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## 1 Lecture 1

Overview of the math part of the lecture. FV = Friedli, Velenik.

1. Ising model: existence of thermodynamic limit of pressure/free energy. We will go over Peierl's argument and phase transitions. *Book references:* FV Chapter 3.
2. Gibbs measures in infinite volume. DLR conditions: Debrushin, Lanford, Ruelle. *Book references:* FV Chapter 6.
3. Mermin-Wagner theorem, absence of continuous symmetry breaking in  $d = 1, 2$ . *Book references:* FV Chapter 9.
4. If time admits: reflection positivity & existence of symmetry breaking in  $d = 3$ . *Book references:* FV Chapter 10.

### 1.1 The Ising Model

#### 1.1.1 The Model

**Notation:** we will denote by  $\Lambda \Subset \mathbb{Z}^d$  that  $\Lambda \subset \mathbb{Z}^d$ , and that  $\Lambda$  is finite, and non-empty. Often, it will be the case that  $\Lambda = \{1, \dots, L\}^d$ ,  $L \in \mathbb{N}$ ; i.e., that  $\Lambda$  consists of a square grid (a “lattice”).

**Configuration space.** We denote a configuration space by  $\Omega_\Lambda := \{-1, +1\}^\Lambda$ . Here,

$$\omega \in \Omega_\Lambda, \omega = (\omega_i)_{i \in \Lambda}, \omega_i \in \{\pm 1\}. \quad (1)$$

More verbosely:  $\Omega_\Lambda$  is the set of functions assigning either  $+1$  or  $-1$  onto each vertex of the lattice, and each  $\omega$  is an individual “configuration” which is a collection of  $\{\pm 1\}$  assigned to each vertex  $i \in \Lambda$ .

We also let  $h \in \mathbb{R}$  be an external magnetic field. Now, we are in place to define the “**Ising Hamiltonian**”, denoted by  $\mathcal{H}_{\Lambda;h}$ :

$$\begin{aligned} \mathcal{H}_{\Lambda;h} : \Omega_\Lambda &\rightarrow \mathbb{R}, \\ \omega = (\omega_i)_{i \in \Lambda} &\mapsto \mathcal{H}_{\Lambda;h}(\omega) = - \sum_{\substack{\{i,j\} \subset \Lambda \\ i,j \text{ n.n.}}} \omega_i \omega_j - h \sum_{i \in \Lambda} \omega_i \end{aligned}$$

where “n.n.” means “nearest neighbors in  $\mathbb{Z}^d$ ”.

A natural question that arises is: what is (are) the minimizer(s) of  $\mathcal{H}$ ?

$$\begin{aligned} h > 0 : \omega_i &= 1 \ \forall i, \\ h < 0 : \omega_i &= -1 \ \forall i, \\ h = 0 : & 2 \text{ minimizers, all } +1 \text{ or all } -1. \end{aligned}$$

We also introduce now the **partition function** (Zustandssamme):

$$Z_\Lambda = \sum_{\omega \in \Omega_\Lambda} e^{-\beta \mathcal{H}_{\Lambda,h}(\omega)}. \quad (2)$$

where  $\beta = \frac{1}{T}$  is the **inverse temperature**.

With this, we may then define the **Gibbs measure**, which is a probability measure on  $\Omega_\Lambda$ :

$$\mu_{\Lambda;\beta,h}(\{\omega\}) = \frac{1}{Z_\Lambda(\beta,h)} e^{-\beta \mathcal{H}_{\Lambda,h}(\omega)} \quad (3)$$

Let's take note of a couple important regimes in different values of the parameter  $\beta$ . First, when  $\beta = 0$ , we get a uniform distribution: it's completely flat. When  $\beta \rightarrow \infty$ , then the measure concentrates on the minimizer(s).

Now, we can define **pressure/free energy**:

$$\psi_\Lambda(\beta, h) := \frac{1}{\beta|\Lambda|} \log Z_\Lambda(\beta, h). \quad (4)$$

This quantity is dependent upon the system size, inverse temperature, and magnetic field, as we might intuitively expect.

We can also define the **total magnetization**, which is the map:

$$M_\Lambda : \Omega_\Lambda \rightarrow \mathbb{R}, \quad (5)$$

$$\omega \mapsto M_\Lambda(\omega) = \sum_{i \in \Lambda} \omega_i. \quad (6)$$

Physically, this is just the sum of the spins on each vertex of the lattice  $\Lambda$ .

**Observation.** We now make an observation regarding the dependence of  $\psi_\Lambda$  on  $h$ :

$$\frac{\partial}{\partial h} \psi_\Lambda(\beta, h) = \frac{1}{|\Lambda|} \sum_{\omega \in \Omega_\Lambda} M_\Lambda(\omega) e^{-\beta \mathcal{H}_{\Lambda,h}(\omega)} \frac{1}{Z_\Lambda(\beta, h)} \quad (7)$$

$$= \frac{1}{|\Lambda|} \langle M_\Lambda \rangle_{\Lambda, \beta, h} \quad (8)$$

$$=: m_\Lambda(\beta, h), \quad (9)$$

where  $m_\Lambda(\beta, h)$  is the **average magnetization** per unit volume.

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