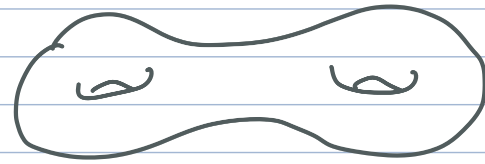
Office Hours: W 4-5:30; Room TBD ~ on website.

Exam: August 5



RG ⇝ curved spaces in $\mathbb{R}^n$

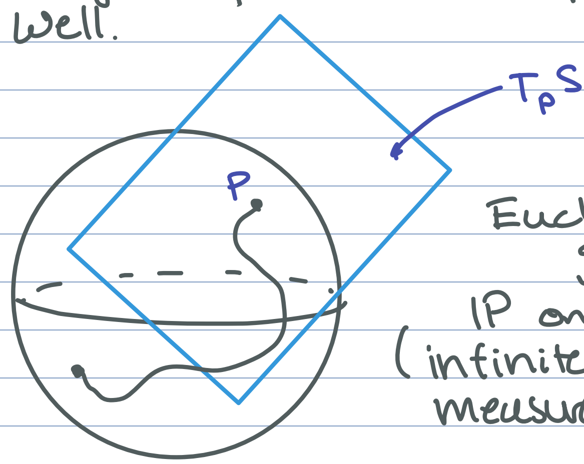
Motivating questions: what properties are intrinsic vs extrinsic?

$\mathbb{R}^n$: natl. metric. But consider:

close in $\mathbb{R}^3$, not in S

⇝ intrinsic geometry a priori not captured by ambient Euclidean metric well.

Inner product:
presents a solutn.



Altnt. approach:
Eucl. IP
{ restricts
IP on each T_pS, $p \in S$.
(infinitesimal length measurement).

This allows us to measure len of C^1-paths $c: I \to S$ by integrating the velocity.

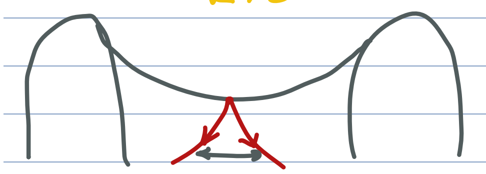
Consider the set of all possible paths, consider the infimum over length.

Measure dist, ∠, volume, etc.

Questn: ∃? shortest paths? ⇝ △s ~ volume of balls?

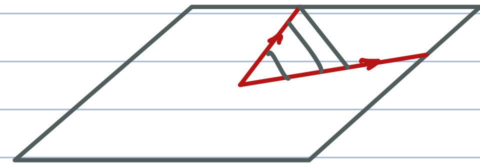
Key observation: (Rie, GB) local geometry is to a large extent determined by a ptwise defined invariant, the "curvature" k, eg := divergence rate of shortest path

$k>0$



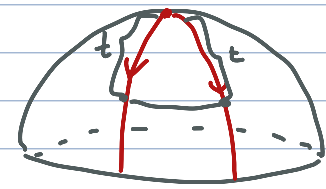
superlinear divergence

$k=0$



Shortest pats diverge @ linear rate

$k<0$



Sublinear rt of divergence

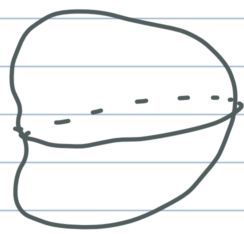
Curvature vs Global Shape.

Gauß-Bonnett (1948): $\Sigma \subset \mathbb{R}^3$ embedded 2D closed sfc,

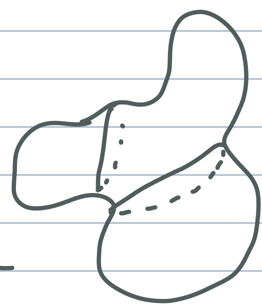
$$\int_\Sigma k \, dA = \chi(\Sigma) \in \mathbb{Z} \quad \text{topological invariant}$$

induced measure from infinitesimal length given by IP (above)

Eg. Σ

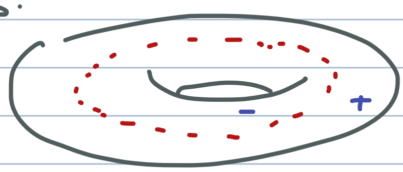


$\cong$



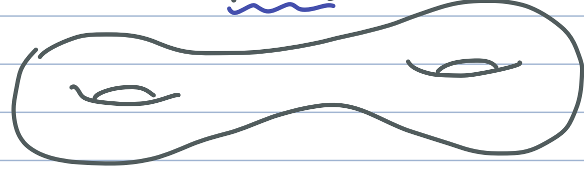
$\chi(\Sigma) = 2$

Torus:



$\chi = 0$

$\chi = -2$



$\ulcorner \chi(\Sigma) = 2 - 2g$, g = genus

end motivation

Now we forget about ambient space and work instead on a smooth manifold M ($\rightsquigarrow$ all mfld will be smooth, without bdy).

$\rightsquigarrow$ tangent bundle TM is also a smooth mf,

$\pi : TM \to M$ natural footpt projn.

Recall: A (smooth) vfield on M is a (smooth) section of π, ie. a (smooth) map $X: M \to TM$ s.t. $\pi \circ X = id_M$.

Def: A (C^k-)Riemannian metric g on M is a family $g = \{g_p\}_{p \in M}$ of inner products (pos. def. symmetric bilinear)

$$g_p : T_pM \times T_pM \longrightarrow \mathbb{R}$$

s.t. for any smooth vector fields X, Y on M, the fct

$$f: M \to \mathbb{R}, \quad p \longmapsto g_p(X_p, Y_p)$$

is smooth (C^k-differentiable). The pair $(M,g) :=$ (C^k-) Riem. mf.

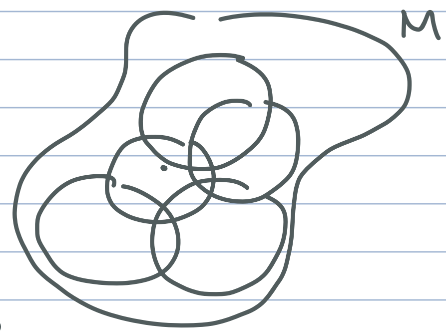
Remarks: a) Given a C^∞-atlas of M, then a family of inner products $g = \{g_p\}_{p \in M}$ is a (C^k-) RieMet iff for any chart (U, x) in this atlas the fct

$$U \to \mathbb{R}, \quad p \mapsto g_p\left(\frac{\partial}{\partial x_j}\Big|_p, \frac{\partial}{\partial x_k}\Big|_p\right)$$

is smooth. $\rightsquigarrow$ exercise class.

b) An open subset U inside a (C^k-) Riem mf (M, g) then $(U, g|_U)$ is also a (C^k-) Riem mf, where

$$g|_U := \{g_p\}_{p \in U}.$$



$\ulcorner$ A: atlas of M

$\rightsquigarrow A|_U = \{(U \cap V, x|_U) \mid (V, x) \in A\}$ is an atlas of U. $\lrcorner$

c) We sometimes write $\langle \cdot, \cdot \rangle$ instead of g (resp $\langle \cdot, \cdot \rangle_p$ instead of g_p).

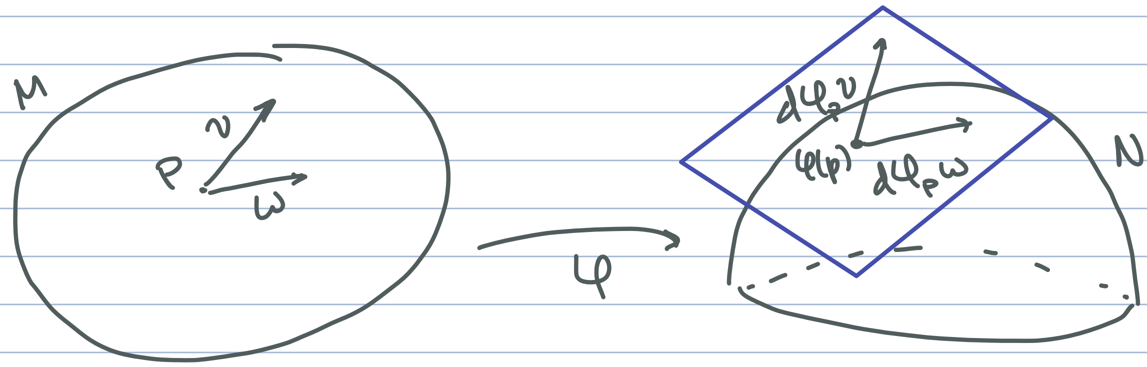
d) Sometimes, also indefinite families of inner products are considered, so-called semi-Riem metrics/mfs, eg in the theory of relativity where you have Lorentzian metrs, i.e. semi-Riem. metrics of signature $(1, n)$ like the Lorentzian metric

$$x_0 y_0 - \sum_{i=1}^{n} x_i y_i \quad \text{on } \mathbb{R}^{n+1}.$$

Any open subset U of a Eucl. VS $(V, \langle\cdot,\cdot\rangle)$ is a RMf with Rm $g_p = \langle\cdot,\cdot\rangle_p$, $p \in U$.

Prop: Let M a smooth mf, (N,h) a Riem mf, $\varphi: M \to N$ a smooth (C^{k+1}-)immersion. Then (N,h) induces a Riem metric g on M via

$$g_p(v,w) = h_{\varphi(p)}(d\varphi_p v, d\varphi_p w); \quad p \in M; \; v,w \in T_pM$$



This Riemannian metric is called the pullback of h via φ, denoted $\varphi^* h$. The immersion $\varphi: (M,g) \longrightarrow (N,h)$ is called a Riemannian immersion

Proof: Since $d\varphi_p : T_pM \to T_{\varphi(p)}N$ is linear & injective, $\forall p \in M$, g_p is bilinear and pos. def. (as $h_{\varphi(p)}$ is).

Symmetry of $h_{\varphi(p)} \Rightarrow$ symm. of $g_p \rightsquigarrow g_p$ an IP.

Smoothness. Let's first assume that φ is a (local) diffeo. Then a smooth vector field X on M induces a smooth vfield on N, namely

$$(\varphi^* X) : q \longmapsto d\varphi_{\varphi^{-1}(q)} X_{\varphi^{-1}(q)}$$

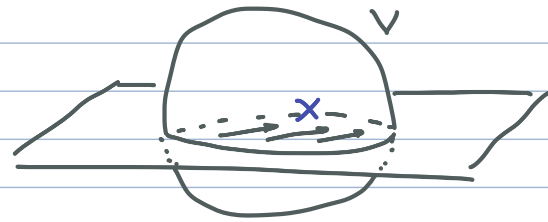
and we have that

$$p \longmapsto g_p(X_p, Y_p) = h_p((\varphi^*X)_{\varphi(p)}, (\varphi^*Y)_{\varphi(p)})$$
$$= ((h.((\varphi^*X).,(\varphi^*Y).)\circ\varphi)(p)$$

For the general case, one can (now) apply the local structure for immersions and assume that

$$M = U \subset \mathbb{R}^n \text{ open}, \quad N = V \subset \mathbb{R}^{n+k} \text{ open}.$$

and $\varphi = i : U \hookrightarrow V$ the inclusion.



$\rightsquigarrow$ extend vfield X on U to

$$\tilde{X}(p) = (X(p), 0) \text{ on } V.$$

($\rightsquigarrow$ details next Wednesday). $\square$

Remarks: • By Whitney's immersion thm, any smooth manifold can be immersed (actually embedded) into some $\mathbb{R}^n$. $\rightsquigarrow$ inherits a Riem metric from $\mathbb{R}^N$.

• By Nash's embedding any $(C^{k\geq 3})$-Riem mf can be isometrically immersed (embedded) into some $\mathbb{R}^N$.

Note: if g, h are Riem metrics on M, then also $(g+h)_p = g_p + h_p$ defines a Riem metric on M. If

$$f : M \to \mathbb{R}$$

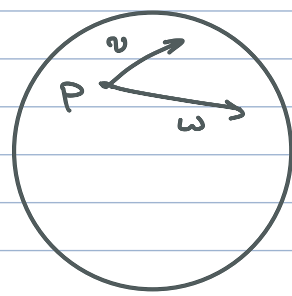
is a pos. smooth function, then

$$(f \cdot g)_p = f_p g_p$$

defines a Riem metric on M.

Example. $B_1^n(0)$ together w/ the Riem metric g

$$g_p(v, w) = \frac{1}{(1 - \langle p, p \rangle)^2} \langle v, w \rangle_p$$



defines a Riem mfld isometric to hyperbolic space.