 2pt function in momentum space:

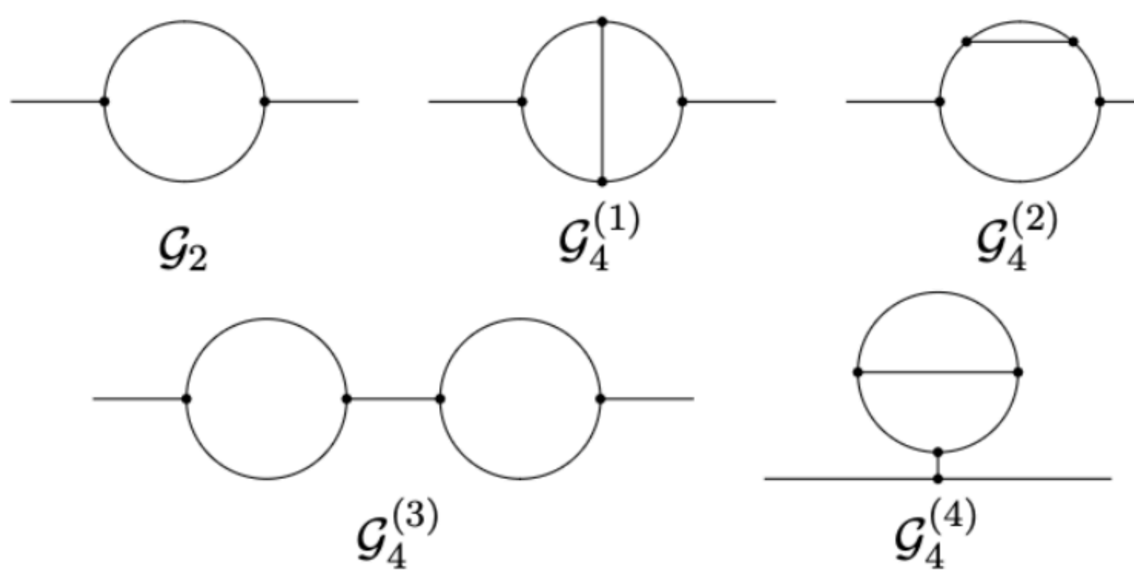
$$\langle\varphi_0(p)\varphi_0(-p)\rangle = \frac{1}{p^2} + \frac{1}{p^2}\Sigma(p^2)\frac{1}{p^2} + \frac{1}{p^2}\Sigma(p^2)\frac{1}{p^2}\Sigma(p^2)\frac{1}{p^2} + \cdots = \frac{1}{p^2-\Sigma(p^2)}$$

where $\Sigma(p^2)$ is the sum of all 1PI diagrams, $\Sigma(p^2) = \sum_{k=1}^{\infty} g_0^{2k}\Sigma_k(p^2)$ (k-loop ones). At $\mathcal{O}(g^4)$, the irreducible diagrams $\frac{1}{p^2}\Sigma_2(p^2)\frac{1}{p^2}$ will contribute as well as reducible $\frac{1}{p^2}\Sigma_1(p^2)\frac{1}{p^2}\Sigma_1(p^2)\frac{1}{p^2}$.

- Masless propagator in position space:

$$\mathbb{G}_d(x,y) = \frac{C_d}{|x-y|^{d-2}}, \quad C_d = \frac{\Gamma(\frac{d}{2}-1)}{4\pi^{d/2}}.$$

- 2pt function can be calculated in position space using Mellin-Barnes representation with MB.m [Czakon'06; Smirnov A., Smirnov V.'09; Smirnov V.'12; Belitsky, Smirnov A., Smirnov V.'23].



- $\langle\phi^i(x)\phi^j(y)\rangle = Z_\phi^{-1}\langle\phi_0^i(x)\phi_0^j(y)\rangle$ and $\langle\sigma(x)\sigma(y)\rangle = Z_\sigma^{-1}\langle\sigma_0(x)\sigma_0(y)\rangle$ are finite $\to Z_\phi, Z_\sigma$. Irreducible $\mathcal{G}_4^{(3)}$ is important!

Renormalization of cubic $O(N)$ model

Flat-space warm-up

- Integral appearing in $\mathcal{G}_2$:

$$\int d^d x_3 \frac{1}{x_{13}^{2a} x_{23}^{2b}} = \frac{C_{a,b}}{x_{12}^{2a+2b-d}}, \quad C_{a,b} \equiv \pi^{\frac{d}{2}} \frac{\Gamma(d/2-a)\Gamma(d/2-b)\Gamma(a+b-d/2)}{\Gamma(a)\Gamma(b)\Gamma(d-a-b)} .$$

- MB representation of integral:

$$\int \frac{d^d x_0}{x_{01}^{2\gamma_1} x_{02}^{2\gamma_2} x_{03}^{2\gamma_3}} = x_{12}^{d-2(\gamma_1+\gamma_2+\gamma_3)} \int_{i\mathbb{R}} \frac{dz_1 dz_2}{(2\pi i)^2} S(\gamma_1,\gamma_2,\gamma_3;z_1,z_2) \frac{x_{13}^{2z_1} x_{23}^{2z_2}}{x_{12}^{2(z_1+z_2)}},$$

where $S(\gamma_1,\gamma_2,\gamma_3;z_1,z_2) = \pi^{\frac{d}{2}} \dfrac{\Gamma(-z_1)\Gamma(-z_2)\Gamma(\frac{d}{2}-\gamma_1-\gamma_3-z_1)\Gamma(\frac{d}{2}-\gamma_2-\gamma_3-z_2)\Gamma(\gamma_3+z_1+z_2)\Gamma(\sum_i \gamma_i - \frac{d}{2}+z_1+z_2)}{\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(\gamma_3)\Gamma(d-\sum_i \gamma_i)}.$

- Combining these two results:

$$\int \frac{d^d x_3 d^d x_4}{\prod_{1\le i<j\le 4} x_{ij}^{2\gamma_{ij}}} = \left(x_{12}^2\right)^{d-\sum_{i<j}\gamma_{ij}} \mathrm{MB}_2 \left(S(\gamma_{14},\gamma_{24},\gamma_{34};z_1,z_2) C_{\gamma_{13}-z_1,\gamma_{23}-z_2} \right),$$

where $\mathrm{MB}_n = \displaystyle\int_{i\mathbb{R}} \frac{dz_1 \cdots dz_n}{(2\pi i)^n}$.

Renormalization of cubic $O(N)$ model

Flat-space warm-up

- The operators $\phi^i\phi^i$ and σ^2 mix under renormalization. The eigenvalues $\gamma_\pm$ and corresponding eigenoperators $\mathcal{O}^\pm$ of anomalous dimension matrix γ_{ab} in [Fei, Giombi, Klebanov'14]. At fixed point to the leading order:

$$\Delta_+ \equiv d - 2 + \gamma_+ = 2 + \Delta_\sigma,$$

where $\Delta_\sigma = \dfrac{d-2}{2} + \gamma_\sigma$, $\gamma_\sigma \equiv -\dfrac{1}{2}\mu\dfrac{\partial}{\partial\mu}\log Z_\sigma$. The operator $\mathcal{O}^+$ is a descendant of σ (EoM).

- For $N = 0$, the primary $\mathcal{O}^-$ drops out and $\mathcal{O}^+ = \sigma^2$. The 1-loop result:

$$\Delta_{\sigma^2} = d - 2 + \gamma_{\sigma^2}, \quad \gamma_{\sigma^2} = -\frac{2g^2}{3(4\pi)^3}.$$

At fixed point $\Delta_{\sigma^2} = 2 + \Delta_\sigma$, where $\Delta_\sigma = \dfrac{d-2}{2} + \gamma_\sigma$, $\gamma_\sigma = \dfrac{g^2}{12(4\pi)^3}$ (EoM $\partial^2\sigma \sim \sigma^2$).

- Including only irreducible diagrams gives $\gamma_{\sigma^2}^{irred} = -\dfrac{5g^2}{6(4\pi)^3}$ [Macfarlane, Woo'74; Borinsky, Gracey, Kompaniets, Schnetz'21], which leads to $2 + \Delta_\sigma = 4 - \gamma_{\sigma^2}^{irred}$ (instead of descendant relation) and *shadow relation* $\Delta_\sigma + \Delta_{\sigma^2}^{irred} = d = 6 - \epsilon$.

Renormalization of cubic $O(N)$ model

Sphere

- Metric on a sphere: $ds^2 = \Omega^2(x)dx^2$, where $\Omega(x) = \dfrac{2R}{1+x^2}$.

- Cubic $O(N)$ model on a sphere [Giombi, Klebanov'15; Tarnopolsky'17] (curvature couplings [Brown, Collins'80; Hathrell'82; Toms'82; Jack'86]):

$$S = \int d^dx\sqrt{g}\left(\frac{1}{2}(\partial_\mu\phi_0^i)^2 + \frac{1}{2}(\partial_\mu\sigma_0)^2 + \frac{\xi}{2}\mathscr{R}\left(\phi_0^i\phi_0^i + \sigma_0^2\right) + \frac{1}{2}g_{1,0}\sigma_0\phi_0^i\phi_0^i + \frac{1}{3!}g_{2,0}\sigma_0^3 + \frac{\eta_{1,0}}{2}\mathscr{R}\phi_0^i\phi_0^i + \frac{\eta_{2,0}}{2}\mathscr{R}\sigma_0^2 + \kappa_0\mathscr{R}^2\sigma_0 + b_0\mathscr{R}^3\right),$$

where $\xi = \dfrac{d-2}{4(d-1)}$. First 3 terms are invariant under Weyl transformation.

- Dimensional regularization in $d = 6-\epsilon$ ['t Hooft, Veltman'72] and the minimal subtraction (MS) scheme ['t Hooft'73]:

$$\eta_{1,0} = Z_\phi^{-1}Z_{\eta_1}\eta_1, \quad \eta_{2,0} = Z_\sigma^{-1}Z_{\eta_2}\eta_2, \quad \kappa_0 = \mu^{-\frac{\epsilon}{2}}Z_\sigma^{-\frac{1}{2}}Z_\kappa\kappa, \quad b_0 = \mu^{-\epsilon}Z_b b,$$

$$Z_{\eta_1} = 1 + \frac{\delta_{\eta_1}}{\eta_1}, \quad Z_{\eta_2} = 1 + \frac{\delta_{\eta_2}}{\eta_2}, \quad Z_\kappa = 1 + \frac{\delta_\kappa}{\kappa}, \quad Z_b = 1 + \frac{\delta_b}{b}.$$

Renormalization of cubic $O(N)$ model

Sphere

- The propagator of massless scalar on the sphere:

$$G_d(x,y) = \frac{C_d}{D(x,y)^{d-2}},$$

where $C_d = \dfrac{\Gamma(\frac{d}{2}-1)}{4\pi^{d/2}}$, $D(x,y) = \sqrt{\Omega(x)\Omega(y)}\,|x-y|$ ($SO(d+1)$ invariant).

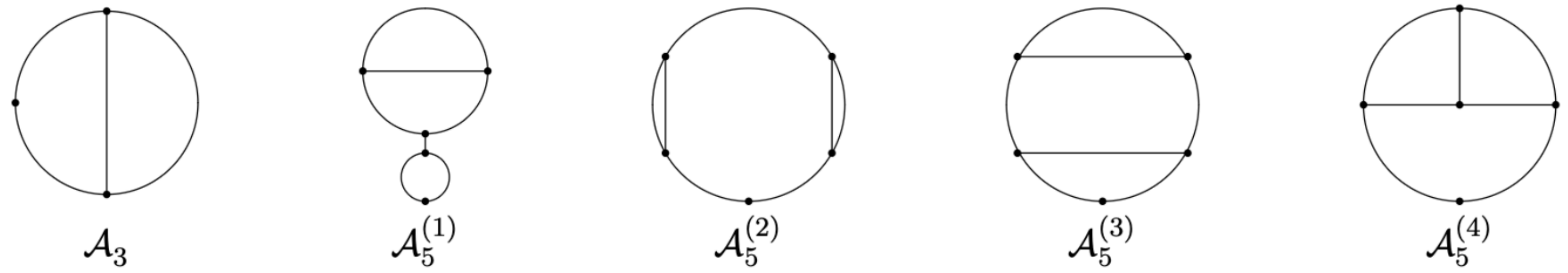
- Integrated 2pt and 3pt functions on a sphere [Drummond'79; Cardy'88; Klebanov, Pufu, Safdi'11]:

$$I_2(\Delta) = \int \frac{d^dx d^dy\ \Omega^d(x)\Omega^d(y)}{D(x,y)^{2\Delta}} = 2^{1+d-2\Delta}\pi^{d+\frac{1}{2}}R^{2(d-\Delta)}\frac{\Gamma(\frac{d}{2}-\Delta)}{\Gamma(\frac{d+1}{2})\Gamma(d-\Delta)},$$

$$I_3(\Delta) = \int \frac{d^dx d^dy d^dz\ \Omega^d(x)\Omega^d(y)\Omega^d(z)}{\left[D(x,y)D(y,z)D(z,x)\right]^{\Delta}} = 8\pi^{\frac{3(1+d)}{2}}R^{3(d-\Delta)}\frac{\Gamma(d-\frac{3\Delta}{2})}{\Gamma(\frac{d+1-\Delta}{2})^3\Gamma(d)}.$$

Renormalization of cubic $O(N)$ model

Sphere: 1pt function



- 1pt function is position independent ($SO(d+1)$ symmetry). Non-curvature contributions:

$$\langle \sigma_0 \rangle = -\left(a_3 \mathcal{A}_3 + a_{5,1}\mathcal{A}_5^{(1)} + a_{5,2}\mathcal{A}_5^{(2)} + a_{5,3}\mathcal{A}_5^{(3)} + a_{5,4}\mathcal{A}_5^{(4)}\right) \frac{C_d I_2(\frac{d-2}{2})}{\text{Vol}(S^d)} + \cdots,$$

where $\text{Vol}(S^d) = \dfrac{2\pi^{(d+1)/2}R^d}{\Gamma((d+1)/2)}$, $C_d = \dfrac{\Gamma(\frac{d}{2}-1)}{4\pi^{d/2}}$ and

$$a_3 = \frac{1}{4}\left(2Ng_{1,0}^3 + Ng_{1,0}^2 g_{2,0} + g_{2,0}^3\right),$$

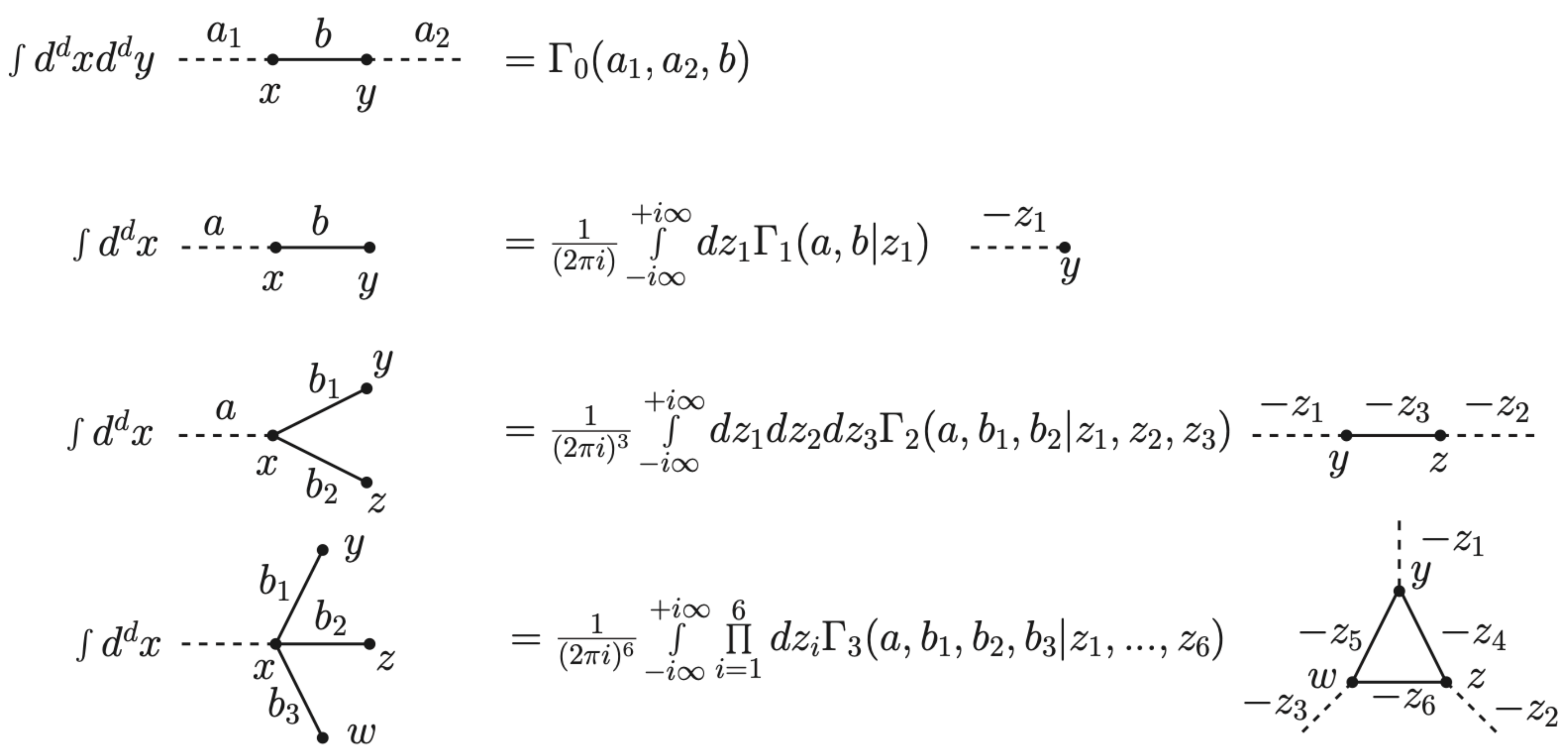
$$a_{5,k} = \begin{cases} \frac{1}{8}\left(2N^2 g_{1,0}^5 + N^2 g_{1,0}^4 g_{2,0} + 2Ng_{1,0}^3 g_{2,0}^2 + 2Ng_{1,0}^2 g_{2,0}^3 + g_{2,0}^5\right), & k=1, \\ \frac{1}{8}\left(4Ng_{1,0}^5 + N^2 g_{1,0}^4 g_{2,0} + 2Ng_{1,0}^2 g_{2,0}^3 + g_{2,0}^5\right), & k=2, \\ \frac{1}{4}\left(N(N+2)g_{1,0}^5 + 2Ng_{1,0}^4 g_{2,0} + Ng_{1,0}^3 g_{2,0}^2 + Ng_{1,0}^2 g_{2,0}^3 + g_{2,0}^5\right), & k=3, \\ \frac{1}{4}\left(2Ng_{1,0}^5 + 3Ng_{1,0}^4 g_{2,0} + 2Ng_{1,0}^3 g_{2,0}^2 + g_{2,0}^5\right), & k=4. \end{cases}$$

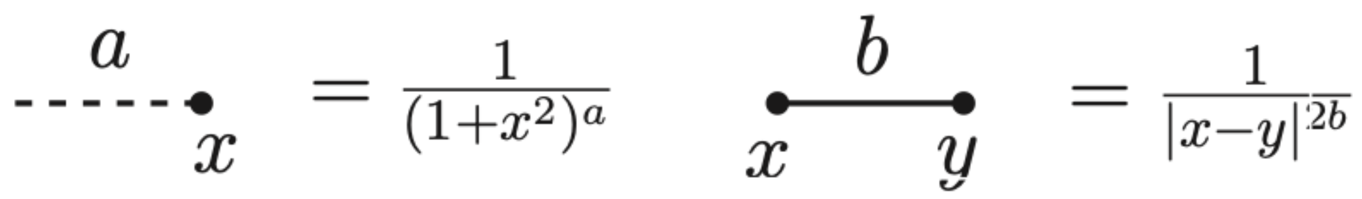
$$\mathcal{A}_3 = (2R)^{8-2d} C_d^4 e^{\gamma_E(d-6)+\frac{\pi^2}{24}(d-6)^2}\pi^d \left(-\frac{1}{8} - \frac{11\epsilon}{32} + \mathcal{O}(\epsilon^2)\right),$$

$$\mathcal{A}_5^{(k)} = (2R)^{14-3d} C_d^7 e^{2\gamma_E(d-6)+\frac{\pi^2}{24}(d-6)^2}\pi^{2d} \begin{cases} \frac{1}{24\epsilon} + \frac{25}{144} + \mathcal{O}(\epsilon), & k=1, \\ \frac{1}{12\epsilon} + \frac{193}{432} + \mathcal{O}(\epsilon), & k=2, \\ -\frac{1}{12\epsilon} - \frac{13}{48} + \mathcal{O}(\epsilon), & k=3, \\ -\frac{1}{4\epsilon} - \frac{73}{48} + \mathcal{O}(\epsilon), & k=4. \end{cases}$$

Renormalization of cubic $O(N)$ model

Sphere: calculation of $\mathscr{A}_3$





$$\mathscr{A}_3 = \int \prod_{i=1}^{2} \left(d^d x_i \Omega^d(x_i) G_d(x_i, x_0)\right) G_d(x_1, x_2)^2 \,.$$

- The integral is independent of x_0 ($SO(d+1)$ symmetry). After $x_i^\mu \to x_i^\mu / x_i^2$ ($i = 1{,}2$) using "Feynman rules" from Appendix B [Fei, Giombi, Klebanov, Tarnopolsky'15]:

$$\mathscr{A}_3 = (2R)^{8-2d} C_d^4 \int \prod_{i=1}^{2} \frac{d^d x_i}{(1+x_i^2)^{3-d/2}} \frac{1}{x_{12}^{2(d-2)}} = (2R)^{8-2d} C_d^4 \,\Gamma_0\left(3 - \frac{d}{2}, 3 - \frac{d}{2}, d-2\right) = -\frac{1}{8(4\pi)^6 R^4} + \mathscr{O}(\epsilon),$$

where $\Gamma_0(a_1, a_2, b) = \dfrac{\pi^d \Gamma(\frac{d}{2} - b)\Gamma(a_1 + b - \frac{d}{2})\Gamma(a_2 + b - \frac{d}{2})\Gamma(a_1 + a_2 + b - d)}{\Gamma(\frac{d}{2})\Gamma(a_1)\Gamma(a_2)\Gamma(a_1 + a_2 + 2b - d)}$.

Renormalization of cubic $O(N)$ model

Sphere: 1pt function

(a) $\mathcal{K}_0$

(b) $\mathcal{K}_1$

- Curvature contribution to 1pt function:

$$\langle \sigma_0 \rangle \supset -\left(\kappa_0 \mathcal{R}^2 \mathcal{K}_0 - \frac{N g_{1,0}\eta_{1,0} + g_{2,0}\eta_{2,0}}{2} \mathcal{R}\mathcal{K}_1 \right) \frac{C_d I_2(\frac{d-2}{2})}{\mathrm{Vol}(S^d)},$$

where

$$\mathcal{K}_0 = 1, \quad \mathcal{K}_1 = \frac{C_d^2 I_2(d-2)}{\mathrm{Vol}(S^d)} = -\frac{2}{(4\pi)^3 R^2 \epsilon} + \mathcal{O}(1)\,.$$

- From finiteness of $\langle \sigma \rangle = Z_\sigma^{-1} \langle \sigma_0 \rangle$ the beta-function:

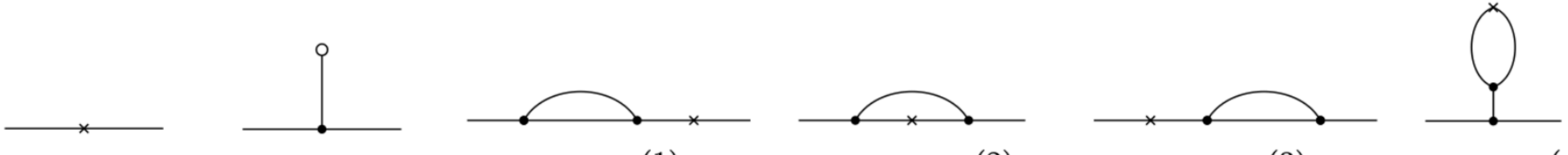
$$\beta_\kappa = \frac{\epsilon}{2}\kappa + \frac{N g_1^2 + g_2^2}{12(4\pi)^3}\kappa - \frac{N\eta_1 g_1 + \eta_2 g_2}{30(4\pi)^3} + \cdots\,.$$

Dots denote $\mathcal{O}(g_1^{n_1} g_2^{n_2}), n_1 + n_2 = 7$ as well as curvature contributions $\mathcal{O}(\eta^2 g, \eta g^3, \kappa g^4)$.

Renormalization of cubic $O(N)$ model

Sphere: 2pt function

- Non-curvature contributions to 2pt are the same as in flat space, so β_{η_1} and β_{η_2} can include only $\mathcal{O}(g_1^{n_1} g_2^{n_2}), n_1 + n_2 = 6$.



(a) $\mathcal{M}_0$ (b) $\mathcal{M}_1$ (c) $\mathcal{M}_2^{(1)}$ (d) $\mathcal{M}_2^{(2)}$ (e) $\mathcal{M}_2^{(3)}$ (f) $\mathcal{M}_2^{(4)}$

- Curvature contributions: $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2^{(1)}, \mathcal{M}_2^{(2)}, \mathcal{M}_2^{(3)}, \mathcal{M}_2^{(4)}$. From finiteness of $\langle \phi^i(x)\phi^j(y)\rangle = Z_\phi^{-1}\langle \phi_0^i(x)\phi_0^j(y)\rangle$ and $\langle \sigma(x)\sigma(y)\rangle = Z_\sigma^{-1}\langle \sigma_0(x)\sigma_0(y)\rangle$ the beta-functions:

$$\beta_{\eta_1} = -\frac{(2\eta_1 + 3\eta_2)g_1^2}{3(4\pi)^3} + \cdots, \quad \beta_{\eta_2} = -\frac{6N\eta_1 g_1^2 - N\eta_2 g_1^2 + 5\eta_2 g_2^2}{6(4\pi)^3} + \cdots.$$

Independent of κ. β_{η_2} can be equivalently obtained from 1pt function.

- Fixed points: $g_1^*, g_2^* = \mathcal{O}(\epsilon^{1/2}), \quad \eta_1^*, \eta_2^* = \mathcal{O}(\epsilon^2), \quad \kappa^* = \mathcal{O}(\epsilon^{3/2})$.

Renormalization of cubic $O(N)$ model

Sphere: comparing $N = 0$ results with old papers

- Our results at $N = 0$ ($g_1 \to 0,\, g_2 \to g,\, \eta_1 \to 0,\, \eta_2 \to \eta$):

$$\beta_\kappa = \frac{\epsilon}{2}\kappa + \frac{\kappa g^2}{12(4\pi)^3} - \frac{\eta g}{30(4\pi)^3} + 0g^3 + 0g^5 + \cdots, \quad \beta_\eta = -\frac{5\eta g^2}{6(4\pi)^3} + 0g^4 + \cdots .$$

- Using the background field method, [Toms'82; Jack'86] obtained:

$$\beta_\kappa = \frac{\epsilon}{2}\kappa + \frac{\kappa g^2}{12(4\pi)^3} - \frac{\eta g}{30(4\pi)^3} - \frac{161 g^3}{2^5 3^4 5^3 (4\pi)^6} + \cdots .$$

- Using the background field method, [Kodaira, Okada'86; Kodaira'86] obtained:

$$\beta_\eta = -\frac{5\eta g^2}{6(4\pi)^3} - \frac{97}{108}\frac{\eta g^4}{(4\pi)^6} - \frac{1}{72}\frac{g^4}{(4\pi)^6} + \cdots,$$

This agrees with [Toms'82] at one-loop, and with [Jack'86] at two-loops, except the latter has the opposite sign for the ηg^2 term.

- In cubic theory $\beta_\eta \neq \gamma_{\sigma^2}\eta$, the renormalization of η receives contribution at leading order from κ vertex in $\mathscr{M}_1$. Correct relation: $\beta_\eta = \gamma_{\sigma^2}^{irred}\eta$.

Sphere free energy

Dimension continuation

- The sphere free energy for a conformally coupled free scalar [Diaz, Dorn'07; Giombi, Klebanov'15]:

$$F_{\text{free}} = \frac{1}{2}\log\det\left(-\nabla^2 + \frac{1}{4}d(d-2)\right) = -\frac{1}{\sin(\pi d/2)\Gamma(1+d)}\int_0^1 du \sin \pi u \Gamma\left(\frac{d}{2}+u\right)\Gamma\left(\frac{d}{2}-u\right).$$

- $\tilde{F} = -\sin(\pi d/2)F$ interpolates between $\tilde{F} = (-1)^{(d+1)/2}F$ in odd dimensions and $\tilde{F} = \pi a/2$ in even.

Weyl anomaly coefficient for free scalar: $a = \frac{1}{3}, \frac{1}{90}, \frac{1}{756}$ in $d = 2,4,6$.

$$\tilde{F}_{\text{free}}(6-\epsilon) = \frac{1}{\Gamma(1+d)}\int_0^1 du \sin \pi u \Gamma\left(\frac{d}{2}+u\right)\Gamma\left(\frac{d}{2}-u\right) = \frac{\pi}{1512} + 0.002042876\epsilon + 0.001064155\epsilon^2 + 0.000396195\epsilon^3 + \mathcal{O}(\epsilon^4).$$

- The leading term in ϵ-expansion for general N in [Giombi, Klebanov'14] ($N = -2$ case in [Fei, Giombi, Klebanov, Tarnopolsky'15]), and the next-to-leading term for large N (without considering curvature vertices) in [Tarnopolsky'16]. We're doing general N.

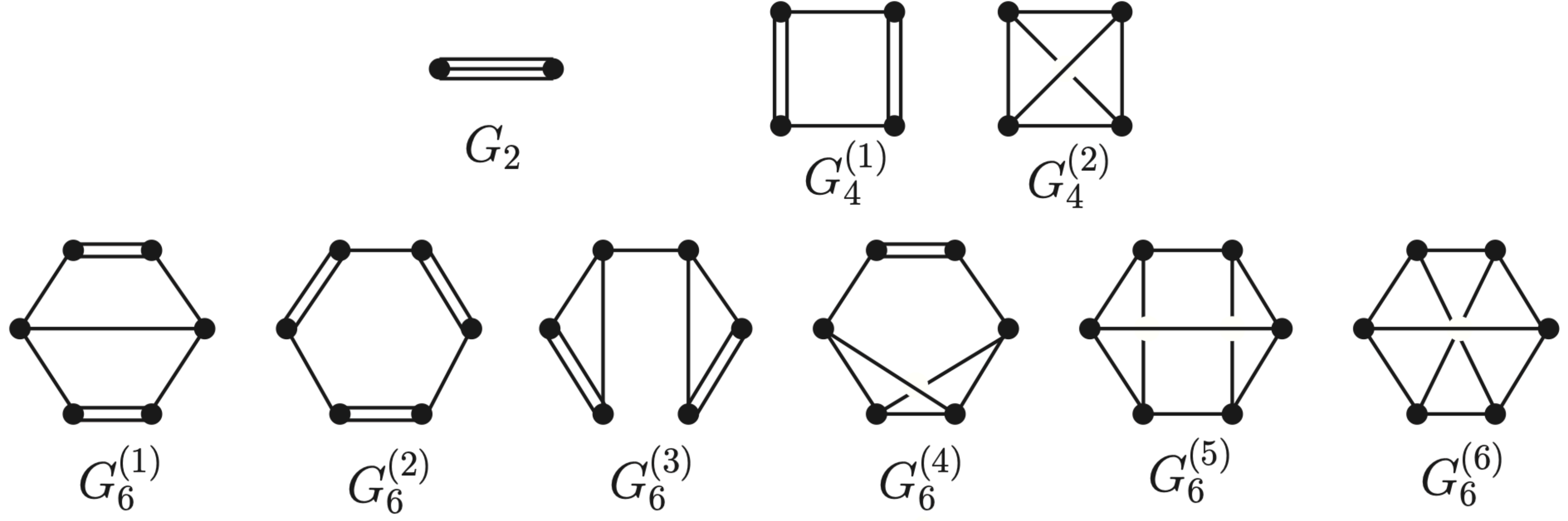
- Integrated connected sphere correlation functions:

$$G_n = \int \prod_{i=1}^{n} d^d x_i \sqrt{g_{x_i}} \langle \varphi_0^3(x_1)\ldots\varphi_0^3(x_n)\rangle_0^{\text{conn}},$$

where $\varphi_0^3 = 3g_{1,0}\sigma_0\phi_0^i\phi_0^i + g_{2,0}\sigma_0^3$.

- To compute F up to order $\epsilon^2 \to$ up to 6th order in the couplings $g_{1,0}$ and $g_{2,0}$. The counterterm in b_0 will remove divergence in 6th order. At the fixed point, b_* then will include a contribution of order ϵ^2.

Sphere free energy



Dimension continuation

- Free energy up to the 6th order:

$$F = (N+1)F_{\text{free}} - \frac{G_2}{2!(3!)^2} - \frac{G_4}{4!(3!)^4} - \frac{G_6}{6!(3!)^6} + b_0 \int d^d x \sqrt{g}\mathscr{R}^3 \,.$$

- Curvature contributions: κ^2, ηg^2, and κg^3 (order $\mathcal{O}(g^6)$) are finite. From the finiteness of F beta-function:

$$\beta_b = \epsilon b + \frac{N+1}{756 \cdot 450(4\pi)^3} + 4b_{61} + \ldots,$$

where $b_{61} = \dfrac{N(2(43N+268)g_1^6 - 12(11N-32)g_1^5 g_2 + (11N+950)g_1^4 g_2^2 + 84 g_1^3 g_2^3 - 44 g_1^2 g_2^4) + 125 g_2^6}{2^{12}3^8 5^3 (4\pi)^{12}}$.

- At fixed point generalized free energy $\tilde{F} = -\sin(\pi d/2)F$:

$$\begin{aligned}
\widetilde{F} = {} & (N+1)\widetilde{F}_{\text{free}} - \frac{(3Ng_1^{*2} + g_2^{*2})(30 + \epsilon(15(\gamma_E + \log(4\pi\mu^2 R^2)) + 56))\epsilon}{2^{10}3^4 5^2 (4\pi)^2} \\
& + \frac{N(9(N-8)(\gamma_E + \log(4\pi\mu^2 R^2)) + 26N - 148)g_1^{*4}\epsilon}{2^{11}3^5 5(4\pi)^5} - \frac{N(3(\gamma_E + \log(4\pi\mu^2 R^2)) + 7)g_1^{*3}g_2^{*}\epsilon}{2^7 3^4 5(4\pi)^5} \\
& + \frac{N(26 + 9\gamma_E + 9\log(4\pi\mu^2 R^2))g_1^{*2}g_2^{*2}\epsilon}{2^{10}3^5 5(4\pi)^5} - \frac{(27(\gamma_E + \log(4\pi\mu^2 R^2)) + 58)g_2^{*4}\epsilon}{2^{11}3^5 5(4\pi)^5} \\
& + \frac{N(2(43N+268)g_1^{*6} - 12(11N-32)g_1^{*5}g_2^{*} + (11N+950)g_1^{*4}g_2^{*2} + 84g_1^{*3}g_2^{*3} - 44g_1^{*2}g_2^{*4}) + 125g_2^{*6}}{2^{12}3^6 5(4\pi)^8}
\end{aligned}$$

Sphere free energy

Long Range Approach

- The long-range model [Fisher, Ma, Nickel'72; Gubser, Jepsen, Parikh, Trundy'17; Giombi, Khanchandani'19; Giombi, Helfenberger, Khanchandani'22]:

$$S = \frac{2^{s-1}\Gamma((d+s)/2)}{\pi^{\frac{d}{2}}\,\Gamma(-s/2)} \int d^d x d^d y \sqrt{g_x}\sqrt{g_y}\, \frac{\varphi_0(x)\varphi_0(y)}{D(x,y)^{d+s}} + \lambda_0 \int d^d x \sqrt{g_x}\; O_0(x),$$

where O_0 is operator of dimension $d-\varepsilon$. d is fixed, $s = s(\varepsilon)$.

- Non-local kinetic term is conformally invariant. There is no local stress-energy tensor.

- Propagator:

$$G_{d,s} = \frac{C_{d,s}}{D(x,y)^{d-s}}, \quad C_{d,s} = \frac{\Gamma((d-s)/2)}{\pi^{d/2} 2^s \Gamma(s/2)}.$$

- In long-range model, $\Delta_\varphi = \dfrac{d-s}{2}$, φ is not renormalized ($Z_\varphi = 1$).

- Crossover from the long-range to short-range fixed points when $s \sim s_*$ [Sak'73,'77; Behan, Rastelli, Rychkov, Zan'17]: $\Delta_\varphi^{SR} = \dfrac{d-s_*}{2}$ (dimension of φ is continuous at the crossover).

$$\Delta_\varphi^{SR} = \frac{d-2}{2} + \gamma_\varphi^{SR} \to s_* = 2 - 2\gamma_\varphi^{SR}.$$

Sphere free energy

Long Range Approach

- Generalized free energy (only for integer d):

$$\tilde{F}^{\mathrm{LR}} = \begin{cases} (-1)^{\frac{d+1}{2}} F^{\mathrm{LR}}, & d \text{ odd}, \\ \frac{\pi}{2} a^{\mathrm{LR}}, & d \text{ even}, \end{cases}$$

where a^{LR} is defined as a coefficient $F^{LR} = (-1)^{d/2} a^{LR} \log R + \cdots$.

$$\tilde{F}^{\mathrm{LR}} = \tilde{F}^{\mathrm{LR}}_{free} + \delta \tilde{F}^{LR}\,.$$

$\tilde{F}^{\mathrm{LR}}_{free}$ is equivalent to $\log\det D(x,y)^{-d-s}$ (2pt function of primary scalars with $\Delta = \dfrac{d+s}{2}$ on a sphere).

- Double trace deformations in large N CFT ↔ transition between quantizations of the dual bulk operator in AdS. Both AdS and CFT calculations [Giombi, Klebanov'14; Gubser, Klebanov'03; Diaz, Dorn'07; Giombi, Klebanov, Pufu, Safdi, Tarnopolsky'13; Sun'20; Fraser-Taliente, Herzog, Shrestha'24]:

$$\tilde{F}^{\mathrm{LR}}_{\mathrm{free}} = \frac{1}{\Gamma(1+d)} \int_0^{\frac{s}{2}} du\ u \sin(\pi u) \Gamma\left(\frac{d}{2}+u\right)\Gamma\left(\frac{d}{2}-u\right).$$

Sphere free energy

Long Range Approach: Quartic theory

- Quartic $O(N)$ model with non-local kinetic term:

$$S = \frac{2^{s-1}\Gamma(\frac{d+s}{2})}{\pi^{\frac{d}{2}}\,\Gamma(-\frac{s}{2})}\int d^dx d^dy \sqrt{g_x}\sqrt{g_y}\frac{\phi_0^i(x)\phi_0^i(y)}{D(x,y)^{d+s}} + \frac{\lambda_0}{4}\int d^dx \sqrt{g_x}\left(\phi_0^i(x)\phi_0^i(x)\right)^2,$$

where $s = \dfrac{d+\varepsilon}{2}$, $\Delta_\phi = \dfrac{d-\varepsilon}{4}$.

- Renormalization:

$$\lambda_0 = \mu^\varepsilon Z_\lambda \lambda, \quad Z_\lambda = 1 + \frac{\delta_\lambda}{\lambda}.$$

- The beta-function [Giombi, Khanchandani'19]:

$$\beta(\lambda) = -\varepsilon\lambda + \frac{2(N+8)}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}\lambda^2 + \frac{8(5N+22)}{(4\pi)^d\Gamma(\frac{d}{2})^2}\left(\gamma_E + 2\psi\left(\frac{d}{4}\right) - \psi\left(\frac{d}{2}\right)\right)\lambda^3 + \mathcal{O}(\lambda^4),$$

where ψ is the digamma function.

- Curvature counterterms: in $d=2$ there is $\mathcal{R}$ but it doesn't affect c.
- In $d=3$:

$$\tilde{F}^{\rm LR} = N\tilde{F}^{\rm LR}_{\rm free} - \frac{N(N+2)}{(N+8)^2}\frac{\pi^2}{576}\left(\varepsilon^3 + \frac{3(5N+22)}{(N+8)^2}(2-\pi+4\log 2)\varepsilon^4\right) + \mathcal{O}(\varepsilon^5).$$

- In $d=2$:

$$c^{\rm LR} = \frac{6}{\pi}N\tilde{F}^{\rm LR}_{\rm free} - \frac{N(N+2)}{8(N+8)^2}\left(\varepsilon^3 + \frac{12(5N+22)\log 2}{(8+N)^2}\varepsilon^4\right) + \mathcal{O}(\varepsilon^5).$$

Sphere free energy

Long Range Approach: Cubic theory

- Cubic $O(N)$ model with non-local kinetic term:

$$S = \frac{2^{s-1}\Gamma(\frac{d+s}{2})}{\pi^{\frac{d}{2}}\,\Gamma(-\frac{s}{2})}\int d^dx d^dy\sqrt{g_x}\sqrt{g_y}\,\frac{\phi_0^i(x)\phi_0^i(y)+\sigma_0(x)\sigma_0(y)}{D(x,y)^{d+s}} + \int d^dx\sqrt{g_x}\left(\frac{g_{1,0}}{2}\sigma_0\phi_0^i\phi_0^i + \frac{g_{2,0}}{6}\sigma_0^3\right),$$

where $s = \dfrac{d+2\varepsilon}{3}$, $\Delta_\phi = \Delta_\sigma = \dfrac{d-\varepsilon}{3}$.

- Renormalization:

$$g_{i,0} = \mu^\varepsilon Z_{g_i} g_i, \quad Z_{g_i} = 1 + \frac{\delta_{g_i}}{g_i}, \quad i = 1,2.$$

- The beta-functions:

$$\beta_1 = -\varepsilon g_1 - \frac{2g_1^2(g_1+g_2)}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}, \quad \beta_2 = -\varepsilon g_2 - \frac{2(Ng_1^3+g_2^3)}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}.$$

There is no non-trivial $N=1$ fixed point.

- For $N = 0, -2$ at fixed point:

$$\widetilde{F}^{\rm LR} = \widetilde{F}^{\rm LR}_{\rm free} + \frac{\varepsilon^2}{288}, \qquad d=3,$$
$$a^{\rm LR} = \frac{2}{\pi}\widetilde{F}^{\rm LR}_{\rm free} + \frac{\Gamma(\frac{7}{6})^3}{144\pi^{3/2}}\varepsilon^2, \qquad d=4,$$
$$\widetilde{F}^{\rm LR} = \widetilde{F}^{\rm LR}_{\rm free} + \frac{\Gamma(\frac{4}{3})^3}{960}\varepsilon^2, \qquad d=5.$$

$$\widetilde{F}^{\rm LR} = -\widetilde{F}^{\rm LR}_{\rm free} - \frac{\varepsilon^2}{432}, \qquad d=3,$$
$$a^{\rm LR} = -\frac{2}{\pi}\widetilde{F}^{\rm LR}_{\rm free} - \frac{\Gamma(\frac{7}{6})^3}{216\pi^{3/2}}\varepsilon^2, \qquad d=4,$$
$$\widetilde{F}^{\rm LR} = -\widetilde{F}^{\rm LR}_{\rm free} - \frac{\Gamma(\frac{4}{3})^3}{1440}\varepsilon^2, \qquad d=5.$$

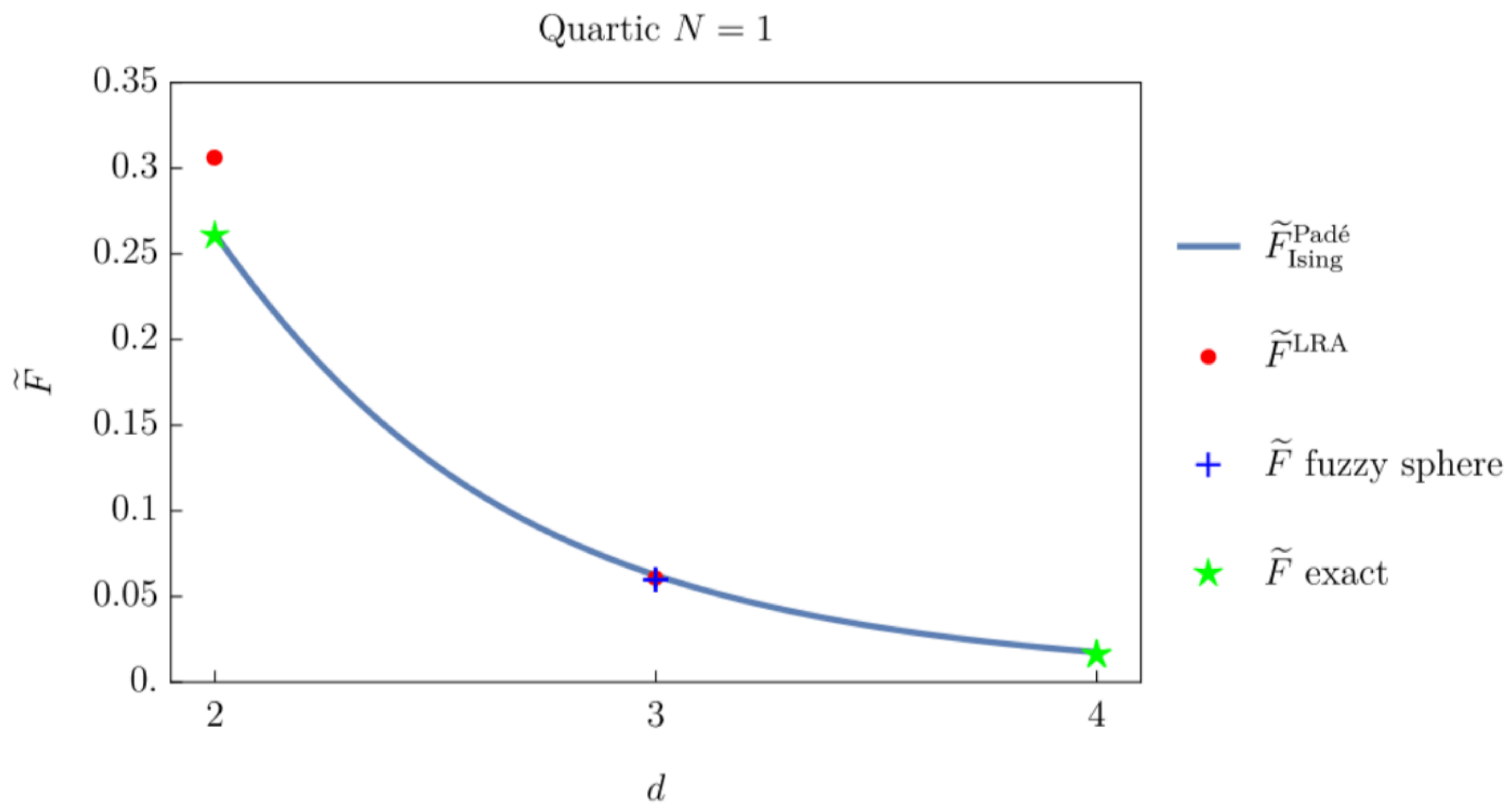
Sphere free energy

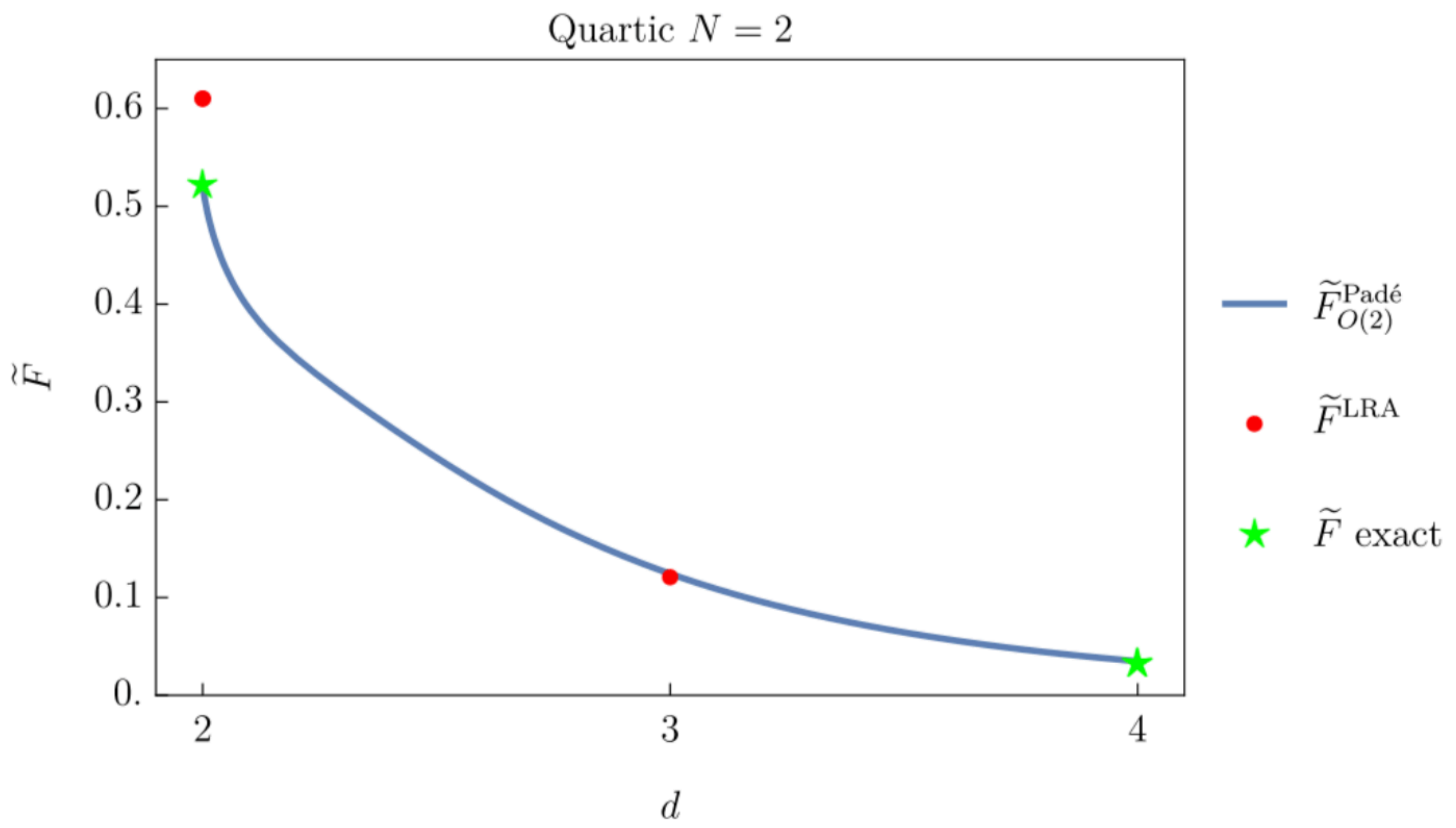
Numerics: Quartic theory

- Two-sided Pade approximants from $4-\epsilon$-expansion are taken from Table 1 in [Fei, Giombi, Klebanov, Tarnopolsky'15].
- In $d=3$, using bootstrap result for Δ_ϕ^{SR} [Henriksson'22], we take $s_* = 3 - 2\Delta_\phi^{SR}$ and $\varepsilon_* = 2s_* - 3$.

For $N=1$, $\Delta_\phi^{SR} = 0.5181$, for $N=2$, $\Delta_\phi^{SR} = 0.5191$.

- In $d=2$, the exact value for $N=1,2$: $\Delta_\phi^{SR} = \frac{1}{8}$ [Nienhuis'82], central charges: $c = \frac{1}{2}, 1$ ($N=1$ case is $M(3,4)$ minimal model).
- Fuzzy sphere result in $d=3$ [Hu, Zhu, He'24].





Sphere free energy

Numerics: Yang-Lee model

- $N=0$ case. In $d=2$, it's a non-unitary minimal model $M(2,5)$ with central charge $c=-\dfrac{22}{5}$ [Fisher'78; Cardy'85]. Results of $6-\epsilon$-expansion:

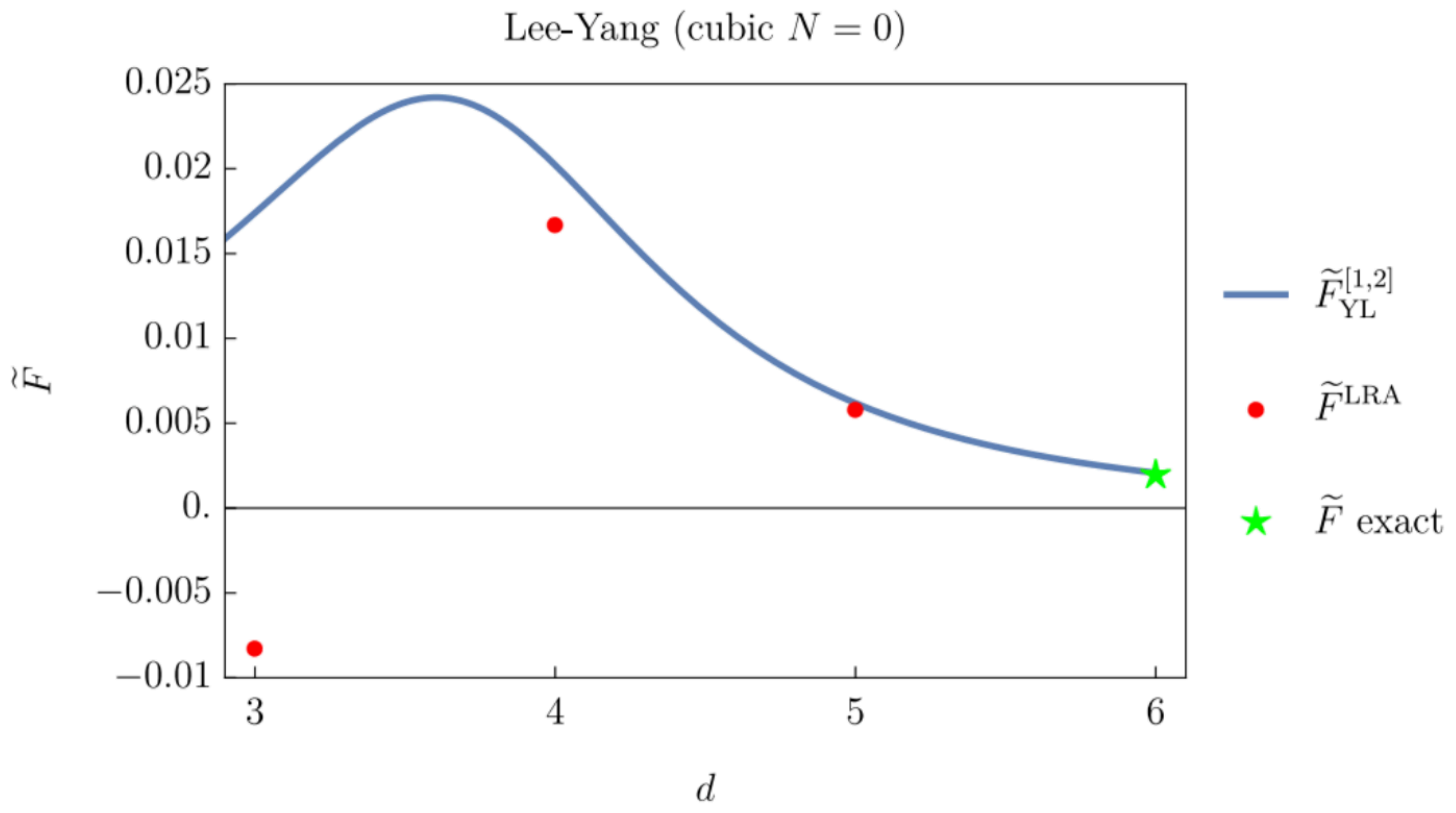
$$\tilde{F}_{\rm YL}(d)=\begin{cases}-\frac{11\pi}{15}, & d=2,\\ \tilde{F}_{\rm free}(6-\epsilon)+\frac{\pi\epsilon^2}{25920}+\frac{397\pi\epsilon^3}{7873200}+\mathcal{O}(\epsilon^4), & d=6-\epsilon,\end{cases}$$

- Pade approximant $\tilde{F}^{[m,n]}(\epsilon)=\dfrac{A_0+A_1\epsilon+\cdots+A_m\epsilon^m}{1+B_1\epsilon+\cdots+B_n\epsilon^n}$, where $m+n\le 3$. The $[1,2]$ Pade has no poles:

$$\tilde{F}_{\rm YL}^{[1,2]}=\frac{0.00207777+0.000500993\epsilon}{1-0.742084\epsilon+0.159126\epsilon^2}.$$

- In $d=3,4,5$, using bootstrap result for Δ_ϕ^{SR} [Gliozzi, Rago'14], we take $s_*=d-2\Delta_\phi^{SR}$ and $\varepsilon_*=\dfrac{3s_*-d}{2}$.

In $d=3$, $\Delta_\phi^{SR}=0.235$, in $d=4$, $\Delta_\phi^{SR}=0.847$, in $d=5$, $\Delta_\phi^{SR}=1.46$.



Sphere free energy

OSp(1|2) model

- The $Sp(N)$ model was proposed in [Fei, Giombi, Klebanov, Tarnopolsky'15]. The action of $Sp(2)$ model on the sphere:

$$S = \int d^d x \sqrt{g} \left(\partial_\mu \theta_0 \partial^\mu \bar{\theta}_0 + \frac{1}{2} (\partial_\mu \sigma_0)^2 + \frac{\xi}{2} \mathcal{R} (\sigma_0^2 + 2\theta_0 \bar{\theta}_0) + g_{1,0} \sigma_0 \theta_0 \bar{\theta}_0 + \frac{1}{6} g_{2,0} \sigma_0^3 + \eta_{1,0} \mathcal{R} \theta_0 \bar{\theta}_0 + \frac{\eta_{2,0}}{2} \mathcal{R} \sigma_0^2 + \kappa_0 \mathcal{R}^2 \sigma_0 + b_0 \mathcal{R}^3 \right),$$

where θ is a complex anticommuting scalar.

- For $g_{2,0} = 2g_{1,0}$, $\eta_{1,0} = \eta_{2,0}$, $\kappa_0 = 0$, the action possesses a sermonic symmetry

$$\delta\theta = \sigma\alpha, \quad \delta\bar{\theta} = \sigma\bar{\alpha}, \quad \delta\sigma = -\alpha\bar{\theta} + \bar{\alpha}\theta$$

that enhances $Sp(2)$ to $OSp(1\,|\,2)$. $OSp(1\,|\,2M)$ theories were studied in [Klebanov'21].

$$S_{OSp(1|2)} = \int d^d x \sqrt{g} \left(\partial_\mu \theta_0 \partial^\mu \bar{\theta}_0 + \frac{1}{2} (\partial_\mu \sigma_0)^2 + \frac{\xi}{2} \mathcal{R} (\sigma_0^2 + 2\theta_0 \bar{\theta}_0) + \frac{g_0}{3} (\sigma_0^2 + 2\theta_0 \bar{\theta}_0)^{\frac{3}{2}} + \frac{\eta_0}{2} \mathcal{R} (\sigma_0^2 + 2\theta_0 \bar{\theta}_0) + b_0 \mathcal{R}^3 \right).$$

- β-functions in $Sp(N)$ theory $\leftrightarrow$ β-functions in $O(N)$ theory by replacement $N \to -N$. $N = -2$ is a $OSp(1\,|\,2)$ theory. $\beta_\kappa = 0$.

- It's a $q \to 0$ limit of the q-state Potts model (describes random spanning forests) [Caracciolo, Jacobsen, Saleur, Sokal, Sportielo'04; Deng, Garoni, Sokal'07; Bauershmidt, Crawford, Helmuth, Swan'19].

Sphere free energy

Numerics: OSp(1|2) model

- $N=-2$ case. In $d=2$, it has central charge $c=-2$. Results of $6-\epsilon$-expansion:

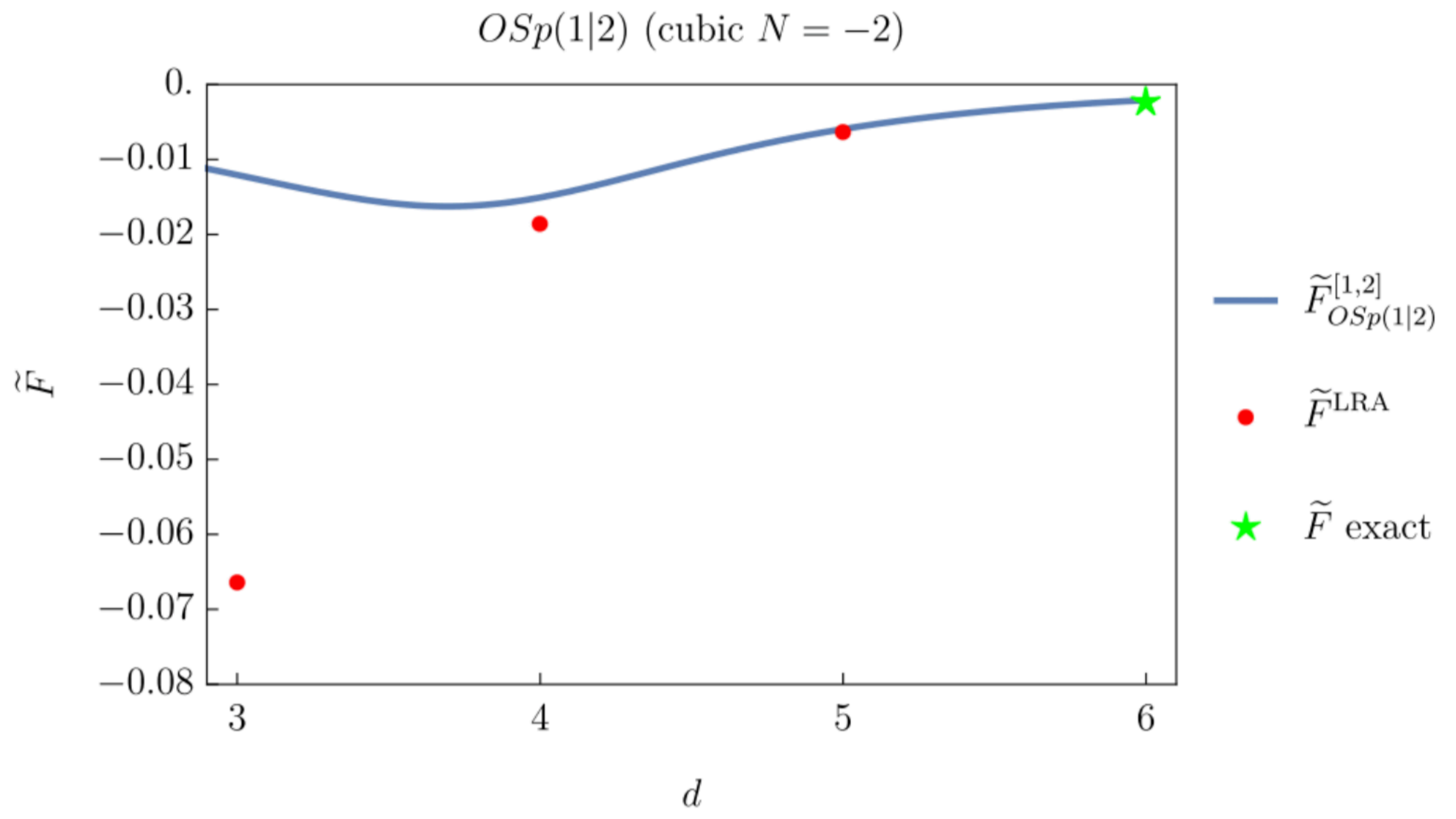
$$\tilde{F}_{\mathrm{OSp}(1|2)}(d)=\begin{cases}-\frac{\pi}{3}, & d=2,\\ -\tilde{F}_{\mathrm{free}}(6-\epsilon)-\frac{\pi\epsilon^2}{43200}-\frac{169\pi\epsilon^3}{5832000}+\mathcal{O}(\epsilon^4), & d=6-\epsilon\,.\end{cases}$$

- The $[1,2]$ Pade approximant:

$$\tilde{F}^{[1,2]}_{OSp(1|2)}=-\frac{0.00207777+0.000539773\epsilon}{1-0.723420\epsilon+0.164109\epsilon^2}\,.$$

- In $d=3,4,5$, using the Monte-Carlo result for Δ_ϕ^{SR} [Deng, Garoni, Sokal'07], we take $s_*=d-2\Delta_\phi^{SR}$ and $\varepsilon_*=\dfrac{3s_*-d}{2}$.

In $d=3$, $\Delta_\phi^{SR}=-0.0838$, in $d=4$, $\Delta_\phi^{SR}=0.920$, in $d=5$, $\Delta_\phi^{SR}=1.46$.



Sphere free energy

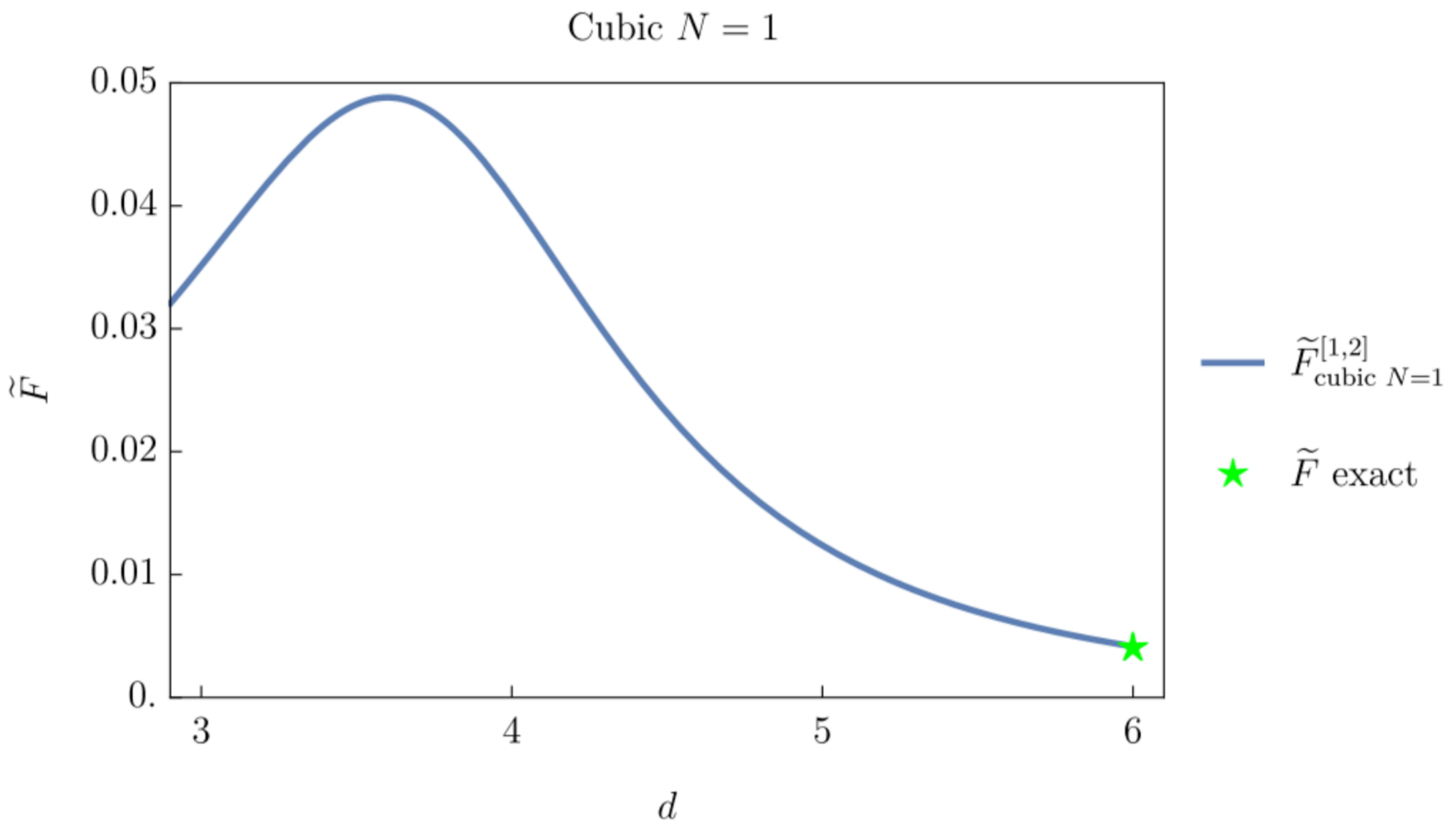
Numerics: N=1 cubic model

- $N=1$ case. In $d=2$, it's a D_5 series version of $M(3,8)$ with central charge $c=-\frac{21}{4}$ [Fei, Giombi, Klebanov, Tarnopolsky'14; Klebanov, Narovlansky, Sun, Tarnopolsky'22; Katsevich, Klebanov, Sun'24]. Results of $6-\epsilon$-expansion:

$$\tilde{F}_{\text{cubic N=1}}(d)=\begin{cases}-\frac{7\pi}{8}, & d=2,\\ 2\tilde{F}_{\text{free}}(6-\epsilon)+\frac{37\pi\epsilon^2}{479040}+\frac{180905801\pi\epsilon^3}{1789221585600}+\mathcal{O}(\epsilon^4), & d=6-\epsilon\,.\end{cases}$$

- The $[1,2]$ Pade has no poles:

$$\tilde{F}^{[1,2]}_{\text{cubic N=1}}=\frac{0.00415555+0.00100299\epsilon}{1-0.741842\epsilon+0.158829\epsilon^2}\,.$$



- In [Fei, Giombi, Klebanov, Tarnopolsky'14; Klebanov, Narovlansky, Sun, Tarnopolsky'22; Katsevich, Klebanov, Sun'24] the RG flow between D series $M(3,10)+\phi_{1,7}\to M(3,8)$ was studied. A D_6 version of $M(3,10)=M(2,5)\otimes M(2,5)$ [Kausch, Takacs, Watts'96; Quella, Runkel, Watts'06].

$$\tilde{F}_{\text{cubic N=1}}-2\tilde{F}_{\text{YL}}=\frac{\pi\epsilon^2}{12934080}+\frac{1018225963\pi\epsilon^3}{3913027607707200}+\mathcal{O}(\epsilon^4)\,.$$

F-theorem violates.

GL description of minimal models

- A series version of unitary models [Zamolodchikov'86]:

$$S_{m+1,m+2} = \int d^d x \left(\frac{1}{2}(\partial_\mu \phi)^2 + \frac{g}{(2m)!}\phi^{2m} \right), \quad g \in \mathbb{R}\,.$$

$\phi \sim \phi_{2,2}$ with holomorphic dimension $h_{2,2} = \dfrac{3}{4(m+1)(m+2)}$.

- For non-unitary models $(2,4m+1)$ [Fisher'78; Cardy'85; Amoruso'16; Zambelli, Zanusso'16; Lencses, Miscioscia, Mussardo, Takacs'24]:

$$S_{2,4m+1} = \int d^d x \left(\frac{1}{2}(\partial_\mu \phi)^2 + \frac{g}{(2m+1)!}\phi^{2m+1} \right), \quad g \in i\mathbb{R}\,.$$

$\phi \sim \phi_{1,2m}$ with holomorphic dimension $h_{2,2} = -m\dfrac{2m-1}{4m+1}$ [Amoruso'16]. $\mathscr{PT}$-symmetry: $\phi \to -\phi$, $i \to -i$.

- For D series versions of $M(3,8)$ and $M(3,10)$ [Fei, Giombi, Klebanov, Tarnopolsky'14; Klebanov, Narovlansky, Sun, Tarnopolsky'22; Katsevich, Klebanov, Sun'24] two-field action:

$$S^D_{3,3\cdot 3\pm 1} = \int d^d x \left(\frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{g_1}{2}\sigma\phi^2 + \frac{g_2}{6}\sigma^3 \right), \quad g_1, g_2 \in i\mathbb{R}\,.$$

$\mathscr{PT}$-symmetry: $\sigma \to -\sigma$, $i \to -i$ and $\mathbb{Z}_2$-symmetry: $\phi \to -\phi$.

- Primary 1-spin field:

$$J_\mu = \sigma\partial_\mu\phi - \phi\partial_\mu\sigma.$$

GL description of minimal models

- Nakayama and Tanaka RG flows [Nakayama, Tanaka'24]:

$$M(kq+I,q)+\phi_{1,2k+1} \to M(kq-I,q)\,.$$

- In particular, there are flows:

$$M(3q-1,q)+i\phi_{1,5} \to M(q+1,q),$$

$$M(3q+1,q)+\phi_{1,7} \to M(3q-1,q)\,.$$

$(3,8)$	$\phi_{1,1}$	$\phi_{1,3}$	$\phi^-_{1,4}$	$\phi_{1,5}$	$\phi_{1,7}$
h	0	$-\frac{1}{4}$	$-\frac{3}{32}$	$\frac{1}{4}$	$\frac{3}{2}$
$\mathbb{Z}_2$	even	even	odd	even	even
$\mathcal{PT}$	even	odd	even	odd	even
GL	1	σ	ϕ	$i\sigma^2+i\phi^2$	$i\phi^2\sigma+i\sigma^3$

$(3,10)$	$\phi_{1,1}$	$\phi_{1,3}$	$\phi^+_{1,5}$	$\phi^-_{1,5}$	$\phi_{1,7}$	$\phi_{1,9}$
h	0	$-\frac{2}{5}$	$-\frac{1}{5}$	$-\frac{1}{5}$	$\frac{3}{5}$	2
$\mathbb{Z}_2$	even	even	even	odd	even	even
$\mathcal{PT}$	even	even	odd	even	even	even
GL	1	$\phi_1\phi_2$	$\phi_1+\phi_2$	$\phi_1-\phi_2$	$i\phi_1\phi_2(\phi_1+\phi_2)$	$T_{1\mu\nu}T_2^{\mu\nu}$

where $\phi_1=(\sigma+\phi)/\sqrt{2}$, $\phi_2=(\sigma-\phi)/\sqrt{2}$.

$$Z^{D_5}_{3,8}=|\chi_{1,1}|^2+|\chi_{1,3}|^2+|\chi_{1,5}|^2+|\chi_{1,7}|^2+|\chi_{1,4}|^2+\chi_{1,2}\bar{\chi}_{1,6}+\chi_{1,6}\bar{\chi}_{1,2}\,,$$

$$Z^{D_6}_{3,10}=|\chi_{1,1}+\chi_{1,9}|^2+|\chi_{1,3}+\chi_{1,7}|^2+2\,|\chi_{1,5}|^2\,.$$

- For odd q for D series versions of $M(q,3q\pm 1)$ [Katsevich, Klebanov, Sun'24]:

$$S^D_{q,3q\pm1}=\int d^dx\left(\frac{1}{2}(\partial_\mu\phi)^2+\frac{1}{2}(\partial_\mu\sigma)^2+\frac{g_1}{(q-1)!}\sigma\phi^{q-1}+\frac{g_2}{6(q-3)!}\sigma^3\phi^{q-3}+\ldots+\frac{g_{(q+1)/2}}{q!}\sigma^q\right).$$

Discussion

- Other theories (quintic, fermionic, …).
- Generalization of c_{eff}-theorem in higher dimensions.