## SHEET 1

### Exercise 3

(a) Let $(X, S)$ be a $\Delta$-complex.

**Claim 0.1.** *A subset $A \subseteq X$ is closed if and only if $\sigma^{-1}(A)$ is closed for all $\sigma \in S$.*

*Proof.* Follows immediately from $\sigma^{-1}(A^c) = \sigma^{-1}(A)^c$ where $c$ is the set-theoretic complement. $\square$

Define $X^{(n)} := \bigcup_{k \leq n} \bigcup_{\sigma \in S_k} \mathrm{im}(\sigma)$, the union of all the images of $k$-simplices for $k \leq n$.

**Claim 0.2.** *$X^{(n)}$ is a closed subspace of $X$.*

*Proof.* If $\sigma \in S_k$, then $\sigma^{-1}(X^{(n)})$ is the union of all faces of $\Delta^k$ of dimension $\leq n$, which is always a closed subset. $\square$

**Claim 0.3.** *A subset $A \subseteq X^{(n)}$ is closed if and only if $\sigma^{-1}(A)$ is closed for all $\sigma \in S_k$ and $k \leq n$.*

*Proof.* Follows from $\sigma^{-1}(A) = \bigcup_\tau \tau^{-1}(A)$, where $\tau$ runs through the faces of $\sigma$ of dimension $\leq n$, and there are only finitely many terms in the union. $\square$

**Claim 0.4.** *The filtration $X^{(0)} \subseteq X^{(1)} \subseteq \cdots \subseteq X$ is a CW-structure on $X$.*

*Proof.* We will check two things:

(1) For every $n \geq 0$ there is a pushout diagram

$$\begin{tikzcd} \coprod_{\sigma \in S_n} \partial \Delta^n \arrow[r] \arrow[d] & \coprod_{\sigma \in S_n} \Delta^n \arrow[d, "\sum_{\sigma \in S_n} \sigma"] \\ X^{(n-1)} \arrow[r] & X^{(n)} \end{tikzcd}$$

(for $n = 0$ this says that $X^{(0)} \cong S_0$ is a discrete space)

(2) A subset $A \subseteq X$ is closed if and only if $A \cap X^{(n)}$ is closed for every $n \geq 0$.

A commutative diagram like the above exists, because of condition (i) in the definition of a $\Delta$-complex. So we get an induced (continuous) map

$$f \colon X^{(n-1)} \coprod_{\coprod_{\sigma \in S_n} \partial \Delta^n} \coprod_{\sigma \in S_n} \Delta^n \to X^{(n)}.$$

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This map is a bijection, precisely because of condition (ii) in the definition of a $\Delta$-complex. To see that the map is actually a homeomorphism, we check that it is closed: So let $A \subseteq X^{(n-1)} \coprod_{\bigsqcup_{\sigma\in S_n} \partial\Delta^n} \bigsqcup_{\sigma\in S_n} \Delta^n$ be a closed subset. Then $f(A)$ is closed if $\sigma^{-1}(f(A))$ is closed for all $\sigma \in S_k$ and $k \leq n$ (by Claim 0.3). But this translates precisely into the condition of $A$ being closed in the quotient topology of $X^{(n-1)} \coprod_{\bigsqcup_{\sigma\in S_n} \partial\Delta^n} \bigsqcup_{\sigma\in S_n} \Delta^n$. By induction (starting at the trivial case $n = -1$) this proves (1).

(2) follows in one direction from Claim 0.2 and in the other direction from Claim 0.3. $\square$

(b) Recall the cellular chain complex:

$$\begin{tikzcd}
C_n^{cell}(X) \arrow[r, equal] \arrow[dd, "\partial^{cell}"] & H_n(X^{(n)}, X^{(n-1)}) \arrow[d, "\partial"] & \bigoplus_{\sigma\in S_n} H_n(\Delta^n, \partial\Delta^n) \arrow[l, "\sum_\sigma \sigma_*"', "\cong"] \arrow[r, equal] & C_n^{\Delta}(X) \\
 & H_{n-1}(X^{(n-1)}) \arrow[d, "j_*"] & & \\
C_{n-1}^{cell}(X) \arrow[r, equal] & H_{n-1}(X^{(n-1)}, X^{(n-2)}) & &
\end{tikzcd}$$

A simplex $\sigma \in S_n$ induces a map $\sigma_*$ as indicated; now note that $H_n(\Delta^n, \partial\Delta^n)$ has a canonical generator, namely the relative homology class represented by $id\colon \Delta^n \to \Delta^n$. We then obtain

$$\partial^{cell}(\sigma_*([id])) = j_*\partial([\sigma]) = j_*\left[\sum_{i=0}^{n}(-1)^i\sigma\circ d^i\right] = \sum_{i=0}^{n}(-1)^i[\sigma\circ d^i]$$

which is precisely what the differential in $C_*^{\Delta}(X)$ would do. In other words, the tautological map $\sum_{\sigma\in S_n}\sigma_*$ indicated in the diagram above is the isomorphism we are looking for.