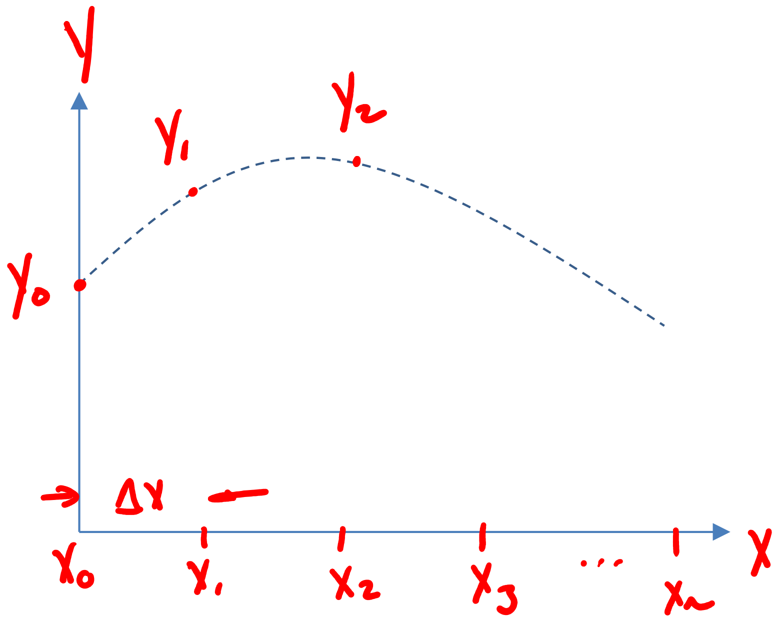
Modified Euler and Runge Kutta Methods

Euler methods for numerical solution of ODEs



$$\frac{dy}{dx} = f(x,y)$$

1. Explicit

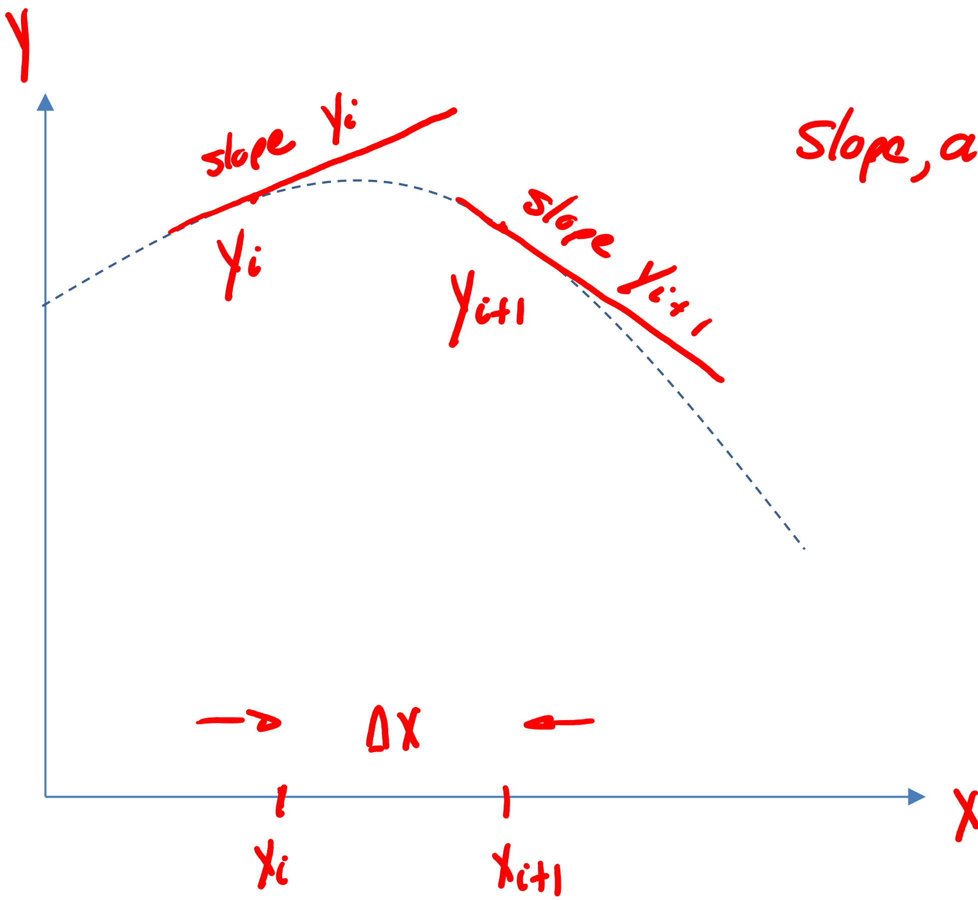
$$y_{i+1} = y_i + \Delta x \cdot f(x_i, y_i)$$

2. Implicit

$$y_{i+1} = y_i + \Delta x \cdot f(x_{i+1}, y_{i+1})$$

3. Heun's method

(average slope)



$$\text{slope,avg} = \frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1})}{2}$$

use explicit Euler method to find

y_{i+1} from slope at y_i

EXAMPLE $\quad \dfrac{dy}{dx} = \dfrac{x^3+1}{y}$ $f(x,y)$ $\quad y(0) = 2 \quad 0 \le x \le 10, \Delta x = 0.5$

$x_0 = 0, \quad y_0 = 2$

left hand slope $\quad f(x_0, y_0) = \left(\dfrac{x^3+1}{y}\right) = \left(\dfrac{0+1}{2}\right) = \dfrac{1}{2}$

y_1 from exp. Euler

$$y_1 = y_0 + \Delta x \cdot f(x_0, y_0) = 2 + 0.5\left(\frac{1}{2}\right) = 2.25$$

right hand slope $\quad f(x_1, y_1) = \left(\dfrac{0.5^3+1}{2.25}\right) = \dfrac{1}{2}$

$\therefore$ average slope $= \dfrac{(1/2 + 1/2)}{2} = 1/2$

use in explicit Euler equation $\quad y_1 = y_0 + \Delta x \cdot (\text{avg slope})$

$$= 2 + (0.5) \cdot (1/2) = 2.25$$

i	x	y_{exact}	y_{Heun}	$y_{Imp.Euler}$
	0	2	2	2
2	1	2.55	2.575	2.63
	2	4.00	4.073	4.33
	3	7.106	7.20	6.97
	4	11.83	11.929	12.02

similar, better accuracy than implicit Euler, but no iterations required

4. Midpoint method (modified Euler)

use explicit Euler to predict y at midpoint of an interval

$$y_{i+1/2} = y_i + \frac{\Delta x}{2} \cdot f(x_i, y_i)$$

then use value to extrapolate (correct) for y_{i+1}

$$y_{i+1} = y_i + \Delta x \cdot f(x_{i+1/2}, y_{i+1/2})$$

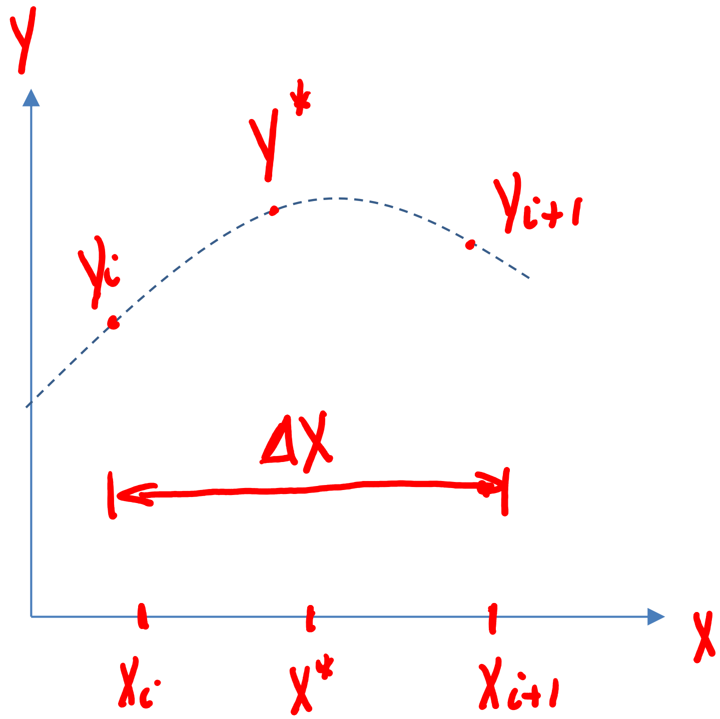
slope at midpoint of interval

5. Runge Kutta method – most widely used,

expresses solution as

$$y_{i+1} = y_i + \Delta x \cdot \left[\begin{array}{c}\text{average slope}\\ \text{in interval}\end{array}\right]$$

weighted average of slopes at x_i, x^*, and x_{i+1}



Based on Taylor series expansion

general form of RK solution

$$y_{i+1} = y_i + \Delta x \cdot \phi(x_i, y_i, \Delta x)$$

where ϕ = increment function

$$= a_1 k_1 + a_2 k_2 + \cdots a_n k_n$$

a_n = constants

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(\underbrace{x_i + p_1 \Delta x, y_i + q_{11} k_1, \Delta x}_{\text{recurrence relationships for weighting factors}})$$

p_1, q_{11} from Taylor Series

Second order

$n = 2$, $E \propto \Delta x^2$

$$y_{i+1} = y_i + \Delta x \cdot \left(\frac{1}{2} k_1 + \frac{1}{2} k_2 \right)$$

$$k_1 = f(x_i, y_i) \leftarrow \text{slope at } x_i \text{, LHS}$$

$$k_2 = f(x_i + \Delta x, \underbrace{y_i + \Delta x \cdot k_1}_{\text{exp. Euler, gives slope at } x_{i+1} \text{, RHS of interval}})$$

Same as Heun's method, average slope

Fourth order - most common, "the" RK formula in most texts

y_{i+1} found based on slopes at 3 points, x_i, x^*, x_{i+1}

$$y_{i+1} = y_i + \frac{\Delta x}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$k_1 \rightarrow k_4$ next lecture