

18.03 Class 4, Feb 10, 2010

First order linear equations: integrating factors

- [1] First order homogeneous linear equations
- [2] Newtonian cooling
- [3] Integrating factor (IF)
- [4] Particular solution, transient, initial condition
- [5] General formula for IF

Definition: A "linear ODE" is one that can be put in the "standard form"

$$\boxed{r(t)x' + p(t)x = q(t)} \quad x = x(t)$$

$r(t)$, $p(t)$ are the "coefficients" [I may have called $q(t)$ also a coefficient also on Monday; this is not correct, fix it if I did.]

The left hand side represents the "system," and the right hand side arises from an "input signal." A solution $x(t)$ is a "system response" or "output signal."

We can always divide through by $r(t)$, to get an equation of the Reduced standard form:

$$\boxed{x' + p(t)x = q(t)} \quad x = x(t) \quad (*)$$

The equation is "homogeneous" if q is the "null signal," $q(t) = 0$. This corresponds to letting the system evolve in isolation: In the bank example, no deposits and no withdrawals. In the RC example, the power source is not providing any voltage increase.

The homogeneous linear equation

$$x' + p(t)x = 0 \quad (*)_h$$

is separable. Here's the solution, in general on the left, with an example (with $p(t) = 2t$) on the right:

$$x' + p(t)x = 0 \quad x' + 2tx = 0$$

$$\text{Separate:} \quad dx/x = -p(t) dt \quad dx/x = -2t dt$$

$$\text{Integrate:} \quad \ln|x| = -\int p(t) dt + c \quad \ln|x| = -t^2 + c$$

$$\text{Exponentiate:} \quad |x| = e^c e^{\{-\int p(t) dt\}} \quad |x| = e^c e^{-t^2}$$

Eliminate the absolute value and reintroduce the lost solution:

$$x = C e^{\{- \int p(t) dt\}}$$

$$x = C e^{\{-t^2\}}$$

In the example, we chose a particular anti-derivative of k , namely kt . That is what I really have in mind to do in general. The constant of integration is taken care of by the constant C .

So the general solution to $(*)_h$ has the form $C x_h$, where x_h is *any* nonzero solution:

$$x_h = e^{\{- \int p(t) dt\}}, \quad x = C x_h$$

We will see that the general case can be solved by an algebraic trick that produces a sequence of two integrations.

[2] Example: Diffusion, e.g. of heat.

About this time of year I start to think about summer. I put my rootbeer in a cooler but it still gets warm. Let's model its temperature by an ODE.

$x(t)$ = root beer temperature at time t .

The greater the temperature difference between inside and outside, the faster $x(t)$ changes.

Simplest ("linear") model of this:

$$x'(t) = k (T_{\text{ext}}(t) - x(t))$$

where $T_{\text{ext}}(t)$ is the "external" temperature. Sanity check: When $T_{\text{ext}}(t) > x(t)$, $x'(t) > 0$ (assuming $k > 0$). We get a linear equation:

$$x' - kx = kT_{\text{ext}}$$

This is "Newton's law of cooling." k could depend upon t and we would still have a linear equation, but let's suppose that we are not watching the process for so long that the insulation of the cooler starts to break down!

Systems and signals analysis:

The system is the cooler.

The output signal = system response is $x(t)$, the temperature in the cooler.

The input signal is the external temperature $T_{\text{ext}}(t)$.

Note that the right-hand side is k times the input signal, not the input signal itself.

What constitutes the input and output signals is a matter of the interpretation of the equation, not of the equation itself.

Question 4.1: k large means

1. good insulation
2. bad insulation

Blank. don't know.

k is small when the insulation is good, large when it is bad.
 It's zero when the insulation is perfect. k is a COUPLING CONSTANT
 When it is zero, the temperature inside the cooler is decoupled from
 the temperature outside. In the construction industry, a number like
 k is pasted on windows; it's called the U-value of the window.

Let's take $k = 1/3$, for example.

Suppose the temperature outside is rising at a constant rate: say

$$T_{\text{ext}} = 60 + 6t \quad (\text{in hours after 10:00})$$

and we need an initial condition: let's say $x(0) = 32$.

So the IVP is $x' + (1/3)x = 20 + 2t$, $x(0) = 32$. (cooler)

This isn't separable: it's something new. We'll describe a method which
 works for ANY first order linear ODE.

[3] Method: Integrating factors (Euler)

This method is based on the product rule for differentiation:

$$(d/dt) (u x) = u x' + u' x$$

For example, suppose we have the equation

$$t x' + 2 x = t$$

(This is not separable; it is linear and in standard form, but not reduced
 standard form.) Here's a *trick*. Multiply both sides by t :

$$t^2 x' + 2t x = t^2$$

The left hand side is now the derivative of a product:

$$(d/dt) (t^2 x) = t^2$$

We can solve this by integrating:

$$t^2 x = t^3/3 + c$$

so $x = t/3 + c t^{-2}$

[In the first lecture, I posed this (with a different righthand side)
 as a flashcard problem, but I did it just after describing the calculation
 of an integrating factor for a *reduced* equation. The reduced equation is
 $x' + 2x/t = 1$, and this has integrating factor t^2 . So it was a poorly
 placed question.]

That was great! The factor t we multiplied by is an "integrating factor."
 I guessed it here. Often you can. The factor to use in the cooler equation
 and other equations may not be so obvious. Here's a calculation, for a linear
 equation in reduced form,

$$x' + p(t)x = q(t)$$

Multiply both sides by u

$$u x' + p u x = u q$$

In order for the right hand side to be $(d/dt)(ux) = ux' + u'x$, the function u must satisfy the differential equation

$$u' = -p u$$

This is separable, and we'll carry out the separation in general in a minute. In the cooler equation, the coefficient $p(t)$ is constant. In that case we have the natural growth equation!

$$u = e^{pt}$$

(I am choosing a value for the constant of integration, because I need just one integrating factor, any one.)

In the case of the cooler problem, $p = 1/3$, so we have:

$$(d/dt) (e^{t/3} x) = (20 + 2t) e^{t/3}$$

Integrate:

$$e^{t/3} x = 60 e^{t/3} + \int 2t e^{t/3} dt$$

Um. Parts: $\int u dv = uv - \int v du$

$$\begin{aligned} u &= 2t, & dv &= e^{t/3} dt \\ du &= 2dt, & v &= 3 e^{t/3} \end{aligned}$$

$$\begin{aligned} e^{t/3} x &= 60 e^{t/3} + 6 t e^{t/3} - 18 e^{t/3} + c \\ &= (42 + 6 t) e^{t/3} + c \end{aligned}$$

Solve for x :

$$x = (42 + 6t) + c e^{-t/3}$$

That's the general solution. Remember, you can check it easily.

u is an "integrating factor."

[4] We still should finish the IVP process:

$$32 = x(0) = 42 + c \quad \text{so} \quad c = -10 :$$

$$x = 42 + 6t - 10 e^{-t/3}$$

We just want one u , not the general u : so the exponent could be any antiderivative of p . In the example, $p = 1/3$ was constant and we took $u = e^{t/3}$.

Note the structure of the general solution:

$$x = x_p + c u^{-1} \quad \text{where}$$

. x_p is a solution, *any solution*. It's called a PARTICULAR SOLUTION but this is a very poor name, because there is nothing particular about it. In this case we chose one with a pretty simple formula -- $x_p = 42 + 6t$.)

. u is an integrating factor.

Very often x_h approaches zero with time, as this one does. It is then called a TRANSIENT. All solutions come to look more and more alike as time goes on. This is a funnel!

I graphed the solutions $42 + 6t$ and x , and some others along with T_{ext} . If the temperature in the cooler is more than 60 degrees at the start, then it declines at first, crosses the nullcline $x = 60 + 6t$ where it is momentarily in equilibrium with the outside, and then rises to become asymptotic to $42 + 6t$ like every other solution.

[5] Let's compute an integrating factor for the general first order linear equation (*) : we are to solve $u' = up$.

This is a separable equation: $du/u = p dt$

$$\ln|u| = \int p dt$$

The constant of integration is in the indefinite integral.

$$|u| = e^{\int p dt}$$

Now there is a choice of sign. Pick one and go with it; say

$$u = e^{\int p dt}$$

That gives you an integrating factor. Any nonzero multiple serves as well.

Note that this is the reciprocal of a solution to the homogeneous equation:

$$u = x_h^{-1}$$

This gets fed into the solution for x :

$$x = u^{-1} \int u q dt$$

and the constant of integration in the integral lets us write

$$x = x_p + c x_h$$

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