Problem solutions
EP "Quantum field theory, string theory and mathematical physics"

Group-theoretical approach in integrable systems
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Содержание

1 Integrable systems and Lax pairs. Symplectic manifolds. 11

2 Vector fields, Lie groups actions, coadjoint orbits. 20

3 Hamiltonian reduction, projection method 31

4 Classical r-matrix structure 41

5 Integrable systems related to Lie algebras 50

6 Integrable systems related to infinite-dimensional Lie algebras 65

Theoretical minimum

Poisson manifold, symplectic form, integrable systems and Lax pair.

- Lagrangian mechanics:
$$S = \int L(q, \dot{q}, t) dt, \quad \delta S = 0 \tag{1}$$
Equations of motion:
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \tag{2}$$

- Hamiltonian mechanics:
$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \quad H(p, q, t) = \sum_{i=1}^{n} p_i \dot{q}_i - L(\dot{q}(q, p, t), q, t) \tag{3}$$
Equations of motion:
$$\begin{cases} \dot{p}_i = -\frac{\partial H}{\partial q_i}, \\ \dot{q}_i = \frac{\partial H}{\partial p_i}. \end{cases} \tag{4}$$

Definition 1. *Poisson bracket* $\{\cdot, \cdot\}$ is a map $C^\infty(M) \times C^\infty(M) \to C^\infty(M)$, which satisfies

1. Anticommutativity: $\{f, g\} = -\{g, f\}$.
2. Bilinearity: $\{\alpha f + \beta g, h\} = \alpha\{f, h\} + \beta\{g, h\}$, $\{f, \alpha g + \beta h\} = \alpha\{f, g\} + \beta\{f, h\}$.
3. Jacobi identity: $\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0$.
4. Leibnitz rule: $\{fg, h\} = f\{g, h\} + \{f, h\}g$.

Definition 2. Smooth manifold M, on which the Poisson bracket is given, is called *a Poisson manifold.*

In canonical coordinates (q_i, p_i):
$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} \right) \tag{5}$$
$$\{p_i, q_j\} = \delta_{ij}, \quad \{q_i, q_j\} = \{p_i, p_j\} = 0 \tag{6}$$
$$\frac{df(p, q, t)}{dt} = \frac{\partial f}{\partial p}\frac{dp}{dt} + \frac{\partial f}{\partial q}\frac{dq}{dt} + \frac{\partial f}{\partial t} = -\frac{\partial f}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial f}{\partial q}\frac{\partial H}{\partial p} + \frac{\partial f}{\partial t} = \{H, f\} + \frac{\partial f}{\partial t} \tag{7}$$
$$\dot{f}(p, q) = \{H, f\} \tag{8}$$

Definition 3. *Symplectic form* is a differential form $\omega \in \Omega^2(M)$, such that

- ω is closed $d\omega = 0$.
- ω is non-degenerate in every point of M ($\forall x \in M \ \forall \xi \neq 0 \hookrightarrow \exists \eta : \omega(\xi, \eta) \neq 0$).

Definition 4. *Poisson bracket* $\{\cdot, \cdot\}$ on (M, ω) is a bilinear operation on differentiable functions, such that
$$\{f, g\} = \omega(v_f, v_g) \tag{9}$$

Definition 5. Integrals of motion F_i and F_j are *integrals in involution*, if $\{F_i, F_j\}$.

Definition 6. *Integrable hamiltonian system* on M ($\dim = 2n$) is a collection of n independent integrals of motion in involution.

Examples:

1. Oscillator $H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}$.

$$\begin{cases} \dot{p} = -m\omega^2 q, \\ \dot{q} = \frac{p}{m}. \end{cases} \to \begin{cases} \ddot{q} + \omega^2 q = 0, \\ \dot{q} = \frac{p}{m}. \end{cases} \to \begin{cases} q(t) = A \sin \omega t + B \cos \omega t, \\ p(t) = m\omega (A \cos \omega t - B \sin \omega t). \end{cases} \tag{10}$$

2. Central field (Kepler problem) $H = \sum\limits_{i=1}^{3} \frac{p_i^2}{2m} + V(r)$.

$$\begin{cases} \dot{p}_i = -\frac{\partial V}{\partial q_i}, \\ \dot{q}_i = \frac{p_i}{m}. \end{cases} \tag{11}$$

Spherical coordinates:

$$\begin{cases} x_1 = r \sin \theta \cos \varphi, \\ x_2 = r \sin \theta \sin \varphi, \\ x_3 = r \cos \theta. \end{cases} \tag{12}$$

Angular momentum $J_{ij} = q_i p_j - q_j p_i$. Integrals of motion:

$$\begin{cases} H = \frac{1}{2} \left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2 \theta} \right) + V(r), \\ J^2 = J_{12}^2 + J_{13}^2 + J_{23}^2 = p_\theta^2 + \frac{p_\varphi^2}{\sin^2 \theta}, \\ J_{12} = p_\varphi. \end{cases} \tag{13}$$

Theorem 1 (Liouville)**.** *The solution for hamiltonian integrable system is obtained by "quadrature".*

Proof. □

Suppose M_f is connected and compact, $M_f \simeq T^n = S^1 \times ... \times S^1$.
Action variables $I_j = \frac{1}{2\pi} \int\limits_{C_j} \alpha$, angle $\theta_k = \frac{\partial S}{\partial I_k}$.

$$\int\limits_{C_j} d\theta_k = \frac{\partial}{\partial I_k} \int\limits_{C_j} dS = \frac{\partial}{\partial I_k} \int\limits_{C_j} \sum_{i=1}^{n} \left(\frac{\partial S}{\partial q_i} dq_i + \frac{\partial S}{\partial I_i} dI_i \right) = \frac{\partial}{\partial I_k} \int\limits_{C_j} \sum_{i=1}^{n} \frac{\partial S}{\partial q_i} dq_i = \\ = \frac{\partial}{\partial I_k} \int\limits_{C_j} \alpha = 2\pi \delta_{jk} \tag{14}$$

How to find such n integrals? There isn't algorithm, but suppose that equations of motion could be written in form

$$\dot{L} = [L, M], \tag{15}$$

where $L, M \in \mathrm{Mat}_{n \times n}$ – *Lax pair.*

Proposition 2. *If equations of motion are $\dot{L} = [L, M]$, then integrals $I_k = \frac{1}{k} TrL^k$ are conserved.*

Proof.

$$\frac{d}{dt}I_k = \frac{1}{k}\mathrm{Tr}(\dot{L}L^{k-1} + ... + L^{k-1}\dot{L}_k) = \mathrm{Tr}(\dot{L}L^{k-1}) = \mathrm{Tr}(LML^{k-1} - ML^k) = \\ = \mathrm{Tr}(ML^k - ML^k) = 0 \quad (16)$$

□

Example:
Calogero-Moser system of interacting particles on a line

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 - \frac{\nu^2}{2}\sum_{i\neq j}\frac{1}{(q_i - q_j)^2} \quad (17)$$

Poisson brackets are canonical. Equations of motion:

$$\begin{cases} \dot{p}_i = -2\nu^2 \sum_{i\neq j} \frac{1}{(q_i-q_j)^2}, \\ \dot{q}_i = p_i \end{cases} \quad (18)$$

Lax matrices can be chosen in the form:

$$L_{ii} = p_i, \quad L_{ij} = \frac{\nu}{q_i - q_j}, \quad i \neq j \quad (19)$$

$$M_{ii} = -\nu\sum_{k\neq i}\frac{1}{(q_i - q_k)^2}, \quad M_{ij} = -\frac{\nu}{(q_i - q_j)^2}, \quad i \neq j \quad (20)$$

So Calogero system has additional integrals of motion

$$\mathrm{Tr}L = \sum_i p_i = P, \quad \frac{1}{2}\mathrm{Tr}L^2 = \frac{1}{2}\sum_{i=1}^{n} p_i^2 - \frac{\nu^2}{2}\sum_{i\neq j}\frac{1}{(q_i - q_j)^2} = H \quad (21)$$

$$\frac{1}{3}\mathrm{Tr}L^3 = \frac{1}{3}\sum_{i=1}^{n} p_i^3 - \nu^2\sum_{i\neq j}\frac{p_i}{(q_i - q_j)^2} \quad (22)$$

The last integral is nontrivial. How to construct such Lax representations and understend if a system is integrable?
Idea: use symmetries to get integrals of motion.

Symplectic geometry, Hamiltonian approach to symmetry.

Definition 7. *Symplectic manifold* is a pair (M, ω), such that

- M – smooth manifold.
- $\omega \in \Omega^2(M)$ – symplectic form.

Dimension of symplectic manifold is even. Symplectic manifold is Poisson, but not vice versa.
Examples:
Let M be a Poisson manifold, then from bilinearity and Leibntz rule in local coordinates

$$\{f, g\}(x) = \sum_{i,j}\pi_{ij}(x)\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} \quad (23)$$

- Anticommutativity: $\pi_{ij}(x) = -\pi_{ji}(x)$.

- Jacobi identity: $\pi_{ik}(x)\frac{\partial}{\partial x_k}\pi_{jl}(x) + \pi_{lk}(x)\frac{\partial}{\partial x_k}\pi_{ij}(x) + \pi_{jk}(x)\frac{\partial}{\partial x_k}\pi_{li}(x) = 0$.

Assume π_{ij} be an invertible matrix. One can define a symplectic form $\omega = -\sum\limits_{i\neq j}(\pi^{-1})_{ij}dx_i \wedge dx_j$.
However, if the Poisson brackets have nontrivial kernel, i.e. there exists a function f: $\{f, \cdot\} = 0$, then this Poisson manifold isn't symplectic.
One can fix the levels of all functions in the kernel of the Poisson brackets and define symplectic structures on these level manifolds called *the symplectic leaves.*
Consider canonical symplectic structure on a cotangent bundle T^*M. Let $\pi : T^*M \to M$ be a projection map

$$\pi(x, \beta) = x, \quad x \in M, \quad \beta \in T^*_x M \tag{24}$$

Choose a point $x \in M$ and a chart $U \subset M$: $x \in U$. Choose local coordinates $q_1, ..., q_n(U)$.
$(dq_1)_x$, ..., $(dq_n)_x$ – basis in $T^*_x M$, then any $\beta \in T^*_x M$ has a form

$$\beta = \sum_{i=1}^{n} p_i(x, \beta)(dq_i)_x \tag{25}$$

So, $p_1, ..., p_n, q_1, ..., q_n$ – basis in $\pi^{-1}(U)$. Using this coordinates, one can write Liouville 1-form

$$\alpha = \sum_{i=1}^{n} p_i dq_i \to \omega = d\alpha, \quad d\omega = d^2\alpha = 0 \tag{26}$$

Theorem 3 (Darboux)**.** *Let (M, ω) – symplectic manifold and $x \in M$, then one can introduce locally around $x \in M$ a system of local coordinates (p_i, q_i), such that $\omega = \sum\limits_{i=1}^{n} dp_i \wedge dq_i$.*

Symplectic form – non-degenerate 2-form, so there is $1:1$ mapping $\Omega^1(M) \leftrightarrow \mathrm{Vect}(M)$ (vector fields on M). Contaction operation

$$\lambda = (\omega, \cdot) = i_v\omega \tag{27}$$

Vector field $v \in \mathrm{Vect}(M)$ defines a local one-parameter group of diffeomorphims

$$\exp(vt) : \mathbb{R} \times M \to M, \quad t \in \mathbb{R}, \quad x \in M \tag{28}$$

$$\begin{cases} \exp(v0)(x) = x, \\ \frac{d}{dt}(\exp(vt)(x)) = v(\exp(vt)(x)) \end{cases} \tag{29}$$

Group properties:

- $\exp(v(t + s)) = \exp(vt)\exp(vs)$.

- $\exp(v(-t)) = (\exp vt)^{-1}$.

Definition 8. Let $v \in \mathrm{Vect}(M)$, then $\forall\lambda \in \Omega^\bullet(M)$ *Lie derivative* L_v is

$$L_v\lambda = \frac{d}{dt}\left(\exp(vt)_*\lambda\right)|_{t=0} \tag{30}$$

Properies of the Lie derivative:

- Cartan formula: $L_v = di_v + i_v d$.

- $L_{[v,u]} = [L_v, L_u]$.

- $[L_v, i_u] = i_{[v,u]}$.

- $L_v\omega(v_1, ..., v_k) = (L_v\omega)(v_1, ..., v_k) + \sum\limits_{i=1}^{k} \omega(v_1, ..., [v, v_i], ..., v_k)$.

Hamiltonian and symplectic vector fields. Lie groups acting on manifolds

Definition 9. Let (M, ω) – a symplectic manifold. Vector field v_H is *hamiltonian* if

$$i_{v_H}\omega = -dH \tag{31}$$

Definition 10. Vector field v is *symplectic* if

$$L_v\omega = 0 \tag{32}$$

Using Cartan formula

$$L_v\omega = di_v\omega + i_v d\omega = d(i_v\omega) = 0 \tag{33}$$

So, $i_v\omega$ is closed form.

Proposition 4. *Any hamiltonian vector field is symplectic.*

Proof. Let v_f is hamiltonian field, then

$$i_{v_f}\omega = -df \tag{34}$$

$i_{v_f}\omega$ is exact form, then $i_{v_f}\omega$ is closed form

$$di_{v_f}\omega = -d^2 f = 0 \tag{35}$$

□

Example of symplectic but not hamiltonian vector field:
Symplectic manifold (M, ω): $M = T^2 = S^1 \times S^1$, $\omega = d\varphi_1 \wedge d\varphi_2$. Symplectic vector field: $v = \frac{\partial}{\partial \varphi_1}$.

$$i_v\omega = d\varphi_2 \to d(i_v\omega) = 0 \tag{36}$$

φ_2 isn't a function on M, so $H \neq \varphi_2$ and v isn't hamiltonian vector field.
If $H^1(M) = 0$, then any symplectic vector field is hamiltonian.

Proposition 5. *If v,u are symplectic vector fields, then their commutator $[v, u]$ is a Hamiltonian vector with hamiltonian $\omega(v, u)$.*

Proof.

$$i_{[v,u]}\omega = L_v i_u\omega - i_u L_v\omega = (di_v + i_v d)i_u\omega = d(i_v i_u\omega) + i_v d(i_u\omega) = d(\omega(u, v)) = -d(\omega(v, u)) \tag{37}$$

$$i_{[v,u]}\omega = -dH, \quad H = \omega(v, u) \tag{38}$$

□

Therefore, if $f, g \in \Omega^0(M)$ and v_f, v_g – coresponding vector fields then $[v_f, v_g] = v_{\{f,g\}}$.
Properties of symplectic vector field:

- $\omega([v_1, v_2], v_3) + \omega([v_2, v_3], v_1) + \omega([v_3, v_1], v_2) = 0$.
- $L_{v_1}\omega(v_2, v_3) + L_{v_2}\omega(v_3, v_1) + L_{v_3}\omega(v_1, v_2) = 0$.
- $\omega([v_1, v_2], v_3) = L_{v_3}\omega(v_1, v_2)$.

Let M is a smooth manifold, then T^*M is symplectic. Let v – a vector field on M, then there exists a unique vector field $\tilde{v}$ on T^*M, which lifts the flow of v. This field is Hamiltonian with

$$H = i_{\tilde{v}}\alpha = \sum p_i v_i(q) \tag{39}$$

Consider Lie groups acting on manifolds. Let G be a Lie group and M – smooth manifold. Action:

$$. : G \times M \to M, \quad (g, x) \to g.x \tag{40}$$

Let $\mathfrak{g} = \mathrm{Lie}(G)$ – Lie algebra of group G.
Consider an element $\xi \in \mathfrak{g}$ and one-parametric subgroup in G, generated by this element $\{e^{\xi t}, t \in \mathbb{R}\}$. This allows to construct *a fundamental vector field* on M:

$$v_\xi(x) = \frac{d}{dt}(e^{\xi t}.x)|_{t=0} \tag{41}$$

Any Lie group can act on itself:

- Left action (left multiplication):
$$(g, h) \to g.h = gh \tag{42}$$
- Right action (right multiplication):
$$(g, h) \to g.h = hg^{-1} \tag{43}$$
- Conjugation:
$$(g, h) \to g.h = ghg^{-1} \tag{44}$$

Definition 11. The derivative at the identity element of G gives an invertible linear map

$$\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}, \quad \mathrm{Ad}_g(\xi) = \frac{d}{dt}(g e^{\xi t} g^{-1})|_{t=0} \tag{45}$$

and defines *the adjoint representation*

$$\mathrm{Ad} : G \to \mathrm{End}(\mathfrak{g}), \quad \mathrm{Ad}(g) = \mathrm{Ad}_g \tag{46}$$

Definition 12. One can also define *the coadjoint representation* of G Ad^* on dual to its Lie algebra $\mathfrak{g}^*$:

$$\mathrm{Ad}^*_g : \mathfrak{g}^* \to \mathfrak{g}^*, \quad \langle \mathrm{Ad}^*_g(\varphi), \xi \rangle = \langle \varphi, \mathrm{Ad}_{g^{-1}}(\xi) \rangle \tag{47}$$

g^{-1} here to have homomorphisms:

$$\mathrm{Ad}^*_g \mathrm{Ad}^*_h = \mathrm{Ad}^*_{gh}, \quad \mathrm{Ad}_g \mathrm{Ad}_h = \mathrm{Ad}_{gh} \tag{48}$$

$$\mathrm{Ad}^* : G \to \mathrm{End}(\mathfrak{g}^*), \quad \mathrm{Ad}^*(g) = \mathrm{Ad}^*_g \tag{49}$$

Definition 13. *Adjoint and coadjoint representations of Lie algebra* $\mathfrak{g}$ can be defined as the infinitesimal versions:

$$\mathrm{ad}_\xi : \mathfrak{g} \to \mathfrak{g}, \quad \mathrm{ad}_\xi(\eta) = [\xi, \eta] \tag{50}$$

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}), \quad \mathrm{ad}(\xi) = \mathrm{ad}_\xi \tag{51}$$

$$\mathrm{ad}^*_\xi : \mathfrak{g}^* \to \mathfrak{g}^*, \quad \langle \mathrm{ad}^*_\xi(\varphi), \eta \rangle = \langle \varphi, -\mathrm{ad}_\xi(\eta) \rangle = -\langle \varphi, [\xi, \eta] \rangle \tag{52}$$

These operations are also homomorphisms:

$$[\mathrm{ad}_\xi, \mathrm{ad}_\eta] = \mathrm{ad}_{[\xi,\eta]}, \quad [\mathrm{ad}^*_\xi, \mathrm{ad}^*_\eta] = \mathrm{ad}^*_{[\xi,\eta]} \tag{53}$$

Consider a Lie algebra $\mathfrak{g}$ and its dual $\mathfrak{g}^*$. $(\mathfrak{g}^*)^* = \mathfrak{g}$, so linear functions on $\mathfrak{g}^*$ are elements of $\mathfrak{g}$ and one can naturally define Poisson brackets on $\mathfrak{g}^*$:

$$\{f_\xi, f_\eta\}(\varphi) = \langle \varphi, [\xi, \eta] \rangle \tag{54}$$

Also we should claim Leibniz rule and define Poisson bracket for polynomials on $\mathfrak{g}^*$. Thus, $\mathfrak{g}^*$ is a Poisson manifold.
For two functions $f, g : \mathfrak{g}^* \to \mathbb{R}$

$$\{f, g\}(\varphi) = \langle \varphi, [df, dg] \rangle \tag{55}$$

Example: $\mathfrak{g} = \mathfrak{so}(3)$.
Commutators:

$$[S_i, S_j] = \epsilon_{ijk} S_k \tag{56}$$

Poisson brackets on $\mathfrak{so}(3)^* \simeq \mathbb{R}^3$:

$$\{S_i, S_j\} = \epsilon_{ijk} S_k \tag{57}$$

$\mathfrak{g}^*$ isn't a symplectic manifold, Poisson bracket is degenerate.

$$C = S_1^2 + S_2^2 + S_3^2, \quad \{C, S_i\} = 0 \tag{58}$$

Proposition 6. *Kernel of the Poisson bracket is the set of Ad^*-invariant functions*

$$f : \mathfrak{g}^* \to \mathbb{R}, \quad f(Ad^*_g(\varphi)) = f(\varphi), \quad \forall g \in G, \varphi \in \mathfrak{g}^* \tag{59}$$

Proof. Let $g = e^{\xi t}$, then

$$\mathrm{Ad}^*_g(\varphi) = \varphi + t\mathrm{ad}^*_\xi(\varphi), \quad t \to 0 \tag{60}$$

$$f(\mathrm{Ad}^*_g(\varphi)) = f(\varphi) + t \langle \mathrm{ad}^*_\xi(\varphi), df \rangle \to \langle \mathrm{ad}^*_\xi(\varphi), df \rangle = 0 \tag{61}$$

$$\langle \mathrm{ad}^*_\xi(\varphi), df \rangle = - \langle \varphi, [\xi, df] \rangle = 0 \tag{62}$$

Then for all linear functions f on $\mathfrak{g}^*$

$$\{f, \cdot\} = 0 \tag{63}$$

$\square$

In order to construct a symplectic manifold, one needs to fix these Ad^*-invariant functions. Consider a coadjoint orbit of G (Ad^*-invariant functions are constants on the coadjoint orbits).
Coadjoint orbit of an element $\varphi \in \mathfrak{g}^* : \mathcal{O}_\varphi \equiv \mathrm{Ad}^*_G = \{\mathrm{Ad}^*_g(\varphi) | g \in G\}$.

Integrable systems related to semisimple Lie algebras

Definition 14. *Lie algebra* $\mathfrak{g}$ – a vector space with the commutation operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} :$ $\forall \xi, \eta, \lambda \in \mathfrak{g}\ \forall a, b \in \mathbb{C} \hookrightarrow$

- linear: $[a\xi + b\eta, \lambda] = a[\xi, \lambda] + b[\eta, \lambda]$.
- skew-symmetric: $[\xi, \eta] = -[\eta, \xi]$.
- Jacobi identity: $[\xi, [\eta, \lambda]] + [\eta, [\lambda, \xi]] + [\lambda, [\xi, \eta]] = 0$.

Denote the basic elements in $\mathfrak{g}$ as t_a, $a = 1, ..., \dim\mathfrak{g}$ and Lie brackets as $[t_a, t_b] = \sum_c f_{ab}^c t_c$, f_{ab}^c – structure constants.
The adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}$:

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}), \quad \mathrm{ad} : \xi \to \mathrm{ad}_\xi \tag{64}$$

$$\mathrm{ad}_\xi(\eta) = [\xi, \eta] \tag{65}$$

The matrix elements of the adjoint representation are f_{ab}^c:

$$\mathrm{ad}_{t_a}(t_b) = [t_a, t_b] = \sum_c f_{ab}^c t_c \to (\mathrm{ad}_{t_a})_b^c = f_{ab}^c \tag{66}$$

One can use the adjoint representation to define a natural bilinear product on $\mathfrak{g}$ – Killing form $(\cdot, \cdot)$:

$$(\xi, \eta) = \mathrm{Tr}(\mathrm{ad}_\xi \mathrm{ad}_\eta) \tag{67}$$

In this basis

$$(\xi, \eta) = \sum_{a,b} \xi^a \eta^b \mathrm{Tr}(\mathrm{ad}_{t_a} \mathrm{ad}_{t_b}) = \sum_{a,b} \xi^a \eta^b \sum_{c,d} f_{ac}^d f_{bd}^c \tag{68}$$

The Killing form is invariant: $(\xi, [\eta, \lambda]) = ([\xi, \eta], \lambda)$:

$$(\xi, [\eta, \lambda]) = \mathrm{Tr}(\mathrm{ad}_\xi \mathrm{ad}_{[\eta,\lambda]}) = \mathrm{Tr}(\mathrm{ad}_\xi [\mathrm{ad}_\eta, \mathrm{ad}_\lambda]) = \mathrm{Tr}([\mathrm{ad}_\xi, \mathrm{ad}_\eta]\mathrm{ad}_\lambda) = ([\xi, \eta], \lambda) \tag{69}$$

Definition 15. A subspace $I \subset \mathfrak{g}$ is called *an ideal* if

$$\forall \xi \in I, \eta \in \mathfrak{g} \hookrightarrow [\xi, \eta] \in I \tag{70}$$

Definition 16. An ideal I is called *abelian* if $\forall \xi, \eta \in I \hookrightarrow [\xi, \eta] = 0$.

Definition 17. A Lie algebra $\mathfrak{g}$ is *semisimple* if it doesn't contain any nontrivial abelian ideal.

Definition 18. A Lie algebra $\mathfrak{g}$ is *simple* if it's nonabelian and its ideals are only $\{0\}$ and $\mathfrak{g}$.

A semisimple Lie algebra is a direct sum of simple ones.

Theorem 7 (Cartan criterion). *$\mathfrak{g}$ is semisimple $\Leftrightarrow$ Killing form is nondegenerate.*

For any simesimple $\mathfrak{g}$ one has $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ (it's important to get examples of not semisimple).
Examples of semisimple Lie algebras: classical Lie algebras $\mathfrak{sl}_n$, $\mathfrak{so}_n$.
Not semisimple Lie algebras: $\{p, q, c : [p, q] = c, [p, c] = [q, c] = 0\}$, $\mathfrak{gl}_n$: $[1, \cdot] = 0$ and $\nexists x, y : [x, y] = 1$. These algebras have nontrivial abelian ideals.

Definition 19. Let $\mathfrak{g}$ be a semisimple Lie algebra, $\xi \in \mathfrak{g}$ is *a semisimple* element if the matrix ad_ξ can be diagonalized.

Definition 20. Let $\mathfrak{g}$ be a semisimple Lie algebra. *A Cartan subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ is a maximal abelian subalgebra: $\forall \xi \in \mathfrak{h}$ is semisimple.
The dimension of $\mathfrak{h}$ is called the rank of $\mathfrak{g}$: $\mathrm{rk}\mathfrak{g} = \dim \mathfrak{h}$.

Definition 21. Denote the basis in $\mathfrak{h}$ as $h_1, ..., h_r$, $r = \mathrm{rk}\,\mathfrak{g}$. All these elements are semisimple and commute with each other, so ad_h can be diagonalized simultaneously and have a basis of common eigenvectors $e_\alpha \in \mathfrak{g}$:

$$\mathrm{ad}_h e_\alpha = \alpha(h) e_\alpha, \quad \forall h \in \mathfrak{h} \tag{71}$$

This defines a map $\alpha : \mathfrak{h} \to \mathbb{C}$, $\alpha \in \mathfrak{h}^*$ – this linear form is called *the root* of the Lie algebra $\mathfrak{g}$. Denote the set of all roots as Δ.

If α is a root, then $-\alpha$ is also a root. and if $\alpha \neq 0$, then the eigenspace is one-dimensional. This provides the Cartan decomposition of $\mathfrak{g}$: $\{h_i\}$ – basis in $\mathfrak{h}$, then $\{h_i, e_\alpha\}$ form a basis in $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \tag{72}$$

The basic example: $\mathfrak{g} = \mathfrak{sl}_n$.
$\mathfrak{h}$ is the Cartan subalgebra of traceless diagonal matrices. Denote $\lambda_i \in \mathfrak{h}^*$:

$$\lambda_i(\mathrm{diag}(a_1, a_2, ..., a_n)) = a_i \tag{73}$$

Then the space of roots is $\Delta = \{\lambda_i - \lambda_j | 1 \leq i \leq n, 1 \leq j \leq n, i \neq j\}$ and the decomposition is

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{i \neq j} \mathfrak{g}_{\lambda_i - \lambda_j}, \quad \mathfrak{g}_{\lambda_i - \lambda_j} = \langle E_{ij} \rangle \tag{74}$$

1 Integrable systems and Lax pairs. Symplectic manifolds.

1. **Lax pair for oscillators.**
Find a Lax pair representation $\dot{L} = [L, M]$ for a one-dimensional harmonic oscillator

$$H = \frac{p^2}{2} + \frac{\omega^2 x^2}{2} \tag{75}$$

Use this ansatz for L-operator

$$L = \begin{pmatrix} p & f(q) \\ f(q) & -p \end{pmatrix} \tag{76}$$

How does the answer change if the anharmonic oscillator is considered instead of the harmonic one?
Solution.
Equations of motion:

$$\begin{cases} \dot{p} = -\frac{\partial H}{\partial q} = -\omega^2 q, \\ \dot{q} = \frac{\partial H}{\partial p} = p \end{cases} \rightarrow \dot{L} = \begin{pmatrix} \dot{p} & \frac{\partial f}{\partial q}\dot{q} \\ \frac{\partial f}{\partial q}\dot{q} & -\dot{p} \end{pmatrix} = \begin{pmatrix} -\omega^2 q & \frac{\partial f}{\partial q} p \\ \frac{\partial f}{\partial q} p & \omega^2 q \end{pmatrix} \tag{77}$$

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$[L, M] = LM - ML = \begin{pmatrix} (c-b)f(q) & (d-a)f(q) + 2bp \\ (a-d)f(q) - 2cp & (b-c)f(q) \end{pmatrix} \tag{78}$$

Comparing (77) and (78), we obtain

$$a = d = 0, \quad b = -c = \frac{1}{2}\frac{\partial f}{\partial q} = \frac{\omega^2 q}{2f(q)} \tag{79}$$

$$\frac{\partial f}{\partial q} = \frac{\omega^2 q}{f(q)} \rightarrow f(q) = \pm\sqrt{\omega^2 q^2 + C} \tag{80}$$

Let $C = 0$, then

$$f(q) = \omega q \rightarrow b = -c = \frac{\omega}{2} \tag{81}$$

Lax pair:

$$\boxed{L = \begin{pmatrix} p & \omega q \\ \omega q & -p \end{pmatrix}, \quad M = \begin{pmatrix} 0 & \frac{\omega}{2} \\ -\frac{\omega}{2} & 0 \end{pmatrix}} \tag{82}$$

Anharmonic oscillator:

$$H = \frac{p^2}{2} + V(q) \tag{83}$$

Equations of motion:

$$\begin{cases} \dot{p} = -\frac{\partial H}{\partial q} = -V'(q), \\ \dot{q} = \frac{\partial H}{\partial p} = p \end{cases} \quad \rightarrow \dot{L} = \begin{pmatrix} \dot{p} & \frac{\partial f}{\partial q}\dot{q} \\ \frac{\partial f}{\partial q}\dot{q} & -\dot{p} \end{pmatrix} = \begin{pmatrix} -V'(q) & \frac{\partial f}{\partial q}p \\ \frac{\partial f}{\partial q}p & V'(q) \end{pmatrix} \tag{84}$$

Comparing (78) and (84), we obtain

$$a = d = 0, \quad b = -c = \frac{1}{2}\frac{\partial f}{\partial q} = \frac{\omega^2 q}{2f(q)} \tag{85}$$

$$\frac{\partial f}{\partial q} = \frac{V'(q)}{f(q)} \rightarrow f(q) = \pm\sqrt{2V(q) + C} \tag{86}$$

Let $C = 0$, then

$$f(q) = \sqrt{2V(q)} \rightarrow b = -c = \frac{V'(q)}{2\sqrt{2V(q)}} \tag{87}$$

Lax pair:

$$\boxed{L = \begin{pmatrix} p & \sqrt{2V(q)} \\ \sqrt{2V(q)} & -p \end{pmatrix}, \quad M = \begin{pmatrix} 0 & \frac{V'(q)}{2\sqrt{2V(q)}} \\ -\frac{V'(q)}{2\sqrt{2V(q)}} & 0 \end{pmatrix}} \tag{88}$$

2. **Rational Ruijsenaars-Schneider system.**
Consider a many-body system on a line in coordinates $\{q_i, p_i\}, 1 \leq i \leq n$ with standard Poisson brackets

$$\{p_i, q_j\} = \delta_{ij}, \quad \{q_i, q_j\} = \{p_i, p_j\} = 0 \tag{89}$$

and Hamilton function

$$H = \sum_{i=1}^{n} e^{p_i/c} \prod_{k \neq i} \frac{q_i - q_k + \eta}{q_i - q_k}, \tag{90}$$

where c and η are constants.

- Write the equations of motion for the system in the form $\ddot{q}_i = \ldots$ (It can be useful to find the expression for $\dot{q}_i$ firstly, and express other quantities via $\dot{q}_i$).
- Show that Hamiltonian equations of motion can be presented in the Lax form

$$\dot{L} = [L, M], \tag{91}$$

where matrices L and M are

$$L_{ij} = \frac{e^{p_j/c}}{q_i - q_j + \eta} \prod_{k \neq j} \frac{q_j - q_k + \eta}{q_j - q_k} \tag{92}$$

$$M_{ij} = -\frac{\dot{q}_j}{q_i - q_j}, \quad i \neq j, \tag{93}$$

$$M_{ii} = -\frac{\dot{q}_i}{\eta} + \sum_{k \neq i} \frac{\eta \dot{q}_k}{(q_i - q_k + \eta)(q_i - q_k)} = -\frac{\dot{q}_i}{\eta} + \sum_{k \neq i} \left(\frac{\dot{q}_k}{q_i - q_k} - \frac{\dot{q}_k}{q_i - q_k + \eta} \right) \tag{94}$$

- Let $\eta = \frac{\nu}{c}$. Investigate the limit $c \to \infty$ while ν remains constant.

Solution.

- $$\{f(p), q_i\} = \frac{\partial f}{\partial p_i}, \quad \{f(q), p_i\} = -\frac{\partial f}{\partial q_i} \tag{95}$$

$$\dot{q}_i = \{H, q_i\} = \{e^{p_i/c}, q_i\} \prod_{k\neq i} \frac{q_i - q_k + \eta}{q_i - q_k} = \frac{e^{p_i/c}}{c} \prod_{k\neq i} \frac{q_i - q_k + \eta}{q_i - q_k} \tag{96}$$

$$H = c \sum_{i=1}^{n} \dot{q}_i \tag{97}$$

$$\dot{p}_i = \{H, p_i\} = e^{p_i/c} \left\{ \prod_{k\neq i} \frac{q_i - q_k + \eta}{q_i - q_k}, p_i \right\} + \sum_{j\neq i} e^{p_j/c} \left\{ \prod_{k\neq j} \frac{q_j - q_k + \eta}{q_j - q_k}, p_i \right\} \tag{98}$$

$$\begin{aligned} \left\{ \prod_{k\neq i} \frac{q_i - q_k + \eta}{q_i - q_k}, p_i \right\} &= \prod_{k\neq i,j} \frac{q_i - q_k + \eta}{q_i - q_k} \sum_{j\neq i} \left\{ \frac{q_i - q_j + \eta}{q_i - q_j}, p_i \right\} = \\ = \prod_{k\neq i,j} \frac{q_i - q_k + \eta}{q_i - q_k} \sum_{j\neq i} \frac{\eta}{(q_i - q_j)^2} &= \prod_{k\neq i} \frac{q_i - q_k + \eta}{q_i - q_k} \sum_{j\neq i} \left(\frac{1}{q_i - q_j} - \frac{1}{q_i - q_j + \eta} \right) \end{aligned} \tag{99}$$

$$\begin{aligned} \left\{ \prod_{k\neq j} \frac{q_j - q_k + \eta}{q_j - q_k}, p_i \right\} &= \prod_{k\neq i,j} \frac{q_j - q_k + \eta}{q_j - q_k} \left\{ \frac{q_j - q_i + \eta}{q_j - q_i}, p_i \right\} = \\ = -\prod_{k\neq i,j} \frac{q_j - q_k + \eta}{q_j - q_k} \frac{\eta}{(q_i - q_j)^2} &= -\prod_{k\neq j} \frac{q_j - q_k + \eta}{q_j - q_k} \left(\frac{1}{q_j - q_i} - \frac{1}{q_j - q_i + \eta} \right) \end{aligned} \tag{100}$$

$$\begin{aligned} \dot{p}_i = e^{p_i/c} \prod_{k\neq i} \frac{q_i - q_k + \eta}{q_i - q_k} \sum_{j\neq i} \left(\frac{1}{q_i - q_j} - \frac{1}{q_i - q_j + \eta} \right) - \\ - \sum_{j\neq i} e^{p_j/c} \prod_{k\neq j} \frac{q_j - q_k + \eta}{q_j - q_k} \left(\frac{1}{q_j - q_i} - \frac{1}{q_j - q_i + \eta} \right) = \\ = \sum_{j\neq i} \left(c\dot{q}_i \left(\frac{1}{q_i - q_j} - \frac{1}{q_i - q_j + \eta} \right) + c\dot{q}_j \left(\frac{1}{q_i - q_j} - \frac{1}{q_i - q_j - \eta} \right) \right) \end{aligned} \tag{101}$$

$$\dot{p}_i = \eta c \left(\frac{1}{q_i - q_j} \left(\frac{\dot{q}_i}{q_i - q_j + \eta} + \frac{\dot{q}_j}{q_j - q_i + \eta} \right) \right) \tag{102}$$

$$\begin{aligned} \ddot{q}_i = \frac{d}{dt} \left(\frac{e^{p_i/c}}{c} \prod_{k\neq i} \frac{q_i - q_k + \eta}{q_i - q_k} \right) = \dot{p}_i \frac{e^{p_i/c}}{c^2} \prod_{k\neq i} \frac{q_i - q_k + \eta}{q_i - q_k} + \frac{e^{p_i/c}}{c} \prod_{k\neq i} \frac{q_i - q_k + \eta}{q_i - q_k} \times \\ \times \sum_{k\neq i} \left(\frac{\dot{q}_i - \dot{q}_k}{q_i - q_k + \eta} - \frac{\dot{q}_i - \dot{q}_k}{q_i - q_k} \right) = \frac{\dot{p}_i \dot{q}_i}{c} + \sum_{k\neq i} \dot{q}_i (\dot{q}_i - \dot{q}_k) \left(\frac{1}{q_i - q_k + \eta} - \frac{1}{q_i - q_k} \right) \end{aligned} \tag{103}$$

$$\boxed{\ddot{q}_i = \sum_{k\neq i} \dot{q}_i \dot{q}_j \left(\frac{2}{q_i - q_k} - \frac{1}{q_i - q_k + \eta} - \frac{1}{q_i - q_k + \eta} \right)} \tag{104}$$

- $$L_{ij}=\frac{e^{p_j/c}}{q_i-q_j+\eta}\prod_{k\neq j}\frac{q_j-q_k+\eta}{q_j-q_k}=\frac{c\dot{q}_j}{q_i-q_j+\eta},\quad L_{ii}=\frac{\dot{q}_i}{\eta}\tag{105}$$

$$\begin{aligned}\dot{L}_{ij}=\frac{c\ddot{q}_j}{q_i-q_j+\eta}-\frac{c\dot{q}_j(\dot{q}_i-\dot{q}_j)}{(q_i-q_j+\eta)^2}=\frac{c\dot{q}_i\dot{q}_j}{q_i-q_j+\eta}\left(\frac{2}{q_j-q_i}-\frac{1}{q_j-q_i+\eta}-\frac{1}{q_j-q_i-\eta}\right)+\\+\sum_{k\neq i,j}\frac{c\dot{q}_j\dot{q}_k}{q_i-q_j+\eta}\left(\frac{2}{q_j-q_k}-\frac{1}{q_j-q_k+\eta}-\frac{1}{q_j-q_k-\eta}\right)-\frac{c\dot{q}_j(\dot{q}_i-\dot{q}_j)}{(q_i-q_j+\eta)^2}\end{aligned}\tag{106}$$

$$\begin{aligned}[L,M]_{ij}=(L_{ii}-L_{jj})M_{ij}+(M_{jj}-M_{ii})L_{ij}+\sum_{k=1}^{n}(L_{ik}M_{kj}-M_{ik}L_{kj})=-\frac{\dot{q}_i-\dot{q}_j}{\eta}\frac{c\dot{q}_j}{q_i-q_j}+\\+\frac{c\dot{q}_j}{q_i-q_j+\eta}\frac{\dot{q}_i-\dot{q}_j}{\eta}+\frac{c\dot{q}_j}{q_i-q_j+\eta}\left(\frac{\dot{q}_i}{q_j-q_i}-\frac{\dot{q}_i}{q_j-q_i+\eta}-\frac{\dot{q}_j}{q_j-q_i}+\frac{\dot{q}_j}{q_i-q_j+\eta}\right)+\\+\sum_{k\neq i,j}\frac{c\dot{q}_j\dot{q}_k}{q_i-q_j+\eta}\left(\frac{1}{q_j-q_k}-\frac{1}{q_j-q_k+\eta}-\frac{1}{q_i-q_k}+\frac{1}{q_i-q_k+\eta}\right)+\\+\sum_{k\neq i,j}\left(\frac{c\dot{q}_j\dot{q}_k}{(q_i-q_k)(q_k-q_j+\eta)}-\frac{c\dot{q}_j\dot{q}_k}{(q_k-q_j)(q_i-q_k+\eta)}\right)\end{aligned}\tag{107}$$

$$\frac{1}{(q_i-q_k)(q_k-q_j+\eta)}=\left(\frac{1}{q_i-q_k}+\frac{1}{q_k-q_j+\eta}\right)\frac{1}{q_i-q_j+\eta}\tag{108}$$

$$\frac{1}{(q_k-q_j)(q_i-q_k+\eta)}=\left(\frac{1}{q_k-q_j}+\frac{1}{q_i-q_k+\eta}\right)\frac{1}{q_i-q_j+\eta}\tag{109}$$

$$\begin{aligned}[L,M]_{ij}=\frac{c\dot{q}_j^2}{\eta(q_i-q_j)}-\frac{c\dot{q}_i\dot{q}_j}{\eta(q_i-q_j)}+\frac{c\dot{q}_i\dot{q}_j}{\eta(q_i-q_j+\eta)}-\frac{c\dot{q}_j^2}{\eta(q_i-q_j+\eta)}+\frac{c\dot{q}_i\dot{q}_j}{(q_i-q_j+\eta)(q_i-q_j+\eta)}-\\-\frac{c\dot{q}_j^2}{(q_i-q_j+\eta)(q_i-q_j)}-\frac{c\dot{q}_i\dot{q}_j}{(q_i-q_j+\eta)(q_i-q_j)}+\frac{c\dot{q}_j^2}{(q_i-q_j+\eta)^2}+\\+\sum_{k\neq i,j}\frac{c\dot{q}_j\dot{q}_k}{q_i-q_j+\eta}\left(\frac{2}{q_j-q_k}-\frac{1}{q_j-q_k+\eta}-\frac{1}{q_j-q_k-\eta}\right)\end{aligned}\tag{110}$$

$$\frac{c\dot{q}_j^2}{\eta(q_i-q_j)}-\frac{c\dot{q}_j^2}{\eta(q_i-q_j+\eta)}=\frac{c\dot{q}_j^2}{(q_i-q_j)(q_i-q_j+\eta)}\tag{111}$$

$$\frac{c\dot{q}_i\dot{q}_j}{\eta(q_i-q_j+\eta)}-\frac{c\dot{q}_i\dot{q}_j}{\eta(q_i-q_j)}=-\frac{c\dot{q}_i\dot{q}_j}{(q_i-q_j)(q_i-q_j+\eta)}\tag{112}$$

$$\begin{aligned}[L,M]_{ij}=\frac{c\dot{q}_j^2}{(q_i-q_j+\eta)^2}+\frac{c\dot{q}_i\dot{q}_j}{q_i-q_j+\eta}\left(\frac{1}{q_i-q_j+\eta}-\frac{2}{q_i-q_j}\right)+\\+\sum_{k\neq i,j}\frac{c\dot{q}_j\dot{q}_k}{q_i-q_j+\eta}\left(\frac{2}{q_j-q_k}-\frac{1}{q_j-q_k+\eta}-\frac{1}{q_j-q_k-\eta}\right)\end{aligned}\tag{113}$$

As seen,

$$\dot{L}_{ij}=[L,M]_{ij}\tag{114}$$

3. **Classical Sklyanin algebra.**
Consider a four-dimensional space with coordinates (S_0, S_1, S_2, S_3). Let J_1, J_2, J_3 be appropriate constants. Check that the operations defined on linear functions as

$$\{S_0, S_i\} = -\{S_i, S_0\} = \epsilon_{ijk} S_j S_k (J_j - J_k), \quad \{S_i, S_j\} = \epsilon_{ijk} S_0 S_k, \quad i, j, k \in \{1, 2, 3\} \tag{115}$$

and on polynomials via the Leibniz rule define Poisson brackets on this space.
Show that these Poisson brackets are degenerate. Namely, find Casimir functions of the form

$$C_1 = \alpha S_0^2 + \beta (S_1^2 + S_2^2 + S_3^2) \tag{116}$$

$$C_2 = \gamma S_0^2 + \delta (J_1 S_1^2 + J_2 S_2^2 + J_3 S_3^2) \tag{117}$$

Solution.
Check 3 axioms:

- Antisymmetry of Poisson brackets $\{f, g\} = -\{g, f\}$ follows from the definition.
- Define $\{\{S_\mu, S_\nu\}, S_\lambda\}$ in such a way that the Leibniz identity $\{f, gh\} = g\{f, h\} + \{f, g\}h$ holds.
- Jacobi identity.

$$\begin{aligned}\{\{S_0, S_1\}, S_2\} = 2\{\epsilon_{123} S_2 S_3 (J_2 - J_3), S_2\} = 2\epsilon_{123} S_2 (J_2 - J_3)\{S_3, S_2\} = \\ = 2\epsilon_{123} S_2 (J_2 - J_3) \epsilon_{321} S_0 S_1 = -2 S_2 S_0 S_1 (J_2 - J_3)\end{aligned} \tag{118}$$

$$\begin{aligned}\{\{S_2, S_0\}, S_1\} = -2\{\epsilon_{231} S_3 S_1 (J_3 - J_1), S_1\} = -2\epsilon_{231} S_1 (J_3 - J_1)\{S_3, S_1\} = \\ = -2\epsilon_{231} S_1 (J_3 - J_1) \epsilon_{312} S_0 S_2 = -2 S_1 S_0 S_2 (J_3 - J_1)\end{aligned} \tag{119}$$

$$\{\{S_1, S_2\}, S_0\} = \{\epsilon_{123} S_0 S_3, S_0\} = -2\epsilon_{123} S_0 \epsilon_{312} S_1 S_2 (J_1 - J_2) = -2 S_0 S_1 S_2 (J_1 - J_2) \tag{120}$$

$$\{\{S_0, S_1\}, S_2\} + \{\{S_2, S_0\}, S_1\} + \{\{S_1, S_2\}, S_0\} = 0 \tag{121}$$

Analogically, for other pairs $(0, 2, 3)$ or $(0, 1, 3)$.

$$\{\{S_0, S_i\}, S_i\} + \{\{S_i, S_i\}, S_0\} + \{\{S_i, S_0\}, S_i\} = \{\{S_0, S_i\}, S_i\} + 0 - \{\{S_0, S_i\}, S_i\} = 0 \tag{122}$$

$$\begin{aligned}\{\{S_0, S_0\}, S_i\} + \{\{S_i, S_0\}, S_0\} + \{\{S_0, S_i\}, S_i\} = \\ = 0 + \{\{S_i, S_0\}, S_0\} - \{\{S_i, S_0\}, S_i\} = 0\end{aligned} \tag{123}$$

$$\{\{S_1, S_2\}, S_3\} = \{\epsilon_{123} S_0 S_3, S_3\} = 2\epsilon_{123}\epsilon_{312} S_1 S_2 (J_1 - J_2) S_3 = 2 S_1 S_2 S_3 (J_1 - J_2) \tag{124}$$

$$\{\{S_3, S_1\}, S_2\} = \{\epsilon_{312} S_0 S_2, S_2\} = 2\epsilon_{312}\epsilon_{213} S_1 S_3 (J_1 - J_3) S_2 = -2 S_1 S_2 S_3 (J_1 - J_3) \tag{125}$$

$$\{\{S_2, S_3\}, S_1\} = \{\epsilon_{231} S_0 S_1, S_1\} = 2\epsilon_{231}\epsilon_{123} S_2 S_3 (J_2 - J_3) S_1 = 2 S_1 S_2 S_3 (J_2 - J_3) \tag{126}$$

$$\{\{S_1, S_2\}, S_3\} + \{\{S_3, S_1\}, S_2\} + \{\{S_2, S_3\}, S_1\} = 0 \tag{127}$$

Find Casimir elements:

$$\begin{aligned}\{S_0, C_1\} = \{S_0, \alpha S_0^2 + \beta(S_1^2 + S_2^2 + S_3^2)\} = \{S_0, \beta(S_1^2 + S_2^2 + S_3^2)\} = \\ = 2\beta(\{S_0, S_1\}S_1 + \{S_0, S_2\}S_2 + \{S_0, S_3\}S_3) = \\ = 4\beta S_1 S_2 S_3 (J_2 - J_3 - (J_1 - J_3) + J_1 - J_2) = 0\end{aligned} \tag{128}$$

$$\begin{aligned}\{S_1, C_1\} = \{S_1, \alpha S_0^2 + \beta(S_1^2 + S_2^2 + S_3^2)\} = \{S_1, \alpha S_0^2 + \beta(S_2^2 + S_3^2)\} = \\ = 2\alpha\{S_1, S_0\}S_0 + 2\beta(\{S_1, S_2\}S_2 + \{S_1, S_3\}S_3) = \\ = -4\alpha S_0 S_2 S_3 (J_2 - J_3) + 4\beta S_0 S_2 S_3 (1 - 1) = -4\alpha S_0 S_2 S_3 (J_2 - J_3)\end{aligned} \tag{129}$$

$$\boxed{\alpha = 0 \to C_1 = \beta(S_1^2 + S_2^2 + S_3^2)} \tag{130}$$

$\{S_2, C_1\}$ and $\{S_3, C_1\}$ give the same condition.

$$\begin{aligned}\{S_0, C_2\} = \{S_0, \gamma S_0^2 + \delta(J_1 S_1^2 + J_2 S_2^2 + J_3 S_3^2)\} = \{S_0, \delta(J_1 S_1^2 + J_2 S_2^2 + J_3 S_3^2)\} = \\ = 2\delta(J_1\{S_0, S_1\}S_1 + J_2\{S_0, S_2\}S_2 + J_3\{S_0, S_3\}S_3) = \\ = 4\delta S_1 S_2 S_3 (J_1(J_2 - J_3) - J_2(J_1 - J_3) + J_3(J_1 - J_2)) = 0\end{aligned} \tag{131}$$

$$\begin{aligned}\{S_1, C_2\} = \{S_1, \gamma S_0^2 + \delta(J_1 S_1^2 + J_2 S_2^2 + J_3 S_3^2)\} = \{S_1, \gamma S_0^2 + \delta(J_2 S_2^2 + J_3 S_3^2)\} = \\ = 2\gamma\{S_1, S_0\}S_0 + 2\delta(J_2\{S_1, S_2\}S_2 + J_3\{S_1, S_3\}S_3) = \\ = -4\gamma S_0 S_2 S_3 (J_2 - J_3) + 4\delta S_0 S_2 S_3 (J_2 - J_3) = 4(\delta - \gamma) S_0 S_2 S_3 (J_2 - J_3)\end{aligned} \tag{132}$$

$$\boxed{\delta = \gamma \to C_2 = \gamma(S_0^2 + J_1 S_1^2 + J_2 S_2^2 + J_3 S_3^2)} \tag{133}$$

4. **Universal Poisson brackets.**
Consider a system with Hamilton function

$$H = \frac{p_1^2 + p_2^2 + p_3^2}{2} \tag{134}$$

and Poisson brackets

$$\{p_i, x_j\} = \delta_{ij}, \quad \{x_i, x_j\} = 0, \quad \{p_i, p_j\} = F_{ij}(x) \tag{135}$$

- Write the equations of motion for the system. What is the physical meaning of these equations?
- Which conditions should be imposed on $F_{ij}(x)$ to make (135) a Poisson bracket.
- Write down the corresponding symplectic form. Check if it is non-degenerate.

Solution.

- Equations of motion:

$$\begin{cases} \frac{dp_i}{dt} = \{H, p_i\}, \\ \frac{dx_i}{dt} = \{H, x_i\}. \end{cases} \tag{136}$$

$$\{H, p_i\} = \frac{1}{2}\left\{p_1^2 + p_2^2 + p_3^2, p_i\right\} = (p_1\{p_1, p_i\} + p_2\{p_2, p_i\} + p_3\{p_3, p_i\}) = \tag{137}$$

$$= p_1 F_{1i}(x) + p_2 F_{2i}(x) + p_3 F_{3i}(x) \tag{138}$$

$$\{H, x_i\} = \frac{1}{2}\{p_1^2 + p_2^2 + p_3^2, x_i\} = (p_1\{p_1, x_i\} + p_2\{p_1, x_i\} + p_3\{p_3, x_i\}) = \tag{139}$$

$$= p_1\delta_{1i} + p_2\delta_{2i} + p_3\delta_{3i} = p_i \tag{140}$$

$$\boxed{\begin{cases} \frac{dp_i}{dt} = p_1 F_{1i}(x) + p_2 F_{2i}(x) + p_3 F_{3i}(x), \\ \frac{dx_i}{dt} = p_i. \end{cases}} \tag{141}$$

Equations of motion are similar to the motion in magnetic field:

$$\ddot{x}_i = \dot{p}_i = p_j F_{ji} = \dot{x}_j F_{ji} \tag{142}$$

Let

$$B_k = \frac{1}{2}\epsilon_{kij} F_{ij} \to F_{ij} = \epsilon_{ijk} B_k \tag{143}$$

$$\ddot{q}_i = \epsilon_{ijk}\dot{x}_j B_k \tag{144}$$

- Using the anticommutative property of Jacobi bracket, we get

$$F_{ij}(x) = \{p_i, p_j\} = -\{p_j, p_i\} = -F_{ji}(x) \to \boxed{F_{ij}(x) = -F_{ji}(x)} \tag{145}$$

Using Jacobi identity, we get

$$\{\{p_i, p_j\}, p_k\} + \{\{p_k, p_i\}, p_j\} + \{\{p_j, p_k\}, p_i\} = 0 \tag{146}$$

$$\{F_{ij}(x), p_k\} + \{F_{ki}(x), p_j\} + \{F_{jk}(x), p_i\} = 0 \tag{147}$$

This identity is equal

$$\boxed{\frac{\partial F_{ij}(x)}{\partial x^k} + \frac{\partial F_{jk}(x)}{\partial x^i} + \frac{\partial F_{ki}(x)}{\partial x^j} = 0 \leftrightarrow dF = 0} \tag{148}$$

We get Bianchi identity.

- Let $x = (\boldsymbol{x}, \boldsymbol{p})$ – vector in the phase space. Poisson bracket $\{f, g\}(x) = \pi_{ij}(x)\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}$ corresponds symplectic form $\omega = A(\pi^{-1})_{ij} dx_i \wedge dx_j$ with non-zero A (for example, $A = -\frac{1}{2}$).

$$\pi = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & F_{12} & F_{13} \\ 0 & -1 & 0 & -F_{12} & 0 & F_{23} \\ 0 & 0 & -1 & -F_{13} & -F_{23} & 0 \end{pmatrix} \to \pi^{-1} = \begin{pmatrix} 0 & F_{12} & F_{13} & -1 & 0 & 0 \\ -F_{12} & 0 & F_{23} & 0 & -1 & 0 \\ -F_{13} & -F_{23} & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\boxed{\omega = \sum_{i=1}^{3} dp_i \wedge dx_i - F_{12}(x)dx_1 \wedge dx_2 - F_{23}(x)dx_2 \wedge dx_3 - F_{31}(x)dx_3 \wedge dx_1} \tag{149}$$

Consider arbitrary non-zero vector $\xi = \xi_i(x, p)\frac{\partial}{\partial x_i} + \tilde{\xi}_i(x, p)\frac{\partial}{\partial p_i}$.

$$\omega(\xi, \cdot) = \sum_{i=1}^{3}\left(\tilde{\xi}_i + \sum_{j\neq i} F_{ij}\xi_j\right) dx_i - \xi_i dp_i.$$

According to the definition of nondegeneracy, we need to find some vector η:

$$\omega(\xi, \eta) \neq 0 \tag{150}$$

Let $\eta = \frac{\partial}{\partial x_i}$, then $\omega(\xi, \eta) = \tilde{\xi}_i + \sum\limits_{j \neq i} F_{ij}\xi_j$. Let $\eta = \frac{\partial}{\partial p_i}$, then $\omega(\xi, \eta) = -\xi_i$. $\xi_i(x,p)$ and $\tilde{\xi}_i(x,p)$ can't be 0 together, because $\xi \neq 0$, so $\omega(\xi, \eta) \neq 0$.

5. **Lagrange top.**
Consider a system with a Hamilton function $H(\vec{S}, \vec{P})$ defined on a six-dimensional space with coordinates S_1, S_2, S_3, P_1, P_2, P_3 and Poisson brackets

$$\{S_i, S_j\} = \epsilon_{ijk}S_k, \quad \{S_i, P_j\} = \epsilon_{ijk}P_k, \quad \{P_i, P_j\} = 0 \tag{151}$$

Denote $\omega_i = \frac{\partial H}{\partial S_i}$ and $h_i = -\frac{\partial H}{\partial P_i}$.

- Write down the equations of motion for this system (in components and in vector form).
- Consider a special form of Hamilton function

$$H = \frac{1}{2}(J_1 S_1^2 + J_2 S_2^2 + J_3 S_3^2) - (h_1 P_1 + h_2 P_2 + h_3 P_3) \tag{152}$$

  Which physical system is described by this Hamilton function?
  Let $J_1 = J_2 = a$, $J_3 = b$, $h_1 = h_2 = 0$ and $h_3 = h$. Show that in this case a scalar product $(\vec{S} \cdot \vec{h})$ is in involution with Hamilton function H, i.e. defines the additional conservation law.
- Show that the Poisson brackets (151) are degenerate and check that there are two Casimir functions

$$C_1 = P_1^2 + P_2^2 + P_3^2, \quad C_2 = S_1P_1 + S_2P_2 + S_3P_3 \tag{153}$$

  Are the Poisson brackets (151) related to any Lie algebra? Describe this Lie algebra.
- Fix the level surface of Casimir functions $C_1 = p_1^2$, $C_2 = ps$, where $p \neq 0$ and s are constants. Show that a change of variables $S_i \hookrightarrow y_i = S_i - \frac{s}{p}p_i$ defines an isomorphism of the level surface with the cotangent bundle T^*S^2.
- Consider another change of variables on the level surface

$$p_1 = p\cos\theta\cos\phi, \quad p_2 = p\cos\theta\sin\phi, \quad p_3 = p\sin\theta \tag{154}$$

$$y_1 = p_\phi\tan\theta\cos\phi, \quad y_2 = p_\phi\tan\theta\sin\phi + p_\theta\cos\phi, \quad y_3 = -p_\phi \tag{155}$$

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \phi \leq 2\pi \tag{156}$$

  Compute the Poisson brackets between coordinates θ, ϕ, p_θ, p_ϕ and show that they are non-degenerate. Find a symplectic form ω, corresponding to these Poisson brackets and calculate the integral of ω over the sphere θ, ϕ. How could these results be interpreted?

6. **Two oscillators.**
Consider a system of two independent oscillators with Hamilton function

$$H = H_1 + H_2 = \frac{\omega_1}{2}(p_1^2 + q_1^2) + \frac{\omega_2}{2}(p_2^2 + q_2^2) \tag{157}$$

and standard Poisson brackets

$$\{p_i, q_j\} = \delta_{ij}, \quad \{p_i, p_j\} = \{q_i, q_j\} = 0 \tag{158}$$

- Describe geometrically the level manifold of Hamilton function $H = E$ for different values of E.
- Describe geometrically the level manifold of functions $H_1 = E_1$ and $H_2 = E_2$ for different values of E_1, E_2. How the level manifolds from the previous point are made up of these level manifolds?
- Let $\omega_1 = \omega_2 = \omega$. Prove that in this case there are three independent conserved quantities

$$I_1 = q_1 q_2 + p_1 p_2, \quad I_2 = p_1 q_2 - p_2 q_1, \quad I_3 = \frac{1}{2}(p_1^2 + q_1^2 - p_2^2 - q_2^2) \tag{159}$$

- Check that these quantities satisfy the condition

$$\omega^2 (I_1^2 + I_2^2 + I_3^2) = H^2 \tag{160}$$

and show that their level manifolds define the Hopf fibration $S^1 \hookrightarrow S^3 \twoheadrightarrow S^2$.

Solution.

- The level manifold:

$$H = \frac{\omega_1}{2}(p_1^2 + q_1^2) + \frac{\omega_2}{2}(p_2^2 + q_2^2) = E \tag{161}$$

In coordinates $\tilde{p}_1 = p_1\sqrt{\frac{\omega_1}{2}}$, $\tilde{q}_1 = q_1\sqrt{\frac{\omega_1}{2}}$, $\tilde{p}_2 = p_2\sqrt{\frac{\omega_2}{2}}$, $\tilde{p}_1 = p_2\sqrt{\frac{\omega_2}{2}}$:

$$\boxed{\tilde{p}_1^2 + \tilde{q}_1^2 + \tilde{p}_2^2 + \tilde{q}_2^2 = E} \tag{162}$$

We get the equation of a sphere S^3 with radius $\sqrt{E}$.

- The level manifold of function H_1:

$$H_1 = \frac{\omega_1}{2}(p_1^2 + q_1^2) = E_1 \tag{163}$$

In coordinates $\tilde{p}_1 = p_1\sqrt{\frac{\omega_1}{2}}$, $\tilde{q}_1 = q_1\sqrt{\frac{\omega_1}{2}}$:

$$\boxed{\tilde{p}_1^2 + \tilde{q}_1^2 = E_1} \tag{164}$$

We get the equation of a circle S^1 with radius $\sqrt{E_1}$.
In coordinates $\tilde{p}_2 = p_2\sqrt{\frac{\omega_2}{2}}$, $\tilde{q}_2 = q_2\sqrt{\frac{\omega_2}{2}}$:

$$\boxed{\tilde{p}_2^2 + \tilde{q}_2^2 = E_2} \tag{165}$$

We get the equation of a circle S^1 with radius $\sqrt{E_2}$.
Summarizing, we get a torus $S^1 \times S^1$.

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$$H = \frac{\omega}{2}(p_1^2 + q_1^2 + p_2^2 + q_2^2) \tag{166}$$

$$\mathrm{rk}\begin{pmatrix} \frac{\partial I_1}{\partial q_1} & \frac{\partial I_1}{\partial p_1} & \frac{\partial I_1}{\partial q_2} & \frac{\partial I_1}{\partial p_2} \\ \frac{\partial I_2}{\partial q_1} & \frac{\partial I_2}{\partial p_1} & \frac{\partial I_2}{\partial q_2} & \frac{\partial I_2}{\partial p_2} \\ \frac{\partial I_3}{\partial q_1} & \frac{\partial I_3}{\partial p_1} & \frac{\partial I_3}{\partial q_2} & \frac{\partial I_3}{\partial p_2} \end{pmatrix} = \mathrm{rk}\begin{pmatrix} q_2 & p_2 & q_1 & p_1 \\ -p_2 & q_2 & p_1 & -q_1 \\ q_1 & p_1 & -q_2 & -p_2 \end{pmatrix} = 3 \tag{167}$$

So, I_1, I_2, I_3 are independent quantities.

$$\dot{I}_1 = \{H, I_1\} = \frac{\omega}{2}(\{p_1^2, q_1q_2\} + \{q_1^2, p_1p_2\} + \{p_2^2, q_1q_2\} + \{q_2^2, p_1p_2\}) =$$
$$= \frac{\omega}{2}(2p_1\{p_1, q_1\}q_2 + 2q_1\{q_1, p_1\}p_2 + 2q_1p_2\{p_2, q_2\} + 2p_1q_2\{q_2, p_2\}) =$$
$$= \omega(p_1q_2 - q_1p_2 + q_1p_2 - p_1q_2) = 0 \quad (168)$$

$$\dot{I}_2 = \{H, I_2\} = \frac{\omega}{2}(\{p_1^2, -p_2q_1\} + \{q_1^2, p_1q_2\} + \{p_2^2, p_1q_2\} + \{q_2^2, -p_2q_1\}) =$$
$$= \frac{\omega}{2}(-2p_1p_2\{p_1, q_1\} + 2q_1\{q_1, p_1\}q_2 + 2p_1p_2\{p_2, q_2\} - 2q_2\{q_2, p_2\}q_1) =$$
$$= \omega(-p_1p_2 + q_1q_2 + p_1p_2 - q_1q_2) = 0 \quad (169)$$

$$\dot{I}_3 = \{H, I_3\} = \frac{\omega}{4}(\{p_1^2, q_1^2\} + \{q_1^2, p_1^2\} + \{p_2^2, -q_2^2\} + \{q_2^2, -p_2^2\}) =$$
$$= \frac{\omega}{2}(4p_1q_1\{p_1, q_1\} + 4q_1p_1\{q_1, p_1\} - 4p_2q_2\{p_2, q_2\} - 4q_2p_2\{q_2, p_2\}) =$$
$$= \omega(p_1q_1 - q_1p_1 - p_2q_2 + q_2p_2) = 0 \quad (170)$$

So, I_1, I_2, I_3 are conserved quantities.

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$$\omega^2(I_1^2 + I_2^2 + I_3^2) = \omega^2\left((q_1q_2 + p_1p_2)^2 + (p_1q_2 - p_2q_1)^2 + \frac{1}{4}(p_1^2 + q_1^2 - p_2^2 - q_2^2)^2\right) =$$
$$= \omega^2\left(q_1^2q_2^2 + p_1^2p_2^2 + 2q_1q_2p_1p_2 + p_1^2q_2^2 + p_2^2q_1^2 - 2p_1q_2p_2q_1 + \right.$$
$$\left. + \frac{1}{4}(p_1^4 + q_1^4 + p_2^4 + q_2^4 + 2p_1^2q_1^2 - 2p_1^2p_2^2 - 2p_1^2q_2^2 - 2q_1^2p_2^2 - 2q_1^2q_2^2 + 2p_2^2q_2^2)\right) =$$
$$= \omega^2\left(\frac{1}{4}(p_1^4 + q_1^4 + p_2^4 + q_2^4) + \frac{1}{2}(p_1^2q_1^2 + p_1^2p_2^2 + p_1^2q_2^2 + p_2^2q_1^2 + q_1^2q_2^2 + p_2^2q_2^2)\right) =$$
$$= \frac{\omega^2}{4}(p_1^2 + q_1^2 + p_2^2 + q_2^2)^2 = H^2 \quad (171)$$

$$\boxed{H^2 = \omega^2(I_1^2 + I_2^2 + I_3^2)} \quad (172)$$

2 Vector fields, Lie groups actions, coadjoint orbits.

1. **Lie derivative.**
   Consider a vector field v on a smooth manifold M and define two operations on differential forms on M: contraction

$$i_v : \Omega^n(M) \to \Omega^{n-1}(M),\ i_v\lambda = \lambda(v, \cdot, \cdot, ...) \quad (173)$$

and Lie derivative

$$\mathcal{L}_v : \Omega^n(M) \to \Omega^n(M),\ \mathcal{L}_v\lambda = \frac{d}{dt}(\exp(vt)^*\lambda)\Big|_{t=0} \quad (174)$$

- Show that these operations satisfy the following properties:

$$\mathcal{L}_v = di_v + i_v d, \tag{175}$$

$$\mathcal{L}_{[v,u]} = [\mathcal{L}_v, \mathcal{L}_u], \tag{176}$$

$$[\mathcal{L}_v, i_u] = i_{[v,u]}. \tag{177}$$

- Show also that Lie derivative is a derivation with respect to contraction, i.e.

$$\mathcal{L}_v \lambda(v_1, ..., v_k) = (\mathcal{L}_v \lambda)(v_1, ..., v_k) + \sum_{i=1}^{k} \lambda(v_1, ..., [v, v_i], ..., v_k) \tag{178}$$

- Let v_1, v_2, v_3 be symplectic vector fields which conserve the symplectic form ω. Show that

$$\omega([v_1, v_2], v_3) = -\mathcal{L}_{v_3}\omega(v_1, v_2), \tag{179}$$

$$\omega([v_1, v_2], v_3) + \omega([v_2, v_3], v_1) + \omega([v_3, v_1], v_2) = 0, \tag{180}$$

$$\mathcal{L}_{v_1}\omega(v_2, v_3) + \mathcal{L}_{v_2}\omega(v_3, v_1) + \mathcal{L}_{v_3}\omega(v_1, v_2) = 0. \tag{181}$$

Solution.
Lie derivative:

$$(\mathcal{L}_v T)^{j_1...j_s}_{i_1...i_q} = \xi^k \frac{\partial T^{j_1...j_s}_{i_1...i_q}}{\partial x^k} - T^{kj_2...j_s}_{i_1...i_q} \frac{\partial \xi^{j_1}}{\partial x^k} - ... - T^{j_1...j_{s-1}k}_{i_1...i_q} \frac{\partial \xi^{j_s}}{\partial x^k} + T^{j_1...j_s}_{ki_2...i_q} \frac{\partial \xi^k}{\partial x^{i_1}} + ... + \\ + T^{j_1...j_s}_{i_1...i_{q-1}k} \frac{\partial \xi^k}{\partial x^{i_q}} \tag{182}$$

- – Show Cartan identity by induction.
  Let $\lambda = f \in C^\infty(M)$ and $v = \xi^i \frac{\partial}{\partial x^i}$, then

$$\mathcal{L}_v f = \xi^i \frac{\partial f}{\partial x^i} = v(f) = df(v) = i_v df \tag{183}$$

  Let $\lambda = dx^i$. Lie derivative commutates with external differential:

$$d\mathcal{L}_v = \mathcal{L}_v d \rightarrow \mathcal{L}_v \lambda = \mathcal{L}_v dx^i = d\mathcal{L}_v x^i = di_v dx^i = di_v \lambda \tag{184}$$

$$d\lambda = d^2 x^i = 0 \tag{185}$$

  Show that if Cartan identity is true for differential forms $\alpha \in \Omega^s(M)$ and $\beta \in \Omega^p(M)$:

$$\mathcal{L}_v \alpha = i_v d\alpha + di_v \alpha, \quad \mathcal{L}_v \beta = i_v d\beta + di_v \beta, \tag{186}$$

  then this identity is true for $\omega = \alpha \wedge \beta$. Leibniz rules for external differential and contraction:

$$d\omega = d\alpha \wedge \beta + (-1)^s \alpha \wedge d\beta, \quad i_v \omega = i_v \alpha \wedge \beta + (-1)^s \alpha \wedge i_v \beta \tag{187}$$

$$i_v d\omega = i_v(d\alpha \wedge \beta + (-1)^s \alpha \wedge d\beta) = i_v d\alpha \wedge \beta + (-1)^{s-1} d\alpha \wedge i_v \beta + \\ + (-1)^s i_v \alpha \wedge d\beta + \alpha \wedge i_v d\beta \tag{188}$$

$$di_v\omega = d(i_v\alpha \wedge \beta + (-1)^s \alpha \wedge i_v\beta) = di_v\alpha \wedge \beta + (-1)^{s-1} i_v\alpha \wedge d\beta + \\ + (-1)^s d\alpha \wedge i_v\beta + \alpha \wedge di_v\beta \quad (189)$$

$$i_v d\omega + di_v\omega = i_v d\alpha \wedge \beta + di_v\alpha \wedge \beta + \alpha \wedge i_v d\beta + \alpha \wedge di_v\beta = \mathcal{L}_v\alpha \wedge \beta + \alpha \wedge \mathcal{L}_v\beta \quad (190)$$

$$i_v d\omega + di_v\omega = \mathcal{L}_v\omega \quad (191)$$

Thus, Cartan identity is true for $\lambda \in \Omega^n(M)$:

$$\boxed{\mathcal{L}_v = di_v + i_v d} \quad (192)$$

– Show, that $\mathcal{L}_{[v,u]} = [\mathcal{L}_v, \mathcal{L}_u]$ by induction.
Let $\lambda = f \in C^\infty(M)$, then $\mathcal{L}_v f = v(f)$.

$$\mathcal{L}_{[v,u]} f = [v,u](f) = v(u(f)) - u(v(f)) \quad (193)$$

$$[\mathcal{L}_v, \mathcal{L}_u] f = (\mathcal{L}_v\mathcal{L}_u - \mathcal{L}_u\mathcal{L}_v) f = \mathcal{L}_v u(f) - \mathcal{L}_u v(f) = v(u(f)) - u(v(f)) \quad (194)$$

$$\mathcal{L}_{[v,u]} f = [\mathcal{L}_v, \mathcal{L}_u] f \quad (195)$$

Let $\omega = df \in \Omega^1(M)$, then

$$\mathcal{L}_{[v,u]} df = d\mathcal{L}_{[v,u]} f = d([v,u](f)) = d(v(u(f))) - d(u(v(f))) \quad (196)$$

$$[\mathcal{L}_v, \mathcal{L}_u] df = (\mathcal{L}_v\mathcal{L}_u - \mathcal{L}_u\mathcal{L}_v) df = \mathcal{L}_v d(\mathcal{L}_u f) - \mathcal{L}_u d(\mathcal{L}_v f) = \mathcal{L}_v d(u(f)) - \mathcal{L}_u d(v(f)) = \\ = d\mathcal{L}_v(u(f)) - d\mathcal{L}_u(v(f)) = d(v(u(f))) - d(u(v(f))) \quad (197)$$

$$\mathcal{L}_{[v,u]} df = [\mathcal{L}_v, \mathcal{L}_u] df \quad (198)$$

Let w is a vector field, then

$$\mathcal{L}_v w = [v, w] \quad (199)$$

$$[\mathcal{L}_u, \mathcal{L}_v] w = (\mathcal{L}_u\mathcal{L}_v - \mathcal{L}_v\mathcal{L}_u) w = \mathcal{L}_u[v,w] - \mathcal{L}_v[u,w] = [u,[v,w]] - [v,[u,w]] = \\ = [u,[v,w]] + [v,[w,u]] = -[w,[u,v]] = [[u,v],w] = \mathcal{L}_{[u,v]} w \quad (200)$$

Show, that identity is true for 2 arbitrary tensors α and β:

$$\mathcal{L}_{[u,v]}\alpha = [\mathcal{L}_u, \mathcal{L}_v]\alpha, \quad \mathcal{L}_{[u,v]}\beta = [\mathcal{L}_u, \mathcal{L}_v]\beta, \quad (201)$$

then this identity is true for $\omega = \alpha \otimes \beta$. We will now show that our formula is correct for the tensor $\alpha \otimes \beta$. Use Leibniz rule:

$$\mathcal{L}_u\mathcal{L}_v(\alpha \otimes \beta) = \mathcal{L}_u(\mathcal{L}_v\alpha \otimes \beta + \alpha \otimes \mathcal{L}_v\beta) = \\ = \mathcal{L}_u\mathcal{L}_v\alpha \otimes \beta + \mathcal{L}_u\alpha \otimes \mathcal{L}_v\beta + \mathcal{L}_v\alpha \otimes \mathcal{L}_u\beta + \alpha \otimes \mathcal{L}_u\mathcal{L}_v\beta \quad (202)$$

$$[\mathcal{L}_u, \mathcal{L}_v](\alpha \otimes \beta) = [\mathcal{L}_u, \mathcal{L}_v]\alpha \otimes \beta + \alpha \otimes [\mathcal{L}_u, \mathcal{L}_v]\beta = \mathcal{L}_{[u,v]}\alpha \otimes \beta + \alpha \otimes \mathcal{L}_{[u,v]}\beta = \\ = \mathcal{L}_{[u,v]}(\alpha \otimes \beta) \quad (203)$$

Thus, this identity is true for all tensors:

$$\boxed{\mathcal{L}_{[v,u]} = [\mathcal{L}_v, \mathcal{L}_u]} \quad (204)$$

– Show, that $[\mathcal{L}_v, i_u] = i_{[v,u]}$ by induction.
For a function $f \in C^1(M, \mathbb{R})$:

$$[\mathcal{L}_v, i_u]f = \mathcal{L}_v i_u f - i_u \mathcal{L}_v f = 0, \quad i_{[v,u]}f = 0 \tag{205}$$

For a 1-form α:

$$\mathcal{L}_v i_u \alpha = \mathcal{L}_v(\alpha(u)) = v(\alpha(u)),\ i_u \mathcal{L}_v \alpha = i_u d i_v \alpha + i_u i_v d\alpha = u(\alpha(v)) + d\alpha(v, u) \tag{206}$$

$$d\alpha(v, u) = v(\alpha(u)) - u(\alpha(v)) - \alpha([v, u]) \tag{207}$$

$$[\mathcal{L}_v, i_u]\alpha = \alpha([v, u]) \tag{208}$$

$$i_{[v,u]}\alpha = \alpha([v, u]) \tag{209}$$

$$[\mathcal{L}_v, i_u](\omega \wedge \eta) = [\mathcal{L}_v, i_u]\omega \wedge \eta + (-1)^k \omega \wedge [\mathcal{L}_v, i_u]\eta, \tag{210}$$

where ω is assumed to be a k-form, and η is an arbitrary form (follows from Leibniz's rules for the Lie derivative and the contraction).

$$\boxed{[\mathcal{L}_v, i_u] = i_{[v,u]}} \tag{211}$$

- Show, that

$$\mathcal{L}_v \lambda(v_1, ..., v_k) = (\mathcal{L}_v \lambda)(v_1, ..., v_k) + \sum_{i=1}^{k} \lambda(v_1, ..., [v, v_i], ..., v_k) \tag{212}$$

$$\begin{aligned}
\mathcal{L}_v \lambda(v_1, ..., v_k) &= \mathcal{L}_v(i_{v_1} ... i_{v_k} \lambda) = \\
&= (i_{[v,v_1]} + i_{v_1} \mathcal{L}_v)(i_{v_2} ... i_{v_k} \lambda) = (i_{[v,v_1]} i_{v_2} ... i_{v_k} \lambda) + i_{v_1}(\mathcal{L}_v)(i_{v_2} ... i_{v_k} \lambda) = \\
&= (i_{[v,v_1]} i_{v_2} ... i_{v_k} \lambda) + i_{v_1}(i_{[v,v_2]} + i_{v_2} \mathcal{L}_v)(i_{v_3} ... i_{v_k} \lambda) = \\
&= ... = (i_{v_1}, ..., i_{v_k})(\mathcal{L}_v \lambda) + \sum_{i=1}^{k} (i_{v_1} ... i_{[v,v_j]} ... i_{v_k} \lambda) = \\
&= (\mathcal{L}_v \lambda)(v_1, ..., v_k) + \sum_{i=1}^{k} \lambda(v_1, ..., [v, v_i], ..., v_k)
\end{aligned} \tag{213}$$

- – Show, that $\omega([v_1, v_2], v_3) = -\mathcal{L}_{v_3}\omega(v_1, v_2)$ for symplectic ω.

$$\mathcal{L}_{v_3}\omega = 0 \tag{214}$$

$$\begin{aligned}
\mathcal{L}_{v_3}\omega(v_1, v_2) &= (\mathcal{L}_{v_3}\omega)(v_1, v_2) + \omega([v_3, v_1], v_2) + \omega(v_1, [v_3, v_2]) = \\
&= \omega([v_3, v_1], v_2) + \omega([v_2, v_3], v_1)
\end{aligned} \tag{215}$$

Now we use identity from the next –:

$$\omega([v_1, v_2], v_3) + \omega([v_2, v_3], v_1) + \omega([v_3, v_1], v_2) = 0 \tag{216}$$

$$\boxed{\mathcal{L}_{v_3}\omega(v_1, v_2) = -\omega([v_1, v_2], v_3)} \tag{217}$$

– v_i is a symplectic vector field, so

$$\mathcal{L}_{v_i}\omega = 0 \tag{218}$$

ω is a symplectic form, so $d\omega = 0$.

$$d\omega(v_1, v_2, v_3) = v_1(\omega(v_2, v_3)) - v_2(\omega(v_1, v_3)) + v_3(\omega(v_1, v_2)) - \\ - \omega([v_1, v_2], v_3) + \omega([v_1, v_3], v_2) - \omega([v_2, v_3], v_1) = 0 \tag{219}$$

$$v_1(\omega(v_2, v_3)) = \mathcal{L}_{v_1}(\omega(v_2, v_3)) = (\mathcal{L}_{v_1}\omega)(v_2, v_3) + \omega([v_1, v_2], v_3) + \omega(v_2, [v_1, v_3]) = \\ = \omega([v_1, v_2], v_3) + \omega([v_3, v_1], v_2) \tag{220}$$

$$v_2(\omega(v_1, v_3)) = \mathcal{L}_{v_2}(\omega(v_1, v_3)) = (\mathcal{L}_{v_2}\omega)(v_1, v_3) + \omega([v_2, v_1], v_3) + \omega(v_1, [v_2, v_3]) = \\ = -\omega([v_1, v_2], v_3) - \omega([v_2, v_3], v_1) \tag{221}$$

$$v_3(\omega(v_1, v_2)) = \mathcal{L}_{v_3}(\omega(v_1, v_2)) = (\mathcal{L}_{v_3}\omega)(v_1, v_2) + \omega([v_3, v_1], v_2) + \omega(v_1, [v_3, v_2]) = \\ = \omega([v_3, v_1], v_2) + \omega([v_2, v_3], v_1) \tag{222}$$

$$\boxed{\omega([v_1, v_2], v_3) + \omega([v_2, v_3], v_1) + \omega([v_3, v_1], v_2) = 0} \tag{223}$$

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$$\mathcal{L}_{v_1}\omega(v_2, v_3) + \mathcal{L}_{v_2}\omega(v_3, v_1) + \mathcal{L}_{v_3}\omega(v_1, v_2) = \\ = -\omega([v_2, v_3], v_1) - \omega([v_1, v_3], v_2) - \omega([v_1, v_2], v_3) \tag{224}$$

$$\boxed{\mathcal{L}_{v_1}\omega(v_2, v_3) + \mathcal{L}_{v_2}\omega(v_3, v_1) + \mathcal{L}_{v_3}\omega(v_1, v_2) = 0} \tag{225}$$

2. Let M be a smooth manifold and T^*M – its cotangent bundle equipped with canonical symplectic form $\omega = d\alpha$. Let v be a vector field on M, and $\tilde{v}$ a vector field on T^*M which lifts the flow of v

$$\exp(\tilde{v}t)(x, \beta) = (\exp(vt)x, \exp(vt)_*\beta) \tag{226}$$

- Find the expression for the vector field $\tilde{v}$ in local coordinates p, q for the vector field $v = \sum\limits_i v_i \frac{\partial}{\partial q_i}$.
- Show that the flow of the vector field $\tilde{v}$ preserves the Liouville 1-form and the symplectic form
$$\mathcal{L}_{\tilde{v}}\alpha = 0, \ \mathcal{L}_{\tilde{v}}\omega = 0.$$
- Show that this vector field is Hamiltonian with $H = i_{\tilde{v}}\alpha = \sum_i p_i v_i(x)$.
- Consider a Lie group G acting on the manifold M as
$$G \times M \to M : (g, x) \mapsto g.x$$
this action can be naturally lifted to the action of G on the cotangent bundle T^*M
$$G \times T^*M \to T^*M : (g, (x, \beta)) \mapsto (g.x, g_*\beta).$$
Show that the lifted action is Hamiltonian and find the momentum map.

Solution.

- Infinitesimal form:

$$\tilde{v}(q, p_i dq^i) = (q(0) + tv(x) + \mathcal{O}(t), p_i(0)d(q^i(0) - tv^i(x) + \mathcal{O}(t))) \tag{227}$$

$$q^i(t) \approx q^i(0) + tv^i(x) \rightarrow \frac{\partial q^i(t)}{\partial t} = v^i(x) \tag{228}$$

$$p_i d\left(q^i(0) - tv^i(x) + \mathcal{O}(t)\right) \approx \left(p_i(0) - tp_k \frac{\partial v^k(x)}{\partial q^i}\right) dq^i \tag{229}$$

$$p_i(t) = p_i(0) - tp_k(0)\frac{\partial v^k(x)}{\partial q^i} \rightarrow \frac{\partial p_i(t)}{\partial t} = -p_k(0)\frac{\partial v^k(x)}{\partial q^i} \tag{230}$$

$$\begin{aligned}\tilde{v}(f) = \frac{df(x,\beta)}{dt} &= \frac{\partial f}{\partial q^i}\frac{\partial q^i(t)}{\partial t} + \frac{\partial f}{\partial p_i}\frac{\partial p_i(t)}{\partial t} = \frac{\partial f}{\partial q^i}\frac{\partial q^i(t)}{\partial t} + \frac{\partial f}{\partial p_i}\frac{\partial p_i(t)}{\partial t} = \\ &= \left(v^i(x)\frac{\partial}{\partial q^i} + \left(-p_k(0)\frac{\partial v^k(x)}{\partial q^i}\right)\frac{\partial}{\partial p_i}\right) f = \left(\tilde{v}^i \frac{\partial}{\partial q^i} + \tilde{v}_{i+n}\frac{\partial}{\partial p_i}\right) f\end{aligned} \tag{231}$$

$$\boxed{\tilde{v} = v^i \frac{\partial}{\partial q^i} - p_j \frac{\partial v^j}{\partial q^i}\frac{\partial}{\partial p_i}} \tag{232}$$

- Liouville 1-form:

$$\alpha = p_i dq^i \rightarrow \omega = dp_i \wedge dq^i \tag{233}$$

$$\begin{aligned}\mathcal{L}_{\tilde{v}}\alpha = (di_{\tilde{v}} + i_{\tilde{v}}d)\alpha &= d(p_j v^i \delta^j_i) - v^j \delta^i_j dp_i + \left(-p^j \frac{\partial v_j}{\partial q^i}\right) dq^i = \\ &= v^i dp_i + p_i \frac{\partial v^i}{\partial q^j} dq^j - v^i dp_i - p^j \frac{\partial v_j}{\partial q^i} dq^i = 0\end{aligned} \tag{234}$$

$$\begin{aligned}\mathcal{L}_{\tilde{v}}\omega &= (di_{\tilde{v}} + i_{\tilde{v}}d)\omega = di_{\tilde{v}}\omega = di_{\tilde{v}}(dp_i \wedge dq^i) = \\ &= d\left(-v^i dp_i + \left(-p^j \frac{\partial v_j}{\partial q^i}\right) dq^i\right) = -dv^i \wedge dp_i - \frac{\partial v_j}{\partial q^i} dp^j \wedge dq^i - p^j d\left(\frac{\partial v_j}{\partial q^i} dq^i\right) = \\ &= -dv^i \wedge dp_i + dv_j \wedge dp^j - p^j d(dv_j) = 0\end{aligned} \tag{235}$$

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$$i_{\tilde{v}}\omega = -v^i dp_i + \left(-p^j \frac{\partial v_j}{\partial q^i}\right) dq^i \tag{236}$$

$$dH = v^i dp_i + p_i \frac{\partial v^i}{\partial q^j} dq^j = v^i dp_i + p^j \frac{\partial v_j}{\partial q^i} dq^i \tag{237}$$

$$i_{\tilde{v}}\omega = -dH \tag{238}$$

As seen, vector field $\tilde{v}$ is Hamiltonian with $H = i_{\tilde{v}}\alpha = \sum_i p_i v_i(q)$.

3.

4. **Coadjoint orbits of $GL(N)$.**
Consider the coadjoint orbits of $GL(N)$ passing through the diagonal element

$$S = g^{-1}\Lambda g, \quad \Lambda = (\lambda_1, ..., \lambda_N) \tag{239}$$

- Find the dimension of the coadjoint orbit for $\lambda_i \neq \lambda_j$ for all $i \neq j$.
- Find the dimension of the coadjoint orbit for the diagonal element

$$\Lambda = \mathrm{diag}(\underbrace{\mu_1, ..., \mu_1}_{n_1}, ..., \underbrace{\mu_k, ..., \mu_k}_{n_k}) \tag{240}$$

Which nontrivial orbit has minimal dimension?

- Deduce that from canonical Poisson brackets

$$\{\xi_{i\alpha}, \eta_{j\beta}\} = \delta_{ij}\delta_{\alpha\beta},\ \{\xi_{i\alpha}, \xi_{j\beta}\} = 0,\ \{\eta_{i\alpha}, \eta_{j\beta}\} = 0, i, j = 1, ..., N,\ \alpha, \beta = 1, ..., K \tag{241}$$

follows the Poisson–Lie brackets for $\mathfrak{gl}^*(N)$ between elements

$$f_{S_{ij}} = \sum_{\alpha=1}^{K} \xi_{i\alpha}\eta_{j\alpha} \tag{242}$$

Solution.

- Coadjoint orbit:

$$O_\Lambda = \{\mathrm{Ad}^*_g(\Lambda) = g^{-1}\Lambda g | g \in GL(N)\} \tag{243}$$

Let find the stabilizer of matrices with $\lambda_i \neq \lambda_j$ for all $i \neq j$. We should find matices g:

$$\Lambda = g^{-1}\Lambda g \to g\Lambda = \Lambda g \tag{244}$$

Commutating matrices have one set of the eigenvectors. Eigenvectors of the Λ: $\{(1, 0, ..., 0)^T, (0, 1, ..., 0)^T, ..., (0, 0, ..., 1)^T\}$. So, g is a diagonal matrix. The dimension of diagonal matrices:

$$\dim(Stab) = N \tag{245}$$

Dimension of $GL(N)$:

$$\dim(GL(N)) = N^2 \tag{246}$$

Dimension of orbit:

$$\boxed{\dim(Orb) = N^2 - N} \tag{247}$$

- The eigenspace associated with a block with μ_i is a vector space of dimension n_i. After any action on λ we must not leave the corresponding vector space. This is true for any i. Therefore we have that the stabilizer λ consists only of block matrices, where the blocks have the corresponding size

$$g = \begin{pmatrix} \boxed{G_1} & 0 & \dots & 0 \\ 0 & \boxed{G_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \boxed{G_k} \end{pmatrix}, \tag{248}$$

where G_i is a matrix of size $n_i \times n_i : \sum\limits_{i=1}^{k} n_i = N$. The dimension of such matrices is

$$\dim(Stab) = \sum_{i=1}^{k} n_i^2 \tag{249}$$

$$\boxed{\dim(Orb) = N^2 - \sum_{i=1}^{k} n_i^2} \tag{250}$$

In case $k = 1$:

$$\sum_{i=1}^{1} n_i^2 = n_1^2, \quad \sum_{i=1}^{1} n_i = n_1 = N \to \dim(Orb) = 0 \tag{251}$$

So, the trivial orbit, consisting from the identity matrix, has 0 dimension. To find a nontrivial orbit ($k \geq 2$) with minimal dimension we should maximize $\sum\limits_{i=1}^{k} n_i^2$ with $\sum\limits_{i=1}^{k} n_i = N$.

Let $S_k = \sum\limits_{i=1}^{k} n_i^2$. Compare S_k with S_{k-1}

$$S_k = \sum_{i=1}^{k} n_i^2, \quad S_{k-1} = \sum_{i=1}^{k-1} m_i^2 \tag{252}$$

Let $m_i = n_i$ for $i < k-1$ and $m_{k-1} = n_{k-1} + n_k$, than

$$S_{k-1} - S_k = 2n_{k-1}n_k > 0 \tag{253}$$

For any set $\{n_1, ...n_k\}$ find a set $\{m_1, ..., m_{k-1}\}$:

$$S_{k-1}(\{m_1, ..., m_{k-1}\}) > S_k(\{n_1, ...n_k\}) \tag{254}$$

We take $k = 2$ and $n_1 = 1$, $n_2 = N - 1$:

$$S_2 = 1 + (N-1)^2 \tag{255}$$

$$\boxed{\dim(Orb) = N^2 - (1 + (N-1)^2) = 2N - 2} \tag{256}$$

- $$\begin{aligned}\{f_{S_{ij}}, f_{S_{kl}}\} = \{\xi_{i\alpha}\eta_j^\alpha, \xi_{k\beta}\eta_l^\beta\} = \xi_{k\beta}\{\xi_{i\alpha}\eta_j^\alpha, \eta_l^\beta\} + \{\xi_{i\alpha}\eta_j^\alpha, \xi_{k\beta}\}\eta_l^\beta = \\ = \xi_{k\beta}\xi_{i\alpha}\{\eta_j^\alpha, \eta_l^\beta\} + \xi_{k\beta}\{\xi_{i\alpha}, \eta_l^\beta\}\eta_j^\alpha + \xi_{i\alpha}\{\eta_j^\alpha, \xi_{k\beta}\}\eta_l^\beta + \{\xi_{i\alpha}, \xi_{k\beta}\}\eta_j^\alpha\eta_l^\beta = \\ = \xi_{k\beta}\eta_j^\alpha\delta_{il}\delta_\alpha^\beta - \xi_{i\alpha}\eta_l^\beta\delta_{jk}\delta_\beta^\alpha = \xi_{k\alpha}\eta_j^\alpha\delta_{il} - \xi_{i\alpha}\eta_l^\alpha\delta_{jk} = f_{S_{kj}}\delta_{il} - f_{S_{il}}\delta_{jk}\end{aligned} \tag{257}$$

S_{ij} are elements of the algebra $\mathfrak{gl}(N)$ or an equally likely linear function on $\mathfrak{gl}^*(N)$. If we take the differential of this function by definition

$$f_{S_{ij}}(\xi + \Delta\xi) = f_{S_{ij}}(\xi) + \langle \Delta\xi, df_{S_{ij}}(\xi) \tag{258}$$

On the other hand,

$$f_{S_{ij}}(\xi + \Delta\xi) = \langle \xi + \Delta\xi, S_{ij} \rangle = \langle \xi, S_{ij} \rangle + \langle \Delta\xi, S_{ij} \rangle \tag{259}$$

We obtain

$$df_{S_{ij}}(\xi) = S_{ij} \in \mathfrak{gl}(N).$$

Therefore,

$$\{f_{S_{ij}}, f_{S_{kl}}\}(\xi) = \langle \xi, [S_{ij}, S_{kl}] \rangle = \langle \xi, S_{kj} \rangle \delta_{il} - \langle \xi, S_{il} \rangle \delta_{kj} \tag{260}$$

$$d(\{f_{S_{ij}}, f_{S_{kl}}\}(\xi)) = S_{kj}\delta_{il} - S_{il}\delta_{kj} \tag{261}$$

$$\boxed{\{f_{S_{ij}}, f_{S_{kl}}\} = f_{S_{kj}}\delta_{il} - f_{S_{il}}\delta_{kj}} \tag{262}$$

5.

6. **Coadjoint orbits of $SL(2, \mathbb{R})$.**
Consider a group $G = SL(2, \mathbb{R})$ defined as

$$G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1, a, b, c, d \in \mathbb{R} \right\} \tag{263}$$

and its Lie algebra

$$\mathfrak{g} = \left\{ \begin{pmatrix} x & y - z \\ y + z & -x \end{pmatrix} : x, y, z \in \mathbb{R} \right\} \tag{264}$$

One can also identify a dual space $\mathfrak{g}^* \cong \mathfrak{g}$ via the pairing $\langle \phi, X \rangle = \mathrm{Tr}(\phi X)$.

- Show that a generic coadjoint orbit of G can be identified with a level set of the function

$$f(x, y, z) = x^2 + y^2 - z^2 \tag{265}$$

- How many coadjoint orbits of G are contained in the singular level set $f(x, y, z) = 0$?
- Show that symplectic forms on coadjoint orbits of G in cylindrical coordinates can be presented in the form

$$\omega = dz \wedge d\theta, \quad x = \rho\cos\theta, \quad y = \rho\sin\theta \tag{266}$$

Solution.

- The basis of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ is consists of:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{267}$$

$$\begin{pmatrix} x & y - z \\ y + z & -x \end{pmatrix} = y\sigma_x - z\sigma_y + x\sigma_z \tag{268}$$

Let σ_x^*, σ_y^*, σ_z^* is the dual basis of $\mathfrak{sl}^*(2, \mathbb{R})$. Isomorphism $\varphi : \mathfrak{g} \to \mathfrak{g}^*$ maps $\varphi(\sigma_x) = \sigma_x^*$, $\varphi(\sigma_y) = -\sigma_y^*$, $\varphi(\sigma_z) = \sigma_z^*$.
Consider

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}) \tag{269}$$

Every matrix $A \in \mathfrak{sl}(2, \mathbb{R})$ can be reduced to one of the following normal forms ($\lambda > 0$):

(a)

$$\begin{pmatrix} x & y-z \\ y+z & -x \end{pmatrix} = S \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix} S^{-1} = \begin{pmatrix} \lambda(bd-ac) & \lambda(a^2-b^2) \\ -\lambda(c^2-d^2) & -\lambda(bd-ac) \end{pmatrix} \tag{270}$$

$$\begin{cases} x = \lambda(bd-ac), \\ y-z = \lambda(a^2-b^2), \\ y+z = -\lambda(c^2-d^2); \end{cases} \to \begin{cases} x = \lambda(bd-ac), \\ y = \frac{\lambda}{2}(a^2-b^2-c^2+d^2), \\ z = \frac{\lambda}{2}(b^2-a^2+d^2-c^2). \end{cases} \tag{271}$$

$$f(x,y,z) = x^2+y^2-z^2 = \lambda^2(bc-ad)^2 = \lambda^2 \tag{272}$$

Coadjoint orbit:

$$\mathcal{O}_{\lambda\sigma_x^*} = \{y\sigma_x^* + z\sigma_y^* + x\sigma_z^* | x^2+y^2-z^2 = \lambda^2\} \tag{273}$$

It's a hyperboloid of 2 sheet (elliptic hyperboloid).

(b)

$$\begin{pmatrix} x & y-z \\ y+z & -x \end{pmatrix} = S \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} S^{-1} = \begin{pmatrix} -\lambda(ac+bd) & \lambda(a^2+b^2) \\ -\lambda(c^2+d^2) & \lambda(ac+bd) \end{pmatrix} \tag{274}$$

$$\begin{cases} x = -\lambda(ac+bd), \\ y-z = \lambda(a^2+b^2) \geq 0, \\ y+z = -\lambda(c^2+d^2) \leq 0; \end{cases} \to \begin{cases} x = -\lambda(ac+bd), \\ y = \frac{\lambda}{2}(a^2+b^2-c^2-d^2), \\ z = -\frac{\lambda}{2}(a^2+b^2+c^2+d^2) \leq 0; \end{cases} \tag{275}$$

$$f(x,y,z) = x^2+y^2-z^2 = -\lambda^2(bc-ad)^2 = -\lambda^2 \tag{276}$$

Coadjoint orbit:

$$\mathcal{O}_{-\lambda\sigma_y^*} = \{y\sigma_x^* + z\sigma_y^* + x\sigma_z^* | x^2+y^2-z^2 = \lambda^2 | z \leq 0\} \tag{277}$$

It's a lower part of a hyperboloid of 1 sheet.

(c)

$$\begin{pmatrix} x & y-z \\ y+z & -x \end{pmatrix} = S \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} S^{-1} = \begin{pmatrix} \lambda(ad+bc) & -2\lambda ab \\ 2\lambda cd & -\lambda(ad+bc) \end{pmatrix} \tag{278}$$

$$\begin{cases} x = \lambda(ad+bc), \\ y-z = -2\lambda ab, \\ y+z = 2\lambda cd; \end{cases} \to \begin{cases} x = \lambda(ad+bc), \\ y = \lambda(cd-ab), \\ z = \lambda(ab+cd). \end{cases} \tag{279}$$

$$f(x,y,z) = x^2+y^2-z^2 = \lambda^2(bc-ad)^2 = \lambda^2 \tag{280}$$

Coadjoint orbit:

$$\mathcal{O}_{\lambda\sigma_z^*} = \{y\sigma_x^* + z\sigma_y^* + x\sigma_z^* | x^2+y^2-z^2 = \lambda^2\} \tag{281}$$

It's a hyperboloid of 2 sheet (elliptic hyperboloid).

(d)

$$\begin{pmatrix} x & y-z \\ y+z & -x \end{pmatrix} = S \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} S^{-1} = \begin{pmatrix} \lambda bd & -\lambda b^2 \\ \lambda d^2 & -\lambda bd \end{pmatrix} \tag{282}$$

$$\begin{cases} x = \lambda bd, \\ y-z = -\lambda b^2 \leq 0, \\ y+z = \lambda d^2 \geq 0; \end{cases} \to \begin{cases} x = \lambda bd, \\ y = \frac{\lambda}{2}(d^2-b^2), \\ z = \frac{\lambda}{2}(b^2+d^2) \geq 0. \end{cases} \tag{283}$$

$$f(x,y,z)=x^2+y^2-z^2=0 \tag{284}$$

Coadjoint orbit:

$$\{y\sigma_x^*+z\sigma_y^*+x\sigma_z^*|x^2+y^2-z^2=0|z\geq 0\} \tag{285}$$

It's a upper part of a cone.

(e)

$$\begin{pmatrix} x & y-z \\ y+z & -x \end{pmatrix} = S\begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} S^{-1} = \begin{pmatrix} -\lambda ac & \lambda a^2 \\ -\lambda c^2 & \lambda ac \end{pmatrix} \tag{286}$$

$$\begin{cases} x=-\lambda ac, \\ y-z=\lambda a^2\geq 0, \\ y+z=-\lambda c^2\leq 0; \end{cases} \rightarrow \begin{cases} x=-\lambda ac, \\ y=\frac{\lambda}{2}(a^2-c^2), \\ z=-\frac{\lambda}{2}(a^2+c^2)\geq 0. \end{cases} \tag{287}$$

$$f(x,y,z)=x^2+y^2-z^2=0 \tag{288}$$

Coadjoint orbit:

$$\{y\sigma_x^*+z\sigma_y^*+x\sigma_z^*|x^2+y^2-z^2=0|z\leq 0\} \tag{289}$$

It's a lower part of a cone.

(f)

$$\begin{pmatrix} x & y-z \\ y+z & -x \end{pmatrix} = S\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} S^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{290}$$

$$\begin{cases} x=0, \\ y-z=0, \\ y+z=0; \end{cases} \rightarrow \begin{cases} x=0, \\ y=0, \\ z=0. \end{cases} \tag{291}$$

$$f(x,y,z)=x^2+y^2-z^2=0 \tag{292}$$

Coadjoint orbit:

$$\{y\sigma_x^*+z\sigma_y^*+x\sigma_z^*|x=y=z=0\} \tag{293}$$

It's a one point $(0,0,0)$.

- As seen above, the 3 conjugate orbits $SL(2,\mathbb{R})$ are contained in the singular level set $f(x,y,z)=0$.: an upper and a lower parts of a cone and one point $(0,0,0)$.
- Konstant-Kirillov form:

$$\omega=\omega_{\mu\nu}(x)dx^\mu\wedge dx^\nu, \quad \omega_{\mu\nu}=-\omega_{\nu\mu} \tag{294}$$

$$\{f,g\}=\frac{1}{2}\omega_{\mu\nu}\frac{\partial f}{\partial x^\mu}\frac{\partial g}{\partial x^\nu} \tag{295}$$

Polar coordinates on hyperboloid:

$$\begin{cases} \rho=\sqrt{x^2+y^2}=\sqrt{z^2+1}, \\ \tan\theta=\frac{y}{x} \end{cases} \tag{296}$$

Poisson brackets:

$$\{x_i,x_j\}=\epsilon_{ijk}x_k \tag{297}$$

$$\{z,\tan\theta\}=\left\{z,\frac{y}{x}\right\}=\frac{1}{x}\{z,y\}-\frac{y}{x^2}\{z,x\}=-\frac{1}{x}x-\frac{y}{x^2}y=-1-\tan^2\theta=-\frac{1}{\cos^2\theta} \tag{298}$$

$$\{f,g\} = \left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial \theta} - \frac{\partial g}{\partial z}\frac{\partial f}{\partial \theta}\right)\{z,\theta\} \tag{299}$$

$$\{z, \tan\theta\} = -\frac{1}{\cos^2\theta}\{z,\theta\} \to \{z,\theta\} = 1 \tag{300}$$

$$\boxed{\omega = \frac{1}{2}dz \wedge d\theta - \frac{1}{2}d\theta \wedge dz = dz \wedge d\theta} \tag{301}$$

3 Hamiltonian reduction, projection method

1. **Free particle?**
Let M be a four-dimensional phase space $T^*\mathbb{R}$ with the canonical symplectic structure

$$\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2 \tag{302}$$

Consider a system of a free particle with Hamilton function defined on this phase space

$$H = \frac{1}{2}(p_1^2 + p_2^2) \tag{303}$$

- Show that the diagonal action of $G = U(1) = SO(2)$ on the planes (q_1, q_2) and (p_1, p_2) is Hamiltonian and find the momentum map $\mu(p, q)$ of this action.
- Check that the Hamilton function is invariant with respect to this action and find the corresponding Hamiltonian vector field v_H. Find integral curves $(q(t), p(t))$ of v_H with initial conditions $(q(0), p(0))$.
- Show that for $\mu(p, q) = l \neq 0$ the reduced phase space is $M_l \simeq \mathbb{R}_+ \times \mathbb{R}$.
- Find the expression for the reduced Hamilton function H_l. Check that the projection $r(t), p_r(t)$ of the integral curve found above is indeed an integral curve of the Hamiltonian vector field v_{H_l}. What is the mechanical interpretation of this result?

Solution.

-
$$g = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \in G = SO(2) \tag{304}$$

$$g.\begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 & 0 \\ -\sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & \cos\varphi & \sin\varphi \\ 0 & 0 & -\sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} q_1\cos\varphi + q_2\sin\varphi \\ q_2\cos\varphi - q_1\sin\varphi \\ p_1\cos\varphi + p_2\sin\varphi \\ p_2\cos\varphi - p_1\sin\varphi \end{pmatrix} \tag{305}$$

$$\xi = \begin{pmatrix} 0 & \varphi \\ -\varphi & 0 \end{pmatrix} \in \mathfrak{so}(2) \tag{306}$$

$$v_\xi(t) = \frac{d}{dt}(e^{\xi t}.x)|_{t=0} \tag{307}$$

$$v_\xi = q_2\frac{\partial}{\partial q_1} - q_1\frac{\partial}{\partial q_2} + p_2\frac{\partial}{\partial p_1} - p_1\frac{\partial}{\partial p_2} \tag{308}$$

$$\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2 \tag{309}$$

Check that action is weakly Hamiltonian:

$$i_{v_\xi}\omega = -q_2dp_1 + q_1dp_2 + p_2dq_1 - p_1dq_2 = -d(p_1q_2 - p_2q_1) = -dH_\xi \tag{310}$$

$$H_\xi = p_1 q_2 - p_2 q_1 \tag{311}$$

Check that the action is Hamiltonian:

$$\forall g \in SO(2) \forall \xi \in \mathfrak{so}(2) \hookrightarrow g_* H_\xi = H_{\mathrm{Ad}_g(\xi)} = H_\xi \tag{312}$$

$$\begin{aligned} H_\xi = p_1 q_2 - p_2 q_1 \to (p_1 \cos\varphi + p_2 \sin\varphi)(q_2 \cos\varphi - q_1 \sin\varphi) - \\ - (p_2 \cos\varphi - p_1 \sin\varphi)(q_1 \cos\varphi + q_2 \sin\varphi) = p_1 q_2 - p_2 q_1 \end{aligned} \tag{313}$$

Momentium map:

$$\boxed{\mu(p,q) = p_1 q_2 - p_2 q_1} \tag{314}$$

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$$H = \frac{1}{2}(p_1^2 + p_2^2) \to \frac{1}{2}((p_1 \cos\varphi + p_2 \sin\varphi)^2 + (p_2 \cos\varphi - p_1 \sin\varphi)^2) = \frac{1}{2}(p_1^2 + p_2^2) \tag{315}$$

Hamiltonian vector field v_H:

$$i_{v_H}\omega = -dH = -p_1 dp_1 - p_2 dp_2 \tag{316}$$

$$\boxed{v_H = p_1 \frac{\partial}{\partial q_1} + p_2 \frac{\partial}{\partial q_2}} \tag{317}$$

$$\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{p}_1 \\ \dot{p}_2 \end{pmatrix} = v_H \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \\ 0 \\ 0 \end{pmatrix} \to \boxed{\begin{cases} q_1(t) = p_1(0)t + q_1(0), \\ q_2(t) = p_2(0)t + q_2(0), \\ p_1(t) = p_1(0), \\ p_2(t) = p_2(0). \end{cases}} \tag{318}$$

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$$\mu(p,q) = p_1 q_2 - p_2 q_1 = l \neq 0 \tag{319}$$

$$M_l = \mu^{-1}(l)/G_l, \quad G_l = \{g \in G = SO(2) | \mathrm{Ad}_g^* l = l\} = G = SO(2) \tag{320}$$

Polar coordinates:

$$\begin{cases} q_1 = r\cos\varphi, \\ q_2 = r\sin\varphi \end{cases} \tag{321}$$

$$\begin{aligned} p_1 dq_1 + p_2 dq_2 = p_1(dr\cos\varphi - r\sin\varphi d\varphi) + p_2(dr\sin\varphi + r\cos\varphi d\varphi) = \\ = dr(p_1\cos\varphi + p_2\sin\varphi) + rd\varphi(p_2\cos\varphi - p_1\sin\varphi) = p_r dr + p_\varphi d\varphi \end{aligned} \tag{322}$$

$$\begin{cases} p_r = p_1\cos\varphi + p_2\sin\varphi, \\ p_\varphi = p_2 r\cos\varphi - p_1 r\sin\varphi = p_2 q_1 - p_1 q_2 = -l \end{cases} \tag{323}$$

$$\mu^{-1}(l) = \{(q_1, p_1, q_2, p_2 | p_1 q_2 - p_2 q_1 = l)\} = \{(r, p_r, \varphi, p_\varphi) | p_\varphi = -l\} \tag{324}$$

$$\boxed{M_l = \{(r, p_r, p_\varphi) | p_\varphi = -l\} \simeq \mathbb{R}_+ \times \mathbb{R}} \tag{325}$$

- Find the expression for the reduced Hamilton function H_l.

$$\begin{cases} p_1 = p_r \cos\varphi - \frac{p_\varphi \sin\varphi}{r}, \\ p_2 = p_r \sin\varphi + \frac{p_\varphi \cos\varphi}{r} \end{cases} \tag{326}$$

$$H = \frac{p_1^2 + p_2^2}{2} = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) \to \boxed{H_l = \frac{1}{2}\left(p_r^2 + \frac{l^2}{r^2}\right)} \tag{327}$$

$$r(t) = \sqrt{q_1^2(t) + q_2^2(t)} = \sqrt{(p_1(0)t + q_1(0))^2 + (p_2(0)t + q_2(0))^2} \tag{328}$$

$$\boxed{r(t) = \sqrt{2H(0)t^2 + 2r(0)p_r(0)t + r^2(0)}} \tag{329}$$

$$\begin{cases} \cos\varphi(t) = \frac{p_1(0)t + q_1(0)}{\sqrt{2H(0)t^2 + 2r(0)p_r(0)t + r^2(0)}}, \\ \sin\varphi(t) = \frac{p_2(0)t + q_2(0)}{\sqrt{2H(0)t^2 + 2r(0)p_r(0)t + r^2(0)}} \end{cases} \tag{330}$$

$$\boxed{p_r(t) = \frac{2H(0)t + r(0)p_r(0)}{\sqrt{2H(0)t^2 + 2r(0)p_r(0)t + r^2(0)}}} \tag{331}$$

$$\begin{aligned} \omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2 = \\ = \left(dp_r \cos\varphi - p_r \sin\varphi d\varphi - \frac{dp_\varphi \sin\varphi}{r} - \frac{p_\varphi \cos\varphi d\varphi}{r} + \frac{p_\varphi \sin\varphi dr}{r^2}\right) \wedge \\ \wedge (dr \cos\varphi - r \sin\varphi d\varphi) + \left(dp_r \sin\varphi + p_r \cos\varphi d\varphi + \frac{dp_\varphi \cos\varphi}{r} - \frac{p_\varphi \sin\varphi d\varphi}{r} - \frac{p_\varphi \cos\varphi dr}{r^2}\right) \wedge \\ \wedge (dr \sin\varphi + r \cos\varphi d\varphi) = dp_r \wedge dr + dp_\varphi \wedge d\varphi \end{aligned} \tag{332}$$

$$i_{v_{H_l}}\omega = -dH_l = -p_r dp_r + \frac{l^2}{r^3} dr \tag{333}$$

$$\boxed{v_{H_l} = p_r \frac{\partial}{\partial r} + \frac{l^2}{r^3}\frac{\partial}{\partial p_r} + \frac{l}{r^2}\frac{\partial}{\partial \varphi}} \tag{334}$$

$$\begin{pmatrix} \dot{r} \\ \dot{\varphi} \\ \dot{p}_r \\ \dot{p}_\varphi \end{pmatrix} = v_H \begin{pmatrix} r \\ \varphi \\ p_r \\ p_\varphi \end{pmatrix} = \begin{pmatrix} p_r \\ \frac{l}{r^2} \\ \frac{l^2}{r^3} \\ 0 \end{pmatrix} \to \boxed{\begin{cases} r(t) = \sqrt{c_1 t^2 + c_2 t + c_3}, \\ p_r(t) = \frac{c_1 t + \frac{c_2}{t}}{\sqrt{c_1 t^2 + c_2 t + c_3}} \end{cases}} \tag{335}$$

As seen, the projection of the integral curve is indeed an intergral curve of the vector field v_{H_l}. Physical meaning that when projected, the motion effectively becomes one-dimensional and potential energy $V_{\text{eff}} = \frac{l^2}{2r^2}$ appears.

2. **Geodesic moving.**
   Consider a particle with mass moving on the geodesics on a two-dimensional sphere $x_0^2 + x_1^2 + x_2^2 = 1$.

   - Write explicitly the geodesic equation on the sphere and the generic form of geodesic line.

- Consider the projection map

$$q = \pi(x) = \arccos x_0 \tag{336}$$

Find the Hamilton function $H(p,q)$ which describes the motion in the system after projection.

- Use the form of geodesic line to solve explicitly the equations of motion for the system after projection.

Solution.

- Lagrangian of a particle with mass moving on a sphere:

$$L = \frac{m\dot{\boldsymbol{x}}^2}{2} = \frac{m}{2}(\dot\theta^2 + \sin^2\theta\dot\varphi^2) \tag{337}$$

Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} = 0 \to \begin{cases} \ddot\theta - \dot\varphi^2 \sin\theta\cos\theta = 0, \\ \ddot\varphi \sin^2\theta + 2\dot\varphi\dot\theta\sin\theta\cos\theta = 0 \end{cases} \tag{338}$$

$$\frac{\partial L}{\partial \varphi} = 0 \to p_\varphi = \frac{\partial L}{\partial \dot\varphi} = m\sin^2\theta\dot\varphi = \text{const} \tag{339}$$

$$\dot\varphi = \frac{p_\varphi}{m\sin^2\theta} \tag{340}$$

Legendre transformation:

$$H = p_\varphi\dot\varphi + p_\theta\dot\theta - L = \frac{m}{2}(\dot\theta^2 + \sin^2\theta\dot\varphi^2) = \text{const} \tag{341}$$

$$H = \frac{m}{2}\left(\dot\theta^2 + \frac{p_\varphi^2}{m^2\sin^2\theta}\right) \to \dot\theta = \sqrt{\frac{2H}{m} - \frac{p_\varphi^2}{m^2\sin^2\theta}} \tag{342}$$

$$\int \frac{d\theta}{\sqrt{\frac{2H}{m} - \frac{p_\varphi^2}{m^2\sin^2\theta}}} = \int dt = t - t_0 \tag{343}$$

$$\begin{aligned} \int \frac{d\theta}{\sqrt{\frac{2H}{m} - \frac{p_\varphi^2}{m^2\sin^2\theta}}} &= \pm\int \frac{\sin\theta d\theta}{\sqrt{\frac{2H}{m}\sin^2\theta - \frac{p_\varphi^2}{m^2}}} = \pm\int \frac{d\cos\theta}{\sqrt{\frac{2H}{m} - \frac{p_\varphi^2}{m^2} - \frac{2H}{m}\cos^2\theta}} = \\ &= \pm m\int \frac{\frac{d\cos\theta}{\sqrt{2mH - p_\varphi^2}}}{\sqrt{1 - \frac{2mH}{2mH - p_\varphi^2}\cos^2\theta}} = \pm\sqrt{\frac{m}{2H}}\arcsin\left(\frac{\cos\theta}{\sqrt{1 - \frac{p_\varphi^2}{2mH}}}\right) \end{aligned} \tag{344}$$

$$\boxed{\cos\theta = \pm\sqrt{1 - \frac{p_\varphi^2}{2mH}}\sin\left(\sqrt{\frac{2H}{m}}(t - t_0)\right)} \tag{345}$$

$$\int d\varphi = \int \frac{p_\varphi dt}{m\left(1 - \left(1 - \frac{p_\varphi^2}{2mH}\right)\sin^2\left(\sqrt{\frac{2H}{m}}(t - t_0)\right)\right)} = \varphi - \varphi_0 \tag{346}$$

$$\int \frac{p_\varphi dt}{m\left(1-\left(1-\frac{p_\varphi^2}{2mH}\right)\sin^2\left(\sqrt{\frac{2H}{m}}(t-t_0)\right)\right)} = \int \frac{d\left(\sqrt{\frac{p_\varphi^2}{2mH}}\tan\left(\sqrt{\frac{2H}{m}}(t-t_0)\right)\right)}{1+\frac{p_\varphi^2}{2mH}\tan^2\left(\sqrt{\frac{2H}{m}}(t-t_0)\right)} = $$
$$= \arctan\left(\sqrt{\frac{p_\varphi^2}{2mH}}\tan\left(\sqrt{\frac{2H}{m}}(t-t_0)\right)\right) \quad (347)$$

$$\boxed{\varphi = \varphi_0 + \arctan\left(\sqrt{\frac{p_\varphi^2}{2mH}}\tan\left(\sqrt{\frac{2H}{m}}(t-t_0)\right)\right)} \quad (348)$$

Geodesic curves – circles of large diameter.

- $$x_0 = \cos\theta \to q = \mathrm{arccos} x_0 = \theta, \quad p_\theta = p \quad (349)$$

  So, hamiltonian after projection will only have terms with θ:

$$\boxed{H(p,q) = \frac{p^2}{2m} + \frac{p_\varphi^2}{2m\sin^2 q}} \quad (350)$$

- Parametrization of geodesic line:

$$\boldsymbol{r}(t) = \boldsymbol{r}_0 \cos t + \dot{\boldsymbol{r}}_0 \sin t \quad (351)$$

$$|\boldsymbol{r}_0| = |\boldsymbol{r}(0)| = 1, \quad |\dot{\boldsymbol{r}}_0| = |\dot{\boldsymbol{r}}(0)| = 1, \quad \boldsymbol{r}_0 \cdot \dot{\boldsymbol{r}}_0 = 0 \quad (352)$$

$$\boxed{\begin{cases} q(t) = \cos\theta(t) = z_0\cos t + \dot{z}_0 \sin t, \\ p(t) = p_\theta(t) = \sqrt{2mH - \frac{p_\varphi^2}{\sin^2 q(t)}} \end{cases}} \quad (353)$$

3. **Kepler problem – 1.**
Consider the standard Kepler problem in a three-dimensional space with the Hamilton function
$$H = \frac{p^2}{2} - \frac{\alpha}{r} \quad (354)$$
where $p^2 = p_1^2 + p_2^2 + p_3^2$, $r = \sqrt{q_1^2+q_2^2+q_3^2}$ and the coupling constant $\alpha > 0$. The equations of motion of this system are given by

$$\begin{cases} \dot{q}_i = p_i, \\ \dot{p}_i = -\frac{\alpha}{r^3}q_i \end{cases} \quad (355)$$

- Find the integrals of motion for the system using Laplace method. Let $I(p,q)$ be an integral of motion, then

$$\{H, I(p,q)\} = \sum_{i=1}^{3}\left(\frac{\partial I}{\partial q_i}p_i - \frac{\partial I}{\partial p_i}\frac{\alpha}{r^3}q_i\right) = 0 \quad (356)$$

  Expand $I(p,q)$ in homogeneous polynomials in p_i of degree $k \geq 0$

$$I(p,q) = \sum_{k=0}^{\infty} I_k(p,q) \quad (357)$$

Write the system of equations on I_k, which follows from the conservation of I. Show that this system is finite if one assumes I to be a polynomial in p_i of degree n. Solve these finite systems for $n = 1, 2$ and write the corresponding integrals of motion. How many functionally independent conservation laws are obtained in this way?

- Consider the angular momentum vector and the Laplace vector

$$\boldsymbol{l} = [\boldsymbol{q} \times \boldsymbol{p}], \quad \boldsymbol{A} = [\boldsymbol{l} \times \boldsymbol{p}] + \frac{\alpha}{r}\boldsymbol{q} \tag{358}$$

Show that the components of these vector are integrals of motion. Check that the Poisson brackets between the components correspond to the Lie algebra $\mathfrak{so}(4)$. Namely, check that

$$\{l_i, l_j\} = \epsilon_{ijk} l_k, \quad \{l_i, A_j\} = \epsilon_{ijk} A_k, \quad \{A_i, A_j\} = -2H\epsilon_{ijk} l_k \tag{359}$$

and make an appropriate change of variables to show this Poisson algebra corresponds to $\mathfrak{so}(4)$ Lie algebra. These calculations shows that the Kepler problem has $\mathfrak{so}(4)$ symmetry.

Solution.

- Expand $I(p, q)$ in homogeneous polynomials in p_i of degree $k \geq 0$

$$I(p, q) = \sum_{k=0}^{\infty} I_k(p, q), \quad I_k(\lambda p, q) = \lambda^k I_k(p, q) \tag{360}$$

$I(p, q)$ is an integral of motion, so

$$\{H, I(p, q)\} = \left\{H, \sum_{k=0}^{\infty} I_k(p, q)\right\} = \sum_{i=1}^{3}\left(p_i \frac{\partial}{\partial q_i}\sum_{k=0}^{\infty} I_k - \frac{\alpha q_i}{r^3}\frac{\partial}{\partial p_i}\sum_{k=0}^{\infty} I_k\right) = 0 \tag{361}$$

The system of equations on I_k:

$$\boxed{\begin{cases} \sum\limits_{i=1}^{3} q_i \frac{\partial I_1}{\partial p_i} = 0, \\ \sum\limits_{i=1}^{3}\left(p_i \frac{\partial I_{k-1}}{\partial q_i} - \frac{\alpha q_i}{r^3}\frac{\partial I_{k+1}}{\partial p_i}\right) = 0, \quad k \geq 1 \end{cases}} \tag{362}$$

This system is finite if one assumes I to be a polynomial in p_i of degree n:

$$I(p, q) = \sum_{k=0}^{n} I_k(p, q) \tag{363}$$

$$\boxed{\begin{cases} \sum\limits_{i=1}^{3} q_i \frac{\partial I_1}{\partial p_i} = 0, \\ \sum\limits_{i=1}^{3}\left(p_i \frac{\partial I_{k-1}}{\partial q_i} - \frac{\alpha q_i}{r^3}\frac{\partial I_{k+1}}{\partial p_i}\right) = 0, \quad k \in \{1, ..., n-1\}, n > 1 \\ \sum\limits_{i=1}^{3} p_i \frac{\partial I_{n-1}}{\partial q_i} = 0, \\ \sum\limits_{i=1}^{3} p_i \frac{\partial I_n}{\partial q_i} = 0. \end{cases}} \tag{364}$$

Consider cases:

– $n=1$.

$$\begin{cases} \sum_{i=1}^{3} q_i \frac{\partial I_1}{\partial p_i} = 0, \\ \sum_{i=1}^{3} p_i \frac{\partial I_0}{\partial q_i} = 0, \\ \sum_{i=1}^{3} p_i \frac{\partial I_1}{\partial q_i} = 0. \end{cases} \tag{365}$$

$I_0 = \text{const}, \ I_1 = f_1(q)p_1 + f_2(q)p_2 + f_3(q)p_3$.

$$\begin{cases} \sum_{i=1}^{3} q_i f_i(q) = q_1 f_1(q) + q_2 f_2(q) + q_3 f_3(q) = 0, \\ \sum_{i=1}^{3} p_i \frac{\partial (f_1(q)p_1 + f_2(q)p_2 + f_3(q)p_3)}{\partial q_i} = p_1(\frac{\partial f_1}{\partial q_1}p_1 + \frac{\partial f_2}{\partial q_1}p_2 + \frac{\partial f_3}{\partial q_1}p_3) + \\ +p_2(\frac{\partial f_1}{\partial q_2}p_1 + \frac{\partial f_2}{\partial q_2}p_2 + \frac{\partial f_3}{\partial q_2}p_3) + p_3(\frac{\partial f_1}{\partial q_3}p_1 + \frac{\partial f_2}{\partial q_3}p_2 + \frac{\partial f_3}{\partial q_3}p_3) = 0. \end{cases} \tag{366}$$

From the first equation $f_i(q) = f_{i1}q_1 + f_{i2}q_2 + f_{i3}q_3$.

$$\begin{cases} q_1(f_{11}q_1 + f_{12}q_2 + f_{13}q_3) + q_2(f_{21}q_1 + f_{22}q_2 + f_{23}q_3) + \\ +q_3(f_{31}q_1 + f_{32}q_2 + f_{33}q_3) = 0, \\ p_1(f_{11}p_1 + f_{21}p_2 + f_{31}p_3) + p_2(f_{12}p_1 + f_{22}p_2 + f_{32}p_3) + \\ +p_3(f_{13}p_1 + f_{23}p_2 + f_{33}p_3) = 0. \end{cases} \tag{367}$$

$f_{11} = f_{22} = f_{33} = 0, \ f_{12} + f_{21} = 0, \ f_{13} + f_{31} = 0, \ f_{23} + f_{32} = 0$.

$$f_{12} = -f_{21} = c_1, \quad f_{13} = -f_{31} = c_2, \quad f_{23} = -f_{32} = c_3 \tag{368}$$

$$I_0 = \text{const}, \quad I_1 = c_1(q_2p_1 - q_1p_2) + c_2(q_3p_1 - q_1p_3) + c_3(q_3p_2 - q_2p_3) \tag{369}$$

$$\boxed{I = I_0 - c_1 l_3 + c_2 l_2 - c_3 l_1} \tag{370}$$

We have 3 functionally independent conservation laws.

– $n=2$.

$$\begin{cases} \sum_{i=1}^{3} q_i \frac{\partial I_1}{\partial p_i} = 0, \\ \sum_{i=1}^{3} \left(p_i \frac{\partial I_0}{\partial q_i} - \frac{\alpha q_i}{r^3} \frac{\partial I_2}{\partial p_i} \right) = 0, \\ \sum_{i=1}^{3} p_i \frac{\partial I_1}{\partial q_i} = 0, \\ \sum_{i=1}^{3} p_i \frac{\partial I_2}{\partial q_i} = 0. \end{cases} \tag{371}$$

Equations on I_1 remain the same.

$$I_1 = -c_1 l_3 + c_2 l_2 - c_3 l_1 \tag{372}$$

$$\begin{cases} \sum_{i=1}^{3} \left(p_i \frac{\partial I_0}{\partial q_i} - \frac{\alpha q_i}{r^3} \frac{\partial I_2}{\partial p_i} \right) = 0, \\ \sum_{i=1}^{3} p_i \frac{\partial I_2}{\partial q_i} = 0. \end{cases} \tag{373}$$

$I_0 = f(q),\ I_2 = \sum_{i,j} a_{ij}(q) p_i p_j,\ a_{ij} = a_{ji}.$

$$\sum_{i=1}^{3} \left(p_i \frac{\partial I_0}{\partial q_i} - \frac{\alpha q_i}{r^3} \frac{\partial I_2}{\partial p_i} \right) = \sum_{i=1}^{3} \left(p_i \frac{\partial f}{\partial q_i} - \frac{2\alpha q_i}{r^3} \sum_j a_{ij} p_j \right) = \\ = \sum_{i=1}^{3} p_i \left(\frac{\partial f}{\partial q_i} - \frac{2\alpha}{r^3} \sum_j a_{ij} q_j \right) = 0 \quad (374)$$

$$\sum_{i=1}^{3} p_i \frac{\partial I_2}{\partial q_i} = \sum_{i=1}^{3} p_i^3 \frac{\partial a_{ii}}{\partial q_i} + \sum_{i \neq j} p_i^2 p_j \left(\frac{\partial a_{ii}}{\partial q_j} + 2 \frac{\partial a_{ij}}{\partial q_i} \right) + \\ + p_1 p_2 p_3 \left(\frac{\partial a_{12}}{\partial q_3} + \frac{\partial a_{13}}{\partial q_2} + \frac{\partial a_{23}}{\partial q_1} \right) = 0 \quad (375)$$

$$\begin{cases} \frac{\partial f}{\partial q_i} - \frac{2\alpha}{r^3} \sum_j a_{ij} q_j = 0, \\ \frac{\partial a_{ii}}{\partial q_i} = 0, \\ \frac{\partial a_{ii}}{\partial q_j} + 2 \frac{\partial a_{ij}}{\partial q_i} = 0, \\ \frac{\partial a_{12}}{\partial q_3} + \frac{\partial a_{13}}{\partial q_2} + \frac{\partial a_{23}}{\partial q_1} = 0; \end{cases} \quad (376)$$

$$\frac{\partial f}{\partial q_i} = \frac{2\alpha}{r^3} \sum_j a_{ij} q_j \to \frac{\partial^2 f}{\partial q_j \partial q_i} = \frac{2\alpha}{r^3} \sum_k \frac{\partial a_{ik}}{\partial q_j} q_k + \frac{2\alpha}{r^3} a_{ij} - \frac{6\alpha q_j}{r^5} \sum_k a_{ik} q_k \quad (377)$$

$$\frac{\partial^2 f}{\partial q_j \partial q_i} - \frac{\partial^2 f}{\partial q_i \partial q_j} = \frac{2\alpha}{r^3} \sum_k \left(\frac{\partial a_{ik}}{\partial q_j} - \frac{\partial a_{jk}}{\partial q_i} \right) q_k - \frac{6\alpha}{r^5} \sum_k (a_{ik} q_j - a_{jk} q_i) q_k = 0 \quad (378)$$

a_{ij} is homogeneous over q:

$$a_{ij}(q) = \alpha_{ij} + \sum_k \alpha_{ijk} q_k, \quad a_{ijk} = a_{jik} \quad (379)$$

$$\frac{2\alpha}{r^3} \sum_k \left(\alpha_{ikj} - \alpha_{jki} \right) q_k - \frac{6\alpha}{r^5} \sum_k (\alpha_{ik} q_j - \alpha_{jk} q_i) q_k - \\ - \frac{6\alpha}{r^5} \sum_{k,l} (\alpha_{ikl} q_j - \alpha_{jkl} q_i) q_k q_l = 0 \quad (380)$$

$$\frac{2\alpha}{r^5} \sum_{k,l} \left(\alpha_{ikj} - \alpha_{jki} \right) q_k q_l q_l - \frac{6\alpha}{r^5} \sum_k (\alpha_{ik} q_j - \alpha_{jk} q_i) q_k - \\ - \frac{6\alpha}{r^5} \sum_{k,l} (\alpha_{ikl} q_j - \alpha_{jkl} q_i) q_k q_l = 0 \quad (381)$$

$$\alpha_{ij} = a \delta_{ij} \quad (382)$$

$$\begin{cases} \alpha_{ijk} = \alpha_{jik}, \\ \alpha_{ikl} = \delta_{ik} a_l \text{ or } \alpha_{ikl} = \delta_{il} a_j, \\ \alpha_{iii} = 0, \\ \alpha_{iij} + 2\alpha_{iji} = 0, \\ \alpha_{123} + \alpha_{132} + \alpha_{231} = 0; \end{cases} \quad (383)$$

$$\begin{cases} \alpha_{ii1} = a_1, \alpha_{ii2} = a_2, \alpha_{ii3} = a_3, \\ \alpha_{i1i} = \alpha_{1ii} = -\frac{a_1}{2}, \\ \alpha_{i2i} = \alpha_{2ii} = -\frac{a_2}{2}, \\ \alpha_{i3i} = \alpha_{3ii} = -\frac{a_3}{2}, \\ \alpha_{iii} = \alpha_{ijk} = 0. \end{cases} \tag{384}$$

$$\begin{gathered} I_2 = \sum_{i,j} a_{ij}(q)p_ip_j = (a+a_2q_2+a_3q_3)p_1^2+(a+a_1q_1+a_3q_3)p_2^2+(a+a_1q_1+a_2q_2)p_3^2- \\ -(a_2q_1+a_1q_2)p_1p_2-(a_3q_2+a_2q_3)p_2p_3-(a_3q_1+a_1q_3)p_1p_3 = \\ = a(p_1^2+p_2^2+p_3^2)+a_1(q_1(p_2^2+p_3^2)-p_1(q_2p_2+q_3p_3))+ \\ +a_2(q_2(p_1^2+p_3^2)-p_2(q_1p_1+q_3p_3))+a_3(q_3(p_1^2+p_2^2)-p_3(q_1p_1+q_2p_2)) = \\ = ap^2+a_1(q_1p^2-p_1\boldsymbol{qp})+a_2(q_2p^2-p_2\boldsymbol{qp})+a_3(q_3p^2-p_3\boldsymbol{qp}) \end{gathered} \tag{385}$$

$$I_2 = ap^2 - a_1[\boldsymbol{l}\times\boldsymbol{p}]_1 - a_2[\boldsymbol{l}\times\boldsymbol{p}]_2 - a_3[\boldsymbol{l}\times\boldsymbol{p}]_3 \tag{386}$$

$$\begin{cases} \frac{\partial f}{\partial q_1} = \frac{2\alpha}{r^3}\sum_j a_{1j}q_j = \frac{\alpha}{r^3}(2aq_1+a_2q_1q_2+a_3q_1q_3-a_1q_2^2-a_1q_3^2), \\ \frac{\partial f}{\partial q_2} = \frac{2\alpha}{r^3}\sum_j a_{2j}q_j = \frac{\alpha}{r^3}(2aq_2+a_1q_1q_2+a_3q_2q_3-a_2q_1^2-a_2q_3^2), \\ \frac{\partial f}{\partial q_3} = \frac{2\alpha}{r^3}\sum_j a_{3j}q_j = \frac{\alpha}{r^3}(2aq_3+a_1q_1q_3+a_2q_2q_3-a_3q_1^2-a_3q_2^2); \end{cases} \tag{387}$$

$$\begin{cases} \frac{\partial f}{\partial q_1} = \alpha(\frac{2aq_1}{r^3}+\frac{a_1q_1+a_2q_2+a_3q_3}{r^3}q_1-\frac{a_1}{r}), \\ \frac{\partial f}{\partial q_2} = \alpha(\frac{2aq_2}{r^3}+\frac{a_1q_1+a_2q_2+a_3q_3}{r^3}q_2-\frac{a_2}{r}), \\ \frac{\partial f}{\partial q_3} = \alpha(\frac{2aq_3}{r^3}+\frac{a_1q_1+a_2q_2+a_3q_3}{r^3}q_3-\frac{a_3}{r}). \end{cases} \tag{388}$$

$$I_0 = f(q) = -\frac{2a\alpha}{r} - \frac{\alpha}{r}(a_1q_1+a_2q_2+a_3q_3) \tag{389}$$

$$\begin{gathered} I = I_0+I_1+I_2 = -\frac{2a\alpha}{r} - \frac{\alpha}{r}(a_1q_1+a_2q_2+a_3q_3) - c_1l_3+c_2l_2-c_3l_1+ \\ +ap^2-a_1[\boldsymbol{l}\times\boldsymbol{p}]_1-a_2[\boldsymbol{l}\times\boldsymbol{p}]_2-a_3[\boldsymbol{l}\times\boldsymbol{p}]_3 = 2a\left(\frac{p^2}{2}-\frac{\alpha}{r}\right)-c_1l_3+c_2l_2-c_3l_1- \\ -a_1\left([\boldsymbol{l}\times\boldsymbol{p}]_1+\frac{\alpha}{r}q_1\right)-a_2\left([\boldsymbol{l}\times\boldsymbol{p}]_2+\frac{\alpha}{r}q_2\right)-a_3\left([\boldsymbol{l}\times\boldsymbol{p}]_3+\frac{\alpha}{r}q_3\right) \end{gathered} \tag{390}$$

$$\boxed{I = 2aH - c_1l_3 + c_2l_2 - c_3l_1 - a_1A_1 - a_2A_2 - a_3A_3} \tag{391}$$

$$\mathrm{rg}\left(\frac{\partial(H,\boldsymbol{l},\boldsymbol{A})}{\partial(\boldsymbol{q},\boldsymbol{p})}\right) = \mathrm{rg}\begin{pmatrix} \frac{\partial H}{\partial q_1} & \frac{\partial H}{\partial q_2} & \frac{\partial H}{\partial q_3} & \frac{\partial H}{\partial p_1} & \frac{\partial H}{\partial p_2} & \frac{\partial H}{\partial p_3} \\ \frac{\partial l_1}{\partial q_1} & \frac{\partial l_1}{\partial q_2} & \frac{\partial l_1}{\partial q_3} & \frac{\partial l_1}{\partial p_1} & \frac{\partial l_1}{\partial p_2} & \frac{\partial l_1}{\partial p_3} \\ \frac{\partial l_2}{\partial q_1} & \frac{\partial l_2}{\partial q_2} & \frac{\partial l_2}{\partial q_3} & \frac{\partial l_2}{\partial p_1} & \frac{\partial l_2}{\partial p_2} & \frac{\partial l_2}{\partial p_3} \\ \frac{\partial l_3}{\partial q_1} & \frac{\partial l_3}{\partial q_2} & \frac{\partial l_3}{\partial q_3} & \frac{\partial l_3}{\partial p_1} & \frac{\partial l_3}{\partial p_2} & \frac{\partial l_3}{\partial p_3} \\ \frac{\partial A_1}{\partial q_1} & \frac{\partial A_1}{\partial q_2} & \frac{\partial A_1}{\partial q_3} & \frac{\partial A_1}{\partial p_1} & \frac{\partial A_1}{\partial p_2} & \frac{\partial A_1}{\partial p_3} \\ \frac{\partial A_2}{\partial q_1} & \frac{\partial A_2}{\partial q_2} & \frac{\partial A_2}{\partial q_3} & \frac{\partial A_2}{\partial p_1} & \frac{\partial A_2}{\partial p_2} & \frac{\partial A_2}{\partial p_3} \\ \frac{\partial A_3}{\partial q_1} & \frac{\partial A_3}{\partial q_2} & \frac{\partial A_3}{\partial q_3} & \frac{\partial A_3}{\partial p_1} & \frac{\partial A_3}{\partial p_2} & \frac{\partial A_3}{\partial p_3} \end{pmatrix} = 5 \tag{392}$$

We obtain 5 functionally independent conservation laws.

- $$l_i = \epsilon_{ijk} q_j p_k, \quad A_i = \epsilon_{ijk} l_j p_k + \frac{\alpha}{r} q_i \tag{393}$$

Poisson brackets:

$$\{q_i, p_j\} = \delta_{ij}, \quad \{q_i, q_j\} = \{p_i, p_j\} = 0 \tag{394}$$

– Check that $\{l_i, l_j\} = \epsilon_{ijk} l_k$:

$$\{l_i, q_j\} = \epsilon_{ij'k}\{q_{j'} p_k, q_j\} = -\epsilon_{ij'k} q_{j'} \delta_{kj} = -\epsilon_{ij'j} q_{j'} = \epsilon_{ijk} q_k \tag{395}$$

$$\{l_i, p_j\} = \epsilon_{ij'k}\{q_{j'} p_k, p_j\} = \epsilon_{ij'k} p_k \delta_{j'j} = \epsilon_{ijk} p_k \tag{396}$$

$$\begin{aligned}\{l_i, l_j\} &= \{l_i, \epsilon_{jkl} q_k p_l\} = \epsilon_{jkl}(\{l_i, q_k\} p_l + q_k \{l_i, p_l\}) = \\ &= \epsilon_{jkl}\epsilon_{ikm} q_m p_l + \epsilon_{jkl}\epsilon_{ilm} q_k p_m = (\delta_{ji}\delta_{lm} - \delta_{jm}\delta_{il}) q_m p_l - \\ &- (\delta_{ji}\delta_{km} - \delta_{jm}\delta_{ik}) q_k p_m = \delta_{ij} q_l p_l - q_j p_i - \delta_{ij} q_k p_k + q_i p_j = \\ &= q_i p_j - q_j p_i \end{aligned} \tag{397}$$

$$\epsilon_{ijk} l_k = \epsilon_{ijk}\epsilon_{klm} q_l p_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) q_l p_m = q_i p_j - q_j p_i \tag{398}$$

$$\boxed{\{l_i, l_j\} = \epsilon_{ijk} l_k} \tag{399}$$

– Check that $\{l_i, A_j\} = \epsilon_{ijk} A_k$:

$$\{q_i, f(q)\} = 0, \quad \{p_i, f(q)\} = -\frac{\partial f}{\partial q_i} \tag{400}$$

$$\left\{q_i, \frac{\alpha}{r}\right\} = 0, \quad \left\{p_i, \frac{\alpha}{r}\right\} = \frac{\alpha q_i}{r^3} \tag{401}$$

$$\left\{l_i, \frac{\alpha}{r}\right\} = \epsilon_{ijk}\left\{p_j q_k, \frac{\alpha}{r}\right\} = \epsilon_{ijk}\frac{\alpha}{r^3} q_j q_k = 0 \tag{402}$$

$$\begin{aligned}\{l_i, A_j\} &= \left\{l_i, \epsilon_{jkl} l_k p_l + \frac{\alpha}{r} q_j\right\} = \epsilon_{jkl}(\{l_i, l_k\} p_l + l_k \{l_i, p_l\}) + \\ &+ \left\{l_i, \frac{\alpha}{r}\right\} q_j + \frac{\alpha}{r}\{l_i, q_j\} = \epsilon_{jkl}\epsilon_{ikm} l_m p_l + \epsilon_{jkl} l_k \epsilon_{ilm} p_m + \frac{\alpha}{r}\epsilon_{ijk} q_k = \\ &= (\delta_{ji}\delta_{lm} - \delta_{jm}\delta_{li}) l_m p_l - (\delta_{ji}\delta_{km} - \delta_{jm}\delta_{ki}) l_k p_m + \frac{\alpha}{r}\epsilon_{ijk} q_k = \\ &= \delta_{ij} l_l p_l - l_j p_i - \delta_{ij} l_k p_k + l_i p_j + \frac{\alpha}{r}\epsilon_{ijk} q_k = l_i p_j - l_j p_i + \frac{\alpha}{r}\epsilon_{ijk} q_k \end{aligned} \tag{403}$$

$$\begin{aligned}\epsilon_{ijk} A_k &= \epsilon_{ijk}\epsilon_{klm} l_l p_m + \frac{\alpha}{r}\epsilon_{ijk} q_k = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{lj}) l_l p_m + \frac{\alpha}{r}\epsilon_{ijk} q_k = \\ &= l_i p_j - l_j p_i + \frac{\alpha}{r}\epsilon_{ijk} q_k \end{aligned} \tag{404}$$

$$\boxed{\{l_i, A_j\} = \epsilon_{ijk} A_k} \tag{405}$$

– Check that $\{A_i, A_j\} = -2H\epsilon_{ijk} l_k$:

$$\begin{aligned}\{p_i, A_j\} &= \left\{p_i, \epsilon_{jkl} l_k p_l + \frac{\alpha}{r} q_j\right\} = \epsilon_{jkl}\{p_i, l_k\} p_l + \left\{p_i, \frac{\alpha}{r}\right\} q_j + \frac{\alpha}{r}\{p_i, q_j\} = \\ &= \epsilon_{jkl}\epsilon_{ikm} p_m p_l + \frac{\alpha q_i q_j}{r^3} - \frac{\alpha}{r}\delta_{ij} = (\delta_{ji}\delta_{lm} - \delta_{jm}\delta_{il}) p_m p_l + \frac{\alpha q_i q_j}{r^3} - \frac{\alpha}{r}\delta_{ij} = \\ &= \delta_{ij} p^2 - p_i p_j + \frac{\alpha q_i q_j}{r^3} - \frac{\alpha}{r}\delta_{ij} \end{aligned} \tag{406}$$

$$
\begin{aligned}
\{q_i, A_j\} = \left\{q_i, \epsilon_{jkl} l_k p_l + \frac{\alpha}{r} q_j\right\} = \epsilon_{jkl}(\{q_i, l_k\} p_l + l_k \{q_i, p_l\}) = \\
= \epsilon_{jkl}\epsilon_{ikm} q_m p_l + \epsilon_{jkl} l_k \delta_{il} = (\delta_{ji}\delta_{lm} - \delta_{jm}\delta_{li}) q_m p_l + \epsilon_{ijk} l_k = \\
= \delta_{ij} q_l p_l - p_i q_j + \epsilon_{ijk} l_k = \delta_{ij} \boldsymbol{qp} - p_i q_j + \epsilon_{ijk} l_k
\end{aligned} \tag{407}
$$

$$
\begin{aligned}
\left\{\frac{\alpha}{r}, A_j\right\} = \left\{\frac{\alpha}{r}, \epsilon_{jkl} l_k p_l + \frac{\alpha}{r} q_j\right\} = \epsilon_{jkl}\left(\left\{\frac{\alpha}{r}, l_k\right\} p_l + l_k \left\{\frac{\alpha}{r}, p_l\right\}\right) = \\
= -\epsilon_{jkl} l_k \frac{\alpha q_l}{r^3}
\end{aligned} \tag{408}
$$

$$
\begin{aligned}
\{A_i, A_j\} = \left\{\epsilon_{ikl} l_k p_l + \frac{\alpha}{r} q_i, A_j\right\} = \epsilon_{ikl}(\{l_k, A_j\} p_l + l_k \{p_l, A_j\}) + \\
+ \left\{\frac{\alpha}{r}, A_j\right\} q_i + \frac{\alpha}{r}\{q_i, A_j\} = \\
= \epsilon_{ikl}\left(\epsilon_{kjm} A_m p_l + l_k\left(\delta_{lj} p^2 - p_l p_j + \frac{\alpha q_l q_j}{r^3} - \frac{\alpha}{r}\delta_{lj}\right)\right) - \epsilon_{jkl} l_k \frac{\alpha q_l q_i}{r^3} + \\
+ \frac{\alpha}{r}(\delta_{ij} \boldsymbol{qp} - p_i q_j + \epsilon_{ijk} l_k) = -(\delta_{ij}\delta_{lm} - \delta_{im}\delta_{lj}) A_m p_l - \epsilon_{ijk} l_k p^2 - \\
- \epsilon_{ikl} l_k p_l p_j + \epsilon_{ikl} \frac{\alpha}{r^3} q_j q_l l_k + 2\epsilon_{ijk} l_k \frac{\alpha}{r} - \epsilon_{jkl} \frac{\alpha}{r^3} q_i q_l l_k + \frac{\alpha}{r} q_l p_l \delta_{ij} - \frac{\alpha}{r} p_i q_j = \\
= -\delta_{ij} A_l p_l + A_i p_j - 2H \epsilon_{ijk} l_k - \epsilon_{ikl} l_k p_l p_j + \frac{\alpha}{r^3} l_k q_l (\epsilon_{ikl} q_j - \epsilon_{jkl} q_i) + \\
+ \frac{\alpha}{r} q_l p_l \delta_{ij} - \frac{\alpha}{r} p_i q_j = -\delta_{ij} \epsilon_{lkm} l_k p_m p_l - \frac{\alpha}{r}\delta_{ij} q_l p_l + \epsilon_{ikl} l_k p_l p_j + \frac{\alpha}{r} q_i p_j - \\
- 2H \epsilon_{ijk} l_k - \epsilon_{ikl} l_k p_l p_j + \frac{\alpha}{r^3} l_k q_l (\epsilon_{ikl} q_j - \epsilon_{jkl} q_i) + \frac{\alpha}{r} q_l p_l \delta_{ij} - \frac{\alpha}{r} p_i q_j = \\
= -2H \epsilon_{ijk} l_k + \frac{\alpha}{r^3} l_k q_l (\epsilon_{ikl} q_j - \epsilon_{jkl} q_i)
\end{aligned} \tag{409}
$$

$$
\begin{aligned}
\frac{\alpha}{r^3} l_k q_l (\epsilon_{ikl} q_j - \epsilon_{jkl} q_i) = \frac{\alpha}{r^3} \epsilon_{kmn} q_m p_n q_l (\epsilon_{ikl} q_j - \epsilon_{jkl} q_i) = \\
= -\frac{\alpha}{r^3}(\delta_{im}\delta_{ln} - \delta_{in}\delta_{lm}) q_m p_n q_l q_j + \frac{\alpha}{r^3}(\delta_{mj}\delta_{nl} - \delta_{ml}\delta_{nj}) q_m p_n q_l q_i = \\
= -\frac{\alpha}{r^3} q_i q_j p_l q_l + \frac{\alpha}{r^3} p_i q_j q_l q_l + \frac{\alpha}{r^3} q_i q_j p_l q_l - \frac{\alpha}{r^3} q_i p_j q_l q_l = 0
\end{aligned} \tag{410}
$$

$$
\boxed{\{A_i, A_j\} = -2H \epsilon_{ijk} l_k} \tag{411}
$$

Change variables:

$$
l_i \to l_i, \quad A_i \to \frac{u_i}{\sqrt{-2H}} \tag{412}
$$

$$
\boxed{\{l_i, l_j\} = \epsilon_{ijk} l_k, \quad \{l_i, u_j\} = \epsilon_{ijk} u_k, \quad \{u_i, u_j\} = \epsilon_{ijk} l_k} \tag{413}
$$

Poisson algebra corresponds to $\mathfrak{so}(4)$ Lie algebra. Thus, the Kepler problem has $\mathfrak{so}(4)$ symmetry.

4 Classical r-matrix structure

1. **Classical r-matrix for oscillator.**
   Consider a classical one-dimensional harmonic oscillator

$$
H = \frac{p^2}{2} + \frac{\omega^2 q^2}{2} \tag{414}
$$

with the Lax operator from Problem 1, Task 1

$$L = \begin{pmatrix} p & \omega q \\ \omega q & -p \end{pmatrix} \tag{415}$$

- Find the classical r-matrix for this L-operator, i.e. a 4×4 matrix r such that:

  $$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2], \tag{416}$$

  where

  $$L_1 = L \otimes 1, \quad L_2 = 1 \otimes L, \quad \{L_1, L_2\} = \sum_{ij,kl} \{L_{ij}, L_{kl}\} E_{ij} \otimes E_{kl} \tag{417}$$

  $$r_{12} = \sum_{ij,kl} r_{ij,kl} E_{ij} \otimes E_{kl}, \quad r_{21} = \sum_{ij,kl} r_{ij,kl} E_{kl} \otimes E_{ij} \tag{418}$$

- Using the classical r-matrix find the matrix M, such that

  $$\dot{L} = [L, M] \tag{419}$$

  Compare the result with the Problem 1 from Task 1.

Solution.

- $$L = \begin{pmatrix} p & \omega q \\ \omega q & -p \end{pmatrix} = p\sigma_z + \omega q \sigma_x \tag{420}$$

  $$\{L_1, L_2\} = \{p, \omega q\} \sigma_z \otimes \sigma_x + \{\omega q, p\} \sigma_x \otimes \sigma_z = \omega(\sigma_z \otimes \sigma_x - \sigma_x \otimes \sigma_z) \tag{421}$$

  Suppose $r_{12} = r^{yx} \sigma_y \otimes \sigma_x$, $r_{21} = r^{yx} \sigma_x \otimes \sigma_y$, therefore

  $$[r_{12}, L_1] = r^{yx} p [\sigma_y, \sigma_z] \otimes \sigma_x + r^{yx} \omega q [\sigma_y, \sigma_x] \otimes \sigma_x = 2ir^{yx}(p\sigma_x \otimes \sigma_x - \omega q \sigma_z \otimes \sigma_x) \tag{422}$$

  $$[r_{21}, L_2] = 2ir^{yx}(p\sigma_x \otimes \sigma_x - \omega q \sigma_x \otimes \sigma_z) \tag{423}$$

  $$[r_{12}, L_1] - [r_{21}, L_2] = 2i\omega q r^{yx}(\sigma_x \otimes \sigma_z - \sigma_z \otimes \sigma_x) \tag{424}$$

  $$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2] \to r^{yx} = \frac{1}{2iq} \tag{425}$$

  $$\boxed{r_{12} = \frac{1}{2iq} \sigma_y \otimes \sigma_x, \quad r_{21} = \frac{1}{2iq} \sigma_x \otimes \sigma_y} \tag{426}$$

- For the given L-operator:

  $$H_k = \frac{1}{k} \mathrm{Tr} L^k \to \frac{\partial L}{\partial t_k} = [L, M_k], \quad (M_k)_1 = \mathrm{Tr}_2(r_{12} L_2^{k-1}) \tag{427}$$

  $$H = H_2 = \frac{1}{2} \mathrm{Tr} L^2 = \frac{p^2}{2} + \frac{\omega^2 q^2}{2} \tag{428}$$

  $$\begin{aligned} M = \mathrm{Tr}_2(r_{12} L_2) &= \mathrm{Tr}_2 \left(\frac{1}{2iq} (\sigma_y \otimes \sigma_x)(1 \otimes (p\sigma_z + \omega q \sigma_x)) \right) = \\ &= \mathrm{Tr}_2 \left(\frac{1}{2iq} (p\sigma_y \otimes \sigma_x \sigma_z + \omega q \sigma_y \otimes \sigma_x^2) \right) = \frac{1}{2iq} p \sigma_y \mathrm{Tr}(\sigma_x \sigma_z) + \frac{\omega}{2i} \sigma_y \mathrm{Tr}(\mathbf{1}) = \frac{\sigma_y}{i} \end{aligned} \tag{429}$$

  $$\boxed{M = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}} \tag{430}$$

2. **Spectral parameter.**
Consider a classical Euler top with three different components of the inverse of inertia tensor

$$H = \frac{1}{2}\sum_{a=1}^{3} J_a S_a^2, \quad \{S_a, S_b\} = \sum_c \epsilon_{abc} S_c, \quad J_1 \neq J_2 \neq J_3 \neq J_1 \tag{431}$$

Define the 3×3 matrices

$$S = \sum_{i,j} S_{ij} E_{ij}, \quad S_{ij} = \sum_k \epsilon_{ijk} S_k, \quad \Omega = \sum_{ij} \Omega_{ij} E_{ij}, \quad \Omega_{ij} = \sum_k \epsilon_{ijk} J_k S_k \tag{432}$$

- Check that the equations of motion can be presented in the form $\dot{S} = [S, \Omega]$, but this Lax representation is not provide any nontrivial conservation laws (Casimir only).
- Let K be a diagonal matrix with elements $K_i = \frac{1}{2}(J_j^{-1} + J_k^{-1} - J_i^{-1})$ (all indices different). Check that $S = K\Omega + \Omega K$ and that the top has the Lax representation with spectral parameter
$$L(z) = S + zK^2, \quad M(z) = \Omega + zK \tag{433}$$
- Show that $\mathrm{Tr}L(z)$ and $\mathrm{Tr}L^2(z)$ do not provide nontrivial integrals of motion, but $\mathrm{Tr}L^3(z)$ provides – its expansion in z contains the Hamilton function H.

Solution.

- Consider the matrix S:

$$S = \sum_{i,j} S_{ij} E_{ij} = \sum_{i,j,k} \epsilon_{ijk} E_{ij} S_k = \begin{pmatrix} 0 & S_3 & -S_2 \\ -S_3 & 0 & S_1 \\ S_2 & -S_1 & 0 \end{pmatrix} \tag{434}$$

The equations of motion:

$$\dot{S}_i = \{H, S_i\} \tag{435}$$

$$\{H, S_i\} = \frac{1}{2}\sum_{a=1}^{3} J_a \{S_a^2, S_i\} = \sum_{a=1}^{3} J_a S_a \{S_a, S_i\} = -\sum_{a,c} \epsilon_{iac} J_a S_a S_c \tag{436}$$

$$\begin{cases} \dot{S}_1 = (J_3 - J_2) S_2 S_3, \\ \dot{S}_2 = (J_1 - J_3) S_1 S_3, \\ \dot{S}_3 = (J_2 - J_1) S_1 S_2. \end{cases} \tag{437}$$

$$\dot{S} = \sum_{i,j,k} \epsilon_{ijk} \dot{S}_k E_{ij} = \begin{pmatrix} 0 & \dot{S}_3 & -\dot{S}_2 \\ -\dot{S}_3 & 0 & \dot{S}_1 \\ \dot{S}_2 & -\dot{S}_1 & 0 \end{pmatrix} \tag{438}$$

$$\dot{S} = \begin{pmatrix} 0 & (J_2 - J_1) S_1 S_2 & (J_3 - J_1) S_1 S_3 \\ (J_1 - J_2) S_1 S_2 & 0 & (J_3 - J_2) S_2 S_3 \\ (J_1 - J_3) S_1 S_3 & (J_2 - J_3) S_2 S_3 & 0 \end{pmatrix} \tag{439}$$

Consider the matrix Ω:

$$\Omega = \sum_{i,j} \Omega_{ij} E_{ij} = \sum_{i,j,k} \epsilon_{ijk} E_{ij} J_k S_k = \begin{pmatrix} 0 & J_3 S_3 & -J_2 S_2 \\ -J_3 S_3 & 0 & J_1 S_1 \\ J_2 S_2 & -J_1 S_1 & 0 \end{pmatrix} \tag{440}$$

$$\boxed{\dot{S} = [S, \Omega]} \tag{441}$$

Lax representation:

$$\dot{L} = [L, M], \quad L = S, M = \Omega \tag{442}$$

However, this Lax representation os not good:

$$\mathrm{Tr} L = 0, \quad \mathrm{Tr} L^2 = -\sum_a S_a^2 = -C, \tag{443}$$

where C – Casimir element. Tr L^k – also functions of C. Lax representation is not provide any nontrivial conservation laws.

- Let K be a diagonal matrix with elements $K_{ii} = \frac{1}{2}(J_j^{-1} + J_k^{-1} - J_i^{-1})$ (all indices different).

$$K = \begin{pmatrix} \frac{1}{2}(J_2^{-1} + J_3^{-1} - J_1^{-1}) & 0 & 0 \\ 0 & \frac{1}{2}(J_1^{-1} + J_3^{-1} - J_2^{-1}) & 0 \\ 0 & 0 & \frac{1}{2}(J_1^{-1} + J_2^{-1} - J_3^{-1}) \end{pmatrix} \tag{444}$$

$$K\Omega + \Omega K = \begin{pmatrix} 0 & S_3 & -S_2 \\ -S_3 & 0 & S_1 \\ S_2 & -S_1 & 0 \end{pmatrix} = S \tag{445}$$

$$\boxed{S = K\Omega + \Omega K} \tag{446}$$

Check that the top has the Lax representation with spectral parameter

$$L(z) = S + zK^2, \quad M(z) = \Omega + zK \tag{447}$$

$$\dot{L}(z) = \dot{S} = [S, \Omega] \tag{448}$$

$$\begin{aligned} [L(z), M(z)] = [S, \Omega] + z[S, K] + z[K^2, \Omega] = [S, \Omega] + z[K\Omega + \Omega K, K] + z[K^2, \Omega] = \\ = [S, \Omega] + z(K\Omega K + \Omega K^2 - K^2\Omega - K\Omega K + K^2\Omega - \Omega K^2) = [S, \Omega] \end{aligned} \tag{449}$$

$$\boxed{\dot{L}(z) = [L(z), M(z)]} \tag{450}$$

-

$$L(z) = S + zK^2 \tag{451}$$

$$L(z) = \begin{pmatrix} \frac{z}{4}(J_2^{-1} + J_3^{-1} - J_1^{-1})^2 & S_3 & -S_2 \\ -S_3 & \frac{z}{4}(J_1^{-1} + J_3^{-1} - J_2^{-1})^2 & S_1 \\ S_2 & -S_1 & \frac{z}{4}(J_1^{-1} + J_2^{-1} - J_3^{-1})^2 \end{pmatrix} \tag{452}$$

$$\mathrm{Tr} L(z) = \frac{3z}{4}\left(\frac{1}{J_1^2} + \frac{1}{J_2^2} + \frac{1}{J_3^2}\right) - \frac{z}{2}\left(\frac{1}{J_1 J_2} + \frac{1}{J_1 J_3} + \frac{1}{J_2 J_3}\right) \tag{453}$$

$$\begin{aligned} \mathrm{Tr} L^2(z) = -2(S_1^2 + S_2^2 + S_3^2) + \frac{z^2}{16}\left[\left(\frac{1}{J_1} + \frac{1}{J_2} - \frac{1}{J_3}\right)^4 + \left(-\frac{1}{J_1} - \frac{1}{J_2} + \frac{1}{J_3}\right)^4 + \right. \\ \left. + \left(\frac{1}{J_1} + \frac{1}{J_2} + \frac{1}{J_3}\right)^4\right] \end{aligned} \tag{454}$$

$$\begin{aligned} \mathrm{Tr} L^3(z) = -\frac{3}{2J_1^2 J_2^2 J_3^2}((J_1^2 + J_2^2 + J_3^2)(S_1^2 + S_2^2 + S_3^2) - 2J_1 J_2 J_3(J_1 S_1^2 + J_2 S_2^2 + J_3 S_3^2)) + \\ + f(J_1, J_2, J_3) \end{aligned} \tag{455}$$

Expansion of $T(z)$ in z contains the Hamilton function H.

3. **Exercises with permutation matrices.**
Denote the standard basis in $\mathrm{Mat}_{N\times N}$ as $\{E_{ab}|a,b=1,...,N\}$, the matrix elements of these matrices are $(E_{ab})_{ij}=\delta_{ai}\delta_{bj}$.

- Consider a permutation operator $P\in\mathrm{Mat}_{N\times N}^{\otimes 2}$ defined by its action on two N-dimensional vectors
$$P(a\otimes b)=b\otimes a \tag{456}$$
Show that in the standard basis the permutation operator has the form
$$P=\sum_{i,j=1}^{N}E_{ij}\otimes E_{ji} \tag{457}$$

- Consider permutation operators in the tensor product of K vector spaces, defined as
$$P_{ij}(v_1\otimes...\otimes v_i\otimes...\otimes v_j\otimes...\otimes v_K)=(v_1\otimes...\otimes v_j\otimes...\otimes v_i\otimes...\otimes v_K) \tag{458}$$
Write the representation of this operator in $\mathrm{Mat}_{N\times N}^{\otimes K}$ and check the following formulas in this representation (consider all indices i,j,k are distinct)
$$P_{ij}P_{ij}=1,\quad P_{ij}P_{jk}=P_{jk}P_{ik}=P_{ik}P_{ij},\quad P_{ij}P_{ik}P_{jk}=P_{jk}P_{ik}P_{ij} \tag{459}$$

- Let $\hbar$ and $\{z_i|i=1,...,K\}$ be arbitrary constants. Consider matrices
$$R_{ij}(z_i,z_j)=\frac{1}{\hbar}+\frac{P_{ij}}{z_i-z_j} \tag{460}$$
Show that the matrices defined above satisfy the quantum Yang–Baxter equation
$$R_{ij}(z_i,z_j)R_{ik}(z_i,z_k)R_{jk}(z_j,z_k)=R_{jk}(z_j,z_k)R_{ik}(z_i,z_k)R_{ij}(z_i,z_j) \tag{461}$$
and unitarity condition
$$R_{ij}(z_i,z_j)R_{ji}(z_j,z_i)\propto 1 \tag{462}$$

- Consider an operator $R_{ij}(z_i,z_j)$ satisfying the quantum Yang–Baxter equation as a series in $\hbar$
$$R_{ij}(z_i,z_j)=\frac{1}{\hbar}+r_{ij}(z_i,z_j)+\mathcal{O}(\hbar) \tag{463}$$
Show that the operator r_{ij} then satisfies the classical Yang–Baxter equation
$$[r_{ij}(z_i,z_j),r_{ik}(z_i,z_k)]+[r_{ij}(z_i,z_j),r_{jk}(z_j,z_k)]+[r_{ik}(z_i,z_k),r_{jk}(z_j,z_k)]=0 \tag{464}$$

Solution.

- Consider tensor products:
$$a\otimes b=\begin{pmatrix} a_1\begin{pmatrix} b_1\\ \vdots\\ b_n\end{pmatrix}\\ \vdots\\ a_n\begin{pmatrix} b_1\\ \vdots\\ b_n\end{pmatrix}\end{pmatrix},\quad b\otimes a=\begin{pmatrix} b_1\begin{pmatrix} a_1\\ \vdots\\ a_n\end{pmatrix}\\ \vdots\\ b_n\begin{pmatrix} a_1\\ \vdots\\ a_n\end{pmatrix}\end{pmatrix} \tag{465}$$

Suppose, that $P = \sum\limits_{i,j=1}^{N} E_{ij} \otimes E_{ji}$. Then

$$P(a \otimes b)_{kN+l} = \left(\sum_{i,j=1}^{N} E_{ij} \otimes E_{ji} \right) (a \otimes b)_{kN+l} = \sum_{i,j=1}^{N} (E_{ij}a)_k (E_{ji}b)_l = \\ = \sum_{i,j,p,q} (E_{ij})_{kp} a_p (E_{ji})_{lq} b_q = \sum_{i,j,p,q} \delta_{ik}\delta_{jp} a_p \delta_{jl}\delta_{iq} b_q = a_l b_k = (b \otimes a)_{kN+l} \quad (466)$$

$$\boxed{P = \sum_{i,j=1}^{N} E_{ij} \otimes E_{ji}} \quad (467)$$

- $$P_{ij}(v_1 \otimes ... \otimes v_i \otimes ... \otimes v_j \otimes ... \otimes v_K) = (v_1 \otimes ... \otimes v_j \otimes ... \otimes v_i \otimes ... \otimes v_K) \quad (468)$$

$$\boxed{P_{ij} = \sum_{k,l=1}^{N} \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \underbrace{E_{kl}}_{i} \otimes \cdots \otimes \underbrace{E_{lk}}_{j} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}} \quad (469)$$

$$P_{ij}^2(v_1 \otimes ... \otimes v_i \otimes ... \otimes v_j \otimes ... \otimes v_K) = P_{ij}(v_1 \otimes ... \otimes v_j \otimes ... \otimes v_i \otimes ... \otimes v_K) = \\ = v_1 \otimes ... \otimes v_i \otimes ... \otimes v_j \otimes ... \otimes v_K \quad (470)$$

$$\boxed{P_{ij}P_{ij} = 1} \quad (471)$$

$$P_{ij}P_{jk}(v_1 \otimes ... \otimes v_i \otimes ... \otimes v_j \otimes ... \otimes v_k \otimes ... \otimes v_K) = P_{ij}(v_1 \otimes ... \otimes v_i \otimes ... \otimes v_k \otimes ... \otimes v_j \otimes ... \otimes v_K) = \\ = (v_1 \otimes ... \otimes v_k \otimes ... \otimes v_i \otimes ... \otimes v_j \otimes ... \otimes v_K) \quad (472)$$

$$P_{jk}P_{ik}(v_1 \otimes ... \otimes v_i \otimes ... \otimes v_j \otimes ... \otimes v_k \otimes ... \otimes v_K) = P_{jk}(v_1 \otimes ... \otimes v_k \otimes ... \otimes v_j \otimes ... \otimes v_i \otimes ... \otimes v_K) = \\ = (v_1 \otimes ... \otimes v_k \otimes ... \otimes v_i \otimes ... \otimes v_j \otimes ... \otimes v_K) \quad (473)$$

$$P_{ik}P_{ij}(v_1 \otimes ... \otimes v_i \otimes ... \otimes v_j \otimes ... \otimes v_k \otimes ... \otimes v_K) = P_{ik}(v_1 \otimes ... \otimes v_j \otimes ... \otimes v_i \otimes ... \otimes v_k \otimes ... \otimes v_K) = \\ = (v_1 \otimes ... \otimes v_k \otimes ... \otimes v_i \otimes ... \otimes v_j \otimes ... \otimes v_K) \quad (474)$$

$$\boxed{P_{ij}P_{jk} = P_{jk}P_{ik} = P_{ik}P_{ij}} \quad (475)$$

$$P_{ij}P_{ik}P_{jk}(v_1 \otimes ... \otimes v_i \otimes ... \otimes v_j \otimes ... \otimes v_k \otimes ... \otimes v_K) = P_{ij}P_{ik}(v_1 \otimes ... \otimes v_i \otimes ... \otimes v_k \otimes ... \otimes v_j \otimes ... \otimes v_K) = \\ = P_{ij}(v_1 \otimes ... \otimes v_j \otimes ... \otimes v_k \otimes ... \otimes v_i \otimes ... \otimes v_K) = \\ = (v_1 \otimes ... \otimes v_k \otimes ... \otimes v_j \otimes ... \otimes v_i \otimes ... \otimes v_K) \quad (476)$$

$$P_{jk}P_{ik}P_{ij}(v_1 \otimes ... \otimes v_i \otimes ... \otimes v_j \otimes ... \otimes v_k \otimes ... \otimes v_K) = P_{jk}P_{ik}(v_1 \otimes ... \otimes v_j \otimes ... \otimes v_i \otimes ... \otimes v_k \otimes ... \otimes v_K) = \\ = P_{jk}(v_1 \otimes ... \otimes v_k \otimes ... \otimes v_i \otimes ... \otimes v_j \otimes ... \otimes v_K) = \\ = (v_1 \otimes ... \otimes v_k \otimes ... \otimes v_j \otimes ... \otimes v_i \otimes ... \otimes v_K) \quad (477)$$

$$\boxed{P_{ij}P_{ik}P_{jk} = P_{jk}P_{ik}P_{ij}} \quad (478)$$

- $$R_{ij}(z_i, z_j) = \frac{1}{\hbar} + \frac{P_{ij}}{z_i - z_j} \tag{479}$$

$$\begin{aligned} R_{ij}(z_i, z_j)R_{ik}(z_i, z_k)R_{jk}(z_j, z_k) &= \left(\frac{1}{\hbar} + \frac{P_{ij}}{z_i - z_j}\right)\left(\frac{1}{\hbar} + \frac{P_{ik}}{z_i - z_k}\right)\left(\frac{1}{\hbar} + \frac{P_{jk}}{z_j - z_k}\right) = \\ &= \frac{1}{\hbar^3} + \frac{1}{\hbar^2}\left(\frac{P_{ij}}{z_i - z_j} + \frac{P_{ik}}{z_i - z_k} + \frac{P_{jk}}{z_j - z_k}\right) + \\ &+ \frac{1}{\hbar}\left(\frac{P_{ij}}{z_i - z_j}\frac{P_{ik}}{z_i - z_k} + \frac{P_{ik}}{z_i - z_k}\frac{P_{jk}}{z_j - z_k} + \frac{P_{ij}}{z_i - z_j}\frac{P_{jk}}{z_j - z_k}\right) + \\ &+ \frac{P_{ij}}{z_i - z_j}\frac{P_{ik}}{z_i - z_k}\frac{P_{jk}}{z_j - z_k} \end{aligned} \tag{480}$$

$$\begin{aligned} R_{jk}(z_j, z_k)R_{ik}(z_i, z_k)R_{ij}(z_i, z_j) &= \left(\frac{1}{\hbar} + \frac{P_{jk}}{z_j - z_k}\right)\left(\frac{1}{\hbar} + \frac{P_{ik}}{z_i - z_k}\right)\left(\frac{1}{\hbar} + \frac{P_{ij}}{z_i - z_j}\right) = \\ &= \frac{1}{\hbar^3} + \frac{1}{\hbar^2}\left(\frac{P_{jk}}{z_j - z_k} + \frac{P_{ik}}{z_i - z_k} + \frac{P_{ij}}{z_i - z_j}\right) + \\ &+ \frac{1}{\hbar}\left(\frac{P_{jk}}{z_j - z_k}\frac{P_{ik}}{z_i - z_k} + \frac{P_{ik}}{z_i - z_k}\frac{P_{ij}}{z_i - z_j} + \frac{P_{ij}}{z_i - z_j}\frac{P_{jk}}{z_j - z_k}\right) + \\ &+ \frac{P_{jk}}{z_j - z_k}\frac{P_{ik}}{z_i - z_k}\frac{P_{ij}}{z_i - z_j} \end{aligned} \tag{481}$$

$$\begin{aligned} &R_{ij}(z_i, z_j)R_{ik}(z_i, z_k)R_{jk}(z_j, z_k) - R_{jk}(z_j, z_k)R_{ik}(z_i, z_k)R_{ij}(z_i, z_j) = \\ &= \frac{1}{\hbar}\left(\frac{P_{ij}P_{ik} - P_{ik}P_{ij}}{(z_i - z_j)(z_i - z_k)} + \frac{P_{ik}P_{jk} - P_{jk}P_{ik}}{(z_i - z_k)(z_j - z_k)} + \frac{P_{ij}P_{jk} - P_{jk}P_{ij}}{(z_i - z_j)(z_j - z_k)}\right) + \\ &+ \frac{P_{ij}P_{ik}P_{jk} - P_{jk}P_{ik}P_{ij}}{(z_j - z_k)(z_i - z_k)(z_i - z_j)} \end{aligned} \tag{482}$$

Using formulas from previous item, we obtain

$$\begin{aligned} &R_{ij}(z_i, z_j)R_{ik}(z_i, z_k)R_{jk}(z_j, z_k) - R_{jk}(z_j, z_k)R_{ik}(z_i, z_k)R_{ij}(z_i, z_j) = \\ &= \frac{[P_{ij}, P_{ik}]}{\hbar}\left(\frac{1}{(z_i - z_j)(z_i - z_k)} + \frac{1}{(z_i - z_k)(z_j - z_k)} - \frac{1}{(z_i - z_j)(z_j - z_k)}\right) = 0 \end{aligned} \tag{483}$$

$$\boxed{R_{ij}(z_i, z_j)R_{ik}(z_i, z_k)R_{jk}(z_j, z_k) - R_{jk}(z_j, z_k)R_{ik}(z_i, z_k)R_{ij}(z_i, z_j) = 0} \tag{484}$$

So, matrices R_{ij} satisfy the quantum Yang–Baxter equation.

$$\begin{aligned} R_{ij}(z_i, z_j)R_{ji}(z_j, z_i) &= \left(\frac{1}{\hbar} + \frac{P_{ij}}{z_i - z_j}\right)\left(\frac{1}{\hbar} + \frac{P_{ji}}{z_j - z_i}\right) = \\ &= \frac{1}{\hbar^2} + \frac{1}{\hbar}\left(\frac{P_{ij}}{z_i - z_j} + \frac{P_{ji}}{z_j - z_i}\right) + \frac{P_{ij}}{z_i - z_j}\frac{P_{ji}}{z_j - z_i} = \frac{1}{\hbar^2} - \frac{1}{(z_i - z_j)^2} \propto 1 \end{aligned} \tag{485}$$

So, matrices R_{ij} satisfy unitary condition.

- $$R_{ij}(z_i,z_j)=\frac{1}{\hbar}+r_{ij}(z_i,z_j)+\mathcal{O}(\hbar)=\frac{1}{\hbar}+r_{ij}(z_i,z_j)+q_{ij}(z_i,z_j)\hbar+\mathcal{O}(\hbar^2) \tag{486}$$

$$\begin{aligned}R_{ij}(z_i,z_j)R_{ik}(z_i,z_k)R_{jk}(z_j,z_k)&=\left(\frac{1}{\hbar}+r_{ij}(z_i,z_j)+q_{ij}(z_i,z_j)\hbar+\mathcal{O}(\hbar^2)\right)\times\\ \times\left(\frac{1}{\hbar}+r_{ik}(z_i,z_k)+q_{ik}(z_i,z_k)\hbar+\mathcal{O}(\hbar^2)\right)&\left(\frac{1}{\hbar}+r_{jk}(z_j,z_k)+q_{jk}(z_j,z_k)\hbar+\mathcal{O}(\hbar^2)\right)=\\ =\frac{1}{\hbar^3}&+\frac{1}{\hbar^2}(r_{ij}(z_i,z_j)+r_{ik}(z_i,z_k)+r_{jk}(z_j,z_k))+\\ +\frac{1}{\hbar}(r_{ij}(z_i,z_j)r_{ik}(z_i,z_k)&+r_{ik}(z_i,z_k)r_{jk}(z_j,z_k)+r_{ij}(z_i,z_j)r_{jk}(z_j,z_k)+\\ &+q_{ij}(z_i,z_j)+q_{ik}(z_i,z_k)+q_{jk}(z_j,z_k))+\mathcal{O}(1)\end{aligned} \tag{487}$$

$$\begin{aligned}R_{jk}(z_j,z_k)R_{ik}(z_i,z_k)R_{ij}(z_i,z_j)&=\left(\frac{1}{\hbar}+r_{jk}(z_j,z_k)+q_{jk}(z_j,z_k)\hbar+\mathcal{O}(\hbar^2)\right)\times\\ \times\left(\frac{1}{\hbar}+r_{ik}(z_i,z_k)+q_{ik}(z_i,z_k)\hbar+\mathcal{O}(\hbar^2)\right)&\left(\frac{1}{\hbar}+r_{ij}(z_i,z_j)+q_{ij}(z_i,z_j)\hbar+\mathcal{O}(\hbar^2)\right)=\\ =\frac{1}{\hbar^3}&+\frac{1}{\hbar^2}\left(r_{ij}(z_i,z_j)+r_{ik}(z_i,z_k)+r_{jk}(z_j,z_k)\right)+\\ +\frac{1}{\hbar}(r_{jk}(z_j,z_k)r_{ik}(z_i,z_k)&+r_{ik}(z_i,z_k)r_{ij}(z_i,z_j)+r_{jk}(z_j,z_k)r_{ij}(z_i,z_j)+\\ &+q_{ij}(z_i,z_j)+q_{ik}(z_i,z_k)+q_{jk}(z_j,z_k))+\mathcal{O}(1)\end{aligned} \tag{488}$$

$$\begin{aligned}R_{ij}(z_i,z_j)R_{ik}(z_i,z_k)R_{jk}(z_j,z_k)&-R_{jk}(z_j,z_k)R_{ik}(z_i,z_k)R_{ij}(z_i,z_j)=\\ =\frac{1}{\hbar}([r_{ij}(z_i,z_j),r_{ik}(z_i,z_k)]&+[r_{ik}(z_i,z_k),r_{jk}(z_j,z_k)]+\\ &+[r_{ij}(z_i,z_j),r_{jk}(z_j,z_k)])+\mathcal{O}(1)=0\end{aligned} \tag{489}$$

  We obtain

$$\boxed{[r_{ij}(z_i,z_j),r_{ik}(z_i,z_k)]+[r_{ij}(z_i,z_j),r_{jk}(z_j,z_k)]+[r_{ik}(z_i,z_k),r_{jk}(z_j,z_k)]=0} \tag{490}$$

  So, matrices r_{ij} satisfy the classical Yang–Baxter equation.

4. **Higher flows.**
Consider the Calogero–Moser system with Lax operator

$$L=\sum_{i=1}^{n}p_iE_{ii}+\sum_{i\neq j}\frac{\nu}{q_i-q_j}E_{ij}, \tag{491}$$

where ν is a constant and p_i,q_j have the canonical Poisson brackets.

- Compute three first conservation laws using the L-operator

$$H_1=\mathrm{Tr}\ L,\quad H_2=\frac{1}{2}\mathrm{Tr}\ L^2,\quad H_3=\frac{1}{3}\mathrm{Tr}\ L^3 \tag{492}$$

- Write the canonical equations of motion for coordinates and momenta in these three cases

$$\frac{dp_i}{dt_k} = \{H_k, p_i\}, \quad \frac{dq_i}{dt_k} = \{H_k, q_i\} \tag{493}$$

- Check that the matrix

$$r_{12} = -\sum_{i\neq j} \frac{1}{q_i - q_j} E_{ij} \otimes E_{ji} - \sum_{i\neq j} \frac{1}{q_i - q_j} E_{ii} \otimes E_{ij} \tag{494}$$

is the classical r-matrix for the Calogero–Moser Lax operator L.

- Compute three M-operators, corresponding to the Hamiltonians written above

$$\frac{dL}{dt_k} = \{H_k, L\} = [L, M_k] \tag{495}$$

- Check explicitly that the second and the third flows commute

$$\left[\frac{d}{dt_2} + M_2, \frac{d}{dt_3} + M_3\right] = 0 \tag{496}$$

Solution.

-

$$L = \sum_{i=1}^{n} p_i E_{ii} + \sum_{i\neq j} \frac{\nu}{q_i - q_j} E_{ij} \tag{497}$$

$$\boxed{H_1 = \mathrm{Tr} L = \sum_i p_i} \tag{498}$$

$$L^2 = \sum_{i,j=1}^{n} p_i p_j E_{ii} E_{jj} + \sum_{k=1}^{n} \sum_{i\neq j} \frac{\nu p_k}{q_i - q_j} (E_{kk} E_{ij} + E_{ij} E_{kk}) + \\ + \sum_{i\neq j} \sum_{k\neq l} \frac{\nu^2}{(q_i - q_j)(q_k - q_l)} E_{ij} E_{kl} \tag{499}$$

$$E_{ij} E_{kl} = \delta_{il} \delta_{jk} E_{ii} \tag{500}$$

$$L^2 = \sum_{i=1}^{n} p_i^2 E_{ii} - \sum_{i\neq j} \frac{\nu^2}{(q_i - q_j)^2} E_{ii} \tag{501}$$

$$\boxed{H_2 = \frac{1}{2} \mathrm{Tr} L^2 = \frac{1}{2} \sum_i p_i^2 - \frac{\nu^2}{2} \sum_{i\neq j} \frac{1}{(q_i - q_j)^2}} \tag{502}$$

$$L^3 = \sum_{i,j,k=1}^{n} p_i p_j p_k E_{ii} E_{jj} E_{kk} + \sum_{k,l=1}^{n} \sum_{i\neq j} \frac{\nu p_k p_l}{q_i - q_j} (E_{kk} E_{ll} E_{ij} + E_{ll} E_{ij} E_{kk} + E_{ij} E_{kk} E_{ll}) + \\ + \sum_{m=1}^{n} \sum_{i\neq j} \sum_{k\neq l} \frac{\nu^2 p_m}{(q_i - q_j)(q_k - q_l)} (E_{mm} E_{ij} E_{kl} + E_{ij} E_{mm} E_{kl} + E_{ij} E_{kl} E_{mm}) + \\ + \sum_{i\neq j} \sum_{k\neq l} \sum_{m\neq p} \frac{\nu^3}{(q_i - q_j)(q_k - q_l)(q_m - q_p)} E_{ij} E_{kl} E_{mp} = \\ = \sum_{i=1}^{n} p_i^3 E_{ii} - 3\nu^2 \sum_{i\neq j} \frac{p_i}{(q_i - q_j)^2} E_{ii} + \nu^3 \sum_{i\neq j\neq l} \frac{\nu^3}{(q_i - q_j)(q_j - q_l)(q_l - q_i)} E_{ii} \tag{503}$$

$$\boxed{H_3 = \frac{1}{3}\mathrm{Tr}L^3 = \frac{1}{3}\sum_i p_i^3 - \nu^2 \sum_{i\neq j} \frac{p_i}{(q_i-q_j)^2}} \tag{504}$$

- $$\frac{dp_i}{dt_k} = \{H_k, p_i\}, \quad \frac{dq_i}{dt_k} = \{H_k, q_i\} \tag{505}$$

  Consider cases:

  - $k=1$.

$$\begin{cases} \frac{dp_i}{dt_1} = \{H_1, p_i\} = \{\sum_j p_j, p_i\}, \\ \frac{dq_i}{dt_1} = \{H_1, q_i\} = \{\sum_j p_j, q_i\}; \end{cases} \tag{506}$$

$$\boxed{\begin{cases} \frac{dp_i}{dt_1} = 0, \\ \frac{dq_i}{dt_1} = 1. \end{cases}} \tag{507}$$

  - $k=2$.

$$\begin{cases} \frac{dp_i}{dt_2} = \{H_2, p_i\} = \{\sum_j \frac{p_j^2}{2} - \frac{\nu^2}{2}\sum_{j\neq k} \frac{1}{(q_j-q_k)^2}, p_i\}, \\ \frac{dq_i}{dt_2} = \{H_2, q_i\} = \{\sum_j \frac{p_j^2}{2} - \frac{\nu^2}{2}\sum_{j\neq k} \frac{1}{(q_j-q_k)^2}, q_i\}; \end{cases} \tag{508}$$

$$\boxed{\begin{cases} \frac{dp_i}{dt_1} = -2\nu^2 \sum_{j\neq i} \frac{1}{(q_i-q_j)^3}, \\ \frac{dq_i}{dt_1} = p_i. \end{cases}} \tag{509}$$

  - $k=3$.

$$\begin{cases} \frac{dp_i}{dt_3} = \{H_3, p_i\} = \{\sum_k \frac{p_k^3}{3} - \nu^2\sum_{k\neq j} \frac{p_k}{(q_k-q_j)^2}, p_i\}, \\ \frac{dq_i}{dt_3} = \{H_3, q_i\} = \{\sum_k \frac{p_k^3}{3} - \nu^2\sum_{k\neq j} \frac{p_k}{(q_k-q_j)^2}, q_i\}; \end{cases} \tag{510}$$

$$\boxed{\begin{cases} \frac{dp_i}{dt_3} = -2\nu^2 \sum_{k\neq i} \frac{p_i+p_k}{(q_i-q_k)^3}, \\ \frac{dq_i}{dt_3} = p_i - \nu^2 \sum_{k\neq i} \frac{1}{(q_i-q_k)^2}. \end{cases}} \tag{511}$$

5 Integrable systems related to Lie algebras

1. **Weyl group.**
   Let $\mathfrak{g}$ be a simple finite–dimensional complex Lie algebra and R its root system. Weyl group W is generated by all the reflections with respect to all roots

$$w_\alpha(\beta) = \beta - \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\alpha \tag{512}$$

   - Prove that Weyl group W is finite.
   - Consider a simple Lie algebra $\mathfrak{g} = \mathfrak{sl}(n)$, prove that the Weyl group of $\mathfrak{sl}(n)$ is isomorphic to the symmetric group S_{n-1}.

   An abstract crystallographic root system Δ is a collection of the following data:

(a) A finite-dimensional Euclidean space E and a finite set of its vectors Δ, which span the whole space E.

(b) The only scalar multiples of a root $\alpha \in \Delta$ are α itself and $-\alpha \in \Delta$.

(c) For any two roots $\alpha, \beta \in \Delta$ it follows that $w_\alpha(\beta)$ belongs to the root system Δ.

(d) For any two roots $\alpha, \beta \in \Delta$, the number $\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}$ is integer.

- Define a coroot by the formula $\alpha^\vee = \frac{2\alpha}{\langle\alpha,\alpha\rangle}$ and denote the set of coroots by $\Delta^\vee$. Prove that $\Delta^\vee$ is again a root system.
- Describe the dual root system $\Delta^\vee$ for the case of a root system of $\mathfrak{g} = \mathfrak{sl}(n)$.
- Prove that Weyl groups of Δ and $\Delta^\vee$ are isomorphic.

Solution.

- $$\mathrm{ad}_h e_\alpha = \alpha(h)e_\alpha, \quad \alpha(h) = (H_\alpha, h) \tag{513}$$

  Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, $\{h_\alpha\}$ is a basis in $\mathfrak{h}$. Prove that W keeps the root system.
  Finite-dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$ ($[e,f] = h$, $[h,f] = -2f$, $[h,e] = 2e$):

  $$V = \underset{n\in\mathbb{Z}}{\oplus} V(n) \tag{514}$$

  For $V(n)$:

  $$ev^k = (n+1-k)v^{k-1}, \quad fv^k = (k+1)v^{k+1}, \quad hv^k = (n-2k)v^k \tag{515}$$

  $$ev^0 = 0, \quad fv^n = 0 \tag{516}$$

  If α is root, then $-\alpha$ is also a root. Let be $\mathfrak{g} = \mathfrak{h} \oplus \underset{\alpha\in\Delta}{\oplus} \mathfrak{g}_\alpha$ and $e_\alpha \in \mathfrak{g}_\alpha$, $f_\alpha \in \mathfrak{g}_{-\alpha}$, $h_\alpha \in \mathfrak{h}$:

  $$(e_\alpha, f_\alpha) = \frac{2}{\langle\alpha,\alpha\rangle}, \quad h_\alpha = \frac{2H_\alpha}{\langle\alpha,\alpha\rangle} \tag{517}$$

  $$([e_\alpha, f_\alpha], h) = [h, (e_\alpha, f_\alpha)] = \alpha(h)(e_\alpha, f_\alpha) = (H_\alpha, h)(e_\alpha, f_\alpha) \tag{518}$$

  $$[e_\alpha, f_\alpha] = H_\alpha(e_\alpha, f_\alpha) = h_\alpha \tag{519}$$

  $$[h_\alpha, f_\alpha] = -\alpha(h_\alpha)f_\alpha = -(H_\alpha, h_\alpha)f_\alpha = -2f_\alpha \tag{520}$$

  $$[h_\alpha, e_\alpha] = \alpha(h_\alpha)e_\alpha = (H_\alpha, h_\alpha)e_\alpha = 2e_\alpha \tag{521}$$

  So, $\mathfrak{g}$ is a representation of $\mathfrak{sl}_2(\mathbb{C})$. Let be $e_\beta \in \mathfrak{g}_\beta$, then

  $$\mathrm{ad}_{h_\alpha} e_\beta = \beta(h_\alpha)e_\beta = (h_\alpha, H_\beta)e_\beta = \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}e_\beta, \quad \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z} \tag{522}$$

  $$\mathrm{ad}_{f_\alpha} e_\beta = [f_\alpha, e_\beta] \subset \mathfrak{g}_{\beta-\alpha} \to \mathrm{ad}^n_{f_\alpha} e_\beta = [f_\alpha, ...[f_\alpha, e_\beta]] \subset \mathfrak{g}_{\beta-\alpha n} \tag{523}$$

  Let be $n = \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}$. If $\alpha, \beta \in \Delta$, then $w_\alpha(\beta) = \beta - \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\alpha \in \Delta$. So, W keeps the root system.
  Δ is a finite set, so W is a finite group.

- Prove that the Weyl group of $\mathfrak{sl}(n)$ is isomorphic to the symmetric group S_n.

$$S_n = \{\sigma_1, \sigma_2, ..., \sigma_{n-1} | \sigma_i^2 = 1, (\sigma_i \sigma_{i+1})^3 = e, [\sigma_i, \sigma_j] = 0, |i-j| > 1\} \tag{524}$$

Let be $\beta = \lambda_k - \lambda_l \in \Delta$. Consider the comparison σ_i with $w_{\alpha_i}(\beta)$, $\alpha_i = \lambda_i - \lambda_{i+1}$ – simple root.

$$\langle \lambda_i, \lambda_j \rangle = \delta_{ij} \to \langle \alpha_i, \alpha_i \rangle = 2, \quad \langle \alpha_i, \alpha_j \rangle = -1, \quad |i-j| = 1 \tag{525}$$

$$\begin{aligned} w_{\alpha_i}^2(\beta) = w_{\alpha_i}(\beta - \langle \alpha_i, \beta \rangle \alpha_i) = \beta - \langle \alpha_i, \beta \rangle \alpha_i - \langle \alpha_i, \beta \rangle \alpha_i + \langle \alpha_i, \beta \rangle \langle \alpha_i, \alpha_i \rangle \alpha_i = \\ = \beta - \langle \alpha_i, \beta \rangle \alpha_i - \langle \alpha_i, \beta \rangle \alpha_i + 2 \langle \alpha_i, \beta \rangle \alpha_i = \beta \end{aligned} \tag{526}$$

Let be $|i-j| = 1$, than

$$\begin{aligned} w_{\alpha_i} w_{\alpha_j} w_{\alpha_i}(\beta) &= w_{\alpha_i} w_{\alpha_j}(\beta - \langle \alpha_i, \beta \rangle \alpha_i) = \\ &= w_{\alpha_i}(\beta - \langle \alpha_i, \beta \rangle \alpha_i - \langle \alpha_j, \beta \rangle \alpha_j + \langle \alpha_i, \beta \rangle \langle \alpha_j, \alpha_i \rangle \alpha_j) = \\ &= w_{\alpha_i}(\beta - \langle \alpha_i, \beta \rangle \alpha_i - \langle \alpha_j, \beta \rangle \alpha_j - \langle \alpha_i, \beta \rangle \alpha_j) = \\ &= \beta - \langle \alpha_i, \beta \rangle \alpha_i - \langle \alpha_j, \beta \rangle \alpha_j - \langle \alpha_i, \beta \rangle \alpha_j - \langle \alpha_i, \beta \rangle \alpha_i + 2 \langle \alpha_i, \beta \rangle \alpha_i - \\ &- \langle \alpha_j, \beta \rangle \alpha_i - \langle \alpha_i, \beta \rangle \alpha_i - \langle \alpha_i, \beta \rangle \alpha_i = \\ &= \beta - \langle \alpha_j, \beta \rangle \alpha_j - \langle \alpha_i, \beta \rangle \alpha_j - \langle \alpha_j, \beta \rangle \alpha_i - \langle \alpha_i, \beta \rangle \alpha_i - \langle \alpha_i, \beta \rangle \alpha_i = \\ &= w_{\alpha_j} w_{\alpha_i} w_{\alpha_j}(\beta) \end{aligned} \tag{527}$$

$$(w_{\alpha_i} w_{\alpha_{i+1}})^3(\beta) = \beta \tag{528}$$

Let be $|i-j| > 1$, than

$$\begin{aligned} w_{\alpha_i} w_{\alpha_j}(\beta) = w_{\alpha_i}(\beta - \langle \alpha_j, \beta \rangle \alpha_j) = \beta - \langle \alpha_j, \beta \rangle \alpha_j - \langle \alpha_i, \beta \rangle \alpha_i + \langle \alpha_j, \beta \rangle \langle \alpha_i, \alpha_j \rangle \alpha_i = \\ = \beta - \langle \alpha_j, \beta \rangle \alpha_j - \langle \alpha_i, \beta \rangle \alpha_i = w_{\alpha_j} w_{\alpha_i}(\beta) \end{aligned} \tag{529}$$

$$(w_{\alpha_i} w_{\alpha_j})^2(\beta) = w_{\alpha_i} w_{\alpha_j} w_{\alpha_j} w_{\alpha_i}(\beta) = \beta \tag{530}$$

$$\boxed{W \simeq S_n} \tag{531}$$

- $$\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle} \subset \Delta^\vee \tag{532}$$

Prove that $\Delta^\vee$ is a system of roots.

(a) $\Delta^\vee$ spans the space E.

(b) If $\alpha^\vee \in \Delta^\vee$, then $-\alpha^\vee \in \Delta^\vee$.

(c) Let $\alpha^\vee, \beta^\vee \in \Delta^\vee$, then

$$\begin{aligned} w_{\alpha^\vee}(\beta^\vee) = \beta^\vee - \frac{2 \langle \alpha^\vee, \beta^\vee \rangle}{\langle \alpha^\vee, \alpha^\vee \rangle} \alpha^\vee = \frac{2\beta}{\langle \beta, \beta \rangle} - \frac{2 \langle \alpha, \beta \rangle \langle \alpha, \alpha \rangle^2}{\langle \alpha, \alpha \rangle \langle \beta, \beta \rangle \langle \alpha, \alpha \rangle} \frac{2\alpha}{\langle \alpha, \alpha \rangle} = \\ = \frac{2\beta}{\langle \beta, \beta \rangle} - \frac{2 \langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \frac{2\alpha}{\langle \alpha, \alpha \rangle} = \frac{2\beta}{\langle \beta, \beta \rangle} - \frac{4 \langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle \langle \beta, \beta \rangle} \alpha \end{aligned} \tag{533}$$

$$(w_\alpha(\beta))^\vee = \frac{2 \left(\beta - \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \alpha \right)}{\langle \beta - \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \alpha, \beta - \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \alpha \rangle} \tag{534}$$

$$\left\langle \beta - \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\alpha, \beta - \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\alpha \right\rangle = \langle\beta,\beta\rangle - \frac{4\langle\alpha,\beta\rangle^2}{\langle\alpha,\alpha\rangle} + \frac{4\langle\alpha,\beta\rangle^2}{\langle\alpha,\alpha\rangle} = \langle\beta,\beta\rangle \quad (535)$$

$$(w_\alpha(\beta))^\vee = \frac{2\beta}{\langle\beta,\beta\rangle} - \frac{4\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle\langle\beta,\beta\rangle}\alpha \quad (536)$$

$$w_{\alpha^\vee}(\beta^\vee) = (w_\alpha(\beta))^\vee \subset \Delta^\vee \quad (537)$$

(d)

$$\frac{2\langle\alpha^\vee,\beta^\vee\rangle}{\langle\alpha^\vee,\alpha^\vee\rangle} = \frac{2\langle\alpha,\beta\rangle\langle\alpha,\alpha\rangle^2}{\langle\alpha,\alpha\rangle\langle\beta,\beta\rangle\langle\alpha,\alpha\rangle} = \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle\langle\beta,\beta\rangle} \in \mathbb{Z} \quad (538)$$

- A root system of $\mathfrak{g} = \mathfrak{sl}(n)$:

$$\Delta = \{\lambda_i - \lambda_j | 1 \le i, j \le n, i \ne j\} \quad (539)$$

$$(\lambda_i - \lambda_j)^\vee = \frac{2(\lambda_i - \lambda_j)}{\langle\lambda_i - \lambda_j, \lambda_i - \lambda_j\rangle} = \lambda_i - \lambda_j \quad (540)$$

$$\boxed{\Delta^\vee = \Delta} \quad (541)$$

- Prove that Weyl groups of Δ and $\Delta^\vee$ are isomorphic.

$$\varphi : (\beta \to w_\alpha(\beta)) \to (\beta^\vee \to w_{\alpha^\vee}(\beta^\vee) = (w_\alpha(\beta))^\vee) \quad (542)$$

φ is a homomorphism, since

$$(w_{\alpha_2}(w_{\alpha_1}(\beta)))^\vee = w_{\alpha_2^\vee}(w_{\alpha_1}(\beta))^\vee = w_{\alpha_2^\vee}(w_{\alpha_1^\vee}(\beta^\vee)) \quad (543)$$

Since φ is a bijection, then φ is isomorphism.

2. **$\mathfrak{g}_2$ Lie algebra.**
Let V be a three–dimensional complex vector space. Let $\mathfrak{sl}(V)$ be traceless matrices on V and consider the following vector space $\mathfrak{g}_2 = V^* \oplus \mathfrak{sl}(V) \oplus V$ with an antisymmetric bracket on it

$$[A,B] := \begin{cases} AB - BA, & A, B \in \mathfrak{sl}(V) \\ A(B), & A \in \mathfrak{sl}(V), B \in V \\ -A(B), & A \in \mathfrak{sl}(V), B \in V^* \\ A \otimes B - \frac{1}{3}B(A)\cdot 1, & A \in V, B \in V^* \end{cases} \quad (544)$$

- Describe the bracket as a matrix commutator with the help of 7×7 matrices.
- Prove that the bracket satisfies all the Lie algebra axioms, thus making $\mathfrak{g}_2$ a Lie algebra.
- Check that diagonal matrices in $\mathfrak{sl}(V)$ form the Cartan subalgebra of $\mathfrak{g}_2$, and describe the root system of $\mathfrak{g}_2$. Which lengths do the simple roots have? Which angles are between the simple roots?
- Is $\mathfrak{g}_2$ a semisimple Lie algebra? Is it simple?
- Write down a Cartan matrix and draw the Dynkin diagram for $\mathfrak{g}_2$ algebra.

Solution.

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- Diagonal matrices in $\mathfrak{sl}(V)$:

$$h = \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix}, \quad h_1 + h_2 + h_3 = 0 \tag{545}$$

All matrices h commutate, so subalgebra of this matrices is abelian. Let be $A \in \mathfrak{sl}(V)$, then

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad a_{11} + a_{22} + a_{33} = 0 \tag{546}$$

$$\mathrm{ad}_h A = [h, A] = \begin{pmatrix} 0 & a_{12}(h_1 - h_2) & a_{33}(h_1 - h_3) \\ a_{21}(h_2 - h_1) & 0 & a_{23}(h_2 - h_3) \\ a_{31}(h_3 - h_1) & a_{32}(h_3 - h_2) & 0 \end{pmatrix} \tag{547}$$

$$\mathrm{ad}_h e_\alpha = \alpha(h) e_\alpha \tag{548}$$

For $e_\alpha = E_{ij} : \alpha(h) = h_i - h_j$. Let be $\lambda_i \in \mathfrak{h}^* :$

$$\lambda_i(h) = h_i \to \alpha = \lambda_i - \lambda_j \tag{549}$$

The space of roots of $\mathfrak{sl}(V)$:

$$\Delta = \{\pm(\lambda_1 - \lambda_2), \pm(\lambda_1 - \lambda_3) = \pm(2\lambda_1 + \lambda_2), \pm(\lambda_2 - \lambda_3) = \pm(\lambda_1 + 2\lambda_2)\} \tag{550}$$

For $e_\alpha \in \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \subset V$:

$$\mathrm{ad}_h e_\alpha = h(e_\alpha) = \alpha(h) e_\alpha \to \alpha(h) = h_i \to \alpha = \lambda_i \tag{551}$$

For $e_\alpha \in \left\{ \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \right\} \subset V^*$:

$$\mathrm{ad}_h e_\alpha = -h(e_\alpha) = \alpha(h) e_\alpha \to \alpha(h) = -h_i \to \alpha = -\lambda_i \tag{552}$$

The space of roots of $V \oplus V^*$:

$$\Delta = \{\pm\lambda_1, \pm\lambda_2, \pm\lambda_3 = \mp(\lambda_1 + \lambda_2)\} \tag{553}$$

Thus, matrices h form Cartan subalgebra (ad_h is diagonalizable for all $h \in \mathfrak{h}$).
Simple roots:

$$\boxed{\prod = \{\lambda_1 - \lambda_2, \lambda_2\}} \tag{554}$$

Positive roots:

$$\Delta_+ = \{\lambda_1, \lambda_2, \lambda_1 + \lambda_2, \lambda_1 - \lambda_2, 2\lambda_1 + \lambda_2, \lambda_1 + 2\lambda_2\} \tag{555}$$

Negative roots:

$$\Delta_- = \{-\lambda_1, -\lambda_2, -\lambda_1 - \lambda_2, \lambda_2 - \lambda_1, -2\lambda_1 - \lambda_2, -\lambda_1 - 2\lambda_2\} \tag{556}$$

A map $\mathfrak{h} \to \mathfrak{h}^*$:

$$\lambda_i(h) = (h_{\lambda_i}, h) = \mathrm{Tr}(\mathrm{ad}_{h_i} \mathrm{ad}_h) \tag{557}$$

Nondegenerate bilinear form on $\mathfrak{h}^*$:

$$\langle \lambda_i, \lambda_j \rangle = (h_i, h_j) = \mathrm{Tr}(\mathrm{ad}_{h_i}\mathrm{ad}_{h_j}) \tag{558}$$

$$\lambda_i(h) = \mathrm{Tr}(h_i h) = ((h_i)_{11}h_1 + (h_i)_{22}h_2 + ((h_i)_{11} + (h_i)_{22})(h_1 + h_2)) = h_i \tag{559}$$

$$\lambda_1(h) = ((h_1)_{11}h_1 + (h_1)_{22}h_2 + ((h_1)_{11} + (h_1)_{22})(h_1 + h_2)) = h_1 \tag{560}$$

$$\begin{cases} 2(h_1)_{11} + (h_1)_{22} = 1, \\ 2(h_1)_{22} + (h_1)_{11} = 0; \end{cases} \rightarrow (h_1)_{11} = \frac{2}{3}, (h_1)_{22} = -\frac{1}{3} \tag{561}$$

$$h_1 = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \tag{562}$$

$$\lambda_2(h) = ((h_2)_{11}h_1 + (h_2)_{22}h_2 + ((h_2)_{11} + (h_2)_{22})(h_1 + h_2)) = h_2 \tag{563}$$

$$\begin{cases} 2(h_2)_{11} + (h_2)_{22} = 0, \\ 2(h_2)_{22} + (h_2)_{11} = 1; \end{cases} \rightarrow (h_2)_{11} = -\frac{1}{3}, (h_2)_{22} = \frac{2}{3} \tag{564}$$

$$h_2 = \begin{pmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \tag{565}$$

$$\langle \lambda_i, \lambda_j \rangle = \mathrm{Tr}(h_i h_j) \tag{566}$$

$$\langle \lambda_1, \lambda_1 \rangle = \langle \lambda_2, \lambda_2 \rangle = \frac{1+4+1}{9} = \frac{2}{3}, \quad \langle \lambda_1, \lambda_2 \rangle = \frac{-2-2+1}{9} = -\frac{1}{3} \tag{567}$$

$$\langle \lambda_1 - \lambda_2, \lambda_1 - \lambda_2 \rangle = \langle \lambda_1, \lambda_1 \rangle + \langle \lambda_2, \lambda_2 \rangle - 2\langle \lambda_1, \lambda_2 \rangle = \frac{4}{3} + \frac{2}{3} = 2 \tag{568}$$

Length of $\lambda_1 - \lambda_2$ is $\sqrt{2}$ and length of λ_2 is $\sqrt{\frac{2}{3}}$.

$$\langle \lambda_1 - \lambda_2, \lambda_2 \rangle = -\frac{1}{3} - \frac{2}{3} = -1 \tag{569}$$

Angle between simple roots:

$$\cos\alpha = \frac{\langle \lambda_1 - \lambda_2, \lambda_2 \rangle}{\sqrt{\langle \lambda_1 - \lambda_2, \lambda_1 - \lambda_2 \rangle \langle \lambda_2, \lambda_2 \rangle}} = \frac{-1}{\sqrt{2}\sqrt{\frac{2}{3}}} = -\frac{\sqrt{3}}{2} \rightarrow \boxed{\alpha = \frac{5\pi}{6}} \tag{570}$$

- Prove, that group $\mathfrak{g}_2$ is simple. Suppose the contrary, let he algebra have a nontrivial ideal I:

$$i = v + A + \tilde{v} \in I, \quad v \in V, A \in \mathfrak{sl}(V), \tilde{v} \in V^* \tag{571}$$

$\forall v, u \in V \hookrightarrow \exists A \in \mathfrak{sl}(V) : [A, v] = u$, so

$$V \oplus V^* \subset I \tag{572}$$

$\forall A \in \mathfrak{sl}(V) \hookrightarrow \exists v, u \in V : [v, u] = A$, so

$$\mathfrak{g}_2 = V^* \oplus \mathfrak{sl}(V) \oplus V = I \tag{573}$$

Thus, $\mathfrak{g}_2$ is simple. So, $\mathfrak{g}_2$ is semisimple.

- Cartan matrix:

$$a_{ij} = \frac{2\langle\alpha_j, \alpha_i\rangle}{\langle\alpha_i, \alpha_i\rangle} \tag{574}$$

$$a_{11} = a_{22} = 2, \quad a_{12} = \frac{-2}{\frac{2}{3}} = -3, \quad a_{21} = \frac{-2}{2} = -1 \tag{575}$$

$$\boxed{A = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}} \tag{576}$$

Dynkin diagram:

G_2

3. **Dynkin diagrams and Cartan matrices.**
Cartan matrix is a matrix with elements

$$a_{ij} = \frac{2\langle\alpha_j, \alpha_i\rangle}{\langle\alpha_i, \alpha_i\rangle}, \tag{577}$$

where $\prod = \{\alpha_1, ..., \alpha_r\}$ is a collection of simple roots of a root system. Dynkin diagram is a finite graph with vertices representing simple roots and two vertices are connected by 0 edges of roots are orthogonal, 1 if the angle between roots is $\frac{2\pi}{3}$, 2 if the angle is $\frac{3\pi}{4}$, 3 if the angle is $\frac{5\pi}{6}$. Additionally, we orient those edges connecting the simple roots of different lengths from the long one to the short one.

- Prove that for a complex semisimple Lie algebra the Cartan matrix is nondegenerate.
- Prove that Cartan matrix a symmetrisable positively defined matrix, i.e. it can be written as a product of diagonal matrix with positive elements and a symmetric positively defined matrix.
- Describe all Cartan matrices of rank 2 and draw the corresponding Dynkin diagrams on the plane. Find lengths of all roots and angles between all roots. Name the obtained root systems.
- Describe explicitly the Weyl groups for the root systems constructed above for the Cartan matrices of rank 2.
- Prove that Dynkin diagram without multiple edges, i.e. two vertices are connected by either 0 or 1 edge, can't have cycles and vertices with degree ≥ 4.

Solution.

- Cartan matrix:

$$A = (a_{ij}), \quad a_{ij} = \frac{2\langle\alpha_j, \alpha_i\rangle}{\langle\alpha_i, \alpha_i\rangle} \tag{578}$$

By multiplying the i line of Cartan matrix A by the positive number $\frac{\langle\alpha_i,\alpha_i\rangle}{2}$, it becomes the matrix $A = \langle\alpha_j, \alpha_i\rangle$ which has positive determinant because the simple roots span a Euclidean space.

- $$A = DS, \quad D_{ij} = \frac{\delta_{ij}}{2}\langle\alpha_i, \alpha_i\rangle, \quad S_{ij} = \langle\alpha_j, \alpha_i\rangle \tag{579}$$

where D – diagonal matrix with positive elements $\frac{\langle\alpha_i,\alpha_i\rangle}{2} > 0$ and S – symmetric Gram matrix $\langle\alpha_i, \alpha_j\rangle$. Gram matrix is nondegenerate.

-
$$a_{ii} = \frac{2\langle \alpha_i, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} = 2, \quad a_{ij} \leq 0, \quad i \neq j \tag{580}$$

If $a_{ij} = 0$, then $a_{ji} = 0$, because in this case $\langle \alpha_i, \alpha_j \rangle = 0$.
Cartan matrix of rank 2:

$$A = \begin{pmatrix} 2 & a \\ b & 2 \end{pmatrix} \to \det A = 4 - ab > 0 \tag{581}$$

Consider possible cases:

- $a = b = 0$.
  Cartan matrix:

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \to \langle \alpha_1, \alpha_1 \rangle = c_1^2, \quad \langle \alpha_2, \alpha_2 \rangle = c_2^2, \quad \langle \alpha_1, \alpha_2 \rangle = 0 \tag{582}$$

  Length of roots:

$$\boxed{|\alpha_1| = \sqrt{\langle \alpha_1, \alpha_1 \rangle} = c_1, \quad |\alpha_2| = \sqrt{\langle \alpha_2, \alpha_2 \rangle} = c_2} \tag{583}$$

  Angle between roots:

$$\boxed{\cos\alpha = \frac{\langle \alpha_1, \alpha_2 \rangle}{\sqrt{\langle \alpha_1, \alpha_1 \rangle \langle \alpha_2, \alpha_2 \rangle}} = 0 \to \alpha = \frac{\pi}{2}} \tag{584}$$

  Dynkin diagram for $A_1 \times A_1 \simeq D_2 \leftarrow \mathfrak{so}(4)$:

$$A_1 \quad \bullet$$

$$A_1 \quad \bullet$$

- $a = b = -1$.
  Cartan matrix:

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \to \langle \alpha_1, \alpha_1 \rangle = \langle \alpha_2, \alpha_2 \rangle = c^2, \quad \langle \alpha_1, \alpha_2 \rangle = -\frac{c^2}{2} \tag{585}$$

  Length of roots:

$$\boxed{|\alpha_1| = \sqrt{\langle \alpha_1, \alpha_1 \rangle} = |\alpha_2| = \sqrt{\langle \alpha_2, \alpha_2 \rangle} = c} \tag{586}$$

  Angle between roots:

$$\boxed{\cos\alpha = \frac{\langle \alpha_1, \alpha_2 \rangle}{\sqrt{\langle \alpha_1, \alpha_1 \rangle \langle \alpha_2, \alpha_2 \rangle}} = -\frac{1}{2} \to \alpha = \frac{2\pi}{3}} \tag{587}$$

  Dynkin diagram for $A_2 \leftarrow \mathfrak{sl}(3)$:

$$A_2 \quad \bullet\!\!-\!\!\bullet$$

- $a = -1, b = -2$.
  Cartan matrix:

$$A = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \to \langle \alpha_1, \alpha_1 \rangle = 2c^2, \quad \langle \alpha_2, \alpha_2 \rangle = c^2, \quad \langle \alpha_1, \alpha_2 \rangle = -c^2 \tag{588}$$

  Length of roots:

$$\boxed{|\alpha_1| = \sqrt{\langle \alpha_1, \alpha_1 \rangle} = \sqrt{2}c, \quad |\alpha_2| = \sqrt{\langle \alpha_2, \alpha_2 \rangle} = c} \tag{589}$$

  Angle between roots:

$$\boxed{\cos\alpha = \frac{\langle \alpha_1, \alpha_2 \rangle}{\sqrt{\langle \alpha_1, \alpha_1 \rangle \langle \alpha_2, \alpha_2 \rangle}} = -\frac{1}{\sqrt{2}} \to \alpha = \frac{3\pi}{4}} \tag{590}$$

  Dynkin diagram for $B_2 \leftarrow \mathfrak{so}(5)$:

B_2

- $a = -1, b = -3$.
  Cartan matrix:

$$A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \to \langle \alpha_1, \alpha_1 \rangle = 3c^2, \quad \langle \alpha_2, \alpha_2 \rangle = c^2, \quad \langle \alpha_1, \alpha_2 \rangle = -\frac{3}{2}c^2 \tag{591}$$

  Length of roots:

$$\boxed{|\alpha_1| = \sqrt{\langle \alpha_1, \alpha_1 \rangle} = \sqrt{3}c, \quad |\alpha_2| = \sqrt{\langle \alpha_2, \alpha_2 \rangle} = c} \tag{592}$$

  Angle between roots:

$$\boxed{\cos\alpha = \frac{\langle \alpha_1, \alpha_2 \rangle}{\sqrt{\langle \alpha_1, \alpha_1 \rangle \langle \alpha_2, \alpha_2 \rangle}} = -\frac{\sqrt{3}}{2} \to \alpha = \frac{5\pi}{6}} \tag{593}$$

  Dynkin diagram for $G_2 \leftarrow \mathfrak{g}_2$:

G_2

- Dihedral group:

$$\mathrm{Dih}_n = \{r, s | r^n = s^2 = (sr)^n = e\} \tag{594}$$

  Let be

$$\alpha_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{595}$$

  Weyl group W is generated by all the reflections with respect to all roots

$$w_{\alpha_i}(\alpha_j) = \alpha_j - \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle}\alpha_i \tag{596}$$

$$w_{\alpha_i}(\alpha_i) = -\alpha_i \tag{597}$$

  Consider cases:

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$$\langle\alpha_1,\alpha_1\rangle = c_1^2, \quad \langle\alpha_2,\alpha_2\rangle = c_2^2, \quad \langle\alpha_1,\alpha_2\rangle = 0 \tag{598}$$

$$w_{\alpha_1}(\alpha_2) = \alpha_2, \quad w_{\alpha_2}(\alpha_1) = \alpha_1 \tag{599}$$

$$w_{\alpha_1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad w_{\alpha_2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{600}$$

$$w_{\alpha_1}^2 = w_{\alpha_2}^2 = (w_{\alpha_1} w_{\alpha_2})^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{601}$$

$$\boxed{W(D_2) = \mathrm{Dih}_2} \tag{602}$$

–

$$\langle\alpha_1,\alpha_1\rangle = \langle\alpha_2,\alpha_2\rangle = c^2, \quad \langle\alpha_1,\alpha_2\rangle = -\frac{c^2}{2} \tag{603}$$

$$w_{\alpha_1}(\alpha_2) = \alpha_2 + \alpha_1, \quad w_{\alpha_2}(\alpha_1) = \alpha_1 + \alpha_2 \tag{604}$$

$$w_{\alpha_1} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad w_{\alpha_2} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \tag{605}$$

$$w_{\alpha_1}^2 = w_{\alpha_2}^2 = (w_{\alpha_1} w_{\alpha_2})^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{606}$$

$$\boxed{W(A_2) = \mathrm{Dih}_3} \tag{607}$$

–

$$\langle\alpha_1,\alpha_1\rangle = 2c^2, \quad \langle\alpha_2,\alpha_2\rangle = c^2, \quad \langle\alpha_1,\alpha_2\rangle = -c^2 \tag{608}$$

$$w_{\alpha_1}(\alpha_2) = \alpha_2 + \alpha_1, \quad w_{\alpha_2}(\alpha_1) = \alpha_1 + 2\alpha_2 \tag{609}$$

$$w_{\alpha_1} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad w_{\alpha_2} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \tag{610}$$

$$w_{\alpha_1}^2 = w_{\alpha_2}^2 = (w_{\alpha_1} w_{\alpha_2})^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{611}$$

$$\boxed{W(B_2) = \mathrm{Dih}_4} \tag{612}$$

–

$$\langle\alpha_1,\alpha_1\rangle = 3c^2, \quad \langle\alpha_2,\alpha_2\rangle = c^2, \quad \langle\alpha_1,\alpha_2\rangle = -\frac{3}{2}c^2 \tag{613}$$

$$w_{\alpha_1}(\alpha_2) = \alpha_2 + \alpha_1, \quad w_{\alpha_2}(\alpha_1) = \alpha_1 + 3\alpha_2 \tag{614}$$

$$w_{\alpha_1} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad w_{\alpha_2} = \begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix} \tag{615}$$

$$w_{\alpha_1}^2 = w_{\alpha_2}^2 = (w_{\alpha_1} w_{\alpha_2})^6 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{616}$$

$$\boxed{W(G_2) = \mathrm{Dih}_6} \tag{617}$$

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8. **Matrix models, orthogonal polynomials and tau-functions.**
Consider a Hermitian one-matrix model with partition function

$$Z_N(t) = c_N \int_{\mathcal{H}_N} DM e^{-\mathrm{tr}V(M)}, \quad V(M) = \sum_{k=0}^{\infty} t_k M^k \tag{618}$$

where c_N is a factor depending only on the size of matrices, which will be fixed further, $\mathcal{H}_N = \{M \in \mathrm{Mat}_N(\mathbb{C}) | M = M^\dagger\}$ is the space of $N \times N$ Hermitian matrices and the integration measure is the standard invariant Haar measure on $\mathcal{H}_N$ given by

$$DM = \prod_{i=1}^{N} dM_{ii} \prod_{1\le i<j\le N} d\mathrm{Re}M_{ij} d\mathrm{Im}M_{ij} \tag{619}$$

- It is known that every Hermitian matrix M can be diagonalized via the unitary transformation, i.e. $M = U^\dagger \Lambda U$ for the unitary matrix U and the diagonal matrix Λ which contains the eigenvalues of M: $\Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_N)$.
Show that the defined integration measure DM can be presented in the form

$$DM = \frac{\mu_{U(N)}}{\mu_{U(1)^N}} \prod_{i=1}^{N} d\lambda_i \prod_{1\le i<j\le N} (\lambda_i - \lambda_j)^2 \tag{620}$$

Hint: use the correspondence between measure and norm, for example, one can use the expression for the norm in the spherical coordinates $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$ to get the integration measure $dr \cdot r d\theta \cdot r \sin\theta d\varphi$.
Thus, the partition function for $c_N = \frac{1}{N!}\frac{\mathrm{Vol}(U(1))^N}{\mathrm{Vol}(U(N))}$ is rewritten as

$$Z_N(t) = \frac{1}{N!} \int_{\mathbb{R}^N} \prod_{i=1}^{N} (d\lambda_i e^{-V(\lambda_i)}) \prod_{1\le i<j\le N} (\lambda_i - \lambda_j)^2 \tag{621}$$

- Consider a family of polynomials $\{\pi_0(x), \pi_1(x), ..., \pi_{N-1}(x)\}$, such that
  (a) $\deg \pi_k = k$,
  (b) The leading coefficients equal to 1: $\pi_k = \sum_{l\le k} \gamma_{kl} x^l$, $\gamma_{kk} = 1$.
  (c) $\langle \pi_k(x), \pi_l(x) \rangle = e^{q_k(t)} \delta_{kl}$, where $q_k(t)$ are some functions of the parameters of the potential $V(x)$, and the scalar product is defined as $\langle f(x), g(x) \rangle = \int_{\mathbb{R}} f(x) g(x) e^{-V(x)} dx$.

  Show that the partition function Z can be rewritten in the form

$$Z_N(t) = \frac{1}{N!} \prod_{i=1}^{N} \int_{\mathbb{R}} d\lambda_i e^{-V(\lambda_i)} \det_{1\le j,k\le N}(\pi_{j-1}(\lambda_k)) \det_{1\le l,m\le N}(\pi_{l-1}(\lambda_m)) = \prod_{k=0}^{N-1} e^{q_k(t)} \tag{622}$$

- Compute the scalar products $\langle x\pi_k(x), \pi_l(x) \rangle$ for $l < k$ and show that the orthogonal polynomials $\pi_k(x)$ satisfy the three-term identity

$$x\pi_k(x) = \pi_{k+1}(x) - p_k(t)\pi_k(x) + R_k(t)\pi_{k-1}(x) \tag{623}$$

  for some coefficients $p_k(t)$, $R_k(t)$ (not depending on x, only on parameters t_1, t_2, ...) of the potential. Show that $R_k(t) = e^{q_k(t) - q_{k-1}(t)}$.

- Compute the derivative of the scalar product $\langle \pi_k, \pi_k \rangle$ with respect to t_1 and use the properties of the orthogonal polynomials to show that $\frac{\partial q_k(t)}{\partial t_1} = p_k(t)$.
- Compute the derivative of the scalar product $\langle \pi_k, \pi_l \rangle$ for $k \neq l$ with respect to t_1 and find the expression for the derivative $\frac{\partial \pi_k(x)}{\partial t_1}$ in terms of p_i, q_i, π_i.

Solution.

- There is a natural volume form on each finite-dimensional inner-product space of dimension n. Each symmetric positively defined $g \in \mathrm{Mat}_n(\mathbb{R})$ defines an inner-product and metric on $\mathbb{R}^n$:

$$\langle x, y \rangle_g = \sum_{j,k=1}^{n} g_{jk} x_j y_k, \quad ds^2 = \sum_{j,k=1}^{n} g_{jk} dx_j dx_k \tag{624}$$

The associated n-dimensional volume form is

$$Dx = \sqrt{\det g} dx_1 ... dx_n \tag{625}$$

The space of Hermitian matrices $\mathcal{H}_N$ is a vector-space of real dimension $n = N^2$, as may be seen by the isomorphism $\mathcal{H}_N \to \mathbb{R}^n$:

$$M \to \xi = (M_{11}, ..., M_{NN}, \mathrm{Re}M_{12}, ..., \mathrm{Re}M_{N-1,N}, \mathrm{Im}M_{12}, ..., \mathrm{Im}M_{N-1,N}) \tag{626}$$

The Hilbert-Schmidt inner product on $\mathcal{H}_N$ is

$$\mathcal{H}_N \times \mathcal{H}_N \to \mathbb{C}, \quad (M, N) \to \mathrm{Tr}(M^\dagger N) \tag{627}$$

The associated infinitesimal length element is

$$ds^2 = \mathrm{Tr}(dM^2) = \sum_{i=1}^{N} dM_{ii}^2 + 2 \sum_{1 \le i < j \le N} d\mathrm{Re}M_{ij}^2 + d\mathrm{Im}M_{ij}^2 \tag{628}$$

Thus, in the coordinates ξ, the metric is an $N^2 \times N^2$ diagonal matrix whose first N entries are 1 and all other entries are 2, so

$$\det g = 2^{N(N-1)} \tag{629}$$

$$DM = 2^{\frac{N(N-1)}{2}} \prod_{i=1}^{N} dM_{ii} \prod_{1 \le i < j \le N} d\mathrm{Re}M_{ij} d\mathrm{Im}M_{ij} \tag{630}$$

The unitary group, $U(N)$ is the group of linear isometries of $\mathbb{C}^N$ equipped with the standard inner-product $\langle x, y \rangle = x^\dagger y$. Thus, $U(n)$ is equivalent to the group of matrices $U \in \mathrm{Mat}_N(\mathbb{C})$ such that $U^\dagger U = I$. The inner-product (628) and volume form (630) are invariant under the transformation $M \to UMU^\dagger$.
The Lie algebra $\mathfrak{u}(N)$

$$\mathfrak{u}(N) = T_I U(N) = \{A \in \mathrm{Mat}_N(\mathbb{C}) | A = -A^\dagger\} \tag{631}$$

$$T_U U(N) = \{UA, A \in \mathfrak{u}(N)\} \tag{632}$$

For $A, \tilde{A} \in \mathfrak{u}(n)$, we define their inner product $\mathrm{Tr}(A^\dagger \tilde{A}) = -\mathrm{Tr}(A\tilde{A})$. This inner-product is natural, because it is invariant under left application of $U(N)$. That is, for two vector

$UA, U\tilde{A} \in T_U U(N)$ we find $\mathrm{Tr}((UA)^\dagger U\tilde{A}) = \mathrm{Tr}(A^\dagger \tilde{A})$. The associated volume form on $U(n)$ is called *Haar measure*. It is unique, upto a normalizing factor, and we write

$$D\tilde{U} = 2^{\frac{N(N-1)}{2}} \prod_{i=1}^{N} dA_{ii} \prod_{1\le i<j\le N} d\mathrm{Re}A_{ij} d\mathrm{Im}A_{ij} \tag{633}$$

However, when viewing diagonalization $M = U\Lambda U^\dagger$ as a change of variables on $\mathcal{H}_N$, it is necessary to quotient out the following degeneracy: $\forall\theta = (\theta_1, ..., \theta_N) \in \mathbb{R}^N$, the diagonal matrix $D = \mathrm{diag}(e^{i\theta_1}, ..., e^{i\theta_N})$ is unitary and $M = U\Lambda U^\dagger \Leftrightarrow M = UD\Lambda D^\dagger U^\dagger$. Thus, for $\mathcal{H}_N$, the measure $D\tilde{U}$ must be replaced a measure on $U(N)/\mathbb{R}^N$.

$$dM = dU\Lambda U^\dagger + Ud\Lambda U^\dagger + U\Lambda dU^\dagger \tag{634}$$

$$UU^\dagger = I \to dUU^\dagger + UdU^\dagger = 0 \to dU^\dagger = -U^{-1}dUU^\dagger = -U^\dagger dUU^\dagger \tag{635}$$

If $U = I$, then

$$dU^\dagger = -dU \tag{636}$$

$$dM = dU\Lambda U^\dagger + Ud\Lambda U^\dagger - U\Lambda U^\dagger dUU^\dagger \tag{637}$$

$$dM = U(d\Lambda + [U^\dagger dU, \Lambda])U^\dagger \tag{638}$$

$$(U^\dagger dU)^\dagger = dU^\dagger U = -U^\dagger dUU^\dagger U = -U^\dagger dU \tag{639}$$

Matrix $U^\dagger dU$ is antihermitian.
Thus, the volume form on the quotient $U(N)/\mathbb{R}^N$ is locally equivalent to a volume form on the subspace of anti-Hermitian matrices consisting of matrices with zero diagonal:

$$DU = 2^{\frac{N(N-1)}{2}} \prod_{1\le i<j\le N} d\mathrm{Re}A_{ij} d\mathrm{Im}A_{ij} \tag{640}$$

Let be A:

$$U^\dagger dU = dA \tag{641}$$

$$\begin{aligned}\mathrm{Tr}(dM)^2 &= \mathrm{Tr}(dM)^\dagger dM = \mathrm{Tr}U(d\Lambda + [U^\dagger dU, \Lambda])^\dagger U^\dagger U(d\Lambda + [U^\dagger dU, \Lambda])U^\dagger = \\ &= \mathrm{Tr}d\Lambda^2 + 2\mathrm{Tr}d\Lambda[dA, \Lambda] + \mathrm{Tr}[dA, \Lambda]^\dagger[dA, \Lambda] = \mathrm{Tr}d\Lambda^2 + \mathrm{Tr}[dA, \Lambda]^\dagger[dA, \Lambda]\end{aligned} \tag{642}$$

$$dA = d\mathrm{Re}A + id\mathrm{Im}A \tag{643}$$

$$\begin{aligned}\mathrm{Tr}[dA, \Lambda]^\dagger[dA, \Lambda] &= \mathrm{Tr}(d\mathrm{Re}A)\Lambda(d\mathrm{Re}A)\Lambda + \mathrm{Tr}\Lambda(d\mathrm{Re}A)\Lambda(d\mathrm{Re}A) - \\ &- \mathrm{Tr}\Lambda(d\mathrm{Re}A)^2\Lambda - \mathrm{Tr}(d\mathrm{Re}A)\Lambda^2(d\mathrm{Re}A) + \\ &+ \mathrm{Tr}(d\mathrm{Im}A)\Lambda(d\mathrm{Im}A)\Lambda + \mathrm{Tr}\Lambda(d\mathrm{Im}A)\Lambda(d\mathrm{Im}A) - \\ &- \mathrm{Tr}\Lambda(d\mathrm{Im}A)^2\Lambda - \mathrm{Tr}(d\mathrm{Im}A)\Lambda^2(d\mathrm{Im}A) = \\ &= 2\sum_{i<j}(\lambda_i - \lambda_j)^2 d\mathrm{Re}A_{ij}^2 + 2\sum_{i<j}(\lambda_i - \lambda_j)^2 d\mathrm{Im}A_{ij}^2\end{aligned} \tag{644}$$

$$ds^2 = \mathrm{Tr}(dM)^2 = \sum_{i=1}^{N} d\lambda_i^2 + 2\sum_{i<j}(\lambda_i - \lambda_j)^2 d\mathrm{Re}A_{ij}^2 + 2\sum_{i<j}(\lambda_i - \lambda_j)^2 d\mathrm{Im}A_{ij}^2 \tag{645}$$

$$DM = 2^{\frac{N(N-1)}{2}} \prod_{i=1}^{N} d\lambda_i \prod_{1\le i<j\le N} (\lambda_i - \lambda_j)^2 d\mathrm{Re}A_{ij} d\mathrm{Im}A_{ij} = \prod_{1\le i<j\le N} (\lambda_i - \lambda_j)^2 D\Lambda DU \tag{646}$$

$$DM = \frac{\mu_{U(N)}}{\mu_{U(1)^N}} \prod_{i=1}^{N} d\lambda_i \prod_{1\leq i<j\leq N} (\lambda_i - \lambda_j)^2 \tag{647}$$

Thus, the partition function for $c_N = \frac{1}{N!}\frac{\mathrm{Vol}(U(1))^N}{\mathrm{Vol}(U(N))}$ is rewritten as

$$\boxed{Z_N(t) = \frac{1}{N!}\int_{\mathbb{R}^N} \prod_{i=1}^{N}(d\lambda_i e^{-V(\lambda_i)}) \prod_{1\leq i<j\leq N} (\lambda_i - \lambda_j)^2} \tag{648}$$

- Vandermonde determinant:

$$\Delta(\Lambda) = \prod_{1\leq i<j\leq N} (\lambda_i - \lambda_j)^2 = \det \begin{pmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_N \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{N-1} & \lambda_2^{N-1} & \dots & \lambda_N^{N-1} \end{pmatrix} \tag{649}$$

$$\pi_k(x) = 1 + \sum_{1\leq l\leq k} \gamma_{kl} x^l \tag{650}$$

By elementary column operations on the Vandermonde determinant:

$$\prod_{1\leq i<j\leq N} (\lambda_i - \lambda_j) = \det \begin{pmatrix} \pi_0(\lambda_1) & \pi_0(\lambda_2) & \dots & \pi_0(\lambda_N) \\ \pi_1(\lambda_1) & \pi_1(\lambda_2) & \dots & \pi_1(\lambda_N) \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{N-1}(\lambda_1) & \pi_{N-1}(\lambda_2) & \dots & \pi_{N-1}(\lambda_N) \end{pmatrix} = \\ = \det_{1\leq j,k\leq N}(\pi_{j-1}(\lambda_k)) \tag{651}$$

$$Z_N(t) = \frac{1}{N!}\prod_{i=1}^{N}\int_{\mathbb{R}} d\lambda_i e^{-V(\lambda_i)} \det_{1\leq j,k\leq N}(\pi_{j-1}(\lambda_k)) \det_{1\leq l,m\leq N}(\pi_{l-1}(\lambda_m)) \tag{652}$$

$$\det_{1\leq j,k\leq N}(\pi_{j-1}(\lambda_k)) = \sum_{\sigma\in S_N} \mathrm{sgn}(\sigma) \prod_{j=1}^{N} \pi_{\sigma_j - 1}(\lambda_j) \tag{653}$$

$$\det_{1\leq j,k\leq N}(\pi_{j-1}(\lambda_k)) \det_{1\leq l,m\leq N}(\pi_{l-1}(\lambda_m)) = \sum_{\sigma\in S_N} \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau) \prod_{j=1}^{N} \pi_{\sigma_j - 1}(\lambda_j)\pi_{\tau_j-1}(\lambda_j) \tag{654}$$

$$\begin{aligned} Z_N(t) &= \frac{1}{N!}\int_{\mathbb{R}^N}\prod_{i=1}^{N} d\lambda_i e^{-V(\lambda_i)} \sum_{\sigma,\tau\in S_N} \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau)\prod_{j=1}^{N}\pi_{\sigma_j-1}(\lambda_j)\pi_{\tau_j-1}(\lambda_j) = \\ &= \frac{1}{N!}\sum_{\sigma,\tau\in S_N}\mathrm{sgn}(\sigma)\mathrm{sgn}(\tau)\int_{\mathbb{R}^N}\prod_{i=1}^{N} d\lambda_i e^{-V(\lambda_i)}\prod_{j=1}^{N}\pi_{\sigma_j-1}(\lambda_j)\pi_{\tau_j-1}(\lambda_j) = \\ &= \frac{1}{N!}\sum_{\sigma,\tau\in S_N}\mathrm{sgn}(\sigma)\mathrm{sgn}(\tau)\prod_{j=1}^{N} e^{q_{\sigma_j-1}(t)}\delta_{\sigma_j-1,\tau_j-1} = \frac{1}{N!}\sum_{\sigma\in S_N}\prod_{j=1}^{N} e^{q_{\sigma_j-1}(t)} \end{aligned} \tag{655}$$

$$\boxed{Z_N(t) = \prod_{k=0}^{N-1} e^{q_k(t)}} \tag{656}$$

- Since $x\pi_k(x)$ is a polynomial of degree $k+1$ it can be expressed as a linear combination of $\pi_j(x)$:

$$x\pi_k(x) = \sum_{j=0}^{k+1} c_{j,k}\pi_j(x) \tag{657}$$

Since $\pi_j(x) = x^j + ...$, we have $c_{k+1,k} = 1$.

$$\langle x\pi_k(x), \pi_l(x)\rangle = \sum_{j=0}^{k+1} c_{j,k} \langle \pi_j(x), \pi_l(x)\rangle = \sum_{j=0}^{k+1} c_{j,k} e^{q_j(t)}\delta_{jl} = c_{l,k}e^{q_l(t)} \tag{658}$$

$$c_{j,k} = e^{-q_j(t)} \langle x\pi_k(x), \pi_j(x)\rangle \tag{659}$$

For $j \in \{0, ..., k-2\}$

$$\langle x\pi_k(x), \pi_j(x)\rangle = \langle \pi_k(x), x\pi_j(x)\rangle = 0 \tag{660}$$

since $x\pi_j$ lies in the span of $\{\pi_0, ..., \pi_{k-1}\}$. Thus, $c_{j,k} = 0$ for $j = 0, ..., k-2$ and we find

$$x\pi_k(x) = \pi_{k+1}(x) + c_{k,k}\pi_k(x) + c_{k-1,k}\pi_{k-1}(x) \tag{661}$$

Let be $p_k(t) = -c_{k,k}$, $R_k(t) = c_{k-1,k}$, then we obtain three-term identity

$$\boxed{x\pi_k(x) = \pi_{k+1}(x) - p_k(t)\pi_k(x) + R_k\pi_{k-1}(x)} \tag{662}$$

$$R_k(t) = c_{k-1,k} = e^{-q_{k-1}(t)} \langle x\pi_k(x), \pi_{k-1}(x)\rangle \tag{663}$$

$$\langle x\pi_k(x), \pi_{k-1}(x)\rangle = \langle \pi_k(x), x\pi_{k-1}(x)\rangle = \langle \pi_k(x), \pi_k(x)\rangle = e^{q_k(t)} \tag{664}$$

$$\boxed{R_k(t) = e^{q_k(t)-q_{k-1}(t)}} \tag{665}$$

- $$\langle \pi_k(x), \pi_k(x)\rangle = \int_{\mathbb{R}} \pi_k^2(x)e^{-V(x)}dx = e^{q_k(t)} \tag{666}$$

$$\frac{\partial}{\partial t_1} \langle \pi_k(x), \pi_k(x)\rangle = -\int_{\mathbb{R}} x\pi_k^2(x)e^{-V(x)}dx = -\langle x\pi_k(x), \pi_k(x)\rangle = -e^{q_k(t)}c_{k,k} = \\ = e^{q_k(t)}p_k(t) \tag{667}$$

$$\frac{\partial}{\partial t_1} \langle \pi_k(x), \pi_k(x)\rangle = e^{q_k(t)}\frac{\partial q_k(t)}{\partial t_1} \tag{668}$$

$$\boxed{\frac{\partial q_k(t)}{\partial t_1} = p_k(t)} \tag{669}$$

- $$\langle \pi_k(x), \pi_l(x)\rangle = \int_{\mathbb{R}} \pi_k(x)\pi_l(x)e^{-V(x)}dx = 0 \tag{670}$$

$$\frac{\partial}{\partial t_1} \langle \pi_k(x), \pi_l(x)\rangle = -\int_{\mathbb{R}} x\pi_k(x)\pi_l(x)e^{-V(x)}dx = -\langle x\pi_k(x), \pi_l(x)\rangle = -c_{l,k}e^{q_l(t)} \tag{671}$$

6 Integrable systems related to infinite-dimensional Lie algebras

1. **Pseudodifferential operators.**
Consider a ring of pseudodifferential operators with elements of the standard form

$$\sum_{k=0}^{\infty} c_k(x)\partial^{N-k} = c_0\partial^N + c_1\partial^{N-1} + ..., \tag{672}$$

where $c_k(x)$ are functions of one variable x, ∂ is a derivative with respect to x, which has the standard commutation rules with functions: $\partial f(x) = f(x)\partial + f'(x)$. ∂^{-1} is a formal inverse of ∂, such that $\partial\partial^{-1} = \partial^{-1}\partial = 1$.

- Find the explicit expression for the commutation rule of ∂^{-1} with an arbitrary function, namely, rewrite the product $\partial^{-1}f(x)$ in the standard form (672).
- Consider a pseudodifferential operator Q with the property $L = Q^2 = \partial^2 + u(x)$. Write down the first five nontrivial coefficients a_0, a_1, a_2, a_3 and a_4 in the expansion of this operator $Q = \partial + \sum\limits_{k\geq 0} a_k\partial^{-k}$.
- Write down the expressions for operators $M_3 = (Q^3)_+$ and $M_5 = (Q^5)_+$, where $()_+$ denotes the positive part of the pseudodifferential operator (all ∂^k, $k < 0$ terms set to zero)

$$\left(\sum_{k=0}^{\infty} c_k(x)\partial^{N-k}\right)_+ = \sum_{k=0}^{N} c_k(x)\partial^{N-x} \tag{673}$$

Show that the equation $\frac{\partial L}{\partial t_3} = [M_3, L]$ is equivalent to the KdV equation, and the equation $\frac{\partial L}{\partial t_5} = [M_5, L]$ can be considered as one of the higher flows of KdV hierarchy (i.e. it commutes with t_3 flow).

Solution.

- Commutation rule ∂ with functions:

$$\partial f(x) = f(x)\partial + f'(x) \tag{674}$$

$$f(x)\partial^{-1} = \partial^{-1}\partial(f(x)\partial^{-1}) = \partial^{-1}(f'(x)\partial^{-1}) + \partial^{-1}f(x) \tag{675}$$

$$\partial^{-1}f(x) = f(x)\partial^{-1} - \partial^{-1}(f'(x)\partial^{-1}) = f(x)\partial^{-1} - f'(x)\partial^{-2} + \partial^{-1}(f''(x)\partial^{-1}) \tag{676}$$

$$\boxed{\partial^{-1}f(x) = \sum_{k=1}^{\infty}(-1)^{k-1}f^{(k-1)}(x)\partial^{-k}} \tag{677}$$

-

$$Q = \partial + \sum_{k\geq 0} a_k\partial^{-k} \tag{678}$$

$$L = Q^2 = \left(\partial + \sum_{k\geq 0} a_k\partial^{-k}\right)\left(\partial + \sum_{k\geq 0} a_k\partial^{-k}\right) = \partial^2 + u(x) \tag{679}$$

In the product of the sums we will leave the terms only up to ∂^{-3}.

$$\begin{aligned}
&\partial(\partial + a_0 + a_1\partial^{-1} + a_2\partial^{-2} + a_3\partial^{-3} + a_4\partial^{-4}) = \\
&\quad = \partial^2 + a_0' + a_1'\partial^{-1} + a_1 + a_2'\partial^{-2} + a_2\partial^{-1} + a_3'\partial^{-3} + a_3\partial^{-2} + a_4\partial^{-3} + ... = \\
&\qquad = \partial^2 + a_0' + a_1 + (a_1' + a_2)\partial^{-1} + (a_2' + a_3)\partial^{-2} + (a_3' + a_4)\partial^{-3} + ...
\end{aligned} \tag{680}$$

$$a_0(\partial + a_0 + a_1\partial^{-1} + a_2\partial^{-2} + a_3\partial^{-3}) = a_0\partial + a_0^2 + a_0a_1\partial^{-1} + a_0a_2\partial^{-2} + a_0a_3\partial^{-3} \quad (681)$$

$$\begin{aligned} &a_1\partial^{-1}(\partial + a_0 + a_1\partial^{-1} + a_2\partial^{-2}) = \\ &= a_1 + a_1(a_0\partial^{-1} - a_0'\partial^{-2} + a_0''\partial^{-3} - ...) + a_1(a_1\partial^{-2} - a_1'\partial^{-3} + ...) + a_1a_2\partial^{-3} + ... = \\ &= a_1 + a_0a_1\partial^{-1} + a_1(a_1 - a_0')\partial^{-2} + a_1(a_0'' - a_1' + a_2)\partial^{-3} + ... \end{aligned} \quad (682)$$

$$\begin{aligned} &a_2\partial^{-2}(\partial + a_0 + a_1\partial^{-1}) = a_2\partial^{-1} + a_2\partial^{-1}(a_0\partial^{-1} - a_0'\partial^{-2} + ...) + a_2\partial^{-1}a_1\partial^{-2} - ... = \\ &= a_2\partial^{-1} + a_2(a_0\partial^{-2} - a_0'\partial^{-3}) - a_2a_0'\partial^{-3} + a_1a_2\partial^{-3} - ... = \\ &= a_2\partial^{-1} + a_0a_2\partial^{-2} + a_2(a_1 - 2a_0')\partial^{-3} + ... \end{aligned} \quad (683)$$

$$a_3\partial^{-3}(\partial + a_0) = a_3\partial^{-2} + a_3\partial^{-2}(a_0\partial^{-1} - ...) = a_3\partial^{-2} + a_0a_3\partial^{-3} + ... \quad (684)$$

$$a_4\partial^{-4}\partial = a_4\partial^{-3} \quad (685)$$

$$\begin{aligned} &\partial^2 + a_0\partial + (a_0' + a_0^2 + 2a_1) + (2a_0a_1 + a_1' + 2a_2)\partial^{-1} + \\ &+ (2a_0a_2 - a_0'a_1 + a_1^2 + a_2' + 2a_3)\partial^{-2} + \\ &+ (a_0''a_1 - 2a_0'a_2 + 2a_0a_3 - a_1a_1' + 2a_1a_2 + a_3' + 2a_4)\partial^{-3} = \partial^2 + u(x) \end{aligned} \quad (686)$$

$$\begin{cases} a_0 = 0, \\ 2a_1 = u(x), \\ a_1' + 2a_2 = 0, \\ a_1^2 + a_2' + 2a_3 = 0, \\ -a_1a_1' + 2a_1a_2 + a_3' + 2a_4 = 0 \end{cases} \rightarrow \begin{cases} a_0 = 0, \\ a_1 = \frac{u(x)}{2}, \\ a_2 = -\frac{u'(x)}{4}, \\ a_3 = \frac{u''(x) - u^2(x)}{8}, \\ a_4 = \frac{6u(x)u'(x) - u'''(x)}{16} \end{cases} \quad (687)$$

$$\boxed{Q = \partial + \frac{u(x)}{2}\partial^{-1} - \frac{u'(x)}{4}\partial^{-2} + \frac{u''(x) - u^2(x)}{8}\partial^{-3} + \frac{6u(x)u'(x) - u'''(x)}{16}\partial^{-4}} \quad (688)$$

- $$M_3 = (Q^3)_+ = (LQ)_+ \quad (689)$$

$$M_3 = \left((\partial^2 + u(x))\left(\partial + \frac{u(x)}{2}\partial^{-1} - \frac{u'(x)}{4}\partial^{-2}\right)\right)_+ \quad (690)$$

$$\boxed{M_3 = \partial^3 + \frac{3u(x)}{2}\partial + \frac{3u'(x)}{4}} \quad (691)$$

$$\frac{\partial L}{\partial t_3} = \partial_3\partial^2 + \frac{\partial u(x)}{\partial t_3} \quad (692)$$

$$\begin{aligned} &[M_3, L] = \left[\partial^3 + \frac{3u(x)}{2}\partial + \frac{3u'(x)}{4}, \partial^2 + u(x)\right] = u'''(x) - u(x)\partial^3 + \frac{3u(x)}{2}\partial^3 - \\ &- \frac{3u''(x)}{2}\partial - 3u'(x)\partial^2 - \frac{3u(x)}{2}\partial^3 + \frac{3u(x)u'(x)}{2} - \frac{3u^2(x)}{2}\partial + \\ &+ \frac{3u'(x)}{4}\partial^2 - \frac{3u'''(x)}{4} = -u(x)\partial^3 - \frac{9}{4}u'(x)\partial^2 - \frac{3}{2}(u''(x) + u^2(x))\partial + \\ &+ \frac{1}{4}(u'''(x) + 6u(x)u'(x)) \end{aligned} \quad (693)$$

$$\frac{\partial L}{\partial t_3} = M_3 \to \boxed{\frac{\partial u(x)}{\partial t_3} = \frac{1}{4}(u'''(x) + 6u(x)u'(x))} \tag{694}$$

$$M_5 = (Q^5)_+ = (L^2Q)_+ \tag{695}$$

$$\begin{aligned} L^2 = (\partial^2 + u(x))(\partial^2 + u(x)) = \partial^4 + u(x)\partial^2 + u''(x) + 2u'(x)\partial + u(x)\partial^2 + u^2(x) = \\ = \partial^4 + 2u(x)\partial^2 + 2u'(x)\partial + u''(x) + u^2(x) \end{aligned} \tag{696}$$

$$\begin{aligned} M_5 = \Big((\partial^4 + 2u(x)\partial^2 + 2u'(x)\partial + u''(x) + u^2(x))\times \\ \times \left(\partial + \frac{u(x)}{2}\partial^{-1} - \frac{u'(x)}{4}\partial^{-2} + \frac{u''(x) - u^2(x)}{8}\partial^{-3} + \frac{6u(x)u'(x) - u'''(x)}{16}\partial^{-4}\right)\Big)_+ = \\ = \partial^5 + 2u'''(x) + 3u''(x)\partial + 2u'(x)\partial^2 + \frac{u(x)}{2}\partial^3 - \frac{3u'''(x)}{2} - u''(x)\partial - \frac{u'(x)}{4}\partial^2 + \\ + \frac{u'''(x) - 2u(x)u'(x)}{2} + \frac{u''(x) - u^2(x)}{8}\partial + \frac{6u(x)u'(x) - u'''(x)}{16} + \\ + 2u(x)\partial^3 + 2u(x)u'(x) + u^2(x)\partial - \frac{u(x)u'(x)}{2} + 2u'(x)\partial^2 + u'(x)u(x) + \\ + (u''(x) + u^2(x))\partial \end{aligned} \tag{697}$$

$$\boxed{M_5 = \partial^5 + \frac{5u(x)}{2}\partial^3 + \frac{15u'(x)}{4}\partial^2 + \frac{25u''(x) + 15u^2(x)}{8}\partial + \frac{15}{8}\left(\frac{u'''(x)}{2} + u'(x)u(x)\right)}$$

$$\frac{\partial L}{\partial t_5} = \partial_5\partial^2 + \frac{\partial u(x)}{\partial t_5} \tag{698}$$

$$\begin{aligned} [M_5, L] = \left[\partial^5 + \frac{5u(x)}{2}\partial^3 + \frac{15u'(x)}{4}\partial^2, \partial^2 + u(x)\right] + \\ + \left[\frac{25u''(x) + 15u^2(x)}{8}\partial + \frac{15}{8}\left(\frac{u'''(x)}{2} + u'(x)u(x)\right), \partial^2 + u(x)\right] = \\ = u^{(5)}(x) - u(x)\partial^5 - 5u'(x)\partial^4 - \frac{5u''(x)}{2}\partial^3 + \frac{5u(x)u'''(x)}{2} - \frac{5u^2(x)}{2}\partial^3 - \\ - \frac{15u'''(x)}{4}\partial^2 - \frac{15u''(x)}{2}\partial + \frac{15u'(x)u''(x)}{4} - \frac{15u'(x)u(x)}{4}\partial^2 - \\ - \frac{25u'''(x) + 30u(x)u'(x)}{8}\partial^2 - \frac{25u^{(4)}(x) + 30(u'(x))^2 + 30u(x)u''(x)}{8}\partial + \\ + \frac{25u''(x) + 15u^2(x)}{8}u'(x) - \frac{25u''(x) + 15u^2(x)}{8}u(x)\partial + \\ + \frac{15}{8}\left(\frac{u'''(x)}{2} + u'(x)u(x)\right)\partial^2 - \frac{15}{8}\left(\frac{u^{(5)}(x)}{2} + u(x)u'''(x) + 3u'(x)u''(x)\right) \end{aligned}$$

$$\boxed{\frac{\partial u(x)}{\partial t_5} = \frac{1}{16}(u^{(5)}(x) + 10u(x)u'''(x) + 20u'(x)u''(x) + 30u^2(x)u'(x))} \tag{699}$$

2. **Bihamiltonian structure.**
Two Poisson brackets structures $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$ are compatible if any linear combination of them $\lambda_1\{\cdot,\cdot\}_1 + \lambda_2\{\cdot,\cdot\}_2$ also has the Poisson brackets structure (i.e. Jacobi identity is

satisfied).
Define two Poisson brackets for KdV hierarchy: let $u(x) = \sum_n u_n x^{-n-2}$ be a series in x, with the dynamical variables u_n as the coefficients, and delta-function is defined as $\delta(x-y) = \sum_n x^n y^{-n-1}$

$$\{u(x), u(y)\}_1 = -\delta'(x-y) \tag{700}$$

$$\{u(x), u(y)\}_2 = -2u(x)\delta'(x-y) - u'(x)\delta(x-y) - \delta'''(x-y) \tag{701}$$

- Rewrite the brackets $\{u(x), u(y)\}_1$ and $\{u(x), u(y)\}_2$ as Poisson brackets on u_k elements.
- Show that the Poisson structures $\{u(x), u(y)\}_1$ and $\{u(x), u(y)\}_2$ are compatible.
- Consider a linear combination $\{\cdot,\cdot\}_\lambda = \{\cdot,\cdot\}_1 - \lambda\{\cdot,\cdot\}_2$. Let $H_\lambda = \sum_k \lambda^k H_k$ be a central element for these brackets $\{H_\lambda, f\}_\lambda = 0$, $\forall f$. Show that $\{H_k, f\}_1 = \{H_{k-1}, f\}_2$ and the coefficients H_k are in involution with respect to the first and the second Poisson brackets $\{H_k, H_l\}_1 = \{H_k, H_l\}_2 = 0$.
- Consider several first Hamiltonians in the KdV hierarchy

$$H_0 = \int u(x)dx, \quad H_1 = \int u^2(x)dx, \quad H_2 = \int (u^3(x) - u'(x)^2)dx \tag{702}$$

Check that they are in involution and check explicitly that

$$\frac{\partial u(x)}{\partial t_1} = \{H_1, u(x)\}_1 = \{H_0, u(x)\}_2, \quad \frac{\partial u(x)}{\partial t_3} = \{H_1, u(x)\}_2 = \{H_2, u(x)\}_1 \tag{703}$$

Solution.

- $$u(x) = \sum_n u_n x^{-n-2} \tag{704}$$

Rewrite the bracket $\{u(x), u(y)\}_1$:

$$\begin{aligned}\{u(x), u(y)\}_1 = \left\{\sum_n u_n x^{-n-2}, \sum_m u_m y^{-m-2}\right\}_1 &= \sum_{n,m} x^{-n-2} y^{-m-2}\{u_n, u_m\}_1 = \\ &= \sum_{l,m} x^{l-1} y^{-m-2}\{u_{-l-1}, u_m\}_1\end{aligned} \tag{705}$$

$$\delta(x-y) = \sum_n x^n y^{-n-1} \to \delta'(x-y) = \sum_n n x^{n-1} y^{-n-1} \tag{706}$$

$$\{u(x), u(y)\}_1 = -\delta'(x-y) \to \boxed{\{u_n, u_m\}_1 = (n+1)\delta_{n+m+2,0}} \tag{707}$$

Rewrite the bracket $\{u(x), u(y)\}_2$:

$$\begin{aligned}\{u(x), u(y)\}_2 = \left\{\sum_n u_n x^{-n-2}, \sum_m u_m y^{-m-2}\right\}_2 &= \sum_{n,m} x^{-n-2} y^{-m-2}\{u_n, u_m\}_2 = \\ &= \sum_{l,m} x^{l-1} y^{-m-2}\{u_{-l-1}, u_m\}_2\end{aligned} \tag{708}$$

$$2u(x)\delta'(x-y)+u'(x)\delta(x-y)+\delta'''(x-y)=2\sum_n u_n x^{-n-2}\sum_m m x^{m-1}y^{-m-1}+$$
$$+\sum_n(-n-2)u_n x^{-n-3}\sum_m x^m y^{-m-1}+\sum_n n(n-1)(n-2)x^{n-3}y^{-n-1}=$$
$$=\sum_{n,m}(n-2m+2)u_n x^{m-n-3}y^{-m-1}-\sum_{m,n}m(m-1)(m-2)x^{-n-2}y^{-m-1}\delta_{m+n,1} \quad (709)$$

$$\{u(x),u(y)\}_2=-2u(x)\delta'(x-y)-u'(x)\delta(x-y)-\delta'''(x-y) \quad (710)$$

$$\boxed{\{u_n,u_m\}_2=u_{n+m}(n-m)+(n^3-n)\delta_{n+m,0}} \quad (711)$$

- Show that the Poisson structures $\{u(x),u(y)\}_1$ and $\{u(x),u(y)\}_2$ are compatible:

$$\{u_k,\{u_l,u_m\}\}+\{u_m,\{u_k,u_l\}\}+\{u_l,\{u_m,u_k\}\}=\{u_k,\lambda_1\{u_l,u_m\}_1+\lambda_2\{u_l,u_m\}_2\}+$$
$$+\{u_m,\lambda_1\{u_k,u_l\}_1+\lambda_2\{u_k,u_l\}_2\}+\{u_l,\lambda_1\{u_m,u_k\}_1+\lambda_2\{u_m,u_k\}_2\}=$$
$$=\lambda_1\{u_k,\lambda_1\{u_l,u_m\}_1+\lambda_2\{u_l,u_m\}_2\}_1+\lambda_2\{u_k,\lambda_1\{u_l,u_m\}_1+\lambda_2\{u_l,u_m\}_2\}_2+$$
$$+\lambda_1\{u_m,\lambda_1\{u_k,u_l\}_1+\lambda_2\{u_k,u_l\}_2\}_1+\lambda_2\{u_m,\lambda_1\{u_k,u_l\}_1+\lambda_2\{u_k,u_l\}_2\}_2+$$
$$+\lambda_1\{u_l,\lambda_1\{u_m,u_k\}_1+\lambda_2\{u_m,u_k\}_2\}_1+\lambda_2\{u_l,\lambda_1\{u_m,u_k\}_1+\lambda_2\{u_m,u_k\}_2\}_2=$$
$$=\lambda_1^2(\{u_k,\{u_l,u_m\}_1\}_1+\{u_m,\{u_k,u_l\}_1\}_1+\{u_l,\{u_m,u_k\}_1\}_1)+$$
$$+\lambda_2^2(\{u_k,\{u_l,u_m\}_2\}_2+\{u_m,\{u_k,u_l\}_2\}_2+\{u_l,\{u_m,u_k\}_2\}_2)+$$
$$+\lambda_1\lambda_2(\{u_k,\{u_l,u_m\}_2\}_1+\{u_k,\{u_l,u_m\}_1\}_2+\{u_m,\{u_k,u_l\}_2\}_1+$$
$$+\{u_m,\{u_k,u_l\}_1\}_2+\{u_l,\{u_m,u_k\}_2\}_1+\{u_l,\{u_m,u_k\}_1\}_2)=$$
$$=\lambda_1\lambda_2(\{u_k,u_{l+m}(l-m)+(l^3-l)\delta_{l+m,0}\}_1+\{u_k,(l+1)\delta_{l+m+2,0}\}_2+$$
$$+\{u_m,u_{k+l}(k-l)+(k^3-k)\delta_{k+l,0}\}_1+\{u_m,(k+1)\delta_{k+l+2,0}\}_2+$$
$$+\{u_l,u_{m+k}(m-k)+(m^3-m)\delta_{m+k,0}\}_1+\{u_l,(m+1)\delta_{m+k+2,0}\}_2)=$$
$$=\lambda_1\lambda_2((l-m)\{u_k,u_{l+m}\}_1+(k-l)\{u_m,u_{k+l}\}_1+(m-k)\{u_l,u_{m+k}\}_1)=$$
$$=\lambda_1\lambda_2\delta_{k+l+m+2,0}((l-m)(k+1)+(k-l)(m+1)+(m-k)(l+1))=0 \quad (712)$$

$$\boxed{\{u_k,\{u_l,u_m\}\}+\{u_m,\{u_k,u_l\}\}+\{u_l,\{u_m,u_k\}\}=0} \quad (713)$$

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$$\{\cdot,\cdot\}_\lambda=\{\cdot,\cdot\}_1-\lambda\{\cdot,\cdot\}_2 \quad (714)$$
$$H_\lambda=\sum_k\lambda^k H_k \quad (715)$$

H_λ is a central element:

$$\{H_\lambda,f\}_\lambda=0,\ \forall f \quad (716)$$

$$\{H_\lambda,f\}_\lambda=\left\{\sum_k\lambda^k H_k,f\right\}_\lambda=\left\{\sum_k\lambda^k H_k,f\right\}_1-\lambda\left\{\sum_k\lambda^k H_k,f\right\}_2=$$
$$=\sum_k\lambda^k\{H_k,f\}_1-\sum_k\lambda^{k+1}\{H_k,f\}_2=$$
$$=\{H_0,f\}_1+\sum_k\lambda^k(\{H_k,f\}_1-\{H_{k-1},f\}_2) \quad (717)$$

$$\boxed{\{H_0,f\}_1=0,\quad \{H_k,f\}_1=\{H_{k-1},f\}_2} \quad (718)$$

$$\{H_0, f\}_1 = 0 \to \forall k \hookrightarrow \{H_0, H_k\}_1 = 0 \tag{719}$$

$$\{H_0, H_k\}_1 = -\{H_k, H_0\}_1 = -\{H_{k-1}, H_0\}_2 = \{H_0, H_{k-1}\}_2 = 0 \tag{720}$$

$$\{H_1, H_k\}_1 = \{H_0, H_k\}_2 = 0 \to ... \tag{721}$$

$$\boxed{\{H_k, H_l\}_1 = \{H_k, H_l\}_2 = 0} \tag{722}$$

- First Hamiltonians in the KdV hierarchy:

$$H_0 = \int u(x)dx, \quad H_1 = \int (u(x))^2 dx, \quad H_2 = \int ((u(x))^3 - (u'(x))^2)dx \tag{723}$$

$$\begin{aligned}\{H_0, H_1\}_1 = \left\{\int u(x)dx, \int (u(y))^2 dy\right\}_1 = \int\int dxdy\,\{u(x), (u(y))^2\}_1 = \\ = -2\int\int dxdyu(y)\delta'(x-y) = -2\int dxu'(x) = -2u(x)|_0^{2\pi} = 0\end{aligned} \tag{724}$$

$$\begin{aligned}\{H_0, H_1\}_2 = \left\{\int u(x)dx, \int (u(y))^2 dy\right\}_2 = \int\int dxdy\,\{u(x), (u(y))^2\}_2 = \\ = -2\int\int dxdyu(y)(2u(x)\delta'(x-y) + u'(x)\delta(x-y) + \delta'''(x-y)) = \\ = -2\int dx(2u(x)u'(x) + u'(x)u(x) + u'''(x)) = 3(u(x))^2|_0^{2\pi} + 2u''(x)|_0^{2\pi} = 0\end{aligned} \tag{725}$$

$$\begin{aligned}\{H_0, H_2\}_1 = \left\{\int u(x)dx, \int ((u(y))^3 - (u'(y))^2)dy\right\}_1 = \int\int dxdy\,\{u(x), (u(y))^3\}_1 - \\ - \int\int dxdy\,\{u(x), (u'(y))^2\}_1 = -\int\int dxdy 3u^2(y)\delta'(x-y) + \\ + \int\int dxdy 2u'(y)\partial_y\delta'(x-y) = -6\int dxu(x)u'(x) - 2\int dxu'''(x) = \\ = -3u^2(x)|_0^{2\pi} - 2u''(x)|_0^{2\pi} = 0\end{aligned} \tag{726}$$

$$\begin{aligned}\{H_0, H_2\}_2 = \left\{\int u(x)dx, \int ((u(y))^3 - (u'(y))^2)dy\right\}_2 = \int\int dxdy\,\{u(x), (u(y))^3\}_2 - \\ -\int\int dxdy\,\{u(x), (u'(y))^2\}_2 = -\int\int dxdy 3(u(y))^2(2u(x)\delta'(x-y) + u'(x)\delta(x-y) + \delta'''(x-y)) + \\ + \int\int dxdy 2u'(y)\partial_y(2u(x)\delta'(x-y) + u'(x)\delta(x-y) + \delta'''(x-y)) = \\ = -12\int dx(u(x))^2u'(x) - 3\int dxu^2(x)u'(x) - 3\int dx(2u(x)u'''(x) + 6u'(x)u''(x)) - \\ - 2\int dxu(x)u'''(x) - 2\int dxu'(x)u''(x) - 2\int dxu^{(4)}(x) = 0\end{aligned} \tag{727}$$

Check that

$$\frac{\partial u(x)}{\partial t_1} = \{H_1, u(x)\}_1 = \{H_0, u(x)\}_2, \quad \frac{\partial u(x)}{\partial t_3} = \{H_1, u(x)\}_2 = \{H_2, u(x)\}_1 \tag{728}$$

$$\{H_1, u(x)\}_1 = \left\{ \int dy (u(y))^2, u(x) \right\}_1 = \int dy 2u(y) \{u(y), u(x)\}_1 = \\ = \int dy 2u(y) \delta'(x-y) = 2u'(x) \quad (729)$$

$$\{H_0, u(x)\}_2 = \left\{ \int dy u(y), u(x) \right\}_2 = \int dy \{u(y), u(x)\}_2 = \\ = \int dy (2u(x)\delta'(x-y) - u'(x)\delta(x-y) - \delta'''(x-y)) = 2u'(x) \quad (730)$$

$$t_1 = \frac{x}{2} \to \frac{\partial u(x)}{\partial t_1} = \{H_1, u(x)\}_1 = \{H_0, u(x)\}_2 \quad (731)$$

$$\{H_1, u(x)\}_2 = \left\{ \int dy (u(y))^2, u(x) \right\}_2 = \int dy 2u(y) \{u(y), u(x)\}_2 = \\ = \int dy 2u(y)(2u(x)\delta'(x-y) - u'(x)\delta(x-y) - \delta'''(x-y)) = \\ = 4u(x)u'(x) - 2u(x)u'(x) - 2u'''(x) = 2u(x)u'(x) - 2u'''(x) \quad (732)$$

$$\{H_2, u(x)\}_1 = \{ \int (dy(u(y))^3 - (u'(y))^2), u(x)\}_1 = 3 \int dy u^2(y) \{u(y), u(x)\}_1 - \\ - 2 \int dy u'(y) \{u'(y), u(x)\}_1 = \int dy (3u^2(y)\delta'(x-y) - 2u'(y)\partial_y \delta'(x-y))) = \\ = 6u(x)u'(x) - 2u'''(x) \quad (733)$$

3. **Virasoro algebra as a central extension.**
Consider the Witt Lie algebra with generators L_n, $n \in \mathbb{Z}$ and Lie brackets

$$[L_n, L_m] = (n-m)L_{n+m} \quad (734)$$

- Check that differential operators $L_n = -x^{n+1}\partial_x$ form the representation of this Lie algebra.
- Show that $\omega(L_n, L_m) = (n^3 - n)\delta_{n+m,0}$ is a Lie algebra 2-cocycle and it can be used to centrally extend the Witt algebra to define Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3 - n}{12}\delta_{n+m,0}c, \quad [c, L_n] = 0 \quad (735)$$

Show that this central extension is unique up to a multiplication on the arbitrary constant.
- Construct any nontrivial representation of the Virasoro algebra (for example for the central charge equal to one).

Solution.

- Differential operators:

$$L_n = -x^{n+1}\partial_x \quad (736)$$

$$[L_n, L_m] = [-x^{n+1}\partial_x, -x^{m+1}\partial_x] = x^{n+1}\partial_x(x^{m+1}\partial_x) - x^{m+1}\partial_x(x^{n+1}\partial_x) = \\ = x^{n+1}((m+1)x^m\partial_x + x^{m+1}\partial_x^2) - x^{m+1}((n+1)x^n\partial_x + x^{n+1}\partial_x^2) = \\ = x^{n+1}(m+1)x^m\partial_x - x^{m+1}(n+1)x^n\partial_x = -(n-m)x^{n+m+1}\partial_x = (n-m)L_{n+m} \quad (737)$$

•

$$[L_m, L_n] = (m-n)L_{m+n} + \lambda_{m,n} \tag{738}$$

$$[L_n, L_m] = -[L_m, L_n] \tag{739}$$

$$(n-m)L_{n+m} + \lambda_{n,m} = -(m-n)L_{m+n} + \lambda_{m,n} \to \lambda_{n,m} = -\lambda_{m,n} \tag{740}$$

Move the generators

$$L_n \to L_n + q_n \tag{741}$$

$$[L_m, L_n] = (m-n)L_{m+n} + \lambda_{m,n} \tag{742}$$

$$[L_m + q_m, L_n + q_n] = (m-n)(L_{m+n} + q_{m+n}) + \lambda_{m,n} \tag{743}$$

$$\lambda_{m,n} \to \lambda_{m,n} + (m-n)q_{m+n} \tag{744}$$

Choose $q_m = -\frac{1}{m}\lambda_{m,0}$ for $m \neq 0$ and $q_0 = -\frac{1}{2}\lambda_{1,-1}$. Then

$$\lambda_{m,0} \to \lambda_{m,0} + mq_m = 0 \quad \forall m \neq 0 \tag{745}$$

$$\lambda_{1,-1} \to \lambda_{1,-1} + 2q_0 = 0 \tag{746}$$

$$[L_m, L_0] = mL_m, \quad [L_1, L_{-1}] = 2L_0 \tag{747}$$

$$[[L_m, L_n], L_0] = [(m-n)L_{n+m} + \lambda_{m,n}, L_0] = (m-n)((m+n)L_{m+n} + \lambda_{m+n,0}) \tag{748}$$

$$[[L_n, L_0], L_m] = [nL_n + \lambda_{n,0}, L_m] = n((n-m)L_{n+m} + \lambda_{n,m}) \tag{749}$$

$$[[L_0, L_m], L_n] = [-mL_m + \lambda_{0,m}, L_n] = -m((m-n)L_{m+n} + \lambda_{m,n}) \tag{750}$$

Consider Jacobi identity:

$$\begin{aligned}[][[L_m, L_n], L_0] + [[L_n, L_0], L_m] + [[L_0, L_m], L_n] = (m-n)\lambda_{m+n,0} + n\lambda_{n,m} - m\lambda_{m,n} = \\ = (m+n)\lambda_{n,m} = 0\end{aligned} \tag{751}$$

In case $m \neq -n$ we have $\lambda_{n,m} = 0$. Therefore, the only non-vanishing central extensions are $\lambda_{n,-n}$ for $|n| \geq 2$.

$$\lambda_{n,m} = \lambda(n)\delta_{m+n,0} \tag{752}$$

$$[[L_{-n+1}, L_n], L_{-1}] = [(-2n+1)L_1 + \lambda_{-n+1,n}, L_{-1}] = (-2n+1)(2L_0 + \lambda_{1,-1}) \tag{753}$$

$$[[L_n, L_{-1}], L_{-n+1}] = [(n+1)L_{n-1} + \lambda_{n,-1}, L_{-n+1}] = (n+1)(2(n-1)L_0 + \lambda_{n-1,1-n}) \tag{754}$$

$$[[L_{-1}, L_{-n+1}], L_n] = [(n-2)L_{-n} + \lambda_{-1,-n+1}, L_n] = (n-2)((-2n)L_0 + \lambda_{-n,n}) \tag{755}$$

Consider Jacobi identity:

$$\begin{aligned}[][[L_{-n+1}, L_n], L_{-1}] + [[L_n, L_{-1}], L_{-n+1}] + [[L_{-1}, L_{-n+1}], L_n] = (-2n+1)\lambda_{1,-1} + (n+1)\lambda_{n-1,1-n} + \\ + (n-2)\lambda_{-n,n} = (n+1)\lambda_{n-1,1-n} - (n-2)\lambda_{n,-n} = 0\end{aligned} \tag{756}$$

We obtain recurrent identity:

$$\lambda_{n,-n} = \frac{n+1}{n-2}\lambda_{n-1,1-n} = ... = C_3^{n+1}\lambda_{2,-2} = \frac{(n+1)n(n-1)}{6}\lambda_{2,-2} \tag{757}$$

We choose $\lambda_{2,-2} = \frac{c}{2}$.

$$\lambda_{m,n} = \frac{c}{12}(m^3 - m)\delta_{m+n,0} \tag{758}$$

We obtain Virasoro algebra:

$$\boxed{[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}} \tag{759}$$

- A highest weight representation of the Virasoro algebra is a representation generated by a primary state:

  $$L_0\Phi_\Delta = \Delta\Phi_\Delta, \quad L_n\Phi_\Delta = 0, \quad n > 0 \tag{760}$$

  Δ is called the conformal dimension of Φ_Δ. A highest weight representation is spanned by eigenstates of L_0. The eigenvalues take the form $\Delta + n$, where the integer $n \geq 0$ is called the level of the corresponding eigenstate:

  $$L_0 L_{-n}\Phi_\Delta = (\Delta + n)L_{-n}\Phi_\Delta \tag{761}$$

  More precisely, a highest weight representation is spanned by L_0-eigenstates of the type $L_{-n_1}L_{-n_2}\cdots L_{-n_k}\Phi_\Delta$ with $0 < n_1 \leq n_2 \leq \cdots n_k$ and $k \geq 0$, whose levels are $N = \sum_{i=1}^{k} n_i$. Any state whose level is not zero is called a descendant state of Φ_Δ.