MATHEMATISCHES INSTITUT

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Topology II

Sheet 1

Let $\Delta^n \subseteq \mathbb{R}^{n+1}$ be the standard n-simplex. For $i = 0, \dots, n$ let $d^i\colon \Delta^{n-1} \to \Delta^n$ be the inclusion of the face opposite the i-th vertex, that is, the map defined by

$$d^i(t_0, \dots, t_{n-1}) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})\,.$$

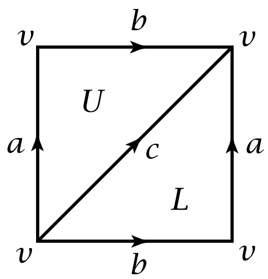
Let X be a topological space. A Δ-*complex structure* on X is a collection of sets $S = \{S_n\}_{n\geq 0}$ where the elements of S_n are maps $\sigma\colon \Delta^n \to X$ satisfying the following conditions:

(i) for every $n \geq 1$, $\sigma \in S_n$ and $0 \leq i \leq n$ we have that $\sigma \circ d^i \in S_{n-1}$

(ii) for every $n \geq 0$ and $\sigma \in S_n$ the map $\sigma|_{\Delta^n \setminus \partial\Delta^n}\colon \Delta^n \setminus \partial\Delta^n \to X$ is injective, and each point of X lies in the image of precisely one such restriction

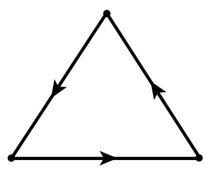
(iii) a subset $A \subseteq X$ is open if and only if $\sigma^{-1}(A)$ is open for every $\sigma \in \bigcup_{n\geq 0} S_n$.

The following figure shows an example of a Δ-complex structure on the 2-torus T:



This Δ-complex structure has $S_0 = \{v\}$, $S_1 = \{a, b, c\}$, $S_2 = \{U, L\}$ and $S_n = \emptyset$ for $n \geq 3$. The arrows and the labels in the picture are meant to tell us how the maps $v\colon \Delta^0 \to T$, $a\colon \Delta^1 \to T$, $U\colon \Delta^2 \to T$ etc. look like. (Recall that the edges of Δ^n carry a natural orientation once an ordering of the $n+1$ vertices has been chosen.)

Exercise 1. Construct Δ-complex structures on the Klein bottle K, on $\mathbb{R}P^2$, on an orientable closed surface Σ_g of genus $g \geq 2$, and on the "dunce hat", i.e., on the following quotient space:



Exercise 2. Let X be a space equipped with a Δ-complex structure S. Let $C_*(X)$ be the singular chain complex of X. Recall that $C_n(X)$ is the free abelian group generated by the set of all maps $\sigma\colon \Delta^n \to X$. For each $n \geq 0$, we let $C_n^\Delta(X) \subseteq C_n(X)$ be the subgroup generated by the set S_n.

(a) Show that $C_*^\Delta(X)$ defines a subcomplex of $C_*(X)$. We denote its homology by $H_*^\Delta(X)$.

(b) Compute $H_*^\Delta(X)$ for $X = K$, $\mathbb{R}P^2$, Σ_g and the dunce hat, using the Δ-complex structures from Exercise 1.

Exercise 3. Let X be a space equipped with a Δ-complex structure.

(a) Show that the Δ-complex structure gives rise to a CW-structure on X.

(b) Let $C_*^{cell}(X)$ denote the cellular chain complex with respect to the CW-structure from (a). Show that there is an isomorphism of chain complexes

$$C_*^\Delta(X) \cong C_*^{cell}(X)\,,$$

The last exercise shows that $H_*^\Delta(X) \cong H_*^{cell}(X) \cong H_*(X)$ (where the last isomorphism was proved in Topology I). In particular, this shows that the homology groups $H_n^\Delta(X)$ are independent of the choice of Δ-complex structure.

This sheet will be discussed in the week of 28 April 2025.