AB Geometrie & Topologie
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Riemannian geometry

Problem set 1

1. *Hyperboloid model of hyperbolic space.* Let b be a nondegenerate symmetric bilinear form on $\mathbb{R}^{n+1}$ of signature $(n,1)$. The form b defines a so-called Lorentzian metric on $\mathbb{R}^{n+1}$. Recall that up to a base change b is given by

$$b(x,y) = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i.$$

However, such a coordinate representation is not needed in the following.

(i) Show that

$$H = \{x \in \mathbb{R}^{n+1} \mid b(x,x) = -1\}$$

is a smooth submanifold of $\mathbb{R}^{n+1}$.

(ii) For any $x \in H$ consider $T_x H$ as a linear subspace of $\mathbb{R}^{n+1}$. Show that

$$T_x H = (\mathbb{R}x)^{\perp_b} := \{v \in \mathbb{R}^{n+1} \mid b(v,x) = 0\}.$$

(iii) For $x \in H$ and $v, w \in T_x H$ we set $h_x(v,w) = b(v,w)$. Show that h_x is an inner product on $T_x H$ and that $h = \{h_x\}_{x \in H}$ defines a Riemannian metric on H.

(iv) For $x \in H$ show that $H \cap T_x H = \emptyset$.

(v) For some fixed $x_0 \in H$ show that

$$H_0 := \{x \in \mathbb{R}^{n+1} \mid b(x,x_0) < 0\}$$

is the connected component of H containing x_0.

(vi) On the open unit ball $B = B_1^n(0) \subset T_{x_0} H$ we consider the Riemannian metric

$$g_p(v,w) = \frac{4}{(1 - b(p,p))^2} b(v,w),$$

where $p \in B$ and $v, w \in T_p B$. The Lorentzian stereographic projection

$$s : H_0 \to B$$

based at $-x_0$ is defined by mapping $x \in H_0$ to the intersection of the line in $\mathbb{R}^{n+1}$ through x and $-x_0$ with $T_{x_0} H$. Show that s is a diffeomorphism and that $s^* g = h$, i.e. that s is an *isometry*. In particular, the pair (H_0, h) is another model for the *hyperbolic space* (note that in the lecture the "4" in the definition of g was missing).

2. *Lie groups.* A Lie group is a smooth manifold G with a group structure such that the group multiplication $G \times G \to G$ is a smooth map.

   (i) Show that the inversion map $G \to G$, $a \mapsto a^{-1}$, of a Lie group G is smooth.

   (ii) Show that $\mathrm{GL}_n(\mathbb{R})$ (with its natural group structure and the manifold structure inherited from $\mathrm{Mat}_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$) is a Lie group.

   (iii) For a nondegenerate quadratic form q on $\mathbb{R}^n$ we consider

$$\mathrm{O}(q) = \{A \in \mathrm{GL}(\mathbb{R}^n) \mid q(Av) = q(v) \text{ for all } v \in \mathbb{R}^n\},$$

   which can equivalently be described in terms of the nondegenerate symmetric bilinear form b corresponding to the quadratic form q as

$$\mathrm{O}(q) = \{A \in \mathrm{GL}(\mathbb{R}^n) \mid b(Av, Aw) = b(v, w) \text{ for all } v, w \in \mathbb{R}^n\}.$$

   Show that

   (a) $\mathrm{O}(q)$ is a smooth submanifold of $\mathrm{GL}(\mathbb{R}^n)$.

   (b) $\mathrm{O}(q)$ is a subgroup $\mathrm{GL}(\mathbb{R}^n)$.

   (c) $\mathrm{O}(q)$ with the manifold structure from (a) and the group structure from (b) is a Lie group.

   Hint: Apply exercise 4 on problem set 0.

   (iv) Show that $O(q)$ is compact, if q is positive definite.

   (v) Show that any Lie group G admits a left invariant Riemannian metric, i.e. a Riemannian metric with respect to which all left translations L_a, $a \in G$, defined as

$$L_a : G \to G, \; b \mapsto ab,$$

   are isometries.

   (vi) (optional in case you are interested) Show that any compact Lie group admits a bi-invariant Riemannian metric, i.e. a Riemannian metric with respect to which all left- and right translations are isometries, e.g. via the following approach:

   (a) Show that there exists a left invariant volume form on G, i.e. a volume form ω such that $L_a^*\omega = \omega$ for all $a \in G$.

   (b) Let g be a left invariant Riemannian metric on G and ω a left invariant volume form on G. Show that

$$\bar{g}_b(u, v) = \int_G g_{ba}\left((dR_a)_b u, (dR_a)_b v\right)\omega, \quad b \in G, \quad u, v \in T_bG$$

   defines a bi-invariant Riemannian metric on G, where $R_a : G \to G$ denotes the a-right translation defined by $b \mapsto ba$.

You can submit your solutions until **Friday** 2.5.2025 at 6 p.m. in Moodle. Solutions will be discussed in the exercise class on Wednesday 7.5.2025.