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1 Lecture 1

Recall that the **tangent bundle** is also a smooth manifold, with the natural footpoint projection given by

$$\pi : TM \to M. \tag{1}$$

A **section** of π is a (smooth) map $X : M \to TM$ such that $\pi \circ X = \mathrm{id}_M$.

Definition 1.0.1: Riemannian Metric & Riemannian Manifold

*A (C^k-)**Riemannian metric** g on M is a family $g = \{g_p\}_{p \in M}$ of inner products (positive definite, symmetric bilinear)*

$$g_p : T_pM \times T_pM \to R \tag{2}$$

such that for any smooth vector fields X, Y on M, the function

$$f : M \to R, \quad p \mapsto g_p(X_p, Y_p) \tag{3}$$

*is smooth (C^k-differentiable). The pair (M, g) is called a **Riemannian manifold** .*

Remarks.

1. Given a C^∞ atlas of M, then a family of inner products $g = \{g_p\}_{p \in M}$ is a (C^k-)Riemannian metric iff for any chart (U, x) in this atlas the function

$$U \to R, \quad p \mapsto g_p\Big(\frac{\partial}{\partial x_j}\Big|_p, \frac{\partial}{\partial x_k}\Big|_p\Big) \tag{4}$$

   is smooth. (To be shown in an exercise class.)

2. Suppose we have an open subset U inside a (C^k-)Riemannian manifold (M, g). Then $(U, g|_U$ is also a (C^k-)Riemannian manifold, where

$$g|_U := \{g_p\}_{p \in U}. \tag{5}$$

3. We sometimes write $\langle \cdot, \cdot \rangle$ instead of g (respectively $\langle \cdot, \cdot \rangle_p$ instead of g_p).

4. Sometimes, also indefinite families of inner products are considered, so-called **semi-Riemannian metrics/manifolds**, e.g. in the theory of relativity where you have Lorentzian metrics, i.e. semi-Riemannian metrics of signature $(1, n)$ like the Lorentzian metric

$$x_0 y_0 - \sum_{i=1}^{n} x_i y_i \text{ on } \mathbb{R}^{n+1}. \tag{6}$$

A simple example of a Riemannian manifold: any open subset U of a Euclidean vector space $(V, \langle\cdot,\cdot\rangle)$ is a Riemannian manifold with Riemannian metric $g_p = \langle\cdot,\cdot\rangle_p,\ p \in U$.

Proposition 1.0.2: Riemannian Immersions

Let M a smooth manifold, (N, h) a Riemannian manifold, $\varphi : M \to N$ a smooth (C^{k+1}-)immersion. Then (N, h) induces a Riemannian metric g on M via

$$g_p(v, w) = h_{\varphi(p)}(d\varphi_p v, d\varphi_p w); \quad p \in M;\ v, w \in T_p M. \tag{7}$$

This Riemannian metric is called the ***pullback of h via ϕ, denoted $\varphi^* h$.*** *The immersion $\varphi : (M, g) \to (N, h)$ is called a* ***Riemannian immersion*** .

Proof. Since $d\varphi_p : T_p M \to T_{\varphi(p)} N$ is linear and injective, $\forall p \in M$, g_p is bilinear and positive definite (as $h_{\varphi(p)}$ is).
Symmetry of $h_{\varphi(p)} \Rightarrow$ symmetry of $g_p \to g_p$ an inner product.
Smoothness. Let's first assume that φ is a (local) diffeomorphism. Then a smooth vector field X on M induces a smooth vector field on N, namely

$$(\varphi^* X) : q \mapsto d\varphi_{\varphi^{-1}(q)} X_{\varphi^{-1}(q)} \tag{8}$$

and we have that

$$p \mapsto g_p(X_p, Y_p) = h_p((\varphi^* X)_{\varphi(p)}, (\varphi^* Y)_{\varphi(p)}) \tag{9}$$
$$= (h((\varphi^* X)_\cdot, (\varphi^* Y)_\cdot) \circ \varphi)(p) \tag{10}$$

For the general case, one can now apply the local structure for immersions and assume that

$$M = U \subset \mathbb{R}^n \text{ open, } N = V \subset \mathbb{R}^{n+k} \text{ open} \tag{11}$$

and $\varphi = i : U \hookrightarrow V$ the inclusion. Next, extend the vector field X on U to

$$\tilde{X}(p) = (X(p), 0) \text{ on } V. \tag{12}$$

($\to$ details next Wednesday). $\square$

Remarks.

1. By Whitney's immersion theorem, any smooth manifold can be immersed (actually embedded) into some $\mathbb{R}^n$. So it inherits a Riemannian metric from $\mathbb{R}^n$.
2. By Nash's embedding, any ($C^{k\geq 3}$-)Riemannian manifold can be isometrically immersed (embedded) into some $\mathbb{R}^n$.

Note. If g, h are Riemannian metrics on M, then also $(g+h)_p = g_p + h_p$ defines a Riemannian metric on M.

$$f : M \to \mathbb{R} \tag{13}$$

is a positive smooth function, then

$$(f \cdot g)_p = f_p g_p \tag{14}$$

defines a Riemannian metric on M.

Example 1.0.3: Balls and Riemannian Metric.

B_1^n together with the Riemannian metric g, given by

$$g_p(v,w) = \frac{1}{(1-\langle p,p\rangle)^2}\langle v,w\rangle_p \tag{15}$$

defines a Riemannian manifold isometric to hyperbolic space.

2 Lecture 2

Definition 2.0.1: Isometry

Let (M,g) and (N,h) be Riemannian manifolds.

1. *A diffeomorphism $\varphi : M \to N$ is called an* ***isometry*** *if $\varphi^* h = g$, i.e.*
$$g_p(v,w) = h_{\varphi(p)}(d\varphi_p v, d\varphi_p w) \tag{16}$$
$\forall p \in M;\ v,w \in T_pM$.
2. *A smooth map $\varphi : M \to N$ is called a* ***local isometry*** *if every point $p \in M$ has an open neighborhood U such that $\varphi|_U : U \to \varphi(U)$ is an isometry.*
3. *The set of all isometries of (M,g)*
$$\mathrm{iso}(M,g) = \{\varphi : M \to M \mid \varphi \text{ is an isometry}\} \tag{17}$$
is a group, the so called ***isometry group****.*

Remarks.

1. If $\varphi : (M, g) \to (N, h)$ is a local isometry, then

$$d\varphi_p : T_pM \to T_pN \tag{18}$$

is a linear isometry.

Example 2.0.2: $\exp$ is a Local Isometry

Consider $\mathbb{R}$ with Euclidean inner product g. Further, consider $S^1 \subset \mathbb{R}^2 \subset \mathbb{C}^1$ with the induced Riemannian metric. Then

$$\exp : \mathbb{R} \to S^1, \quad t \mapsto e^{it} \tag{19}$$

is a local isometry. For

$$\|(d\,\exp)_t\left(\frac{\partial}{\partial t}\right)\| = \left|\frac{d}{dt}\left(e^{it}\right)\right| = |i \cdot e^{it}| = 1 \to d\,\exp_t \neq 0 \tag{20}$$

which shows that $\exp$ is a local isometry, but not an isometry: $\mathbb{R} \ncong S^1$, not even homeomorphic.

Definition 2.0.3: Group action, Isometric action

*Let G be a group, M be a smooth manifold. A **(smooth) action** of G on M is a homomorphism*

$$\phi : G \to \mathrm{Diff}(M), \quad a \mapsto \phi_a; \tag{21}$$

i.e.,

$$\phi_{a_1 a_2}(x) = (\phi_{a_1} \circ \phi_{a_2})(x)\ \forall a_1, a_2 \in G,\ x \in M \tag{22}$$

*where $\phi_e = \mathrm{id}_M$, and $e \in G$ is the neutral element. We sometimes denote such an action as $G \curvearrowright M$, and write $\phi_a(x)$ as ax. If (M, g) is a Riemannian manifold, and $\phi(G) \subset \mathrm{Iso}(M, g)$, then we call this action **isometric**.*

Definition 2.0.4: Free Group Action

*A group action $G \curvearrowright M$ is called **free** if for any $g \in G,\ x \in M,\ gx = x \implies g = e$.*

Definition 2.0.5: Covering Space Action

*A smooth action $\Gamma \curvearrowright M$ of a group Γ on a smooth manifold M is called a **covering space action** if the following holds:*
Every point $x \in M$ has an open neighborhood U such that $\gamma U \cap U = \emptyset\ \forall \gamma \in \Gamma \setminus \{e\}$.
In partuclar, $\Gamma \curvearrowright M$ is free.

Proposition 2.0.6: Quotient of Smooth Manifold by Covering Space Action

Let $\Gamma \curvearrowright M$ be a smooth covering space action of a group Γ on a manifold M. Then M/Γ is a topological manifold with a unique smooth structure with respect to which the projection $M \to M/\Gamma$ is a local diffeomorphism.

Proof. See my live notes for the proof. Please examine it carefully. $\square$

A comment is made on the universal cover of M, and examples are provided in the live notes.

Conversely, we have:

Proposition 2.0.7: Riemannian Metric and Local Isometry

Let $\Gamma \curvearrowright (M, g)$ be a covering space action by isometries on a Riemannian manifold (M, g). Then there exists a unique Riemannian metric $\overline{g}$ on M/Γ such that $\pi : (M, g) \to (M/\Gamma, \overline{g})$ is a local isometry.

Examples are provided, see live notes.

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