1 Part I

1.1 Problem 3: Linear Models

1.1.1 EP 1.5, Problem 33

We write $y(t)$ = mass fo salt in the vat at time t. We wish to express $y(t)$ as a function of t; it will be a differential equation, ince it will involve a rate of change being the rate of change of kilograms of salt in the vat.

Note that our initial condition is $y(0) = 100$ kg of salt. Therefore, we seek to express this problem as an initial value problem, and find a general then particular solution.

The amount of salt exiting the vat at any time t will be:

$$\dot{y}(t) = -0.005y(t) \tag{1}$$
$$\Rightarrow \dot{y} = -0.005y, \tag{2}$$

which is a natural growth equation. So this has a solution

$$y(t) = Ce^{kt} \tag{3}$$

with $k = -0.005$ and $C = 100$. Therefore,

$$y(t) = 100e^{-0.005t} \tag{4}$$

So we see that

$$y(t) = 10 = 100e^{-0.005t} \tag{5}$$
$$\Rightarrow \ln(0.1) = -0.005t \tag{6}$$
$$\frac{\ln(0.1)}{-0.005} = t \tag{7}$$

which is approximately 7.65 minutes.

1.1.2 EP 1.5, Problem 45

First, we write the rate of change of the pollutant in liters per month:

$$\frac{dx}{dt} = +10\frac{L}{m^3} \cdot 2 \times 10^5 \frac{m^3}{\text{month}} - \frac{x(t)}{2 \times 10^6 m^3} \cdot 2 \times 10^5 \frac{m^3}{\text{month}} \tag{8}$$
$$\Longrightarrow \frac{dx}{dt} = 2 \times 10^6 - \frac{x}{10} \text{ in Liters/month.} \tag{9}$$

Now, we recognize this as a first-order linear ordinary differential equation:

$$\dot{x} + kx = Q \tag{10}$$

with $P(t) = k,\ Q(t) = Q$. Therefore, following the method of EP 1.5, we calculate

$$e^{\int P(t)\,dt} = e^{\int k\,dt} = e^{kt}. \tag{11}$$

Notice now that

$$D_t\left[e^{kt}x(t)\right] = e^{kt} \cdot [\text{LHS of (10)}] \tag{12}$$

Therefore:

$$D_t\left[e^{kt}x(t)\right] = e^{kt}Q \tag{13}$$

$$e^{kt}x(t) = \int e^{kt}Q = \frac{Q}{k}e^{kt} + C \tag{14}$$

$$\implies x(t) = \frac{Q}{k} + Ce^{-kt}. \tag{15}$$

Now, with

$$k = \frac{1}{10}, \quad Q = 2 \times 10^6,$$

and the initial conditions given (that is, that $x(0) = 0$), then we have that

$$x(0) = \frac{Q}{k} + C = 0 \tag{16}$$

$$\implies C = -\frac{Q}{k} \tag{17}$$

$$\implies x(t) = 20(1 - e^{-\frac{1}{10}t}) \text{ million liters} \tag{18}$$

For a concentration of $5\frac{L}{m^3}$, then we get

$$5\frac{L}{m^3} \cdot 2 \times 10^6 = 10 \text{ million liters} \Rightarrow \frac{1}{2} = 1 - e^{-\frac{1}{10}t} \tag{19}$$

$$t = -10\ln(0.5) \approx 6.9 \text{ months.} \tag{20}$$