2. Modeling by first order linear ODEs

2.1. **The savings account model.** Modeling a savings account gives a good way to visualize the significance of many of the features of a general first order linear ordinary differential equation.

Write $x(t)$ for the number of dollars in the account at time t. It accrues interest at an interest rate I. This means that at the end of an interest period (say Δt years—perhaps $\Delta t = 1/12$, or $\Delta t = 1/365$) the bank adds $I \cdot x(t) \cdot \Delta t$ dollars to your account:

$$x(t + \Delta t) = x(t) + Ix(t)\Delta t.$$

I has units (years)$^{-1}$. Unlike bankers, mathematicians like to take things to the limit: rewrite our equation as

$$\frac{x(t + \Delta t) - x(t)}{\Delta t} = Ix(t),$$

and suppose that the interest period is made to get smaller and smaller. In the limit as $\Delta t \to 0$, we get

$$\dot{x} = Ix$$

—a differential equation.

In this computation, there was no assumption that the interest rate was constant in time; it could well be a function of time, $I(t)$. In fact it could have been a function of both time and the existing balance, $I(x, t)$. Banks often do make such a dependence—you get a better interest rate if you have a bigger bank account. If x is involved, however, the equation is no longer "linear," and we will not consider that case further here.

Now suppose we make contributions to this savings account. We'll record this by giving the *rate* of savings, q. This rate has units dollars per year, so if you contribute every month then the monthly payments will be $q \Delta t$ with $\Delta t = 1/12$. This payment also adds to your account, so, when we divide by Δt and take the limit, we get

$$\dot{x} = Ix + q.$$

Once again, your rate of payment may not be constant in time; we might have a function $q(t)$. Also, you may withdraw money from the bank, at some rate measured in dollars per year; this will contribute a negative term to $q(t)$, and exert a downward pressure on your bank account.

What we have, then, is the general first order linear ODE:

(1) $$\dot{x} - I(t)x = q(t).$$

2.2. **Linear insulation.** Here is another example of a linear ODE. The linear model here is not as precise as in the bank account example.

A cooler insulates my lunchtime rootbeer against the warmth of the day, but ultimately heat penetrates. Let's see how you might come up with a mathematical model for this process. You can jump right to (2) if you want, but I would like to spend a minute talking about how one might get there, so that you can carry out the analogous process to model other situations.

The first thing to do is to identify relevant parameters and give them names. Let's write t for the time variable, $x(t)$ for the temperature inside the cooler, and $y(t)$ for the temperature outside.

Let's assume **(a)** that the insulating properties of the cooler don't change over time—we're not going to watch this process for so long that the aging of the cooler itself becomes important! These insulating properties probably do depend upon the inside and outside temperatures themselves. Insulation affects the *rate of change* of the temperature: the rate of change at time t of temperature inside depends upon the temperatures in side and outside at time t. This gives us a first order differential equation of the form

$$\dot{x} = F(x, y)$$

Time for the next simplifying assumption: **(b)** that this rate of change depends only on the *difference $y - x$* between the temperatures, and not on the temperatures themselves. This means that

$$\dot{x} = f(y - x)$$

for some function f of *one* variable. If the temperature inside the cooler equals the temperature outside, we expect no change. This means that $f(0) = 0$.

Now, *any* reasonable function has a "tangent line approximation," and since $f(0) = 0$ we have

$$f(z) \simeq kz.$$

When $|z|$ is fairly small, $f(z)$ is fairly close to kz. (From calculus you know that $k = f'(0)$, but we won't use that here.) When we *replace* $f(y - x)$ by $k(y - x)$ in the differential equation, we are **"linearizing"**

the equation. We get the ODE

$$\dot{x} = k(y - x),$$

which is a *linear* equation (first order, inhomogeneous, constant coefficient). The new assumption we are making, in justifying this final simplification, is that **(c)** we will only use the equation when $z = y - x$ is reasonably small—small enough so that the tangent line approximation is reasonably good.

We can write this equation as

(2) $$\dot{x} + kx = ky.$$

The system—the cooler—is represented by the left hand side, and the input signal—the outside temperature—is represented by the right hand side. This is *Newton's law of cooling.*

The constant k is the **coupling constant** mediating between the two temperatures. It will be large if the insulation is poor, and small if it's good. If the insulation is perfect, then $k = 0$. The factor of k on the right might seem odd, but it you can see that it is forced on us by checking units: the left hand side is measured in degrees per hour, so k is measured in units of $(\text{hours})^{-1}$.

We can see some general features of insulating behavior from this equation. For example, the times at which the inside and outside temperatures coincide are the times at which the inside temperature is at a critical point:

(3) $$\dot{x}(t_1) = 0 \qquad \text{exactly when} \qquad x(t_1) = y(t_1).$$

2.3. **System, signal, system response.** A first order linear ODE is in **standard form** when it's written as

(4) $$\dot{x} + p(t)x = q(t).$$

In the bank account example, $p(t) = -I(t)$. This way of writing it reflects a useful "systems and signals" perspective on differential equations, one which you should develop. The left hand side of (4) describes the *system*—the bank, in this instance. Operating without outside influence (that is, without contributions or withdrawals), the system is described by the *homogeneous linear equation*

$$\dot{x} + p(t)x = 0.$$

The right hand side of (4) describes the outside influences. You are "driving" the system, with your contributions and withdrawals. These

constitute the "input signal," to which the bank system reacts. Your bank balance, x, is the "system response."

For our purposes, a **system** is represented by some combination of the dependent variable x and its derivatives; a **signal** is a dependent variable, that is, a function of the independent variable t. Both the input signal q and the system response x are signals. We tend to write the part of the ODE representing the system on the *left*, and the part representing the input signal on the *right*. For more detail on this perspective, see Sections 8, 14 and 24.

This way of thinking takes some getting used to. After all, in these terms the ODE (4) says: the system response x determines the signal (namely, the signal equals $\dot{x} + p(t)x$). The ODE (or more properly the differential *operator*) that represents the system takes the system response and gives you back the input signal—the reverse of what you might have expected. But that is the way it works; the equation gives you conditions on x which make it a response of the system. In a way, the whole objective of solving an ODE is to "invert the system" (or the operator that represents it).

We might as well mention some other bits of terminology, while we're at it. In the equation (4), the function $p(t)$ is a **coefficient** of the equation (the only one in this instance—higher order linear equations have more), and the equation is said to have "constant coefficients" if $p(t)$ is constant. In different but equivalent terminology, if $p(t)$ is a constant then we have a **linear time-invariant**, or **LTI**, system.

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