Contents

1 Lecture 1

Today, we reviewed:

1. Homology.
2. How to construct cohomology.
3. Made a statement about a sequence splitting.
4. The sequence contains $\ker \delta_n$, which we don't understand.

Understanding this $\ker \delta_n$ will be the topic of the next lecture. For more details, see my handwritten notes.

Recall that **homology** is represented as taking a space X, making a chain complex C_i out of it, and then by a purely algebraic manipulation creating a homology H_i. For a cochain complex, we dualize C_i, then take a cohomology H^i.

$$\begin{tikzcd} \text{Space } X \arrow[r, rightsquigarrow] & \text{chain complex } C_i \arrow[r, rightsquigarrow, "\text{purely algebraic}"] \arrow[d, rightsquigarrow, "\text{dualize}"] & \text{homology} H_i \\ & \text{cochain complex } C_i \arrow[r, rightsquigarrow] & \text{cohomology } H^i \end{tikzcd} \tag{1}$$

Suppose now that we have a chain complex C_i of free Abelian groups:

$$\dots C_{n+1} \xrightarrow{\partial} C_n \to C_{n-1} \to \dots. \tag{2}$$

Take an abelian group G (feel free to think about $G = \mathbb{Z}$) and consider the dual

$$(C_n)^* = \mathrm{Hom}(C_n, G), \text{ an additive group.} \tag{3}$$

We have no natural map from $C_n^* \to C_{n-1}^*$, but we do have one from

$$C_n^* \to C_{n+1}^* \tag{4}$$
$$\partial^* : \mathrm{Hom}(C_n, G) \to \mathrm{Hom}(C_{n+1}, G) \tag{5}$$
$$\varphi \mapsto \varphi \circ \partial \tag{6}$$

giving us

$$\dots \to C_{n-1}^* \to C_n^* \xrightarrow{\delta} C_{n+1}^* \to C_{n+2}^* \to \dots \tag{7}$$

Note that we use the sign convention

$$\delta : C_n^* \to C_{n+1}^* \text{ in } (-1)^{n+1}\partial^* \tag{8}$$

which is ***not*** used in Hatcher & Bredon.

Definition 1.0.1: Cochain Complex

*A **cochain complex** $C^\bullet$ is a collection of Abelian groups C^n with codifferentials δ_n such that $\delta_{n+1}\delta_n = 0$.*

Thus,

$$\mathcal{H}om(C_\bullet, G) = \mathrm{Hom}(C_i, G) \text{ with differential } \delta = (-1)^{n+1}\partial^* \tag{9}$$

is a cochain complex, as defined above. Be cautious of Hom vs. $\mathcal{H}om$.

Definition 1.0.2: Cohomology

*For a cochain complex $C^\bullet$, define **cohomology** as*

$$H^i(C^\bullet) = \frac{\ker \delta_i}{\mathrm{im}\delta_{i-1}}. \tag{10}$$

If $C_\bullet$ is a chain complex, then any choice of abelian group G gives a cochain complex

$$\mathcal{H}om(C_\bullet, G), \tag{11}$$

hence a cohomology and thus gohomology groups, denoted as $H^*(C_\bullet, G)$.

If $C_\bullet$ is the singular cochain complex of a space X, then put

$$H^*(X', G) = H^*(\mathcal{H}om(C, G)). \tag{12}$$

Example 1.0.3: Singuluar Cochain Complex

Let $C_\bullet$ be:

$$\dots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0. \tag{13}$$

Then we have:

$$H_*(C_\bullet) = \begin{cases} \mathbb{Z} & n = 0, \\ \mathbb{Z}/2\mathbb{Z} & n = 1, \\ 0 & n = 2, \\ \mathbb{Z} & n = 3. \end{cases} \tag{14}$$

If we let $G = \mathbb{Z}$, and dualize, then we obtain:

$$\dots \longleftarrow 0 \longleftarrow \mathbb{Z} \xleftarrow[0]{} \mathbb{Z} \xleftarrow[\cdot 2]{} \mathbb{Z} \xleftarrow[0]{} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z} \longleftarrow 0. \tag{15}$$

has cohomology:

$$H^*(C_\bullet) = \begin{cases} \mathbb{Z} & n = 0, \\ 0 & n = 1, \\ \mathbb{Z}/2\mathbb{Z} & n = 2, \\ \mathbb{Z} & n = 3. \end{cases} \tag{16}$$

*These are **not** the duals of the homology H_*. Moreover, $G = \mathbb{Z}/2\mathbb{Z}$ has different results.*

Note: Functoriality. If $C_\bullet \to D_\bullet$ is a map of cochain complexes, dualizing gives

$$\mathcal{H}om(D_\bullet, G) \to \mathcal{H}(C_\bullet, G) \tag{17}$$

$$\varphi \mapsto \varphi \circ f \tag{18}$$

which induces a map

$$H^*(\mathcal{H}om(D_\bullet, G)) \to H^*(\mathcal{H}om(C_\bullet, G)). \tag{19}$$

We want to now relate the cohomology of $\mathcal{H}om(C_\bullet, G)$ to the homology of $C_\bullet$.

Lemma 1.0.4: Cohomology and Homology

For any chain complex of free Abelian groups, and any Abelian group G, there is a natural surjection:

$$h : H^n(C_\bullet; G) \to \mathrm{Hom}(H_n(C_\bullet; G)) \tag{20}$$

and the sequence

$$\begin{tikzcd} 0 \arrow[r] & \ker(h) \arrow[r] & H^n(C_\bullet; G) \arrow[r] & \mathrm{Hom}(H_n(C_\bullet; G)) \arrow[l, bend right, dashed] \arrow[r] & 0 \end{tikzcd} \tag{21}$$

splits.

Proof. The proof can be found in my live notes. Please review it in detail. $\square$

Index