Numerical Solution of ODEs

<u>Numerical solution</u> of first order ordinary differential equations

$$\frac{dy}{dx} = f(x, y)$$

<u>"One step" methods</u>

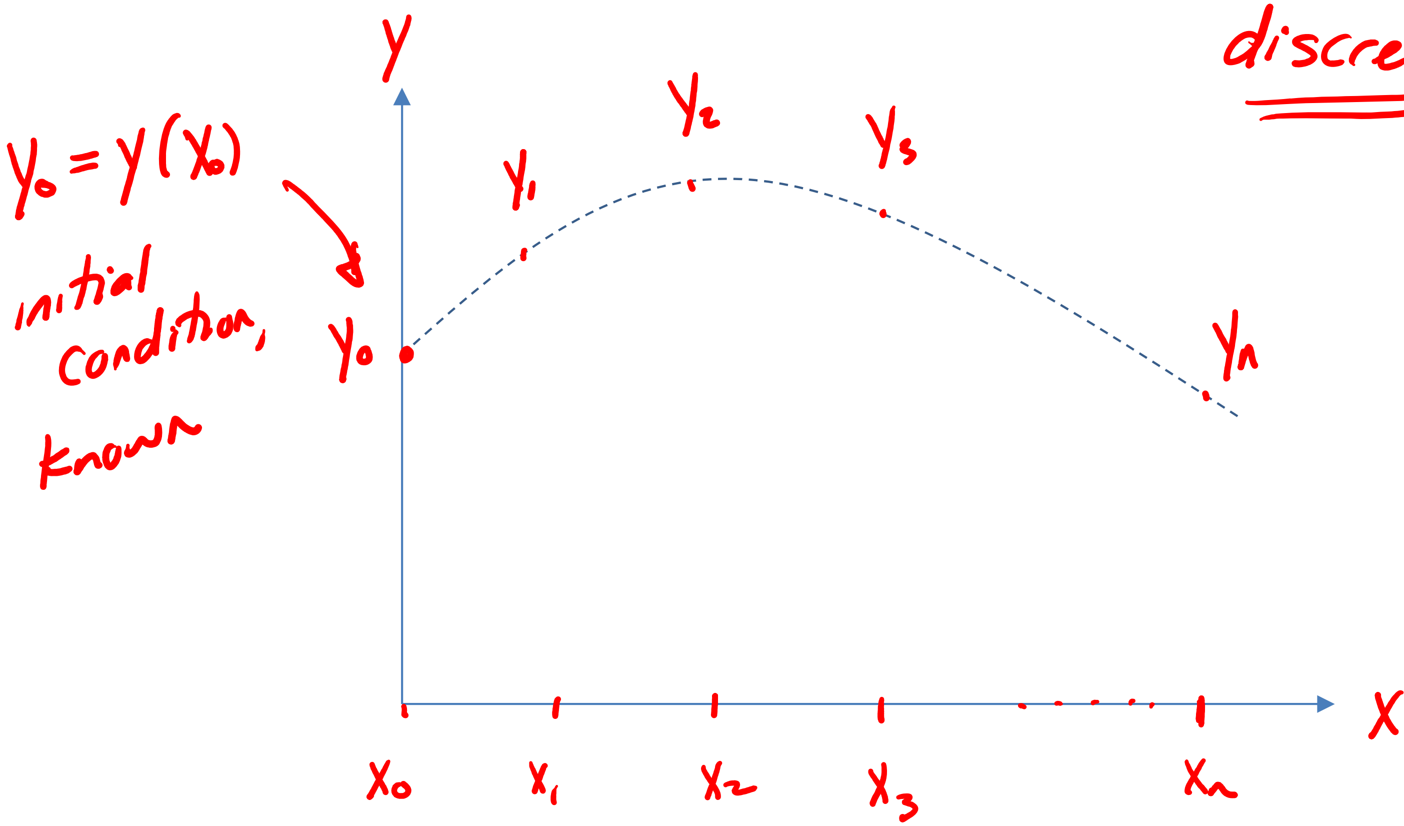
1. Euler's method
2. Improved Euler
3. Runge Kutta

First step – common to all methods

set up a grid of x_i values, Δx spacing

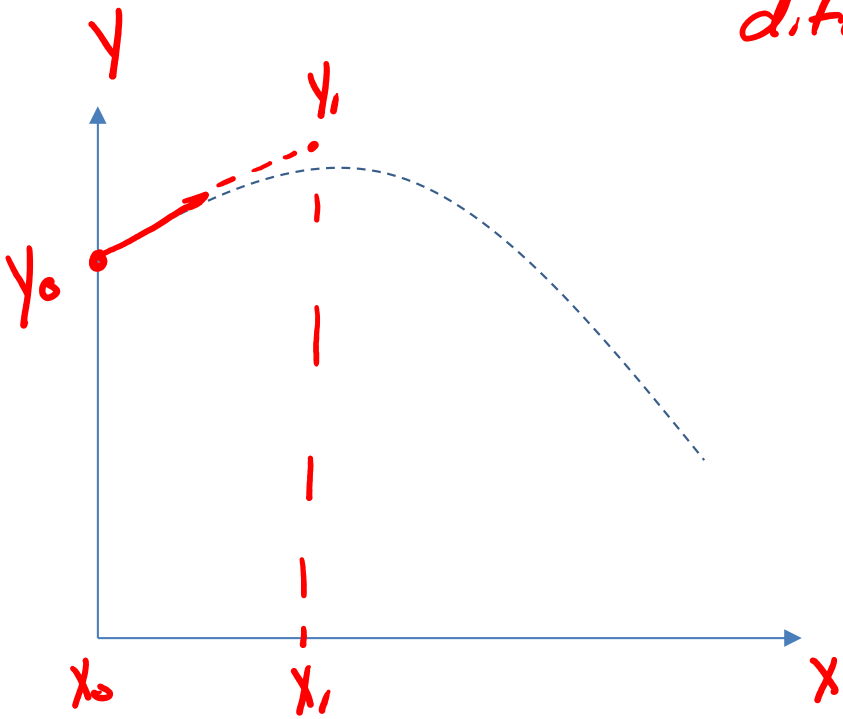
$y_1, y_2 \dots y_n$ unknowns,

discrete solution



Euler's method

– based on first order, forward difference approximation



$$\left.\frac{dy}{dx}\right|_{x_0} \approx \frac{y_1 - y_0}{\Delta x}$$

subs into ODE

$$\frac{y_1 - y_0}{\Delta x} = f(x,y)$$

general form, explicit Euler

$$y_{i+1} = y_i + \Delta x \cdot f(x_i, y_i)$$

<u>EXAMPLE</u> $\frac{dy}{dx} = \underbrace{\frac{x^3+1}{y}}_{f(x,y)}$ $y(0) = 2$ $0 \le x \le 10, \Delta x = 0.5$

Euler eq. $y_{i+1} = y_i + \Delta x \cdot f(x_i, y_i)$

$$y_{i+1} = y_i + 0.5\left(\frac{x_i^3+1}{y_i}\right)$$

i	x_i	y_i	$f(x_i, y_i)$ $\frac{x_i^3+1}{y_i}$	y_{i+1}
0	0	2 (initial condition)	$\frac{1}{2}$	$2 + 0.5(1/2)$ $= 2.25$
1	0.5	2.25	0.5	$2.25 + 0.5(0.5)$ $= 2.5$
2	1.0	2.5	0.8	2.9

continue to $i = 20$, $0 \le x \le 10$

IC

$= \sqrt{\frac{x^4}{2}+2x+4}$

i	x	y	f(x,y)	y_new	y_exact	error
0	0	2.00	0.50	2.25	2.00	0.00
1	0.5	2.25	0.50	2.50	2.24	0.01
2	1	2.50	0.80	2.90	2.55	0.05
3	1.5	2.90	1.51	3.65	3.09	0.19
4	2	3.65	2.46	4.89	4.00	0.35
17	8.5	50.26	12.24	56.38	51.29	1.03
18	9	56.38	12.95	62.85	57.47	1.09
19	9.5	62.85	13.66	69.68	64.00	1.14
20	10	69.68	14.37	76.86	70.88	1.20

y(10)

Error estimate

- method is based on first order approximated,

$$E \propto \Delta x$$

- error accumulates as solution proceeds

- if y_{exact} not available, how do you choose Δx to get an accurate result?

perform convergence study

1. start at "reasonable" # points, solve
2. repeat solution for $\Delta x/2$

$\Delta x = 0.25$

i	x	y	f(x,y)	y_new	y_exact	error
0	0	2.00	0.50	2.25	2.00	0.00
1	0.25	2.25	0.45	2.36	2.12	0.13
2	0.5	2.36	0.48	2.48	2.24	0.12
3	0.75	2.48	0.57	2.63	2.38	0.10
4	1	2.63	0.76	2.82	2.55	0.08
37	9.25	60.13	13.18	63.42	60.69	0.56
38	9.5	63.42	13.53	66.81	64.00	0.57
39	9.75	66.81	13.89	70.28	67.39	0.59
40	x= 10	70.28	14.24	73.84	70.88	0.60

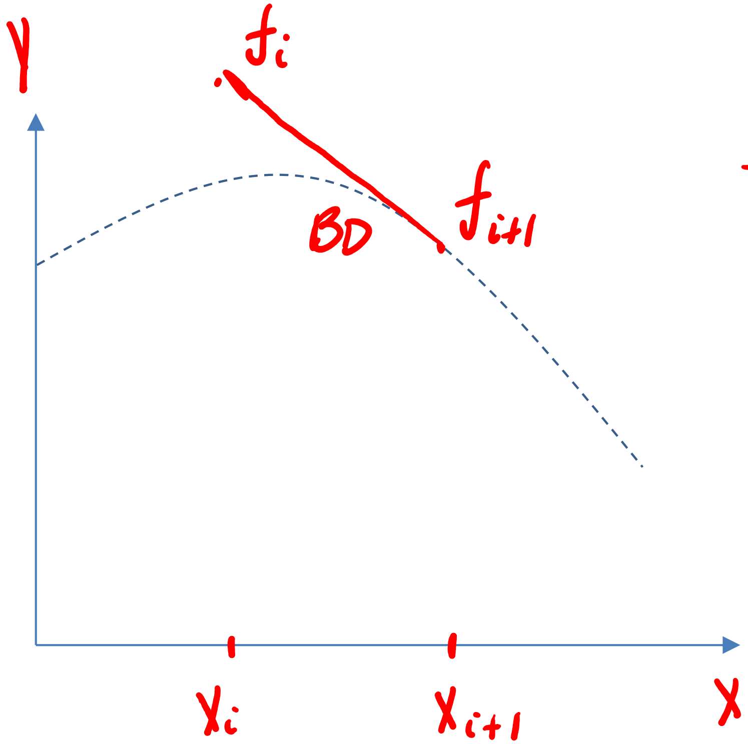
vs. 69.68

Continue process until $\dfrac{|y_{new} - y_{old}|}{y_{new}} \times 100\%$ is less than criteria

→ Solution is "grid independent"

Implicit Euler method

- problem, error accumulates because of forward diff.



- uses backword difference

$$\left.\frac{dy}{dx}\right|_{x_{i+1}} = f(x_{i+1}, y_{+1})$$

$$y_{i+1} = y_i + \Delta x \cdot f(x_{i+1}, y_{i+1})$$

implicit equation, y_{i+1} is on both sides of equation

EXAMPLE $\frac{dy}{dx} = \frac{x^3+1}{y}$ $\underbrace{y(0) = 2}_{y_0}$ $0 \leq x \leq 10, \Delta x = 0.5$

Implicit Euler

for $i=1$

$$y_1 = 2 + 0.5\left(\frac{(0.5)^3+1}{y_1}\right)$$

use iterative solution to find y_1

$$y_1 = 2 + 0.5\left(\frac{(0.5)^3 + 1}{y_1}\right) = 2 + \frac{0.5625}{y_1}$$

Iterative solution guess $y_1 = 2$ (from IC $y_0 = 2$)

<u>iteration #</u>

1 $\quad y_1^{new} = 2 + \frac{0.5625}{2} = 2.281$

2 $\quad y_1^{new} = 2 + \frac{0.5625}{2.281} = 2.247$

3 $\quad = 2.250$

4 $\quad = 2.250$

$$y_2 = y_1 + 0.5\left(\frac{(x_2)^3 + 1}{y_2}\right)$$

x	y_{exact}	$y^{Exp.}_{Euler}$	$y_{Imp.Euler}$
0	2	2	2
1	2.55	2.5	2.63
2	4.00	3.65	4.33
3	7.106	6.58	6.97
4	11.83	11.23	12.02

better than explicit Euler

Notes

- both methods are first order accurate $E \propto \Delta x$

- given Δx, implicit Euler is more accurate than explicit but

requires iteration at each step.