

1 Lecture 1

Recall that the **tangent bundle** is also a smooth manifold, with the natural footpoint projection given by

$$\pi : TM \rightarrow M. \quad (1)$$

A **section** of π is a (smooth) map $X : M \rightarrow TM$ such that $\pi \circ X = \text{id}_M$.

Definition 1.0.1: Riemannian Metric & Riemannian Manifold

A $(C^k\text{-})$ **Riemannian metric** g on M is a family $g = \{g_p\}_{p \in M}$ of inner products (positive definite, symmetric bilinear)

$$g_p : T_p M \times T_p M \rightarrow \mathbb{R} \quad (2)$$

such that for any smooth vector fields X, Y on M , the function

$$f : M \rightarrow \mathbb{R}, \quad p \mapsto g_p(X_p, Y_p) \quad (3)$$

is smooth (C^k -differentiable). The pair (M, g) is called a **Riemannian manifold**.

Remarks.

1. Given a C^∞ atlas of M , then a family of inner products $g = \{g_p\}_{p \in M}$ is a $(C^k\text{-})$ Riemannian metric iff for any chart (U, x) in this atlas the function

$$U \rightarrow \mathbb{R}, \quad p \mapsto g_p\left(\frac{\partial}{\partial x_j}\Big|_p, \frac{\partial}{\partial x_k}\Big|_p\right) \quad (4)$$

is smooth. (To be shown in an exercise class.)

2. Suppose we have an open subset U inside a $(C^k\text{-})$ Riemannian manifold (M, g) . Then $(U, g|_U)$ is also a $(C^k\text{-})$ Riemannian manifold, where

$$g|_U := \{g_p\}_{p \in U}. \quad (5)$$

3. We sometimes write $\langle \cdot, \cdot \rangle$ instead of g (respectively $\langle \cdot, \cdot \rangle_p$ instead of g_p).
4. Sometimes, also indefinite families of inner products are considered, so-called **semi-Riemannian metrics/manifolds**, e.g. in the theory of relativity where you have Lorentzian metrics, i.e. semi-Riemannian metrics of signature $(1, n)$ like the Lorentzian metric

$$x_0 y_0 - \sum_{i=1}^n x_i y_i \text{ on } \mathbb{R}^{n+1}. \quad (6)$$

A simple example of a Riemannian manifold: any open subset U of a Euclidean vector space $(V, \langle \cdot, \cdot \rangle)$ is a Riemannian manifold with Riemannian metric $g_p = \langle \cdot, \cdot \rangle_p$, $p \in U$.

Proposition 1.0.2: Riemannian Immersions

Let M a smooth manifold, (N, h) a Riemannian manifold, $\varphi : M \rightarrow N$ a smooth (C^{k+1}) -immersion. Then (N, h) induces a Riemannian metric g on M via

$$g_p(v, w) = h_{\varphi(p)}(d\varphi_p v, d\varphi_p w); \quad p \in M; v, w \in T_p M. \quad (7)$$

This Riemannian metric is called the **pullback of h via ϕ , denoted φ^*h** . The immersion $\varphi : (M, g) \rightarrow (N, h)$ is called a **Riemannian immersion**.

Proof. Since $d\varphi_p : T_p M \rightarrow T_{\varphi(p)} N$ is linear and injective, $\forall p \in M$, g_p is bilinear and positive definite (as $h_{\varphi(p)}$ is).

Symmetry of $h_{\varphi(p)} \Rightarrow$ symmetry of $g_p \rightarrow g_p$ an inner product.

Smoothness. Let's first assume that φ is a (local) diffeomorphism. Then a smooth vector field X on M induces a smooth vector field on N , namely

$$(\varphi^* X) : q \mapsto d\varphi_{\varphi^{-1}(q)} X_{\varphi^{-1}(q)} \quad (8)$$

and we have that

$$p \mapsto g_p(X_p, Y_p) = h_p((\varphi^* X)_{\varphi(p)}, (\varphi^* Y)_{\varphi(p)}) \quad (9)$$

$$= (h((\varphi^* X)_., (\varphi^* Y)_.) \circ \varphi)(p) \quad (10)$$

For the general case, one can now apply the local structure for immersions and assume that

$$M = U \subset \mathbb{R}^n \text{ open}, N = V \subset \mathbb{R}^{n+k} \text{ open} \quad (11)$$

and $\varphi = i : U \hookrightarrow V$ the inclusion. Next, extend the vector field X on U to

$$\tilde{X}(p) = (X(p), 0) \text{ on } V. \quad (12)$$

(\rightarrow details next Wednesday). \square

Remarks.

1. By Whitney's immersion theorem, any smooth manifold can be immersed (actually embedded) into some \mathbb{R}^n . So it inherits a Riemannian metric from \mathbb{R}^n .
2. By Nash's embedding, any $(C^{k \geq 3})$ -Riemannian manifold can be isometrically immersed (embedded) into some \mathbb{R}^n .

Note. If g, h are Riemannian metrics on M , then also $(g + h)_p = g_p + h_p$

defines a Riemannian metric on M .

$$f : M \rightarrow \mathbb{R} \quad (13)$$

is a positive smooth function, then

$$(f \cdot g)_p = f_p g_p \quad (14)$$

defines a Riemannian metric on M .

Example 1.0.3: Balls and Riemannian Metric.

B_1^n together with the Riemannian metric g , given by

$$g_p(v, w) = \frac{1}{(1 - \langle p, p \rangle)^2} \langle v, w \rangle_p \quad (15)$$

defines a Riemannian manifold isometric to hyperbolic space.

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