Exercise 1

Andrew Winnicki

SoSe2025

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Calculate the probability $\mathbb{P}_{n,E}(X_1 = \varepsilon)$ by counting.

$$\begin{aligned}
\mathbb{P}_{n,E}(X_1 = 1) &= \frac{\#\text{ states with } \omega_1 = 1}{\#\text{ total states}} && (1)\\
&= \frac{\binom{n-1}{E-1}}{\binom{n}{E}} && (2)\\
&= \frac{(n-1)!}{(E-1)!(n-E)!}\left(\frac{n!}{E!(n-E)!}\right)^{-1} && (3)\\
&= \frac{E}{n}. && (4)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbb{P}_{n,E}(X_1 = 0) &= \frac{\#\text{ states with } \omega_1 = 0}{\#\text{ total states}} && (5)\\
&= \frac{\binom{n-1}{E}}{\binom{n}{E}} && (6)\\
&= \frac{(n-1)!}{E!(n-E-1)!}\frac{E!(n-E)!}{n!} && (7)\\
&= \frac{n-E}{n} = 1 - \frac{E}{n}. && (8)
\end{aligned}$$

Now, note that

$$\lim_{n\to\infty} \mathbb{P}_{n,E}(X_1 = 1) = \lim_{n\to\infty} \frac{E_n}{n} = u, \tag{9}$$

$$\lim_{n\to\infty} \mathbb{P}_{n,E}(X_1 = 0) = \lim_{n\to\infty} (1 - \frac{E_n}{n}) = 1 - u. \tag{10}$$

Connect this with the Gibbs distribution by solving for β, first with $\varepsilon = 1$:

$$\frac{e^{-\beta \cdot 1}}{1 + e^{-\beta}} = u \tag{11}$$

$$e^{-\beta} = u(1 + e^{-\beta}) \tag{12}$$

$$e^{-\beta} = \frac{u}{1-u} \tag{13}$$

$$\beta = \ln\left(\frac{1-u}{u}\right). \tag{14}$$

Similarly, for $\beta = 0$, solve:

$$\frac{e^{-\beta \cdot 0}}{1 + e^{-\beta}} = 1 - u \tag{15}$$

$$1 = 1 + e^{-\beta} - u - ue^{-\beta} \tag{16}$$

$$\frac{u}{1-u} = e^{-\beta} \tag{17}$$

$$\beta = \ln\left(\frac{1-u}{u}\right). \tag{18}$$

Therefore, we may write

$$\lim_{n\to\infty} \mathbb{P}_{n,E}(X_1 = 1) = u = \frac{e^{-\beta \cdot 0}}{1 + e^{-\beta}}, \tag{19}$$

$$\lim_{n\to\infty} \mathbb{P}_{n,E}(X_1 = 0) = 1 - u = \frac{e^{-\beta \cdot 1}}{1 + e^{-\beta}} \tag{20}$$

where $\beta = \ln\left(\frac{1-u}{u}\right)$. Therefore, we may again rewrite

$$\lim_{n\to\infty} \mathbb{P}_{n,E}(X_1 = \varepsilon) = \frac{e^{-\beta \cdot \varepsilon}}{1 + e^{-\beta}} = \frac{e^{-\beta \cdot \varepsilon}}{Z(\beta)}, \quad \beta = \ln\left(\frac{1-u}{u}\right). \tag{21}$$

Furthermore, β is uniquely determined because $\ln\left(\frac{1-u}{u}\right)$ is a monotonic function.