Runge Kutta Methods and Higher Order ODEs

4th order Runge Kutta method

$$y_{i+1} = y_i + \frac{\Delta x}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

find y_{i+1} based on x_i, y_i and Δx

using

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{\Delta x}{2}, y_i + \frac{\Delta x}{2} \times k_1\right)$$

$$k_3 = f\left(x_i + \frac{\Delta x}{2}, y_i + \frac{\Delta x}{2} \times k_2\right)$$

$$k_4 = f(x_i + \Delta x, y_i + \Delta x \times k_3)$$

weighting factors

4 factors for 4th order RK.

EXAMPLE $\frac{dy}{dx} = \frac{x^3+1}{y}$ $y(0) = 2$ $0 \le x \le 10, \Delta x = 0.5$

find y_1 compute weighting factors

$k_1 = f(x_0, y_0) = \frac{0+1}{2} = \frac{1}{2}$

$k_2 = f\left(x_0 + \frac{\Delta x}{2}, y_0 + \frac{\Delta x}{2} \cdot k_1\right) = \frac{(0.25)^3+1}{(2+0.25(1/2))} = 0.478$

$k_3 = \frac{(0.25)^3+1}{(2+0.25(0.478))} = 0.479$

$k_4 = 0.502$

$y_1 = 2 + \frac{0.5}{6}\left(0.5 + 2(0.478) + 2(0.479) + 0.502\right)$
$= 2.243$

$x_1 = 0.5 \quad y_1 = 2.243$

find

$$y_2 = y_1 + \frac{\Delta x}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

calculate $k_1 \Rightarrow k_4$

find $y_2 = 2.550$

$\Delta x = 0.5$

x	y	k_1	k_2	k_3	k_4	y_new	y_exact	error
	IC							
0	2.00	0.500	0.478	0.479	0.502	2.243	2.00	0.00E+00
0.5	2.243	0.502	0.600	0.594	0.787	2.550	2.24	3.14E-06
1	2.55	0.784	1.076	1.048	1.423	3.087	2.55	2.95E-05
1.5	3.09	1.417	1.848	1.792	2.259	4.000	3.09	1.55E-04
2	4.00	2.250	2.716	2.648	3.122	5.342	4.00	3.75E-04
2.5	5.34	3.112	3.562	3.497	3.949	7.107	5.34	5.20E-04
8	45.48	11.281	11.647	11.625	11.994	51.294	45.48	2.13E-04
8.5	51.29	11.992	12.358	12.337	12.704	57.468	51.29	1.96E-04
9	57.47	12.703	13.067	13.048	13.414	63.997	57.47	1.80E-04
9.5	64.00	13.413	13.777	13.758	14.123	70.880	64.00	1.66E-04
10	70.8803	14.122	14.486	14.468	14.832	78.119	70.8802	1.54E-04
	RK						exact	

- excellent accuracy, even with $\Delta x = 0.5$
- explicit formulation, no iteration required

All “one step” numerical ODE solution methods can be used to solve higher order ODES

Consider 2nd order IVP

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = e^{-x} \qquad y(0) = 1 \qquad \frac{dy(0)}{dx} = 0$$

let $\frac{dy}{dx} = v$ ① subs into ODE

becomes $\frac{dv}{dx} + xv + y = e^{-x}$ ②

forms set of 2 first order equations

EXAMPLE – solve with explicit Euler method for $\Delta x = 0.5$

$$\frac{dy}{dx} = \underset{f_1}{v} \qquad y(0) = 1$$

$$\frac{dv}{dx} = \underbrace{e^{-x} - xv - y}_{f_2} \qquad v(0) = 0 \quad \text{given as } \frac{dy(0)}{dx} = 0$$

$$y_{i+1} = y_i + \Delta x \cdot f_1(x_i, y_i, v_i)$$

$$= y_i + \Delta x \cdot v_i$$

$$v_{i+1} = v_i + \Delta x \cdot f_2(x_i, y_i, v_i)$$

$$= v_i + \Delta x \cdot (e^{-x_i} - x_i v_i - y_i)$$

start at $i=0$, $x_0 = 0$, $y_0 = 1$, $v_0 = 0$

$$y_1 = 1 + 0.5(0) \qquad v_1 = 0 + 0.5(e^{-0} - 0 - 1)$$

$$y_1 = 1 \qquad v_1 = 0$$

i	x_i	y_i	v_i	y_{i+1}	v_{i+1}
0	0	1	0	1	0
1	0.5	1	0	$1+0.5(0)=1$	$0+0.5(e^{-0.5}-1)$ $=-0.197$
2	1	1	-0.197	0.902	-0.464

$\underbrace{}_{\text{solution}}$

same solution properties as solving first order equations

$E \propto \Delta x$ for exp. Euler

smaller Δx gives higher accuracy, or switch methods

EXAMPLE – solve with 4th order Runge Kutta for $\Delta x = 0.2$ at $x = 0.8$

$$\frac{d^2y}{dx^2} = \sqrt{x+y} \qquad y(0) = 1 \qquad \frac{dy(0)}{dx} = 0$$

$$\frac{dy}{dx} = v \qquad y(0) = 1$$

$$\frac{dv}{dx} = \sqrt{x+y} \qquad v(0) = 0$$

Solve for 8 weighting factors

$$y_{i+1} = y_i + \frac{\Delta x}{6}\left(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1}\right)$$

$$v_{i+1} = v_i + \frac{\Delta x}{6}\left(k_{1,2} + 2k_{2,2} + 2k_{3,2} + k_{4,2}\right)$$

$$\frac{dy}{dx} = \overset{f_1}{v} \quad y(0) = 1 \qquad \frac{dv}{dx} = \overset{f_2}{\sqrt{x+y}} \quad v(0) = 0$$

find y_1 $\quad x_0 = 0, \quad y_0 = 1, \quad v_0 = 0$

find k values in order

$$k_{1,1} = f_1(x_0, y_0, v_0) = 0 \qquad k_{1,2} = \sqrt{0+1} = 1$$

$$k_{2,1} = f_1\left(0 + \frac{\Delta x}{2}, 1 + \frac{\Delta x}{2} \cdot k_{1,1}, 0 + \frac{\Delta x}{2} \cdot k_{1,2}\right)$$

$$= 0 + \frac{0.2}{2}(1) = 0.1$$

$$k_{2,2} = f_2\left(0 + \frac{\Delta x}{2}, 1 + \frac{\Delta x}{2} \cdot k_{1,1}, 0 + \frac{\Delta x}{2} \cdot k_{1,2}\right)$$

$$= \sqrt{\frac{0.2}{2} + 1 + \frac{0.2}{2}(0)} = 1.0488$$

$$\frac{dy}{dx} = \sqrt{x+y} \qquad y(0) = 1$$

$$\frac{dv}{dx} = \sqrt{x+y} \qquad v(0) = 0$$

$k_{1,1} = 0$ $\qquad$ $k_{1,2} = 1$

$k_{2,1} = 0.1$ $\qquad$ $k_{2,2} = 1.0488$

$k_{3,1} = 0.1049$ $\qquad$ $k_{3,2} = 1.054$

$k_{4,1} = 0.2107$ $\qquad$ $k_{4,2} = 1.109$

$$y_1 = 1 + \frac{0.2}{6}\left(0 + 2(0.1) + 2(0.1049) + 0.2107\right)$$

$$= 1.0207 \leftarrow \text{Solution for ODE}$$

$$v_1 = 0 + \frac{0.2}{6}\left(1 + 2(1.0488) + 2(1.054) + 1.109\right)$$

$$= 0.2104 \leftarrow \text{needed to find } y_2$$

Results

$$y(0.8) = 1.3666$$

→ accurate prediction of higher order IVPs

→ complicated! needs computer for calculations

→ can also be applied to systems of equations problems.