AT2 LN2.

Recall:

- cochain complexes. $\cdots \to C^i \xrightarrow{\delta} C^{i+1} \xrightarrow{\delta} \cdots$, $\delta^2 = 0$.
- Cohomology $H^n = \ker \delta_n / \operatorname{im} \delta_{n-1}$
- For a chain cx C_i of free Ab gps, and Ab gp G,

  $\rightsquigarrow$ cochain complex $\mathrm{Hom}(C_\cdot; G)$ which has groups $\mathrm{Hom}(C_i, G)$
  and diff'nl $\delta_i = (-1)^{i+1} \partial^*$

- There is a sequence

$$0 \to \ker(h) \to H^n(C_\cdot; G) \xrightarrow{h} \mathrm{Hom}(H_n(C_\cdot; G)) \to 0$$

which splits.

Goal for today: understand $\ker h$.

Corollary: $\exists$ a split SES $0 \to \mathrm{coker}(i_{n-1}^*) \to H^n(C, G) \to \mathrm{Hom}(H_n(C_\cdot), G) \to 0$, $i_{n-1}^* : B_{n-1} \to Z_{n-1}$ inclusn.

Lemma. ("Hom is Left Exact")

Suppose

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

ses of Ab groups and G a group. Then the dual sequence

$$0 \to \mathrm{Hom}(C; G) \xrightarrow{g^*} \mathrm{Hom}(B; G) \xrightarrow{f^*} \mathrm{Hom}(A; G)$$

is exact. If the original sequence was split, then f^* is surjective.
(?) (there is a $\to 0$ on the right, $\mathrm{Hom}(A; G) \to 0$).

Proof. We may assume $A \subseteq B$, f inclusion, $C = B/A$, g projection (up to replacing seq. w/ isomorphic case).

· injectivity of g^* follows from surj. of g.

$$\begin{tikzcd} B \arrow[r, two heads, "g"] \arrow[rr, bend right, "g^*\varphi"'] & C \arrow[r, "\varphi"] & G \end{tikzcd}$$

· Exactness in the middle: $\psi \in \mathrm{Hom}(B; G)$, $f^*\psi = 0$.

This means that $\psi: B \to G$ which vanishes on A
$\Leftrightarrow$ ψ factors through C.

$$\begin{tikzcd} A \arrow[r] & B \arrow[r, "\psi"] \arrow[d, "g"'] & G \\ & C = B/A \arrow[ur, dotted, "\varphi"'] & \end{tikzcd}$$

$\Leftrightarrow$ $\psi = g^*\varphi$ for $\varphi \in \mathrm{Hom}(C, G)$.

· If $0 \to A \to B \to C \to 0$ is split then any homeo $\varphi: A \to G$ extends to $\bar{\varphi}: B \to G$ (Last time: split $\Rightarrow \exists$ projection $B \to A$).

$\Leftrightarrow$ f^* surj. $\square$

Remarks.

1) $\mathrm{Hom}(\cdot, G)$ contravariant functor, and the property of the lemma is called "left-exactness".

2) $0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$ exact, $G = \mathbb{Z}$. Dualize:

$$0 \to \underset{\substack{\| \\ 0}}{\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}$$ not surjective.

So: Hom in general not exact

Now consider ch cx $C_\bullet$ and $Z_n = \ker(\partial_n)$, $B_n = \mathrm{im}(\partial_n)$ so we get a ses

$$0 \to Z_{n+1} \xrightarrow{1} C_{n+1} \xrightarrow{\partial_{n+1}} B_n \longrightarrow 0. \quad (*)$$

Interpret $Z_\bullet$, $C_\bullet$, $B_\bullet$ as chain complexes with diffntls 0, ∂, 0 to get a SES of chain cxs,

$$0 \to Z_\bullet \to C_\bullet \longrightarrow B_\bullet \longrightarrow 0$$

(here, $(B_\bullet)_n = B_{n-1}$ degree shift)

Observation.

$B_n \subseteq C_n$ subgroup of a free Abelian group.

$\rightsquigarrow B_n$ free Ab $\forall n$

$\rightsquigarrow (*)$ splits.

$\Rightarrow$ Apply Hom, we get

$$0 \to \mathrm{Hom}(B_\bullet, G) \to \mathrm{Hom}(C_\bullet, G) \to \mathrm{Hom}(Z_\bullet, G) \to 0$$

- Just like last semester, a SES of coch cxs gives a LES in cohomology.

Since differential in $B_\bullet$, $Z_\bullet$ is zero, cohomologies of $\mathrm{Hom}(B_\bullet, G)$, $\mathrm{Hom}(Z_\bullet, G)$ are just the cochain groups themselves.

$\Rightarrow$

$$\cdots \to \mathrm{Hom}(Z_{n-1}, G) \xrightarrow[\substack{\text{connecting}\\ \text{map}}]{\rho} \mathrm{Hom}(B_{n-1}, G) \to H^n(C_\bullet, G) \to \mathrm{Hom}(Z_n, G) \to \cdots$$

What is the connecting map? $(B_{n-1} \subseteq Z_{n-1})$ $\hookrightarrow$ up to a sign, ρ is the dual to the inclusion $B_{n-1} \subseteq Z_{n-1}$.

Write $X^* = \mathrm{Hom}(X, G)$

$$\begin{tikzcd}
 & \tilde{\Phi} \arrow[r, mapsto] & \phi \arrow[r, mapsto] & 0 & \\
0 \arrow[r] & B_n^* \arrow[r, "\partial_n^*"] & C_{n+1}^* \arrow[r] & Z_{n+1}^* \arrow[r] & 0 \\
0 \arrow[r] & B_{n-1}^* \arrow[r] \arrow[u, "0"] & C_n^* \arrow[r, "in^*"] \arrow[u, "\delta = (-1)^n \partial^*"'] & Z_n^* \arrow[r] \arrow[u, "0"'] & 0 \\
 & & \tilde{\varphi} \arrow[r, mapsto] & \varphi &
\end{tikzcd}$$

$\rho(\varphi) = \tilde{\Phi}$ (Snake Lemma).

$\varphi : Z_n \to G$ homomorphism.

$\cap$

$\tilde{\varphi} : C_n \to G$ extension: $\tilde{\varphi}_n|_{Z_n} = \varphi$

$\phi = \delta(\tilde{\varphi}) = (-1)^{n+1} \tilde{\varphi} \circ \partial^*$

$\| $

$\partial^* \tilde{\Phi}$

$\Rightarrow \tilde{\Phi}|_{B_n = \mathrm{im}(\partial)} = (-1)^{n+1}\tilde{\varphi}|_{B_n} = (-1)^{n+1}\varphi|_{B_n}$

Extract a SES from the LES

$$\mathrm{Hom}(Z_{n-1}, G) \xrightarrow{i_*} \mathrm{Hom}(B_{n-1}, G) \to H^n(C_\bullet; G) \to \mathrm{Hom}(Z_n, G) \xrightarrow[i^*]{f} \mathrm{Hom}(B_n, G) \to \cdots$$

$$\Rightarrow 0 \to \mathrm{coker}(i^*_{n-1}) \to H^n(C_\bullet, G) \xrightarrow{f} \ker(i^*_n) \to 0$$

$$\ker(i^*_n) = \{\text{homs } \varphi : Z_n \to G \text{ st } \varphi|_{B_n} \equiv 0\}$$
$$\cong \{\text{hom. } \psi : \underbrace{Z_n/B_n}_{\cong\, H_n(C_\bullet)} \to G\} \cong \mathrm{Hom}(H_n(C_\bullet), G)$$

Under the identifications, f becomes the map h from last time.

Observe: this coker is measuring something interesting!

$$0 \to B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \to H_{n-1}(C_\bullet) \to 0 \qquad (*)$$

Apply $\mathrm{Hom}(\cdot, G)$

$$0 \to \mathrm{Hom}(H_{n-1}(C_\bullet), G) \to \mathrm{Hom}(Z_{n-1}, G) \xrightarrow{i^*_{n-1}} \mathrm{Hom}(B_{n-1}, G) \to \mathrm{coker}(i^*_{n-1}) \to 0$$

So: $\mathrm{coker}(i^*_{n-1})$ measures the "non-right-exactness of Hom applied to $(*)$".

Cor: If $H_{n-1}(C_\bullet)$ is free abelian, then

$$H^n(C_\bullet, G) \cong \mathrm{Hom}(H_n(C_\bullet), G).$$

Proof: If $H_{n-1}(C_\bullet)$ free Ab, $\to (*)$ splits $\Rightarrow$ apply Hom $\rightsquigarrow$ ses $\rightsquigarrow$ $\mathrm{coker}(i^*_{n-1}) = 0$. $\square$

Def. H Ab gp. A free resolution of H is an ES

$$\cdots \to F_2 \to F_1 \to F_0 \to H \to 0$$

where all F are Free Ab.

(our sequence (*) is a free resolution of $H_{n-1}(C_\bullet)$.)

Lemma. Any Ab gp H has a free resolution (of length 2).

Proof: $f_0 : \underbrace{\bigoplus_{H \ni h} \mathbb{Z}}_{F_0} \xrightarrow{\pi} H$ hom mapping $1 \in \mathbb{Z}$ to h induced by h.

(check LN, may be nonr.) $0 \to \ker(\pi) \to F_0 \xrightarrow{\pi} H$. $\square$

Corollary. S'pose $\cdots \to F_2 \to F_1 \to F_0 \to H \to 0$ a FR of H, G any Ab gp. Consider the coch cx

$$0 \to \mathrm{Hom}(H,G) \xrightarrow{f_0^*} \mathrm{Hom}(F_0,G) \xrightarrow{f_1^*} \mathrm{Hom}(F_1,G) \to \cdots$$

The cohomology of this is independent of the FR.

We call the first coh group

$$\ker f_2^* / \mathrm{im}\, f_1^* = \mathrm{Ext}^1(H,G)$$

"the derived functor of Hom"

$\rightsquigarrow$ or:

$$0 \to \mathrm{Ext}(H_{n-1}(C_\bullet),G) \to H^n(C_\bullet,G) \to \mathrm{Hom}(H_n(C_\bullet),G) \to 0$$

Technical corollary. S'pose H, H' Ab gps, $\alpha : H \to H'$ hom, $F_\bullet \to H$, $F_\bullet' \to H'$ free resolutions.

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r, "f_2"] \arrow[d, "\alpha_2"] & F_1 \arrow[r, "f_1"] \arrow[d, "\hat\alpha_1\ (\ \alpha_1"] \arrow[dl, "h_1"'] & F_0 \arrow[r, "f_0"] \arrow[d, "\alpha_0"] \arrow[dl, "h_0"'] & H \arrow[r] \arrow[d, "\alpha"] \arrow[dl, "h_{-1}"'] & 0 \\
\cdots \arrow[r] & F_2' \arrow[r, "f_2'"'] & F_1' \arrow[r, "f_1'"'] & F_0' \arrow[r, "f_0'"'] & H' \arrow[r] & 0
\end{tikzcd}$$

Then there are $\alpha_i : F_i \to F_i'$ making the diagram commute.

For any other choice $\hat{\alpha}_i$ there are maps $h_i : F_i \to F_{i+1}'$ s.t.

$$\alpha_i - \hat{\alpha}_i = f_{i+1}' h_i + h_{i-1} f_i .$$

"chain homotopies"

$\Rightarrow \alpha_i$ induce well-defined maps in cohomology.

TC $\Rightarrow$ C (apply for $\alpha = \mathrm{id}$).

"Proof" (sketch): Existence of the α_i. Given α. Want to build α_0.

$$F_0 = \bigoplus_{I_0} \mathbb{Z}, \quad \text{with basis elts } b_i,\ i \in I_0.$$

Look at $\alpha f_0(b_i) \in H'$ and choose some $c_i' \in F_0'$ s.t. $f_1'(c_i') = \alpha f_0(b_i)$. Put $\alpha_0(b_i) := c_i'$.

Since F_0 free, this defines a hom. Continue inductively.

$$b_n \mapsto f_n b_n \mapsto 0$$

$$\begin{tikzcd}
\bigoplus_{I_n} \mathbb{Z} = F_n \arrow[r, "f_n"] \arrow[d, dashed, "\alpha_n"] & F_{n-1} \arrow[r, "f_{n-1}"] \arrow[d] & F_{n-1} \arrow[r] \arrow[d] & \cdots \\
F_n' \arrow[r] & F_{n-1}' \arrow[r] & F_{n-2}' \arrow[r] & \cdots
\end{tikzcd}$$

$$c_n \mapsto \alpha_{n-1} f_n b_n \mapsto 0$$

Continue inductively, put $\alpha_n(b_n) = c_n$ according to