Boundary Value Problems

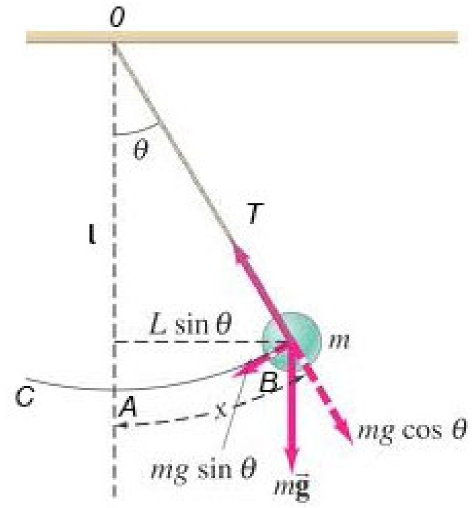
Initial value problem - all conditions at some x value (independent variables)

EXAMPLE Parachutist, jumps from airplane and approaches terminal velocity

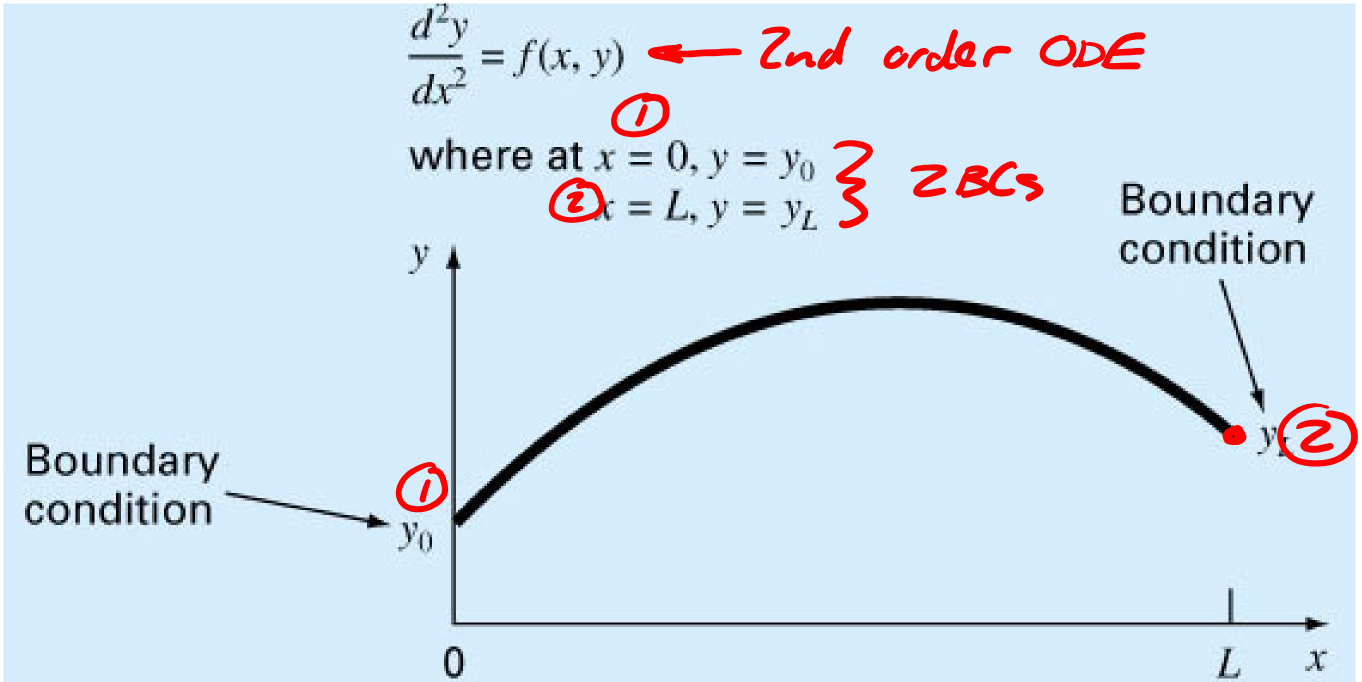
$$v(0) = 0$$

EXAMPLE Oscillation of pendulum

$$\left.\begin{aligned} \theta(0) &= \pi/4 \\ \theta'(0) &= 0 \end{aligned}\right\} \text{both conditions specified at } t=0$$



Boundary value problem – conditions are applied at boundaries



1. Shooting method – convert BVP to an equivalent IVP, conduct a trial-and-error solution (iterative)

EXAMPLE $\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = e^{-x}$ $\underbrace{y(0) = 2 \qquad y(1) = 1}_{2\ \text{BCs}}$

reduce ODE to set of first order equations

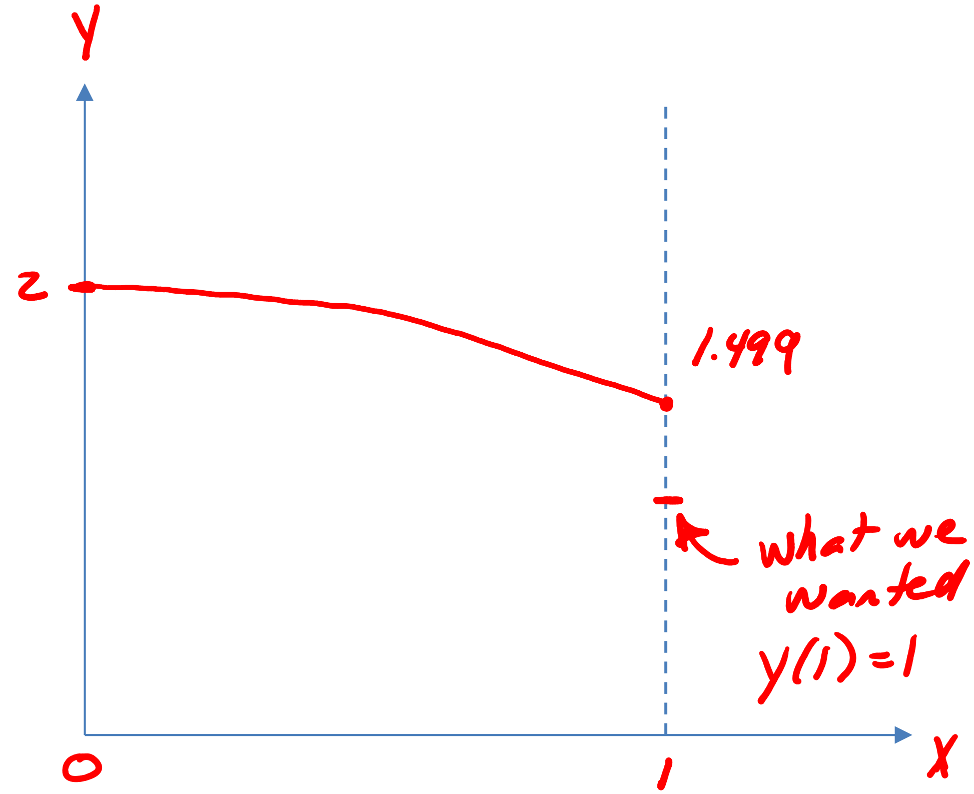
$$\frac{dy}{dx} = V \quad , \quad y(0) = 2$$

$$\frac{dv}{dx} = e^{-x} - xv - y \longleftarrow$$ need an initial value for V, "guess" value $V(0) = 0$

then solve IVP using 4th order RK solution, $\Delta x = 0.5$

4^{th} order RK with $\Delta x = 0.5$

x_i	y_i	v_i
0	2	0* guess
0.5	1.865	
1.0	1.499	



initial guess

$v(0) = \frac{dy(0)}{dx} =$ slope at $x = 0$

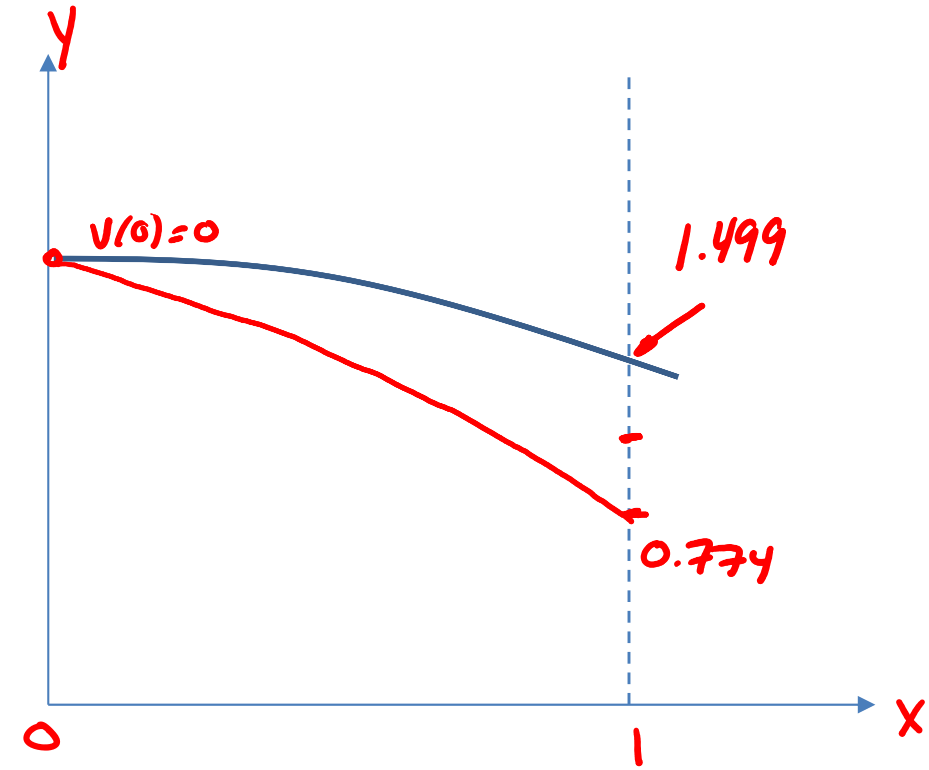
change guessed value for $v(0)$ to negative slope,

$v(0) = -1$, repeat RK solution

Repeat RK with $\Delta x = 0.5$

x_i	y_i	v_i
0	2	-1
0.5	1.405	
1.0	0.774	

* new guess



two approaches

① continue iterations

② interpolate

$$\frac{1.499-1}{0.774-1} = \frac{0-v^*}{-1-v^*}, \quad v^* = -0.6883$$

Repeat RK with $\Delta x = 0.5$

x_i	y_i	v_i
0	2	interpolated value -0.6883
0.5	1.5484	
1.0	1.0001	

verify correct boundary value
$y(1) = 1$

compare these values
∴ grid independent

final step, check grid convergence
use $\Delta x = 0.25$

x_i	y_i	v_i
0	2	same initial value -0.6883
0.25	1.7983	
0.5	1.5481	
0.75	1.2728	
1.0	0.9994	

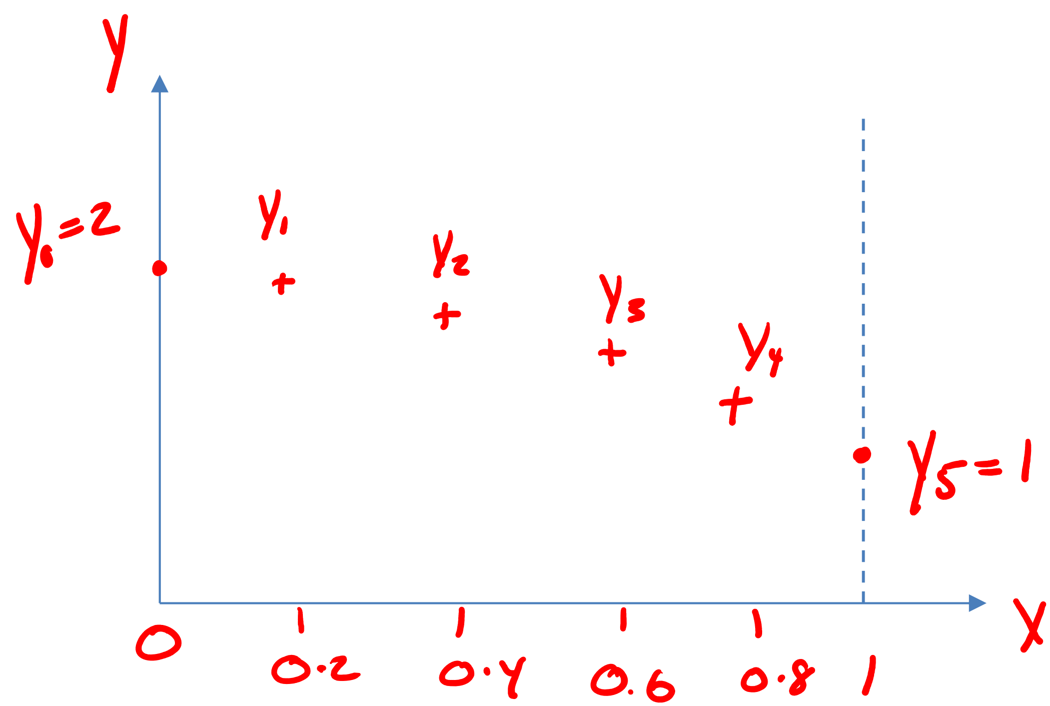
BC ✓

2. Finite difference method – use approximations for derivatives to generate a set of equations

EXAMPLE $$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = e^{-x} \qquad y(0) = 2 \qquad y(1) = 1$$

sub divide region using $\Delta x = 0.2$

Solve for discrete values $y_1 \Rightarrow y_4$



Step 2 - Substitute central difference approximations to derivatives at x_i

$$\frac{d^2y}{dx^2} \cong \frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2} \qquad \frac{dy}{dx} = \frac{y_{i+1} - y_{i-1}}{2\Delta x}$$

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2} + x_i\left(\frac{y_{i+1} - y_{i-1}}{2\Delta x}\right) + y_i = e^{-x_i}$$

multiply both sides by Δx^2

and collect like terms

$$y_{i-1}\left(1 - \frac{x_i}{2}\Delta x\right) + y_i\left(\Delta x^2 - 2\right) + y_{i+1}\left(\frac{x_i}{2}\Delta x + 1\right) = \Delta x^2 e^{-x_i}$$

Step 3 - Form finite difference equations at each non-boundary x_i value

$i=1$, $x_1=0.2$, $\Delta x=0.2$ find y_1

$$y_0\left(1-\frac{0.2}{2}(0.2)\right)+y_1\left(0.2^2-2\right)+y_2\left(\frac{0.2}{2}(0.2)+1\right)=(0.2)^2e^{-0.2}$$

subs boundary condition $y(0)=2$

$$-1.96\,y_1+1.02\,y_2=-1.9273 \leftarrow \text{EQ \#1}$$

$i=2$ $x_2=0.4$ $\Delta x=0.2$, find y_2

$$0.96\,y_1-1.96\,y_2+1.04\,y_3=0.0268 \leftarrow \text{EQ \#2}$$

Step 4 - Solve equations

$$-1.96y_1 + 1.02y_2 = -1.9273$$
$$0.96y_1 - 1.96y_2 + 1.04y_3 = 0.0268$$
$$0.94y_2 - 1.96y_3 + 1.06y_4 = 0.02195$$
$$0.92y_3 - 1.96y_4 = -1.062$$

Solve for $y_1 \to y_4$ using Gauss Seidel

matrix

$$\begin{bmatrix} -1.96 & 1.02 & 0 & 0 \\ 0.96 & -1.96 & 1.04 & 0 \\ 0 & 0.94 & -1.96 & 1.06 \\ 0 & 0 & 0.92 & -1.96 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} = \begin{Bmatrix} -1.9273 \\ 0.0268 \\ 0.02195 \\ -1.062 \end{Bmatrix}$$

"almost" diagonally dominant, banded

Results

$$y_1 = 1.844 \quad y_2 = 1.654 \quad y_3 = 1.441 \quad y_4 = 1.218$$

Exact solution

not usually known

$$y_1 = 1.843 \quad y_2 = 1.653 \quad y_3 = 1.440 \quad y_2 = 1.2176$$

Repeat finite difference solution for $\Delta x = 0.1$ to improve accuracy and prove grid independence. Monitor

$$\varepsilon_a = \frac{(y_{new} - y_{old})}{y_{new}} \times 100\%$$

for convergence