APPM 2360 - Mortgage Report

Anna Hendricks * and Andrew Logue †
University of Colorado Boulder, Boulder, CO 80309

* 109399534
† 109531753

Contents

I. Introduction

Welcome to our report to help you on your decision regarding your mortgage payment. We understand that you do not have the free time or extensive knowledge about loan behavior that our differential equation class has allowed us to obtain. We are therefore here to offer our expertise in helping you choose the very best payment option for you within Boulder's pricey market. In this report, we will break down different scenarios and their implications through calculations regarding types of compound frequencies, duration of the loans, payment per month, down payments, and adjustable rate mortgages. We also demonstrate the general behavior of loan structures by modeling them as graphs, using differential equations. Through the use of the knowledge presented in this report, you can build a broader understanding of your payment options, so that you may pick the absolute best option for your financial situation and preferred payment method.

II. 3.1: Analysis of Fixed Rate Mortgages

1) Effects of Continuous Compounding

Assuming that the interest rate is 3% and the original loan payment is $750,000, we find that after 5 years, you would end up paying the least with a loan compounded once a year. With this compounding you would pay a total of $869,455.56 once the 5 years are up. We deduced this through the compound interest equation:

$$A(t) = A(0)(1 + r/n)^{r*t} \tag{1}$$

We did so by plugging in our values of A(0) = 750000, r = 0.03, t = 5 with n = 1,2,4,12 and recognizing that n = 1 is the cheapest. Compounding continuously results in the highest cost after 5 years ($871,375.68), which we deduced from utilizing the continuous compounding equation along with our previous r and t values:

$$A(t) = A(0)(e)^{r*t} \tag{2}$$

Through the use of MATLAB (appendix A), we graphed the compound interest equation (Eq. 1) with compounding 4 and 12 times a year (n = 4,12) and the continuously compounded equation (Eq. 2), all over a period of t = 0 to t = 30.

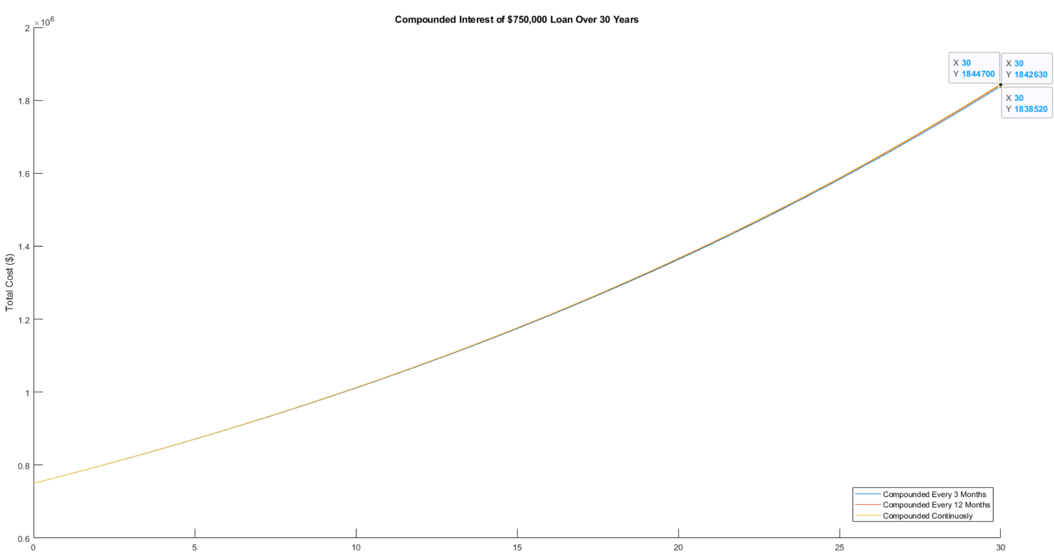

Fig. 1 Compound interest of $750,000 loan over 30 years

2) Equilibrium Solutions

Next we calculate the equilibrium point and stability of the IVP equation 3:

$$A' = rA - 12p, A(0) = A_0 \tag{3}$$

By setting equation 3 equal to zero and solving for A, it was found that $A = \frac{12p}{r}$, and the stability was calculated to be unstable through the use of a phase line. $A = \frac{12p}{r}$ is when the the amount due on the loan will stay the same. Values greater than this will result in the rate of change of the loan increasing, and values below it will cause the rate of change of the loan to be decreasing.

3) Solving IVP to Determine Loan Behavior

We solved the differential equation IVP problem given as equation 3, through the integrating factor process. By moving rA to the left side and choosing p(t) to be -r, we get an integrating factor of e^{-rt}. As a result of the integration factor process, we get $A(t) = \frac{12p}{r} + (A_0 - \frac{12p}{r})e^{-rt}$ which is the equation for the amount remaining on the loan at a given time.

4) Solving for Monthly Payment Values of P

In order to solve for values of monthly payments, we used our A(t) equation from part 3 and plugged in values for each situation and solved for p. The first situation was where payments take place over 10 years with a compound interest rate of 3% and the second was for 30 years at 5%. Plugging in those values with A being set to 0 (amount when loan is fully paid off), we got that the 10 year situation results in a monthly payment of $7,234.30 and the 30 year situation results in a monthly payment of $4,022.55. Depending on how much you want to pay per month, the 10 year option is higher payments over less time, and the 30 year option is lower payments over more time.

5) Total amount paid over 10 year period vs. 30 year period

To find the total amount to be paid for the 10 and 30 year periods, we multiplied the monthly payment price p by 12 months and by the amount of years. This showed us that if you choose the 10 year plan, you will pay a total of $868,116 in the end where as with the 30 year plan, you will pay a total of $1,448,118, almost 600k more.

6) Money Saved from $100,000 Down Payment

By making a $100,000 down payment and paying a loan of $650,000 you will be able to save money at the end of the day. We calculated the new totals by utilizing the solved equation 3 with an A_0 value of 650,000 and solving for p just as we did for the 30 and 10 year period in part 4. Then by calculating the total amount paid as we did in section 5, and adding the initial $100,00 down payment to it we get the new totals. From subtracting these totals from our totals calculated in part 5, we find that the down payment of $100,000 saves us $15,748.30 in the 10 year payment plan and $193,081.51 in the 30 year payment plan. Thus, it's very clear that if possible, placing a down payment will allow you to save money when all is said and done.

7) Advantages/disadvantages of 30-year vs. 10-year mortgages

You may be wondering what to take away from all of this information regarding your loan payment plan options. With the 10 year fixed-rate plan there is less of a time commitment for you, so you get to pay the loan off sooner. On top of this, the total amount paid in the end after interest will be less ($ 868,116 total). However, placing an $ 100,000 down payment will result in smaller sum of money being saved in the end (only $15,748.30), and your monthly payments will be higher ($ 7234.30 per month). The 30 year plan is disadvantageous in that it results in a higher total payment in the end ($ 1,448,118 total). However you get to make smaller payments each month ($ 4022.55 per month), which grants you more time to be accumulating money to pay the loan. In the 30 year plan, you also save more money from an initial down payment ($193,081.51 saved). In conclusion, there are pros and cons to both options depending on your financial situation and preferred payment methods.

III. 3.2: Numerical Solutions

1) Fixed Rate Mortgage

Implementing Euler's method for equation 3 using a step size of 0.5, a $750,000 mortgage having a constant interest rate of 5% with $4,000 monthly payments will be payed off in 30.773 years. This however, is just a model. When compared with the true solution, which shows that the mortgage would be fully payed off in 30.397 years, we see that the model using a step size of 0.5 projects a zero-debt date of 0.376 years greater than that of the true data. The graph shown below in figure 3, created in the MATLAB code in appendix A, reflects these data points.

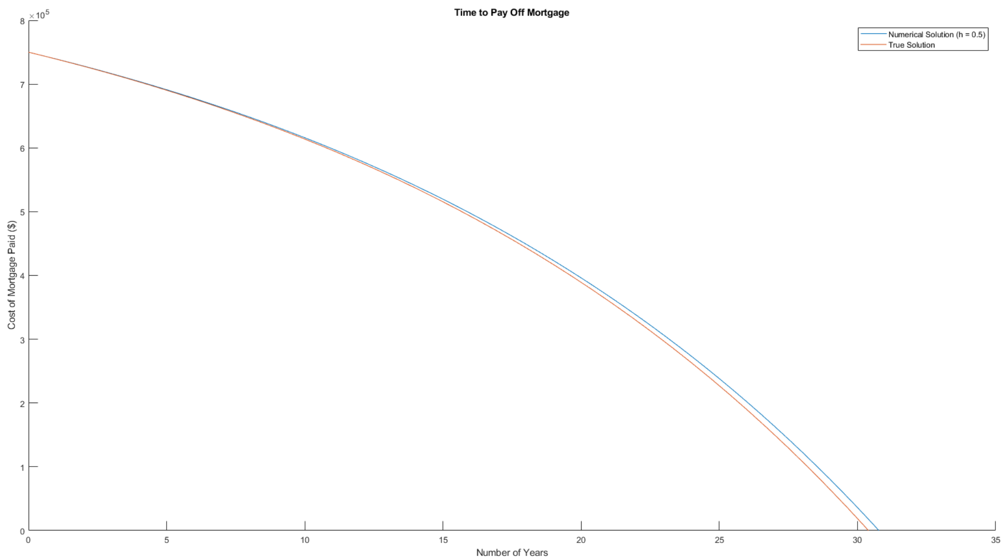

Fig. 2 Time taken to pay off a $750,000 mortgage, modeled with a step size of 0.5, shown against the true solution

Euler's method was applied again for equation 3, this time using a smaller step size of 0.01 but keeping the initial mortgage, interest rate, and monthly payment amount constant. The model using a step size of 0.01 yielded a much more accurate zero-debt date when compared with the true data. The projected amount of time that the mortgage would be paid off was 30.404 years, only 0.007 years greater than the true time to pay off the mortgage of 30.397 years. The graph shown below in figure 4, created in the MATLAB code in appendix A, reflects this trend.

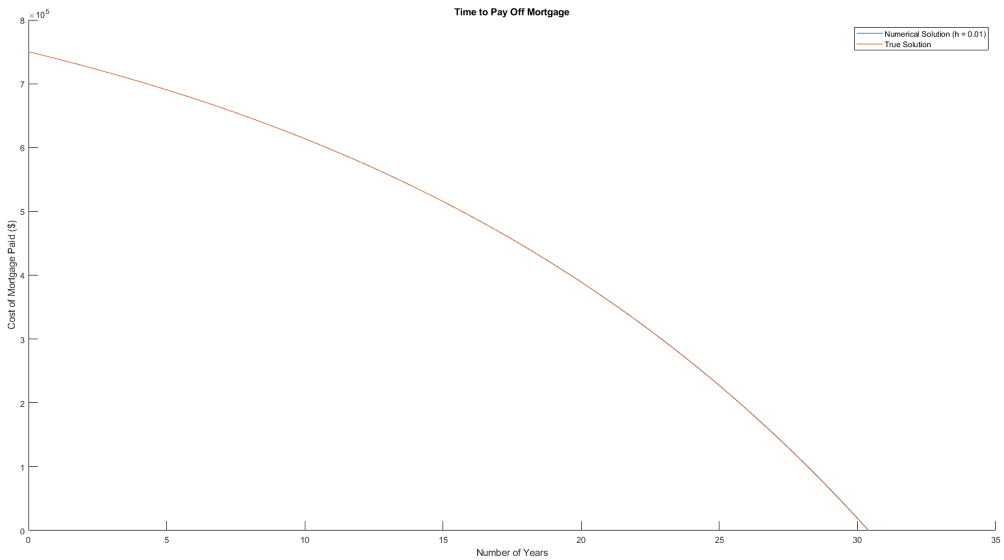

Fig. 3 Time taken to pay off a $750,000 mortgage, modeled with a step size of 0.01, shown against the true solution

From both figure 3, figure 4, and the data collected from them, we can draw the conclusion that as the step size used in the model decreases, the more accurate the model becomes when compared to the true solution to equation 3.

2) Adjustable Rate Mortgage

Using a Euler's method model similar to that shown in figure 4, with a step size of 0.01, we modeled the total time it will take to pay off a $750,000 mortgage when paying either $4,000 or $4,500 monthly. However, instead of the interest being fixed, it alternatively varies with respect to time as shown in equation 4 below. The two payment options for this adjustable rate mortgage are subsequently graphed in figure 4, from which we can see that the mortgage will be fully paid off considerably faster when paying $4,500 monthly rather than paying $4,000 monthly. In fact, when paying $4,500 monthly, you will have the mortgage paid off 12.425 years faster than if you only paid $4,000 a month. This is reflected in figure 4, as the time to pay off the $750,000 mortgage when $p = \$4,000$ is 34.996 years, whereas when $p = \$4,500$ the time to pay off the $750,000 mortgage is just 22.571 years. The calculations used to produced these values were performed in the MATLAB code shown below in appendix A.

$$r(t) = \begin{cases} 0.03 & t \leq 5 \\ 0.03 + 0.015\sqrt{t-5} & t > 5 \end{cases} \tag{4}$$

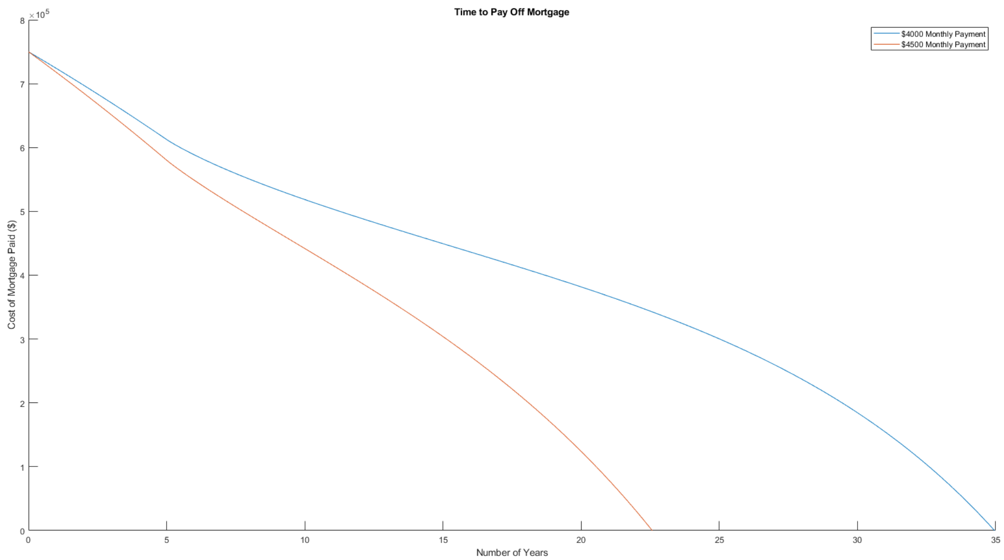

Fig. 4 Time to pay off a $750,000 mortgage with $4000 and $4500 monthly payments with a variable interest rate

In order to determine the total amount of interest paid for each p case, equation 5 (shown below) was used. For the case of the monthly payment being $4,000, $928,080.00 of interest is paid over the 34.996 years it will take you to fully pay off the $750,000 mortgage (appendix A). However, in the case of the monthly payment being $4,500, only $469,320.00 of interest will be paid over the shorter 22.571 years it will take you to fully pay off the same $750,000 mortgage (appendix A). Therefore it can be concluded that it is more cost and time effective to opt into the highest monthly payment plan that you are able to afford. The higher the monthly payment plan, at least when using the variable interest rate from equation 4, the faster the mortgage will be paid off, and with less total interest.

$$totalInterest = p(12t_f) - 750000 \tag{5}$$

IV. Conclusion

After all of our work, we've recognized how each mortgage structure has its pros and cons. However, there are a few behaviors that we found to be true, no matter what the loan structure is. These behaviors include the impact that various variables have on the total amount paid, and time taken to pay off the mortgage. Such variables include monthly payment amounts, interest rates, time to pay back the loan, and down payments. We found that increasing the amount of down payment made, as well as the monthly payment, both have the same impact on the behavior of the mortgage. Increasing each decreases the amount of time required to pay the loan back, as well as the total amount of money paid toward the loan. In general, increasing compounding rate and the amount of time to pay off the loan increases the total sum of the loan payment. Taking more time to pay the loan sets a longer time commitment for you, and will not likely allow you to view other living options until you pay the loan off.

The two mortgage structures in consideration are a fixed interest rate, and an adjustable interest rate loan. The fixed interest rate has more parabolic behavior with respect to the amount left to pay and time that has passed, whereas the adjustable interest rate loan has a more exponential behavior. In order to approximate these mortgage structures, we developed a model in MATLAB in which Euler's method was used to determine the total amount of time it would take to pay off a $750,000 mortgage. We found that it is best to have a small step value when modeling using Euler's method, as shown in the difference between figures 2 and 3. Adjustable rate loans such as the one modeled from equation 4 can either lengthen or shorten the amount of time it will take to pay off your mortgage when compared to a fixed rate loan. In the case of the one previously modeled, it lengthened the amount of time it took to pay back the original loan by nearly five years when compared to the 3% fixed interest rate plan, thus also increasing the total amount of interest that must be paid. For this reason, we recommend choosing the fixed interest rate plan as you will be able to pay it off both quicker and cheaper. Additionally, when examining adjustable rate mortgages and applying the adjustable interest rate shown in equation 4 to our model, we concluded that opting to pay more each month rather than less each month yielded the fastest time to pay off the mortgage, as well as the smallest amount of total interest paid. These findings are clearly reflected in figure 4.

Through our use of differential equations, we learned how to effectively and accurately model mortgage payment plans in MATLAB, so that you now know which monthly payment plan to choose, and how to better model your loan.

V. References

[1] Thaler, E, "*Playing with Mortgages*". Jan, 2021.
[2] Thaler, E, "*Project Guidelines*". Jan, 2021.
[3] MathWorks, "*MATLAB Documentation*". 2021.

VI. Appendix A

MATLAB Code

```
% APPM 2360 Mortgage Project - 2/27/2021
% Andrew Logue
% Anna Hendricks

clear
close all
clc

%% 3.1

r = 0.03; % interest rate (%)
A_0 = 750000; % origional loan ($)
syms t % time (years)

% plot
figure(1)
hold on
% for n = 4
A(t) = A_0*(1+(r/4))^(4*t);
fplot(A(t), [0 30])
% for n = 12
A(t) = A_0*(1+(r/12))^(12*t);
fplot(A(t), [0 30])
% compounding continuosly
A(t) = A_0*exp(r*t);
fplot(A(t), [0 30])

title('Compounded Interest of $750,000 Loan Over 30 Years')
xlabel('Number of Years')
ylabel('Total Cost ($)')
legend('Compounded Every 3 Months', 'Compounded Every 12 Months', ...
    'Compounded Continuosly')


%% 3.2.1 Fixed Rate Mortgage

r = 0.05; % interest rate (%)
p = 4000; % monthly payment ($)
A = 750000; % mortgage ($)
h = 0.5; % step size
i = 1; % indexer
syms t % time (years)

% output Vector
V(i) = A;
% compute money owed after time t
% assume that at the first instance the mortgage becomes negative,
% it has been paid of and is therefore $0
while A > 0
    A = A + h*(r*A - 12*p);
```

```
    i = i+1;
    V(i) = A;
end

% calculate and print when modeled mortgage will be payed off
t_f = interp1(V, 0:h:((length(V) - 1)*h), 0);
fprintf('For a model with h = 0.5, the $750,000 mortgage will be payed off in: %.4f years\n', t_f);

% plot
figure(2)
hold on
% numerical solution
plot(0:h:((length(V) - 1)*h), V)
% true solution
A_true(t) = ((12*p)/r) + (750000 - ((12*p)/r))*exp(r*t);
fplot(A_true(t), [0 31])

title('Time to Pay Off Mortgage')
xlabel('Number of Years')
ylabel('Cost of Mortgage Paid ($)')
ylim([0 800000])
legend('Numerical Solution (h = 0.5)', 'True Solution')

% for true value
x = 0:0.00001:31;
t_f = interp1(((12*p)/r) + (750000 - ((12*p)/r))*exp(r*x), x, 0);
fprintf('For the true solution, the $750,000 mortgage will be payed off in: %.4f years\n', t_f);

clear

r = 0.05; % interest rate (%)
p = 4000; % monthly payment ($)
A = 750000; % mortgage ($)
h = 0.01; % step size
i = 1; % indexer
syms t % time (years)

% output Vector
V(i) = A;
% compute money owed after time t
% assume that at the first instance the mortgage becomes negative,
% it has been paid of and is therefore $0
while A > 0
    A = A + h*(r*A - 12*p);
    i = i+1;
    V(i) = A;
end

% calculate and print when modeled mortgage will be payed off
t_f = interp1(V, 0:h:((length(V) - 1)*h), 0);
fprintf('For a model with h = 0.01, the $750,000 mortgage will be payed off in: %.4f years\n', t_f);
```

```
% plot
figure(3)
hold on
% numerical solution
plot(0:h:((length(V) - 1)*h), V)
% true solution
A_true(t) = ((12*p)/r) + (750000 - ((12*p)/r))*exp(r*t);
fplot(A_true(t), [0 31])

title('Time to Pay Off Mortgage')
xlabel('Number of Years')
ylabel('Cost of Mortgage Paid ($)')
ylim([0 800000])
legend('Numerical Solution (h = 0.01)', 'True Solution')

% for true value
x = 0:0.00001:31;
t_f = interp1(((12*p)/r) + (750000 - ((12*p)/r))*exp(r*x), x, 0);
fprintf('For the true solution, the $750,000 mortgage will be payed off in:
    %.4f years\n', t_f);

%% 3.2.2 Adjustable Rate Mortgage

r = 0.03; % interest rate (%)
p = 4000; % monthly payment ($)
A = 750000; % mortgage ($)
h = 0.01; % step size
i = 1; % indexer
syms t % time (years)
totalInterest = 0; % ($)

% output Vector
V(i) = A;
% compute money owed after time t
% assume that at the first instance the mortgage becomes negative,
% it has been paid of and is therefore $0
while A > 0
    % interest rate remains 3% until past year 5
    if (i-1)*h <= 5
        A = A + h*(r*A - 12*p);
    else
        A = A + h*((r + 0.015*sqrt(i*h - 5))*A - 12*p);
    end
    i = i+1;
    V(i) = A;

end

% calculate and print the total amount of interest
totalInterest = p*((h*(i-1))*12) - 750000;
fprintf('Amount of interest paid with $4000 monthly payments: $%.2f \n',
    totalInterest);
% calculate and print when modeled mortgage will be payed off
```

```
t_f = interp1(V, 0:h:((length(V) - 1)*h), 0);
fprintf('With $4,000 monthly payments, the $750,000 mortgage will be payed off
    in: %.4f years\n', t_f);

% plot
figure(4)
hold on
% for $4000 monthly payments
plot(0:h:((length(V) - 1)*h), V)

clear
r = 0.03; % interest rate (%)
p = 4500; % monthly payment ($)
A = 750000; % mortgage ($)
h = 0.01; % step size
i = 1; % indexer
syms t % time (years)
totalInterest = 0; % ($)

% output Vector
V(i) = A;
% compute money owed after time t
% assume that at the first instance the mortgage becomes negative,
% it has been paid of and is therefore $0
while A > 0
    % interest rate remains 3% until past year 5
    if (i-1)*h <= 5
        A = A + h*(r*A - 12*p);
    else
        A = A + h*((r + 0.015*sqrt(i*h - 5))*A - 12*p);
    end
    i = i+1;
    V(i) = A;

end

% calculate and print the total amount of interest
totalInterest = p*((h*(i-1))*12) - 750000;
fprintf('Amount of interest paid with $4500 monthly payments: $%.2f \n',
    totalInterest);
% calculate and print when modeled mortgage will be payed off
t_f = interp1(V, 0:h:((length(V) - 1)*h), 0);
fprintf('With $4,500 monthly payments, the $750,000 mortgage will be payed off
    in: %.4f years\n', t_f);
% plot
figure(4)
hold on
% for $4500 monthly payments
plot(0:h:((length(V) - 1)*h), V)

title('Time to Pay Off Mortgage')
xlabel('Number of Years')
ylabel('Cost of Mortgage Paid ($)')
ylim([0 800000])
```

```
legend('$4000 Monthly Payment', '$4500 Monthly Payment')
```

VII. Appendix B

3.1 Work

$r = 0.03 \quad A_0 = \$750{,}000 \quad t = 5 \text{ years}$

3.1.1: $n = 1, 2, 4, 12$ = times per year

Using Equation 1 $\left(A(t) = A_0\left(1+\frac{r}{n}\right)^{nt}\right)$:

$n=1: A(5) = 750000\left(1 + .03/1\right)^{(1)(5)} = \$869455.56$

$n=2: A(5) = 750000\left(1 + .03/2\right)^{(2)(5)} = \$870405.62$

$n=4: A(5) = 750000\left(1 + .03/4\right)^{(4)(5)} = \$870888.11$

$n=12: A(5) = 750000\left(1 + .03/12\right)^{(12)(5)} = \$871212.59$

Using Equation 2 $(A(t) = A_0 e^{rt})$:

compounding continuously:

$$A(t) = 750000e^{(.03)(5)}$$
$$= \$871375.682$$

$A' = rA - 12p, \quad A(0) = A_0$

3.1.2: $rA - 12p = 0$

$A = 12p/r \leftarrow$ Equilibrium point

stability:

$\uparrow \leftarrow A > 12p/r, \ A' > 0$

$A = \frac{12p}{r} \rightarrow \bigcirc$

$\downarrow \leftarrow A < 12/r, \ A' < 0$

unstable

3.1.3: $\frac{dA}{dt} - rA = -12p, \quad A(0) = A_0$

$\mu(t) = e^{\int -r\,dt} = e^{-rt}$

$e^{-rt}\left[\frac{dA}{dt} - rA = -12p\right]$

$\int (e^{-rt}A)'\,dt = \int -12p\,e^{-rt}\,dt$

$e^{-rt}A = \frac{12p}{r}e^{-rt} + C$

initial cond: $e^0 A_0 = \frac{12p}{r}e^0 + C$

$\rightarrow C = A_0 - \frac{12p}{r}$

$e^{-rt}A = \frac{12pe^{-rt}}{r} + A_0 - \frac{12p}{r}$

$A(t) = \frac{12p}{r} + \left(A_0 - \frac{12p}{r}\right)e^{rt}$

Fig. 5 Hand calculations and work for section 3.1: 3.1.1 - 3.1.3

3.1.4: $A(t) = \frac{12p}{r} + \left(A_0 - \frac{12p}{r}\right)e^{rt}$

$t = 10$ years, $r = .03$ $A_0 = 750000$

$t = 30$ years, $r = .05$

Finding p for 10 year situation:

$$0 = \frac{12p}{.03} + \left(750000 - \frac{12p}{.03}\right)e^{(.03)(10)}$$

$$0 = 400p + (750000 - 400p)e^{.3}$$

$$0 = 400p + 750000e^{0.3} - 400pe^{0.3}$$

$$-750000e^{0.3} = p(400 - 400e^{0.3})$$

$$\rightarrow p = \$7234.30 \text{ per month}$$

p for 30 year situation:

$$0 = \frac{12p}{.05} + \left(750000 - \frac{12p}{.05}\right)e^{(.05)(30)}$$

$$\rightarrow p = \$4022.55 \text{ per month}$$

3.1.5: 10 year period:

$$(7234.30)(12)(10) = \$868,116.00$$

30 year period:

$$(4022.55)(12)(30) = \$1,448,118$$

3.1.6: 10 year period w/ down payment:

$$0 = 400p + (650000 - 400)e^{.3}$$

$$\rightarrow p = \$6269.73 \text{ per month}$$

downpayment

Total $= p \cdot 12 \cdot 10 = 752367.70 + 100000 = \$852,367.70$

amt saved $= 868,116 - 852,367.70 = \$15,748.30$

30 year period w/ down payment

$$0 = 240p + (650000 - 240p)e^{(.5)(30)}$$

$$\rightarrow p = \$3486.21 \text{ per month}$$

Total $= p \cdot 12 \cdot 30 = \$1,255,036.49$

amt save $= 1,448,118 - 1225036.49$

$= \$193081.51$

Fig. 6 Hand calculations and work for section 3.1: 3.1.4 - 3.1.6

3.2 Work

3.2.1) $A' = rA - 12p$, $A(0) = \$750,000$

$r = 0.05$, $p = \$4000$, $h = 0.5$

$$A_{n+1} = A_n + hA'$$

$A(0)$	$750,000$
$A(0.5)$	$750,000 + (0.5)\cdot(0.05\cdot 750,000 - 12\cdot 4000) = 744,750$
$A(1)$	$744,750 + (0.5)\cdot(0.05\cdot 744,750 - 12\cdot 4000) = 739,368.75$

repeat for $h = 0.01$

3.2.2) $r(t) = \begin{cases} 0.03 & t \leq 5 \\ 0.03 + 0.015\sqrt{t-5} & t > 5 \end{cases}$

$h = 0.01$ $P_1 = \$4000$, $P_2 = \$4500$

for $t \leq 5 \rightarrow A = A + h\left((0.03)A - 12\cdot(P)\right)$

for $t > 5 \rightarrow A = A + h\left((0.03 + 0.015\sqrt{t-5})A - 12\cdot(P)\right)$

Calculate total interest paid:

first 5 years 60 payments of P at $r = 0.03$

Ex/ for $P = \$4,000$ total interest over 5 years $= \$7,200$

formula for total interest:

total Interest $= P\cdot 12t - 750000$

Fig. 7 Hand calculations and work for section 3.2