18.03 Class 1, Feb 3, 2010

Introduction and first methods

[1] Introduction
[2] What and Why
[3] Separable equations
[4] Geometric methods

[1] Welcome to 18.03.

I hope you've picked up an information sheet and syllabus, and a problem set when you came in. All this is also available on the web.

On the PSet you will see reference to:

EP = Edwards and Penney, (6th or 5th ed). The very similar to the each other and to the 4th. I'll try to be sure to give the numbering from both.

Notes = Notes and Exercises. Available from CopyTech in the basement of Building 11, though it is also available (in small pieces) through the course website.

SN = Supplementary Notes. Available in sections through the course website.

The first PSet is due Friday, Feb 12, at 12:45, in the cubbies at 2-106, next to the UMO.

I also hope you went to recitation on Tuesday, where these yellow booklets were handed out. More are available here. Usually I ask students to manufacture the booklet themselves, but the UMO was kind enough to do that for you already. Yea! We'll use them for a primitive but effective form of communication between us. It's private; really I'm the only one who can see the numbers you put up. Today, only one question. More later.

If you need to change recitation, go to the website and follow the link to the "grade management system," which is on Stellar. The sizes are limited.

There are two lecture times for this course, 1:00 and 2:00. You can attend either lecture, BUT you should register in the hour at which you plan to take the hour exams.

The information sheet and the website contain lots of other information. For example, my OFFICE HOURS are Wednesdays 3:15 - 5:15 : e.g., this afternoon.

Any questions?

You should also have picked up the listing of the 10 essential skills at your recitation. Teachers of these courses know the list of skills. They expect you will know how to do these things.

Here's a list of some of the larger courses listing 18.03 as a pre-requisite or co-requisite.

2.001 Mechanics and Materials I

2.003 Dynamics and Vibrations

2.005 Thermal-Fluids Engineering

2.016 Hydrodynamics

3.23 Electrical, Optical, and Magnetic Properties of Materials

6.002 Circuits and Electronics

6.021 Quantitative Physiology: Cells and Tissues

8.04 Quantum Physics I

10.301 Fluid Mechanics

12.005 Applications of Continuum Mechanics to Earth, Atmospheric, and Planetary Sciences

16.01 Unified Engineering

18.100 Analysis I

18.330 Introduction to Numerical Analysis

20.309 Biological Engineering II

22.05 Neutron Science and Reactor Physics

Altogether 140 courses at MIT list 18.03 as a prerequisite or a co-requisite.

[2] A DIFFERENTIAL EQUATION is a relation between a function and its derivatives.

Differential equations form the language in which the basic laws of science are expressed. The science tells us how the system at hand changes "from one instant to the next." The challenge addressed by the theory of differential equations is to take this short-term information and obtain information about long-term overall behavior. So the art and practice of differential equations involves the following sequence of steps: one "models" a system (physical, chemical, biological, economic, or even mathematical) by means of a differential equation; one then attempts to gain information about solutions of this equation; and one then translates this mathematical information back into the scientific context.

```
                              Solve
Differential Equation:   _______________\    Behavior over time
Short term information                  /

            /\
             \                                  /
              \                                /
```

```
     Model     \                               /  Interpret
                \                             /
                 \                           /
                                            \/

                      Physical World
```

A basic example is given by Newton's law, $F = ma$. a = acceleration, the second derivative of x = position. Forces don't effect x directly, but only through its derivatives. This is a second order ODE, and we will study second order ODEs extensively later in the course.

[3] In this first Unit we will study ODEs involving only the first derivative: first order: $y' = F(x,y)$.

Example 1: $y' = 2x$ Solution by integrating: $y = x^2 + c$.

Notice that there are many solutions.
An expression like this, involving a constant (c here) is called the GENERAL SOLUTION. The constant is a "constant of integration."

Example 2: $y' = ky$. General Solution: $y = Ce^{kx}$. MEMORIZE THIS

- it's the central example in this course.

(In fact, a good definition of the exponential function e^x is that it is the solution of the differential equation $y' = y$ such that $y(0) = 1$.)

Q1: What is the general solution to the ODE $dy/dx = 2y+1$?

1. $y = Ce^{2x} - 1$

2. $y = Ce^{x/2} - 2$

3. $x = y^2 + y + c$

4. $y = e^{x/2} + C$

5. $y = Ce^{2x} - 1$

6. $y = Ce^{2x} - 1/2$

7. $y = e^{2x} + c$

8. None of the above

Blank: Don't know

The method: "separation of variables." You studied this in recitation yesterday. Recall the method: Put all the x 's on one side and y 's on the other (if possible):

$dy/(2y+1) = dx$

Integrate both sides:

$(1/2) \ln|2y+1| + c1 = x + c2$

Amalgamate the constants and (if possible) solve for y in terms of x :

$\ln|2y+1| = 2x + c$, $|2y+1| = e^c e^{2x}$, $2y+1 = C e^{2x}$

$y = C e^{2x} - 1/2$

So the answer is: 6

We can check this!: $y' = 2 C e^{2x} = 2(y + 1/2) = 2y+1$

It's a nice feature of differential equations in general: it's easy to check your answer!

Q1.2. Is $y' + xy = x$ separable?

1. Yes

2. No

Blank: don't know.

Well, $y' = x - xy = x(1-y)$ so $dy/(1-y) = x dx$: YES.

We could go on to solve this, but you can do that on your own.

We will see many other methods of solving various types of equations. Unfortunately, most real life equations aren't explicitly solvable, and often you don't actually care as much about the explicit solution as about the general properties. That's one place where today's topic is helpful.

[4] Graphical approach

The ODE $y' = F(x,y)$ specifies a derivative - that is, a slope - at every point in the plane. This is a DIRECTION FIELD or SLOPE FIELD.

Eg $y' = 2x$: I drew some of the direction field. Notice that the slope $F(x,y)$ does not depend on y here: It is invariant under vertical translation.

A SOLUTION of the differential equation is a function whose graph has the given slope at every point it goes through. I drew some. The graphs of solutions are INTEGRAL CURVES. They are vertically nested parabolas. The translation invariance of the direction field is reflected in the fact that a vertical translate of a solution is another solution.

To specify a particular solution of an ODE you have to give an INITIAL CONDITION: when x takes on a certain value, y takes on a specified value.

Eg $y' = y$: I drew some of the direction field. Notice that the slope $F(x,y)$ does not depend on x here: It is invariant under horizontal translation.

Graphs of solutions are now horizontal translates of each other.

Example 3: $y' = y^2 - x$.

This equation does not admit solutions in elementary functions. Nevertheless we can say interesting things about its solutions.

To draw the direction field, find where $F(x,y)$ is constant, say m . This is an ISOCLINE. Eg

$m = 0 : x = y^2$. I drew in the direction field.

$m = 1 : x = y^2 - 1$

$m = -1 : x = y^2 + 1$.

I invoked the Mathlet Isoclines and showed the example.

I drew some solution curves. We have seen in action the

EXISTENCE AND UNIQUENESS THEOREM FOR ODEs:
$y' = F(x,y)$ has exactly one solution such that $y(a) = b$,
for any (a,b) in the region where F is defined.
(The solution $y(x)$ may only exist for x near to a .)

(You actually have to put some technical conditions on F -- see EP.)

The E and U theorem says that there is just one integral curve through each point:

EVERY POINT LIES ON JUST ONE INTEGRAL CURVE: NO CROSSING ALLOWED.

The applet makes it look like many solutions coalesce, but this is just a pixel problem. In reality they are separate, but very close to each other.

Many seem to bunch up along the bottom branch of the parabola. Can we explain this?

I cleared the solutions and drew in just the isoclines $m = -1$ and $m = 0$. Once a solution gets between these two parabolas, it can never escape. The poor thing can't cross the $m = -1$ parabola because it would have to have slope greater than -1 when it does; and it can't cross the $m = 0$ parabola because it would have to have slope less than zero when it does. This is a FUNNEL.

So solutions in that region stay in that region: they are trapped between those two parabolas, which are asymptotic as x ---> infinity. All these solutions become very close to the function - sqrt(x) for large x . This is an ideal situation! - we know, approximately, but with increasing accuracy, about the long term behavior of these solutions, and the answer doesn't depend on initial conditions (as long as you are in this range). This is "stability." (Of course if the solution doesn't get trapped, it's a different story.)

Direction fields let you visualize the qualitative behavior of solutions to differential equations, and this is often what you want to know. But we also want to be able to solve ODEs "analytically," that is, using formulas.