# 1 Lecture 1

Overview of the math part of the lecture. FV = Friedli, Velenik.

1. Ising model: existence of thermodynamic limit of pressure/free energy. We will go over Peierl's argument and phase transitions. *Book references:* FV Chapter 3.

2. Gibbs measures in infinite volume. DLR conditions: Debrushin, Lanford, Ruelle. *Book references:* FV Chapter 6.

3. Mermin-Wagner theorem, absence of continuous symmetry breaking in $d = 1, 2$. *Book references:* FV Chapter 9.

4. If time admits: reflection positivity & existence of symmetry breaking in $d = 3$. *Book references:* FV Chapter 10.

## 1.1 The Ising Model

### 1.1.1 The Model

***Notation:*** we will denote by $\Lambda \Subset \mathbb{Z}^d$ that $\Lambda \subset \mathbb{Z}^d$, and that $\Lambda$ is finite, and non-empty. Often, it will be the case that $\Lambda = \{1, \ldots, L\}^d,\ L \in \mathbb{N}$; i.e., that $\Lambda$ consists of a square grid (a "lattice").

**Configuration space**. We denote a configuration space by $\Omega_\Lambda := \{-1, +1\}^\Lambda$. Here,

$$\omega \in \Omega_\Lambda,\ \omega = (\omega_i)_{i\in\Lambda},\ \omega_i \in \{\pm 1\}. \tag{1}$$

More verbosely: $\Omega_\Lambda$ is the set of functions assigning either $+1$ or $-1$ onto each vertex of the lattice, and each $\omega$ is an individual "configuration" which is a collection of $\{\pm 1\}$ assigned to each vertex $i \in \Lambda$.

We also let $h \in \mathbb{R}$ be an external magnetic field. Now, we are in place to define the "**Ising Hamiltonian**", denoted by $\mathscr{H}_{\Lambda;h}$:

$$\begin{aligned} &\mathscr{H}_{\Lambda;h} : \Omega_\Lambda \to \mathbb{R}, \\ \omega = (\omega_i)_{i\in\Lambda} \mapsto\ &\mathscr{H}_{\Lambda;h}(\omega) = -\sum_{\substack{\{i,j\}\subset\Lambda \\ i,j \text{ n.n.}}} \omega_i\omega_j - h\sum_{i\in\Lambda}\omega_i \end{aligned}$$

where "n.n." means "nearest neighbors in $\mathbb{Z}^d$".

A natural question that arises is: what is (are) the minimizer(s) of $\mathscr{H}$?

$$\begin{aligned} &h > 0 : \omega_i = 1\ \forall i, \\ &h < 0 : \omega_i = -1\ \forall i, \\ &h = 0 : 2 \text{ minimizers, all } +1 \text{ or all } -1. \end{aligned}$$

We also introduce now the **partition function** (Zustandssamme):

$$Z_\Lambda = \sum_{\omega\in\Omega_\Lambda} e^{-\beta\mathscr{H}_{\Lambda;h}(\omega)}. \tag{2}$$

where $\beta = \frac{1}{T}$ is the **inverse temperature**.

With this, we may then define the **Gibbs measure**, which is a probability measure on $\Omega_\Lambda$:

$$\mu_{\lambda;\beta,h}(\{\omega\}) = \frac{1}{Z_\Lambda(\beta,h)} e^{-\beta \mathscr{H}_{\lambda;h}(\omega)} \tag{3}$$

Let's take note of a couple important regimes in different values of the parameter $\beta$. First, when $\beta = 0$, we get a uniform distribution: it's completely flat. When $\beta \to \infty$, then the measure concentrates on the minimzer(s).

Now, we can define **pressure/free energy**:

$$\psi_\Lambda(\beta,h) := \frac{1}{\beta|\Lambda|} \log Z_\Lambda(\beta,h). \tag{4}$$

This quantity is dependent upon the system size, inverse temperature, and magnetic field, as we might intuitively expect.

We can also define the **total magnetization**, which is the map:

$$M_\lambda : \Omega_\Lambda \to \mathbb{R}, \tag{5}$$

$$\omega \mapsto M_\Lambda(\omega) = \sum_{i \in \Lambda} \omega_i. \tag{6}$$

Physically, this is just the sum of the spins on each vertex of the lattice $\Lambda$.

**Observation.** We now make an observation regarding the dependence of $\psi_\Lambda$ on $h$:

$$\frac{\partial}{\partial h}\psi_\Lambda(\beta,h) = \frac{1}{|\Lambda|} \sum_{\omega \in \Omega_\Lambda} M_\Lambda(\omega) e^{-\beta \mathscr{H}_{\Lambda,h}(\omega)} \frac{1}{Z_\Lambda(\beta,h)} \tag{7}$$

$$= \frac{1}{|\Lambda|} \langle M_\Lambda \rangle_{\Lambda,\beta,h} \tag{8}$$

$$=: m_\Lambda(\beta,h), \tag{9}$$

where $m_\Lambda(\beta,h)$ is the **average magnetization** per unit volume.

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