AB Geometrie & Topologie
Dr. Christian Lange
Panagiotis Papadopoulos

Riemannian geometry

Problem set 0

1. (i) Let M be a smooth n-dimensional manifold and $g = \{g_p\}_{p\in M}$ a family of inner products
$$g_p : T_pM \times T_pM \to \mathbb{R}.$$
Show that the following statements are equivalent:

(a) For any pair X, Y of smooth vector fields on M the function $M \to \mathbb{R}$ defined by $p \mapsto g_p(X_p, Y_p)$ is smooth ($\mathcal{C}^k$-differentiable).

(b) There is an atlas of M such that for any chart (U, x) in this atlas the functions $U \to \mathbb{R}$ defined by
$$p \mapsto g_p\left(\left.\frac{\partial}{\partial x_j}\right|_p, \left.\frac{\partial}{\partial x_k}\right|_p\right),$$
$j, k = 1, \dots, n$, are smooth ($\mathcal{C}^k$-differentiable).

(c) For any chart (U, x) of M the functions $U \to \mathbb{R}$ defined by
$$p \mapsto g_p\left(\left.\frac{\partial}{\partial x_j}\right|_p, \left.\frac{\partial}{\partial x_k}\right|_p\right),$$
$j, k = 1, \dots, n$, are smooth ($\mathcal{C}^k$-differentiable).

Remark: Condition (a) has the appealing feature to be coordinate free and independent of choices. Another such definition can be given in terms of smooth sections of a "symmetric 2-tensor bundle" over M, which will be discussed later. However, to actually verify the differentiability of a family of inner products in a concrete situation condition (b) can be more useful.

(ii) Let (M, g) be a ($\mathcal{C}^k$-)Riemannian manifold and $U \subset M$ an open subset. Then $(U, g_{|U})$ is a ($\mathcal{C}^k$-)Riemannian manifold, where $g_{|U}$ is the family of inner products $g_{|U} = \{g_p\}_{p\in U}$.

(iii) Let M be a smooth manifold, (N, h) a Riemannian manifold and $\varphi : M \to N$ a $\mathcal{C}^{k+1}$-immersion. Show that the family of inner products $g = \{g_p\}_{p\in M}$ on M given by
$$g_p(v, w) \mapsto h_{\varphi(p)}(d\varphi_p v, d\varphi_p w)$$
defines a $\mathcal{C}^k$-Riemannian metric on M.

Remark: In the lecture we illustrated that verifying the differentiability of g directly via condition (a) in (i) requires some words. Here use condition (b) instead: observe that the claim is local so that M can be assumed to be an embedded submanifold of N for which there exist submanifold charts, i.e. charts of N that restrict to charts of M (as seen in the previous semester via the local structure theorem for immersions).

We remark that verifying the differentiability based on a definition in terms of smooth sections of a symmetric 2-tensor bundle requires a similar line of arguments.

2. Let M be a smooth manifold. Use a partition of unity to show that there exists a Riemannian metric on M.

   Remark: In contrast, not every smooth manifold admits a Lorentzian metric, e.g. S^2 does not.

3. *Products.* Let M and N be topological manifolds.

   (i) The product topology on $M \times N$ is the coarsest topology (i.e. with the fewest open sets) such that the projections $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ are continuous. Show that $M \times N$ with the product topology is a topological manifold.

   (ii) Assume that $\mathcal{A}_M = \{(U, x)\}$ and $\mathcal{A}_N = \{(V, y)\}$ are smooth structures on M and N, respectively. Show that

   $$\mathcal{A}_M \times \mathcal{A}_N = \{(U \times V, x \times y) \mid (U, x) \in \mathcal{A}_M, (V, y) \in \mathcal{A}_N\}$$

   is a smooth atlas on $M \times N$, where the chart $(U \times V, x \times y)$ is defined as $(x \times y)(p, q) = (x(p), y(q))$ for $p \in U$ and $q \in V$.

   (iii) Show that $\mathcal{A}_M \times \mathcal{A}_N$ induces the unique smooth structure on $M \times N$ with respect to which the projections π_1 and π_2 are smooth.

   (iv) Show that the map

   $$(d\pi_1)_{(p,q)} \times (d\pi_2)_{(p,q)} : T_{(p,q)}(M \times N) \to T_pM \times T_qN$$

   defined by

   $$v \mapsto ((d\pi_1)_{(p,q)}v, (d\pi_2)_{(p,q)}v) =: (v_1, v_2)$$

   is an isomorphism for all $(p, q) \in M \times N$.

   (v) Let g and h be Riemannian metrics on M and N, respectively. Show that

   $$(g \times h)_{(p,q)}(u, v) = g_p(u_1, v_1) + h_q(u_2, v_2),$$

   where $(p, q) \in M \times N$ and $u = (u_1, u_2), v = (v_1, v_2) \in T_pM \times T_qN$, defines a Riemannian metric, the so-called product Riemannian metric on $M \times N$.

4. *Looking back to basic properties of smooth maps.* Let M be a smooth manifold and $N \subset M$ an embedded smooth submanifold.

   (i) Suppose that $F : \Sigma \to M$ is a smooth map from a smooth manifold Σ with image $F(\Sigma) \subset N$. Show that then F is also smooth as a map $F : \Sigma \to N$ into N.

   (ii) Suppose that $W \subset M$ is an embedded smooth submanifold which is a subset of N. Show that then W is also an embedded smooth submanifold of N.

No submission. The material will be discussed in the exercise class on Wednesday, April 30.