Runge Kutta Methods and Higher Order ODEs

4th order Runge Kutta method

$$y_{i+1} = y_i + \frac{\Delta x}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{\Delta x}{2}, y_i + \frac{\Delta x}{2} \times k_1\right)$$

$$k_3 = f\left(x_i + \frac{\Delta x}{2}, y_i + \frac{\Delta x}{2} \times k_2\right)$$

$$k_4 = f(x_i + \Delta x, y_i + \Delta x \times k_3)$$

EXAMPLE $\frac{dy}{dx} = \frac{x^3 + 1}{y}$ $y(0) = 2$ $0 \leq x \leq 10, \Delta x = 0.5$

$x_1 = 0.5 \quad y_1 = 2.243$

x	y	k_1	k_2	k_3	k_4	y_new	y_exact	error
0	2.00	0.500	0.478	0.479	0.502	2.243	2.00	0.00E+00
0.5	2.243	0.502	0.600	0.594	0.787	2.550	2.24	3.14E-06
1	2.55	0.784	1.076	1.048	1.423	3.087	2.55	2.95E-05
1.5	3.09	1.417	1.848	1.792	2.259	4.000	3.09	1.55E-04
2	4.00	2.250	2.716	2.648	3.122	5.342	4.00	3.75E-04
2.5	5.34	3.112	3.562	3.497	3.949	7.107	5.34	5.20E-04
8	45.48	11.281	11.647	11.625	11.994	51.294	45.48	2.13E-04
8.5	51.29	11.992	12.358	12.337	12.704	57.468	51.29	1.96E-04
9	57.47	12.703	13.067	13.048	13.414	63.997	57.47	1.80E-04
9.5	64.00	13.413	13.777	13.758	14.123	70.880	64.00	1.66E-04
10	70.8803	14.122	14.486	14.468	14.832	78.119	70.8802	1.54E-04

All “one step” numerical ODE solution methods

Consider 2nd order IVP

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = e^{-x} \qquad y(0) = 1 \qquad \frac{dy(0)}{dx} = 0$$

EXAMPLE – solve with explicit Euler method for $\Delta x = 0.5$

$$\frac{dy}{dx} = v \quad y(0) = 1$$
$$\frac{dv}{dx} = e^{-x} - xv - y \quad v(0) = 0$$

i	x_i	y_i	v_i	y_{i+1}	v_{i+1}
0	0	1	0	1	0
1	0.5	1	0		

EXAMPLE – solve with 4th order Runge Kutta for $\Delta x = 0.2$ at $x = 0.8$

$$\frac{d^2y}{dx^2} = \sqrt{x + y} \qquad y(0) = 1 \qquad \frac{dy(0)}{dx} = 0$$

$$y_{i+1} = y_i + \frac{\Delta x}{6}\left(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1}\right)$$

$$\frac{dy}{dx} = v \qquad y(0) = 1$$

$$\frac{dv}{dx} = \sqrt{x + y} \qquad v(0) = 0$$

$$\frac{dy}{dx} = \sqrt{x+y} \qquad y(0) = 1 \qquad\qquad \frac{dv}{dx} = \sqrt{x+y} \qquad v(0) = 0$$

$k_{1,1} = 0$ $\qquad$ $k_{1,2} = 1$

$k_{2,1} = 0.1$ $\qquad$ $k_{2,2} = 1.0488$

$k_{3,1} = 0.1049$ $\qquad$ $k_{3,2} = 1.054$

$k_{4,1} = 0.2107$ $\qquad$ $k_{4,2} = 1.109$

Results