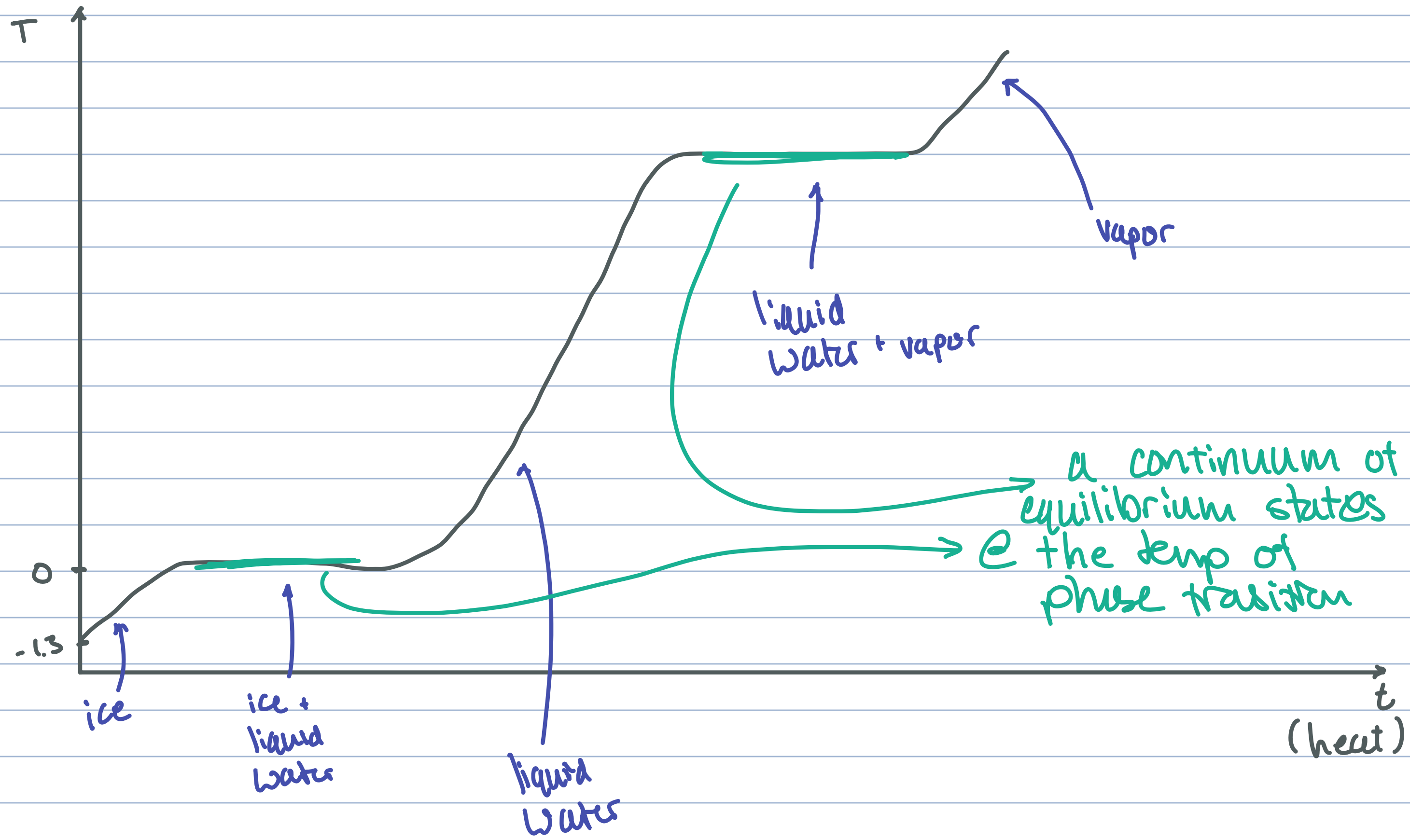
Moodle PW: kasteleyn.

Advertisement: lecture M.Sc. Physics 3 ECTS,

Order in disorder: stochastic geometry in physics.

Friday, 2-4 pm: Office hours (complements on class; background).

Monthly review session.



Motivating question: what is the state of water as a function of time/heat/temperature?

State.

Quantum ~ density operator ρ. Hermitian.

$$\begin{pmatrix} \rho_1 & & & \\ & \rho_2 & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix} \rightsquigarrow \sum \rho_i = \mathrm{tr}\,\rho = 1,$$ to accurately repr. probabilities

Assume thermal equilibrium:

$H =$ Hamiltonian of water

$\beta = \frac{1}{\cancel{k}T}$ inverse temperature

$\rightsquigarrow \quad \rho = \frac{e^{-\beta H}}{\operatorname{tr} e^{-\beta H}}$ } only one state per temperature, contradicting flat parts of graph above.

Why should this make us feel uneasy?

Soln: $e^{-\beta H} / \operatorname{tr} e^{-\beta H}$ is divergent / requires regularization:

not unique, matters.

1. trace $\sum_i e^{-\beta e_i} = \infty$

2. H infinite in thermodynamic limit (some extensive term, eg binding energy)

Phase transition is obtained from details of regularization. Control is required to capture physics of the transition.

The motivation is: how to abstractly treat systems w/ very many dofs. At one end, Ising model ~ at the other, QFT. In CM/SP case, the ∞s arise because of their infinite extent.

Books. Classical: Sabine ~ · SM of Lattice ..., slides.

"FV"

Quantum: Robert ~ · Bratelli, Robinson; 2 vols (we mainly use volume 2).

· Thiring: vol 4

· Robert lecture notes.

Video lectures: Imucast.

Classical Stat Mech.

Overview math part:

1. Ising model: existence of TL of pressure/free energy.

FV Ch3 : Peierl's argument
: Phase transition

2. Gibbs measures in infinite volume.

DLR conditions: Dobrushin, Lanford, Ruelle.
FV Ch 6.

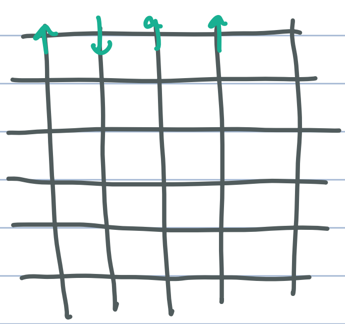
3. Mermin-Wagner thm, absence of cts sym. breaking, $d=1,2$ FV Ch. 9.

(4.) Reflection positivity / ex. of sym. breaking $d=3$
FV Ch 10.

1. The Ising model.

Reference: Hugo Duminil-Copin: 100 years of (critical) Ising model (arXiv 2022).

1.1. The model.



$\uparrow = +1$
$\downarrow = -1$

<u>Notn:</u>

$\Lambda \Subset \mathbb{Z}^d$: $\Lambda \subset \mathbb{Z}^d$ finite, non-empty

Often: $\Lambda = \{1, \dots, L\}^d$, $L \in \mathbb{N} \rightsquigarrow$ cubes.

Configuration space. $\Omega_\Lambda := \{-1, +1\}^\Lambda$.

$$\omega \in \Omega_\Lambda, \quad \omega = (\omega_i)_{i \in \Lambda}, \quad \omega_i \in \{\pm 1\}.$$

$h \in \mathbb{R}$ external magnetic field,

$\mathcal{H}_{\Lambda;h} : \Omega_\Lambda \to \mathbb{R}$,

$$\omega = (\omega_i)_{i\in\Lambda} \longmapsto \mathcal{H}_{\Lambda;h}(\omega) = -\sum_{\substack{\{i,j\}\subset\Lambda \\ i,j \text{ n.n.}}} \omega_i\omega_j - h\sum_{i\in\Lambda}\omega_i$$

n.n.: nearest nbs in $\mathbb{Z}^d$

Minimizer of $\mathcal{H}$?

$h>0$: $\omega_i = 1 \ \forall i$

$h<0$: $\omega_i = -1 \ \forall i$

$h=0$: 2 minimizers, all +1 or all −1

Partition function (Zustandssumme)

$$Z_\Lambda(\beta,h) = \sum_{\omega\in\Omega_\Lambda} e^{-\beta\mathcal{H}_{\Lambda;h}(\omega)}$$

Gibbs measure: prob measure on Ω_Λ

$$\mu_{\Lambda;\beta,h}(\{\omega\}) = \frac{1}{Z_\Lambda(\beta,h)} e^{-\beta\mathcal{H}_{\Lambda;h}(\omega)}$$

$\beta=0$: uniform distribution

$\beta\to\infty$: measure concentrates on minimizer(s).

Pressure / free energy:

$$\psi_\Lambda(\beta,h) := \frac{1}{\beta|\Lambda|}\log Z_\Lambda(\beta,h)$$

Total magnetization:

$$M_\Lambda : \Omega_\Lambda \to \mathbb{R},$$

$$\omega \longmapsto M_\Lambda(\omega) = \sum_{i\in\Lambda}\omega_i$$

Obs: $\frac{\partial}{\partial h}\psi_\Lambda(\beta,h) = \frac{1}{|\Lambda|}\sum_{\omega\in\Omega_\Lambda} M_\Lambda(\omega)\, e^{-\beta\mathcal{H}_{\Lambda,h}(\omega)} \frac{1}{Z_\Lambda(\beta,h)}$

$$= \frac{1}{|\Lambda|}\langle M_\Lambda\rangle_{\Lambda,\beta,h} =: m_\Lambda(\beta,h)$$

average magnetization per unit volume

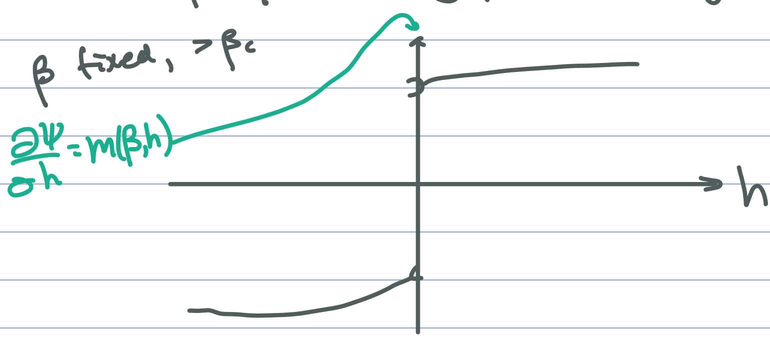
Preview: ① $\psi(\beta,h) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \psi_\Lambda(\beta,h)$

$$\Big(\log Z_\Lambda(\beta,h) \approx \beta|\Lambda|\psi(\beta,h) + \underbrace{\text{correction terms}}^{\ll |\Lambda|} \Big)$$

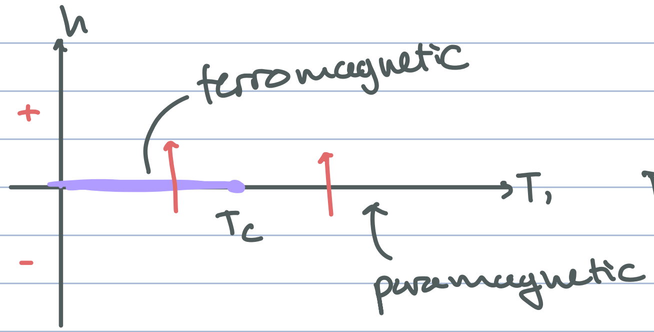
② $\lim_{n\to\infty} m_{\Lambda_n}(\beta,h) \overset{?}{=} \frac{\partial\psi}{\partial h}(\beta,h)$

does it exist?

③ $\exists \beta_c < \infty$ s.t. $\beta > \beta_c$, $\frac{\partial\psi}{\partial h}$ has jump discontinuity at $h=0$.



$d \geq 2$:



$T = \frac{1}{\beta}$

Will there be a jump, step change in something as h is changed from $-$ to $+$, below and above T_c?