Prof. Dr. Sebastian Hensel Winter 2019/20

Topology II

Preparations/Script

Version April 30, 2025

Lecture 1, April 24

1. Cohomology

Our first topic is *cohomology*, which is in many senses dual to homology. Before we dive in, let's ask an important question: why cohomology? There are a few good answers to this:

(1) Many invariants are expressable via cohomology
(2) Cohomology carries extra structure, namely a natural product
(3) The contravariance is actually very good for applications
(4) (Poincaré duality)

To give a first glimpse into this, let us consider a toy example (contrived as it may seem – we lack technology to study "proper", more natural examples). Suppose we consider open subsets $U \subset \mathbb{R}^2$ of the plane with extra structures. For example, a *vector field*, i.e. the assignment of a vector v_x at every point $x \in U$. We could think of this as a continuous map

$$V : U \to \mathbb{R}^2.$$

Another strucure is a *line field*, i.e. a (continuous) choice of a line L_x at every point $x \in U$. If we want to be more formal, this structure is a continuous map

$$f : U \to \mathbb{R}P^1.$$

This phenomenon is fairly common: extra structures on a space U are given by *classifying maps* into suitable spaces.

Say, in our example, we want to figure out if a given line field can be defined by a vector field, i.e. given L, is there a V so that

$$L(x) = \operatorname{span}(V(x)) \quad \forall x \in U.$$

Not every line field is of this type: if the line turns by π as we go around a circle in U this will be impossible! In fact covering space theory can be used to answer it completely. Consider the covering space $p : S^1 \to \mathbb{R}P^1$, and the line field $L : U \to \mathbb{R}P^1$. We exactly ask if L can be lifted to S^1 – if it can, such a lift gives the desired map V, and if V is any vector field defining L (in the sense above), then after normalizing to have norm 1 it is a lift. The covering theorem from last semester answers when this lifting problem is solvable: we want the image of $L_*(\pi_1(U))$ to lie in $p_*(\pi_1(S^1))$. Here, $\pi_1(\mathbb{R}P^1)$ is $\mathbb{Z}$ and the image of $\pi_1(S^1)$ are the even integers. In more elementary terms: we need to look at how the line field turns when moving through U, and our problem is solvable exactly if it never turns an odd number of times.

This is nice and all, but it is not expressed in an object that "lives on U", but rather how the image in $\mathbb{R}P^1$ behaves.
Can we "package" this data into some object on U? The *covariance* of π_1 makes this hard – there is no way to "move" the obstruction from $\mathbb{R}P^1$ to U.
Cohomology works differently – here, the map L induces a cohomology map $L^* : H^*(\mathbb{R}P^2; \mathbb{Z}/2) \to H^*(U; \mathbb{Z}/2)$ going the other way – and there is a class ω in $H^1(\mathbb{R}P^1, \mathbb{Z}/2)$ which exactly measures if a loop turns an odd or even times. We can pull this class back via L^* to an "obstruction class" $L^*\omega \in H^1(U; \mathbb{Z}/2)$ in the cohomology of U, which exactly measures if the line field turns an odd number of times along some loop.
Hence, the (non)vanishing of this class exactly answers if our problem is solvable. Note that this class now lives "in U"!

This example is very contrived, but the general strategy is very powerful: a piece of extra structure on a space X defines a classifying map from X into a universal space, and then the cohomology of that universal space defines (via pullback) "characteristic classes" in the cohomology of X, which obstruct or classify the possible extra structures.

1.1. **Cohomology, Algebraically.** Recall that to compute homology of a space, we defined a *chain complex*, and then defined homology of that complex. The first step was topological, the second is purely algebraic. We now first study how to construct *cohomology* algebraically.
The basic idea is that we want to *dualise*.
So, we are given a chain complex of free Abelian groups

$$\cdots \to C_{n+1} \to C_n \to C_{n-1} \to \cdots$$

with differential $\partial : C_i \to C_{i-1}$. Also suppose that we are given an Abelian group G. We can then define the *dual cochain group*

$$C_n^* = \mathrm{Hom}(C_n, G)$$

as the group of homomorphisms to G. For now, it's fine to think of $G = \mathbb{Z}$ so that "dual" is actually the dual.
How can we turn these groups in a complex? There is no natural way to get from C_n^* to C_{n-1}^*, but there is one to get to C_{n+1}^*:

$$\partial^*(f) = f \circ \partial$$

This would gives us a *cochain complex*:

$$\cdots \to C_{n-1}^* \to C_n^* \to C_{n+1}^* \to \cdots$$

with a *codifferential* ∂^*. Just to be sure, let us define

Definition 1.1. A cochain complex is a collection of Abelian groups C^n together with maps $\delta^n : C^n \to C^{n+1}$

$$\cdots \to C^{n-1} \to C^n \to C^{n+1} \to \cdots$$

so that $\delta^2 = 0$.

Given any cochain complex, we can define its cohomology.

Definition 1.2. If $C^\bullet$ is a cochain complex, we define its *cohomology* as

$$H^n(C^\bullet) = \ker(\delta^n)/\mathrm{im}(\delta^{n-1}).$$

Back to the cochain complex we obtained by dualizing a chain complex. Many sources (e.g. Hatcher and Bredon) do exactly what we described above to define the cohomology of a chain complex. However, it is somewhat more natural to introduce an extra sign (why this is will become clear much later). We thus define the cochain complex

$$\mathcal{H}om(C_\bullet, G)$$

with the same groups as before (i.e. $\mathrm{Hom}(C_n, G)$) but the codifferential

$$\delta = (-1)^{n+1}\partial^* : \mathrm{Hom}(C_{n+1}, G) \to \mathrm{Hom}(C_n, G).$$

Note that *for the groups* $H^n(C^*_\bullet)$ this sign does not mattern. To reiterate, we define

$$H^*(C_\bullet; G) = H^*(\mathcal{H}om(C_\bullet, G)).$$

Note that cohomology with coefficients in G is (clearly) a contravariant functor from chain complexes to groups.

At this stage, an example is in order to get a first impression on how this relates to homology. Our chain complex is the following:

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

We have the following homology:

$$H_0 = \mathbb{Z}, H_1 = \mathbb{Z}/2, H_2 = 0, H_3 = \mathbb{Z}$$

What is the cochain complex (for $G = \mathbb{Z}$)? The groups all stay $\mathbb{Z}, 0$. In fact, we have $f(2x) = 2f(x)$ for any $f : \mathbb{Z} \to \mathbb{Z}$, and thus also the maps are as before:

$$0 \leftarrow \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \leftarrow 0$$

What are the cohomology groups?

$$H^0 = \mathbb{Z}, H^1 = 0, H^2 = \mathbb{Z}/2, H^3 = \mathbb{Z}$$

So, in particular we see that the cohomology groups are *not* the duals of the homology groups!

Our first goal will thus be to understand how exactly the cohomology and homology groups relate.

Lemma 1.3. *For any chain complex C of free Abelian groups, and every Abelian group G, there is a natural surjective map*

$$h : H^n(C; G) \to \mathrm{Hom}(H_n(C), G),$$

and the short exact sequence

$$0 \to \ker(h) \to H^n(C; G) \to \mathrm{Hom}(H_n(C), G) \to 0$$

is split.

It is important to also remember what the map is (which we construct in the proof).

Proof. Denote by $Z_n = \ker \partial \subset C_n$ and $B_n = \mathrm{im}\partial \subset C_n$ the cycles and boundaries of the given chain complex in degree n.
Suppose now $\chi \in H^n(C;G)$ is given. This is represented by a homomorphism

$$\varphi : C_n \to G$$

satisfying $\partial^* \varphi = 0$, or, in other words:

$$\varphi \circ \partial = 0.$$

In particular, we have that $\varphi(B_n) = 0$. We thus have an induced map

$$\overline{\varphi} : Z_n / B_n \to G$$

which defines an element in $\mathrm{Hom}(H_n(C), G)$. The assignment $\varphi \to \overline{\varphi}$ is clearly additive. Furthermore, suppose that $\varphi \in \mathrm{im}\partial^*$. In other words, we have

$$\varphi = \rho \circ \partial$$

for $\rho : C_{n-1} \to G$. Then $\overline{\varphi} = 0$.
As a consequence, the assignment $\varphi \to \overline{\varphi}$ defines the desired map

$$h : H^n(C;G) \to \mathrm{Hom}(H_n(C), G).$$

It remains to show surjectivity and the splitting of the sequence.
To this end, consider the short exact sequence

$$0 \to Z_n \to C_n \xrightarrow{\partial} B_{n-1} \to 0.$$

The group $B_{n-1} \subset C_{n-1}$ is subgroup of a free Abelian group, and thus itself free Abelian. This means that the short exact sequence *splits*, i.e. there is a map

$$\sigma : B_{n-1} \to C_n$$

which is a right inverse to ∂. The map $p = \mathrm{id} - \sigma\partial$ is then a projection

$$p : C_n \to Z_n$$

(restricting to the identity on Z_n). Denote by $\pi : Z_n \to H_n(C)$ the quotient map. We now want to define

$$\mathrm{Hom}(H_n(C), G) \to \ker \partial^*, \varphi \mapsto [\varphi \circ \pi \circ p].$$

Let's see that this makes sense. First, observe that

$$\varphi \circ \pi \circ p : C_n \to G$$

is a homomorphism which vanishes on B_n. Thus, $[\varphi \circ \pi \circ p] \in \ker \partial^*$. Define

$$s : \mathrm{Hom}(H_n(C), G) \to H^n(C;G)$$

to be map obtained from the map above, follows by the natural quotient.

Finally, observe that $hs(\varphi)$ is induced by $\varphi \circ \pi \circ p|_{Z_n}$, seen as a map on $H_n(C)$. As p is the identity on Z_n, this is just φ again. $\square$

Lecture 2, April 29

So, to understand how cohomology and homology relate, we need to understand $\ker(h)$. Our strategy will be to rewrite the sequence in progressively different forms, until this term becomes clearer.
Before we can start, the following is a generally helpful result to know:

Lemma 1.4 ("Hom is left exact"). *Suppose that*

$$0 \to A \to B \to C \to 0$$

is a short exact sequence of Abelian groups, and G is a group. Then the dual sequence

$$0 \to \mathrm{Hom}(C, G) \to \mathrm{Hom}(B, G) \to \mathrm{Hom}(A, G)$$

is exact. If the original short exact sequence is split*, then*

$$0 \to \mathrm{Hom}(C, G) \to \mathrm{Hom}(B, G) \to \mathrm{Hom}(A, G) \to 0$$

is exact.

The extra requirement is necessary: if we consider $0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$. Dualising gives $0 \to 0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to 0$ which is *not* exact.

Proof. We can throughout consider the (isomorphic) sequence where we identify the left group with a subgroup of the middle one, and the right one with the quotient.

$$0 \to A \to B \to B/A = C \to 0$$

Injectivity of $\mathrm{Hom}(C, G) \to \mathrm{Hom}(B, G)$ follows simply because $B \to C$ is surjective.
For exactness in the middle, the kernel of $\mathrm{Hom}(B, G) \to \mathrm{Hom}(A, G)$ is those $\varphi : B \to G$ which vanish on A. These are exactly the ones which factor through $B \to B/A$, or in other words, those in the image of $\mathrm{Hom}(B/A, G) \to \mathrm{Hom}(B, G)$.
Finally, for surjectivity of $\mathrm{Hom}(B, G) \to \mathrm{Hom}(A, G)$ we need to show that any homomorphism $A \to G$ can be extended to B. This is shown using the retraction $p : B \to A$ as above, which exists as the sequence splits. $\square$

The (first) property proved in this lemma is called a "left exact functor".

Back to our setting. Recall that we have a chain complex $C_\bullet$ with cycle and boundary groups $Z_\bullet, B_\bullet$, and we want to try and understand the relation between cohomology and homology.
Here's a trick: we have short exact sequences

$$0 \to Z_{n+1} \to C_{n+1} \to B_n \to 0$$

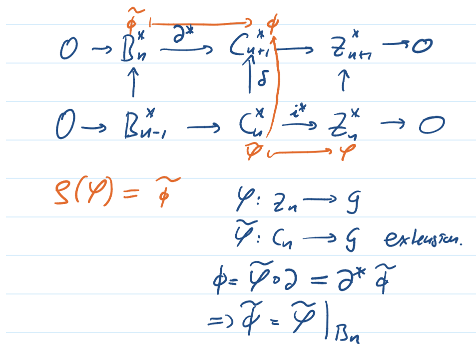

FIGURE 1. The connecting map. Duals are written with a $*$.

We can think of these as a short exact sequence of chain complexes $Z_\bullet, C_\bullet, B_{\bullet-1}$ (with differentials $0, \partial, 0$). As the B_n are free (as subgroups of free Abelian groups), we can apply the last lemma and get short exact sequences

$$0 \to \mathrm{Hom}(B_n, G) \to \mathrm{Hom}(C_{n+1}, G) \to \mathrm{Hom}(Z_{n+1}, G) \to 0.$$

These we can in turn interpret as a short exact sequence of cochain complexes – and such yield long exact sequences (for things like this, chain or cochain doesn't matter, and the result from last semester applies).
The long exact sequence has the form:

$$\cdots \to \mathrm{Hom}(Z_{n-1}, G) \to \mathrm{Hom}(B_{n-1}, G) \to H^n(C; G) \to \mathrm{Hom}(Z_n, G) \to \mathrm{Hom}(B_n, G) \to \cdots$$

(since all coboundary maps for the B, Z terms are 0, computing cohomology just returns these). The only interesting map is the connecting map $\mathrm{Hom}(Z_n, G) \to \mathrm{Hom}(B_n, G)$. Tracing through the diagram (see Figure 1) we see that it is in fact the "obvious" map – the dual i_n^* of the inclusion map.
We can extract a short exact sequence from the long exact sequence:

$$0 \to \mathrm{Coker}(i_{n-1}^*) \to H^n(C; G) \to \ker(i_n^*) \to 0$$

What are the groups here? The kernel (on the right) is maps $Z_n \to G$ which vanish on B_n. This can (as in the proof of Lemma 1.3) be identified with $\mathrm{Hom}(H_n(C); G)$. If we do this, the righthand map in this sequence becomes the map h from that lemma. Hence, we have seen

Corollary 1.5. *There is a split, natural short exact sequence*

$$0 \to \mathrm{Coker}(i_{n-1}^*) \to H^n(C; G) \xrightarrow{h} \mathrm{Hom}(H_n(C); G) \to 0$$

Why is this helpful? The point is that $\mathrm{Coker}(i_{n-1}^*)$ measures something intrinsically interesting. Namely, recall that we have a short exact sequence (by definition)

$$0 \to B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \to H_{n-1}(C) \to 0.$$

Applying $\mathrm{Hom}(\cdot; G)$ to this, we get an exact sequence (by left exactness of Hom)

$$0 \to \mathrm{Hom}(H_{n-1}(C), G) \to \mathrm{Hom}(Z_{n-1}, G) \stackrel{i_{n-1}^*}{\to} \mathrm{Hom}(B_{n-1}, G)$$

So we see that: the kokernel $\mathrm{Coker}(i_{n-1}^*)$ measures the failure of exactness of this sequence (in the only place where it is not automatic)!
In particular, we see that should the original sequence be split, we have exactness everywhere, and thus the cokernel term disappears. This is useful, so let's record it:

Corollary 1.6. *If $H_{n-1}(C)$ is free Abelian, then*

$$h : H^n(C; G) \to \mathrm{Hom}(H_n(C); G)$$

is an isomorphism.

Observe the degree shift! Also, go back to the simple example we did: H_0 was free Abelian, so we could get H^1 by just dualising – but H_1 was torsion, so H^2 was not just dualising...
To explain what happens in general, we need to dive a little bit deeper into commutative algebra (although we won't discuss the most general case here). I hope it has become clear that it is important to understand what happens with exact sequences when applying $\mathrm{Hom}(\cdot, G)$. It will be useful to discuss this a little bit more generally than just for the sequence $0 \to B_{n-1} \stackrel{i_{n-1}}{\to} Z_{n-1} \to H_{n-1}(C) \to 0$.
The key notion is the following:

Definition 1.7. A *free resolution* of an Abelian group H is an exact sequence

$$\cdots F_2 \to F_1 \to F_0 \to H \to 0$$

where each F_i is a free Abelian group.

Lemma 1.8. *Any Abelian group has a free resolution.*

Proof. Choose a set of generators $h_i, i \in I$ of H. We then have a surjective map

$$f_0 : \bigoplus_I \mathbb{Z} \to H,$$

and thus an exact sequence

$$0 \to \ker(f_0) \to \bigoplus_I \mathbb{Z} \to H \to 0.$$

Since subgroups of free Abelian groups are free Abelian, this is already a free resolution. □

Why do we allow "long" free resolutions, if there are always some of "length 2"? First: this idea works more generally, where this statement isn't always true. More importantly: it will turn out that certain properties of H can

be computed with *any* free resolution, so it will be useful to have flexibility to choose whichever we like.
Namely, we will show below:

Corollary 1.9. *Suppose* $\cdots F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$ *is a free resolution, and* G *any Abelian group. Then, consider the (co)chain complex*

$$0 \to \mathrm{Hom}(H, G) \xrightarrow{f_0^*} \mathrm{Hom}(F_0, G) \xrightarrow{f_1^*} \mathrm{Hom}(F_1, G) \xrightarrow{f_2^*} \cdots$$

Then its "first cohomology group"

$$\ker f_2^* / \mathrm{im} f_1^*$$

is independent of the free resolution, and only depends on H *and* G*. It is usually called* $\mathrm{Ext}(H, G)$*.*

The corollary will follow immediately from the following basic result in commutative algebra:

Lemma 1.10. *Suppose that* H, H' *are Abelian groups,* $\alpha : H \to H'$ *is a homomorphism, and* $F_\bullet \to H, F'_\bullet \to H'$ *are free resolutions. Then there are maps* $\alpha_i : F_i \to F'_i$ *which make the following diagram commute:*

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r] \arrow[d, "\alpha_2'"', "\alpha_2"] \arrow[dl, "h_2"'] & F_1 \arrow[r] \arrow[d, "\alpha_1'"', "\alpha_1"] \arrow[dl, "h_1"'] & F_0 \arrow[r] \arrow[d, "\alpha_0'"', "\alpha_0"] \arrow[dl, "h_0"'] & H \arrow[r] \arrow[d, "\alpha"] \arrow[dl, "h_{-1}"'] & 0 \\
\cdots \arrow[r] & F_2' \arrow[r] & F_1' \arrow[r] & F_0' \arrow[r] & H' \arrow[r] & 0
\end{tikzcd}$$

Furthermore, for any other choice of maps $\widehat{\alpha}_i : F_i \to F'_i$ *there is a "chain homotopy" between them, i.e.: maps* $h_i : F_i \to F'_{i+1}$ *so that* $\alpha_i - \widehat{\alpha}_i = f'_{i+1} h_i - h_{i-1} f_i$.

The (straightforward) proof of this is in the (handwritten) file *extension_lemma.pdf*.

Upcoming Lectures

From here, unchecked preparations – read with care...

Back to our discussion of cohomology above. We had seen that

$$0 \to B_{n-1} \overset{i_{n-1}}{\to} Z_{n-1} \to H_{n-1}(C) \to 0$$

is a free resolution of $H_{n-1}(C)$. Thus, as in the corollary, we look at the cochain complex

$$0 \to \mathrm{Hom}(H_{n-1}(C), G) \to \mathrm{Hom}(Z_{n-1}, G) \overset{i_{n-1}^*}{\to} \mathrm{Hom}(B_{n-1}, G) \to 0$$

and get

$$\mathrm{Ext}(H_{n-1}(C), G) = \mathrm{Coker}(i_{n-1}^*)$$

Putting everything together, we then get

Theorem 1.11. *For any chain complex $C_\bullet$ of free Abelian groups, and any Abelian group G, we have short exact sequences*

$$0 \to \mathrm{Ext}(H_{n-1}(C), G) \to H^n(C; G) \to \mathrm{Hom}(H_n(C), G) \to 0$$

In particular, *cohomology groups are determined by homology groups.* We also see that the *integral* homology groups determine cohomology with any coefficients. A theorem like this is usually called "universal coefficient theorem", and it also exists for homology (if you know some category theory, you may guess how it should look – the sequence will go the other way, the "candidate" is the tensor product of integral homology with the coefficient group, and Ext will turn into some kind of dual object, called Tor).
Also, Ext-groups are pretty computable, so this theorem is actually useful to compute cohomology:

Lemma 1.12. *i) For any free Abelian H we have* $\mathrm{Ext}(H, G) = 0$.
ii) For any H, H' we have $\mathrm{Ext}(H \oplus H', G) = \mathrm{Ext}(H, G) \oplus \mathrm{Ext}(H', G)$.
iii) $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}) = G/nG$.

Proof. The key is to choose convenient free resolutions.

(1) Here, $0 \to H \to H \to 0$ is a free resolution, which shows what we want.
(2) Here, suppose we have free resolutions $F_\bullet \to H, F'_\bullet \to H'$. We can then form the free resolution

$$\cdots F_2 \oplus F_2' \to F_1 \oplus F_1' \to F_0 \oplus F_0' \to H \oplus H' \to 0$$

and since taking (co)homology commutes with direct sum of complexes, the result follows.
(3) Here, we use the free resolution

$$0 \to \mathbb{Z} \overset{\cdot n}{\to} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0.$$

To compute Ext, we dualise first to get the cochain complex:

$$0 \to \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, G) \to \mathrm{Hom}(\mathbb{Z}, G) \to \mathrm{Hom}(\mathbb{Z}, G) \to 0$$

The Ext-group we are after is the cokernel of the last nonzero map. To compute it, observe that $\mathrm{Hom}(\mathbb{Z}, G) = G$, and with this identification, the dual of the multiplication-with-n map is again the multiplication-with-n map. Thus, $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}) = G/nG$ as claimed.

□

Corollary 1.13. *If H is a finitely generated Abelian group, and $T \subset H$ is the torsion subgroup, then*

$$\mathrm{Ext}(H, \mathbb{Z}) = T$$

Proof. By the structure theorem of finitely generated Abelian groups, we have $H = \mathbb{Z}^n \oplus \mathbb{Z}/r_1\mathbb{Z} \oplus \oplus\mathbb{Z}/r_k\mathbb{Z}$ for some n and r_i, where $T = \mathbb{Z}/r_1\mathbb{Z} \oplus \oplus\mathbb{Z}/r_k\mathbb{Z}$. Now the corollary follows by applying the lemma. □

Corollary 1.14. *Suppose that C is a chain complex, and suppose that $H_n(C), H_{n-1}(C)$ are finitely generated, with torsion subgroups T_n, T_{n-1}. Then*

$$H^n(C; \mathbb{Z}) = (H_n/T_n) \oplus T_{n-1}.$$

Proof. This follows from above, since $\mathrm{Hom}(H, \mathbb{Z}) = H/T$ for finitely generated H with torsion subgroup T. □

From this, we now completely understand what happened in our initial example.

1.2. **Cohomology, Topologically.** We now begin to discuss cohomology of *spaces*. Before starting the discussion in earnest, we want to motivate a little bit how this notion "shows up naturally" in topological problems.

1.3. **Some toy examples.** For this, we will be looking at *cellular cohomology*, i.e. cohomology defined by the cellular chain complex. We do this first for a graph. That is, we have $C_0(X)$ the free Abelian group with basis corresponding to the vertices of the graph X, and $C_1(X)$ has basis corresponding to edges (with an arbitrary choice of orientation on each).
We then put $C^i(X) = \mathrm{Hom}(C_i(X), \mathbb{Z})$. In other words, functions on vertices or edges. We can describe the differentials explicitly:

$$(\delta^0\varphi)(e) = \varphi(\partial e) = \varphi(e_+) - \varphi(e_-)$$

where $e_,e_+$ are the two endpoints of e. Hence, the kernel of δ^0 simply consists of the functions constant on connected components. Hence, H^1 has a $\mathbb{Z}$ for each connected component. More interestingly, we can think of δ^0 as a sort of "derivative", computing the change in value along the edge.
Let's try to solve equations of the form

$$\psi = \delta^0\varphi$$

which is like finding an antiderivative in analysis. Do two examples: for a tree, this is always possible, but for a nontrivial graph it may not.
We interpret this cohomologically: the kernel of δ^1 is everything (as $C^2 = 0$), and we quotient out the locally constant functions. In this sense, we can see

that solving the above equation is possible exactly iff $[\psi] = 0$ in cohomology. Solutions are unique up to locally constant functions.
This kind of situation is pretty typical: often, cohomology can be used as an *obstruction* to solving equations. You may have seen this already, if you took differentiable manifolds (or from analysis): a differential form is not always the (exterior) derivative of another one – an obvious obstruction is that $d\omega = 0$, but even then it is not necessarily true! The resemblance is not coincidence – for a manifold, exterior derivation actually computes cohomology!

Let's do one more example. We decompose the torus into triangles, and look at first cohomology. Now there is a nontrivial condition to be contained in the kernel of δ^1 ("additivity along triangles"). The cohomology class then again measures the obstruction to realising as the "derivative" of a function.

On the homework, you'll play with a very different application of cohomology to group actions.

1.4. **Simplicial Cohomology, basic properties.** Formally, we will be looking at cohomology of the singular chain complex. More explicitly, if X is a topological space, we let

$$C^n(X;G) = \mathrm{Hom}(C_n(X), G),$$

in other words, functions which assign to any singular n–simplex a value in G. The coboundary can explicitly be seen as

$$\delta\varphi(\sigma) = \varphi(\partial\sigma) = \sum_i (-1)^i \varphi(\sigma|[v_0, \ldots, \hat{v_i}, \ldots, v_{n+1}])$$

Cohomology $H^n(X;G)$ can then be seen as functions on cocycles which vanish on coboundaries (up to equivalence).
Let's observe some basic properties. All of these are really similar to the things we proved about homology last semester, and we can recycle a lot of our hard work.

- From the definitions (or the universal coefficients) we immediately get:
$$H^0(X;G) = \mathrm{Hom}(H_0(X);G)$$
- Actually, since $H_0(X)$ is always free, we also have
$$H^1(X;G) = \mathrm{Hom}(H_1(X);G)$$
- We can define relative cohomology as follows: begin with the sequence for relative homology:
$$0 \to C_n(A) \to C_n(X) \to C_n(X,A) \to 0$$
Observe that it is split: $C_n(X) = C_n(A) \oplus Z$ where Z is spanned by all the singular simplices not in A. Thus, the dual sequence

$$0 \to \mathrm{Hom}(C_n(X,A);G) \to \mathrm{Hom}(C_n(X);G) \to \mathrm{Hom}(C_n(A);G) \to 0$$

is also exact. In fact, it's kind of useful to see this directly, though. The right map is just restriction. It's surjective, since any map from $C_n(A)$ to G can be extended (e.g. by 0) on the singular simplices where it isn't defined. It's kernel is those that restrict to 0 on $C_n(A)$, so it is the image of the left map.

Now, we *define* relative cohomology as the cohomology of $C^n(X, A; G) = \mathrm{Hom}(C_n(X, A); G)$. The fact that we have the short exact sequence above, immegiately gives...

- The long exact sequence for pairs:

$$\ldots H^n(X, A; G) \to H^n(X; G) \to H^n(A; G) \to H^{n+1}(X, A; G) \to \ldots$$

The connecting map here is related to the one in the long homology sequence, see homework.

- *Continuous maps (contravariantly) induce maps in cohomology.*

Recall that if we have a continuous map $f : X \to Y$, we get (by postcomposing) a map

$$f_\sharp : C_n(X) \to C_n(Y)$$

Recall from last semester that we have proved $\partial f_\sharp = f_\sharp \partial$.

Now, dualising, gives

$$f_\sharp^* : C^n(Y) \to C^n(X)$$

and we have $f_\sharp^* \delta = \delta f_\sharp^*$. This shows that $f_\sharp^*$ induces

$$f^* : H^n(Y; G) \to H^n(X; G).$$

Functoriality is obvious from that property for $f_\sharp$.

- *Homotopic maps induce the same map in cohomology.*

We've seen that for homotopic f, g there is a chain homotopy P so that

$$g_\sharp - f_\sharp = \partial P + P \partial.$$

Dualise to get

$$g_\sharp^* - f_\sharp^* = P^* \delta + \delta P^*$$

and so P^* is a cochain homotopy showing $f^* = g^*$.

- *Excision.* Just like for homology excision, the setup is $Z \subset A \subset X$ so that the closure of Z is in the inside of A. Then the inclusion gives an isomorphism

$$i^* : H^n(X, A; G) \to H^n(X - Z, A - Z; G).$$

Again, we can get this by recalling that in the homological proof there were maps

$$\iota : C_n(A + B) \to C_n(X), \rho : C_n(X) \to C_n(A + B)$$

so that $\rho\iota = 1, 1 - \iota\rho = \partial D + D\partial$ (here, ρ uses the small simplices...); recalling that $C_n(A+B)$ is generated by those simplices completely in A, B (where B is the complement of the closure of A).

Dualising gives maps ρ^*, ι^* which induce isomorphisms between the cohomology of $C^n(X)$ and $C^n(A+B)$. That's not quite what we want though, yet.

There is one last diagram (see blackboards) to get what we want.

- *Cellular Cohomology* We can build a cellular cochain complex analogous to what we did in homology, and it indeed computes cohomology. This is done in Hatcher, p.203, and the course blackboards.
- *Mayer-Vietoris* is shown very similarly to excision. We start with the (obviously) short exact sequence:

$$0 \to C^n(A+B;G) \to C^n(A;G) \oplus C^n(B;G) \to C^n(A \cap B;G) \to 0$$

where the leftmost group is the dual of $C_n(A+B) \subset C_n(X)$. Why exact? The left map has coordinates given by restriction, and the right map is difference of the the restrictions.

Since the inclusion $C_n(A+B) \subset C_n(X)$ is a chain homotopy equivalence (that's what we've proved when proving excision), the same is true for the dual, and the left term computes $H^n(X;G)$.

Hence, we get

$$\ldots \to H^n(X;G) \to H^n(A;G) \oplus H^n(B;G) \to H^n(A \cap B;G) \to H^{n+1}(X;G) \to \ldots$$

1.5. **The Cup Product.** Now, we start with something truly new. Throughout, X will be a topological space, and R will be a ring. Think of $\mathbb{Z}$ or $\mathbb{Q}$ if you're unhappy with general rings.
Given cochains $\varphi \in C^k(X;R)$ and $\psi \in C^l(X;R)$ we first want to define a product in $C^{k+l}(X;R)$, simply by the formula

$$(\varphi \cup \psi)(\sigma) = \varphi(\sigma|[v_0, \ldots, v_k])\psi(\sigma|[v_k, \ldots, v_{k+l}])$$

where we use that our cochains have values in a ring, so we can multiply.
Our first goal is to understand how this interacts with the coboundary.

Lemma 1.15.

$$\delta(\varphi \cup \psi) = (\delta\varphi) \cup \psi + (-1)^k \varphi \cup (\delta\psi)$$

for the relations as above.

(by the way, if this formula looks familiar from differential forms, that is not a surprise...)
The proof is a little computation (see Hatcher, Lemma 3.6)

Corollary 1.16. *If φ, ψ are cocycles, then so is $\varphi \cup \psi$. If either is a coboundary, then so is the cup product.*

As such, we obtain a cup product on cohomology. How can we compute such a thing (and is it useful)? To answer this, we take a little detour, to discuss a different kind of cohomology theory, namely simplicial (co)homology. At this point, you may ask why we have to do this – couldn't we compute things with singular or simplicial cohomology? The problem with simplicial is that the chain complex is too big, and any computations are going to be very involved. The problem with cellular is a little bit more subtle: we have defined the cup product on the chain level for singular cohomology – and the *cellular chain complex* is not immediately related to the singular chain complex. Thus, it is not clear how to describe the cup product there. Simplicial cohomology will solve this, and is nice to know anyway.

1.6. **Simplicial (Co)homology.** First, we need to briefly discuss the kind of spaces we work with. These are less general than CW complexes, but more rigid.

Definition 1.17. A Δ*–complex structure on* X consists of maps $\sigma_\alpha : \Delta^n \to X, \alpha \in A_n, n \in \mathbb{N}$ so that:

(1) For all n, α, the restrictions $\sigma_\alpha|_{\mathrm{int}\Delta^n}$ are injective.
(2) Each point $x \in X$ is contained in the image of exactly one $\sigma_\alpha|_{\mathrm{int}\Delta^n}$.
(3) Each restriction of a $\sigma_\alpha, \alpha \in A_n$ to a face is equal to some $\sigma_\beta, \beta \in A_{n-1}$ (under the canonical identification of faces with lower-dimensional simplices).
(4) A set $U \subset X$ is open iff $\sigma_\alpha^{-1}(U)$ is open for all α.

A *subcomplex* is a subset $A \subset X$ which is a Δ–complex with some subset of the σ_α.

You can prove that Δ–complex structures give rise to CW structures, but we won't actually need this.
Next, we define a chain complex. To this end, we let

$$C_n^{\mathrm{simp}} = \bigoplus_{A_n} \mathbb{Z},$$

or more generally

$$C_n^{\mathrm{simp}}(G) = \bigoplus_{A_n} G,$$

and we think of them as formal sums of the simplices σ_α. The boundary operator ∂ which we defined for singular homology makes sense here, since by iii) the boundary terms are again basis vectors. The proofs from last semester show that it is indeed a chain complex.
We can define $H_n^\Delta(X)$ the *simplicial homology* as the homology of this chain complex.
We can define $H_n^\Delta(X, A)$ (for a subcomplex A) in the obvious way, by taking the quotient by the cells in A. As always, there is a long exact sequence of homology groups.
Finally, there is a well-defined map

$$C_n^{\mathrm{simp}}(X) \to C_n(X)$$

which sends a simplex σ_α to the singular simplex it defines in X. Since it commutes with the boundary operators, it induces maps in homology and relative homology.

Theorem 1.18. *For all n, and all pairs (X, A) of a Δ–complex and a subcomplex, the obvious homomorphism*

$$H_n^{\Delta}(X) \to H_n(X)$$

is an isomorphism.The same is true for relative homology groups relative to A.

Proof. We only prove the first bit, and only in the finite-dimensional case. This will suffice for our applications, and the extensions are very similar to arguments we have seen in the CW setting. More details are in the course blackboards.
To begin, we consider the two long exact sequences in homology for two consecutive skeleta. As the boundary map is compatible with the map we have between the chain complexes, and everything is natural, we have the commutative "ladder" as in Hatcher, bottom of page 128.
First, we look at the relative terms. The chain complex $C^{\mathrm{simp}}(X^k, X^{k-1})$ has nonzero terms only in degree k (above, there are no simplices in X^k for dimension reasons, below they are cancelled by X^{k-1}). Thus, a basis for $H^{\Delta}(X^k, X^{k-1})$ is given by the k–cells. These are sent by our comparison map to the (singular, relative) cycles in $H(X^k, X^{k-1}) = H(X^k/X^{k-1})$. As we have seen last semester, this therefore gives an isomorpphism.
So, now we can do induction, on k and n. Then, in the ladder, all terms except for the middle one are isos, hence we are done by the 5-lemma. □

Observe that this is better than the situation we had for cellular homology: here, there is a really nice map comparing the *complexes*, which induces isomorphisms on homology. In particular, by dualising the map, we immediately get a map

$$C^{\bullet}(X) \to C^{\bullet}_{\mathrm{simp}}$$

Corollary 1.19. *The obvious map induces an isomorphism*

$$H^n(X) \to H^n_{\Delta}(X).$$

Proof. This is a consequence of *naturality* in the universal coefficient theorem. This means that the diagram on page 196 of Hatcher commutes (which is essentially clear from construction, see discussion there or the course blackboards). □

Now, we can exploit this. Namely, we can define a cup product on H^n_{Δ} using $C^n_{\mathrm{simp}}(X)$ with the same formula as above, and get

Corollary 1.20. *The obvious isomorphism*

$$H^*(X) \to H^*_{\Delta}(X).$$

respects cup product.

How can this help in examples? Let us study some examples (see course blackboards).

Now that we know that this is useful (and therefore worth understanding properly) let us show some more basic properties. First, let's define relative cup products:

Lemma 1.21 (Relative Cup Products)**.** *The formula above defined cup products*

$$H^k(X;R) \times H^l(X,B;R) \to H^{k+l}(X,B;R)$$
$$H^k(X,A;R) \times H^l(X;R) \to H^{k+l}(X,A;R)$$
$$H^k(X,A;R) \times H^l(X,A;R) \to H^{k+l}(X,A;R)$$

If $A, B \subset X$ are open (or subcomplexes of a CW complex), then there is also a relative cup product

$$H^k(X,A;R) \times H^l(X,B;R) \to H^{k+l}(X,A \cup B;R)$$

Lemma 1.22 (Naturality)**.** *For a map $f : X \to Y$ we have*

$$f^*(\alpha \cup \beta) = f^*\alpha \cup f^*\beta$$

Proof. We show this on chain level, where it is fairly obvious (Prop 3.10 Hatcher) □

More importantly, we need to figure out how noncommutative the cup product is. Namely, we have

Theorem 1.23. *Suppose R is commutative. For any $\alpha \in H^k(X,A;R), \beta \in H^l(X,A;R)$ then we have:*

$$\alpha \cup \beta = (-1)^{kl}\beta \cup \alpha$$

This is Theorem 3.11 in Hatcher.

1.7. **Cup Products and Products.** Next, we will connect the cup product to products of spaces. Doing so will give us a nice way to compute cohomology of products.

The first step is to observe that there is a bilinear map

$$H^*(X;R) \times H^*(Y;R) \to H^*(X \times Y;R)$$

called the *cross product*, which is defined by

$$a \times b = p_1(a)^* \cup p_2^*(b)$$

This induces a homomorphism

$$H^*(X;R) \otimes H^*(Y;R) \to H^*(X \times Y;R)$$

(which is also called cross product).

The next big goal will be Theorem 3.15 from Hatcher – a product formula for cohomology. We use the proof as a "excuse" to discuss some more abstract machinery, which is very typical for algebraic topology. This uses the notions of *categories*, *functors* and *natural transformations*. You've seen categories in the problem set. The key thing to observe is that all of our invariants actually fit this framework:

(1) Fundamental group: topological spaces to groups
(2) Higher homotopy groups: topological spaces to Abelian groups
(3) Homology: topological spaces to graded Abelian groups

Cohomology also fits, we just have to be careful about the direction of arrows: cohomology is a contravariant functor. A *natural transformation* is a map between functors. A really good example is the connecting map in the long exact sequence: Relative homology is a functor on pairs of spaces, and so is (absolute) homology of the subspace. The connecting map is then natural.
In fact, we can now start to make some more sense of the different cohomologies we have seen. Let's restrict to CW complexes. Then we can define a *cohomology theory* as a collection of functors and connecting maps with some properties: homotopy invariance, exactness, excision and additivity.
In this setting, singular and cellular cohomology are now two cohomology theories. To categorically describe in which sense they are "the same", we can use the notion of *natural transformation.*

With this in hand, our strategy for the Künneth formula will be to interpret the left and right hand side as cohomology theories of X (fixing Y), interpret the cross product as a natural transformation between them, and then show a general result that allows to check that such a map is an isomorphism. Besides showing the result, this approach shows that for (co)homology what is really crucial are their formal properties, as opposed to the concrete construction.

2. Poincaré duality

We now come to the last topic of this course, and one of the gems of algebraic topology – Poincare duality.

- So far, things we have done did not depend on *local* topological properties. In fact, most everything was independent under homotopy equivalence. For what comes now, we need spaces with a certain local structure.
- We need the notion of *topological manifolds*. It is very likely that you have seen smooth manifolds before. Our definition is that M is a manifold if it is locally Euclidean, second countable and Hausdorff.
- The *dimension* of a topological manifold is well-defined, by invariance of dimension (equivalently, local homology)
- A manifold is *closed* if it is compact and has no boundary. We'll learn about boundary later, but this notion is very common.
- Here are some examples of manifolds: open subsets of Euclidean space, spheres (in fact, preimages of regular values), projective spaces (in fact, quotients by free group actions).
- As a preview: Poincaré duality will state that for a closed manifold we have $H_i(M; \mathbb{Z}/2) = H^{n-i}(M; \mathbb{Z}/2)$. If the manifold is *orientable* (which we will define below), then this also holds integrally.

- As a warm-up and to see why one might expect a theorem like this, we consider the case of cell-structure. Given a (nice) cell structure (e.g. PL) on a surface or torus we can build a *dual* cell structure. We won't do this formally, but in examples (since we don't attack the general case like this). If we do this, then there is a natural assignment between i–cells of the structure and $(n-i)$–cells of the dual structure. If we work with $\mathbb{Z}/2$–coefficients, we don't have to worry about orientations, and we can use this to identify the chain complexes. The key thing to observe is that under this identification, the boundary operator becomes the coboundary operator.

Orientations, topologically. To define orientations on manifolds, we first observe that for a manifold we have

$$H_n(M|p) := H_n(M, M-p, \mathbb{Z}) = \mathbb{Z}$$

for any point $p \in M$. But, the isomorphism is not unique. An orientation will be a choice of identification, or alternatively, a generator μ_p for each such *local homology group* $H_n(M|p)$. Clearly we have to somehow ensure that the choices of generators are "locally consistent". One way to to that require that for any point $p \in M$ there is a open ball B containing p so taht all $\mu_q, q \in B$ are the image of some $\mu_B \in H_n(M, M-B, \mathbb{Z})$.
This agrees with the usual notion of orientation for $\mathbb{R}^n$ as we do this in linear algebra. We say that M is *orientable* if an orientation exists.

Lemma 2.1. *Any manifold M has a two-sheeted covering space called the* orientation cover*. If M is connected, then it is orientable if and only if the orientation cover is disconnected.*

Proof. Hatcher, around p.235 □

We can define R–orientations in the same way, by choosing compatible generators of local homology with R–coefficients.
There is a slightly different perspective which will be useful below via sections in a covering space (Hatcher 235) Namely, generalising the orientation cover, we can define

$$M_{\mathbb{Z}}, M_R$$

whose points are elements $\alpha_x \in H_n(M, M-x; R)$, and topology as in the case above.

Lemma 2.2. *$M_{\mathbb{Z}}$ has one component homeomorphic to M, and for each $i > 0$ a component homoeomorphic to the orientation cover.*

Now, an orientation is a *section* of this cover, which is a generator at each point.
To understand M_R, observe that we have

$$H_n(M, M-x; R) = H_n(M, M-x) \otimes R,$$

by universal coefficients for homology. Thus, for each $r \in R$ we find a subcovering

$$M_r \subset M_R$$

where the element $\alpha_x = \pm\mu_x \otimes r$ for μ_x any generator of $H_n(M, M - x; \mathbb{Z})$. If $r = -r$ in R, then this M_r is just a copy of M. Otherwise, M_r is a copy of the orientation cover of M. We then have that M is the union of the M_r (which are disjoint, except that $M_{-r} = M_r$. From this, we see that

Lemma 2.3. *If M is orientable, then it is R–orientable for each R. A nonorientable manifold is R–orientable if and only if there is a unit $r \in R$ with $-r = r$.*

In particular, any manifold is $\mathbb{Z}/2$–orientable. We'll focus on $\mathbb{Z}, \mathbb{Z}/2$ for orientablilty (like everyone else).

Our next goal will be to see how to assemble the local orientations of an orientable manifold into a global object, called the *fundamental class* (or *orientation class*) This will be possible due to a very important theorem: Thm 3.26 in Hatcher.

In the homework, you will see how to build the orientation class in case the manifold is also a simplicial complex (as the "sum of all simplices")

One important consequence is also that for noncompact manifolds of dimension n, the largest theoretically possible homology group $H_n(M)$ is in fact always zero.

3. Higher Homotopy Groups

Now that we know that $H_1(X)$ has a more explicit description in terms of homotopy classes of loops, we will explore if a similar description is possible for $H_n, n > 1$ (Hint: in general, no).

We begin with the basic definitions. Higher homotopy groups generalise the fundamental group, where instead of an interval we now map a product of intervals. Namely, consider a topological space X and choose a basepoint p. Now, for some $n \geq 2$, we let

$$\pi_n(X, p)$$

be the set of all maps

$$\gamma : ([0,1]^n, \partial[0,1]^n) \to (X, p)$$

up to homotopies restricting to the identity on $\partial[0,1]^n$.

We define a group operation on $\pi_n(X, p)$ by setting

$$(\gamma + \delta)(x_1, \ldots, x_n) = \begin{cases} \gamma(2x_1, x_2, \ldots, x_n) & \text{if } x_1 \leq 1/2 \\ \delta(2x_1 - 1, x_2, \ldots, x_n) & \text{if } x_1 \geq 1/2 \end{cases}.$$

Just as for the fundamental group, one checks that this is well-defined, has an identity element (the constant map), and inverses are given by reversing the first coordinate.

Here is the first real difference between fundamental groups, and higher homotopy groups:

Lemma 3.1. *For any topological space X, the higher homotopy groups $\pi_n(X,p)$ are Abelian.*

Proof. This is easiest proved "by picture". See Hatcher, page 340, the lecture blackboards, or (ideally) watch me do it in the lecture video 1-3. $\square$

There is a second, useful perspective on $\pi_1(X,p)$: namely, if f is a map representing $[f] \in \pi_1(X,p)$, then f maps the boundary of the cube to a point. Thus, f factors through the map

$$[0,1]^n \to [0,1]^n/\partial[0,1]^n = S^n \to X$$

collapsing the boundary of the cube. As the result is a (n–dimensional) sphere, we have that $\pi_n(X,p)$ is the set of homotopy classes of maps $S^n \to X$ mapping a basepoint $s_0 \in S^n$ to p, up to homotopy relative to s_0. You'll explore this in the problem set.

Our next goal will be to understand how $\pi_n(X,p)$ does (not really) depend on the basepoint of a path-connected space.

Lemma 3.2. *Let $\alpha : [0,1] \to X$ be a path joining $p = \alpha(0)$ to $q = \alpha(1)$. Then there is a map*

$$c_\alpha : \pi_n(X,q) \to \pi_n(X,p).$$

These maps satisfy:

(1) $c_{\mathrm{const}}(f) = f$ for the constant path const,
(2) $c_{\alpha*\beta}(f) = c_\alpha(c_\beta(f))$ if β is another path with $\beta(0) = \alpha(1)$.
(3) $c_\alpha(f+g) = c_\alpha(f) + c_\alpha(g)$.

Proof. The definition of c_α is again best explained by a sketch (Hatcher p. 341, course blackboards or video for 1-4). The map $c_\alpha(f)$ is equal to f on a subcube of half the size, and equal to α on radial segments joining the subcube to the boundary.
With this, property (1) is obvious: simply homotope to make the subcube larger until it becomes all the cube.
Property (2) is similar: $c_\alpha(c_\beta(f))$ has f on a subcube quarter the size, and $\alpha * \beta$ on radial segments (up to reparametrisation). Homotoping to make the sizes correct is straightforwards.
Property (3) is the only one requiring a more complicated homotopy. See Hatcher p. 341 or the lecture notes/video 1-4 for details. $\square$

Corollary 3.3. *For any points p,q in the same path-component of X we have $\pi_n(X,p) \cong \pi_n(X,q)$.*

We finish by noting another conequence of the lemma. Namely, if γ is a loop based at p, then $c_\gamma : \pi_n(X,p) \to \pi_n(X,p)$ is an automorphism of $\pi_n(X)$. In fact, this defines an action of $\pi_1(X,p)$ on $\pi_n(X,p)$ (in other words, the higher homotopy groups are modules over the fundamental group). We don't need this perspective in this course though.

We also observe the following, fairly obvious but useful lemma:

Lemma 3.4. *The assignment $(X, p) \to \pi_n(X, p)$ from pointed topological spaces to (Abelian) groups is a functor for any $n \geq 2$.*

Proof. This is shown exactly like for the fundamental group. □

Here is another big difference between higher homotopy groups and fundamental groups:

Lemma 3.5. *If $p : (Y, y) \to (X, x)$ is a covering space and $n \geq 2$, then the induced map*

$$p_* : \pi_n(Y, y) \to \pi_n(X, x)$$

is an isomorphism.

Proof. First, surjectivity: interpret $[f] \in \pi_n(X, x)$ as a map

$$f : (S^n, s) \to (X, x).$$

Recall that $\pi_1(S^n, s) = 1$, and therefore f lifts to a map $g : X^n \to Y, g(s) = y$. By definition, $p_*([g]) = [f]$.
Next, injectivity: if $p \circ f, p \circ g$ are homotopic, simply lift the homotopy to conclude that f, g are homotopic. □

In particular:

Corollary 3.6. *If X has contractible universal cover $\widetilde{X}$, then*

$$\pi_n(X) = 0, \quad \forall n \geq 2.$$

Proof. Since $\widetilde{X}$ is contractible, it has trivial π_i for all i. By the previous lemma, the corollary follows. □

Although this seems somewhat special, spaces X as in the corollary are very common and so important that they have a name:

Definition 3.7. X is *aspherical* if it has a contractible universal cover.

(Some people dislike that this makes S^1 aspherical, but the notation is very standard). Here are some examples of aspherical spaces: tori, higher genus surfaces, manifolds admitting a complete nonpositive curvature metric.

The next goal will be a long exact pair sequence for homotopy groups. This requires us first to define *relative* homotopy groups.
To this end, let $n \geq 2$ be given, and define

$$I^n = [0, 1]^n,$$

and make the convention that $I^{n-1} \subset I^n$ as the face with $x_n = 0$. Denote by

$$J^n = \overline{I^n - I^{n-1}} = \overline{\{(x_1, \ldots, x_n) \in [0, 1]^n, x_n \neq 0\}}$$

A *map of triples* $f : (I^n, \partial I^n, J^n) \to (X, A, p)$ is then a continuous map $f : I^n \to X$ so that $f(\partial I^n) \subset A, f(J^n) = \{p\}$. Now define

$$\pi_n(X, A, p)$$

to be the set of such maps of triples $f : (I^n, \partial I^n, J^n) \to (X, A, p)$ up to homotopy of such maps.
With the same formula as for $\pi_n(X, p)$ we can equip $\pi_n(X, A, p)$ with a group structure. If $n \geq 3$, this is always Abelian (why is $n = 2$ is here different?).

Lemma 3.8. *$\pi_n(X, A, p)$ can be identified with the set of maps*

$$(D^n, S^{n-1}, s) \to (X, A, p)$$

up to homotopy of such maps. Here, $s \in S^{n-1} = \partial D^n$ is a basepoint.

Proof. This follows since collapsing J^n in I^n gives a map of triples

$$(I^n, \partial I^n, J^n) \to (D^n, \partial D^n, s) = (D^n, S^{n-1}, s).$$

□

This perspective is useful to see when elements are trivial in relative homotopy groups:

Lemma 3.9 (Compression Criterion)**.** *Suppose*

$$f : (D^n, S^{n-1}, s) \to (X, A, p)$$

is a continuous map. Then $[f] = 0 \in \pi_n(X, A, p)$ if and only if it is homotopic relative to S^{n-1} to a map with image contained in A.

Proof. First, suppose that f is homotopic to a map g as in the lemma. Then $[f] = [g] \in \pi_1(X, A, p)$ by definition of the relative homotopy group. Now, choose a homotopy H_t from the identity $D^n \to D^n$ to the constant map with value s (fixing s). Then,

$$g \circ H_t$$

is a homotopy (which has image in A) between g and the constant map with value s. As the homotopy has image in A and maps s to p, it is a homotopy of triples $(D^n, S^{n-1}, s) \to (X, A, p)$. Thus, $[g] = 0$.
Conversely, suppose that $[f] = 0 \in \pi_n(X, A, p)$. This means that there is a homotopy

$$F : [0, 1] \times D^n \to X,$$

between f and the constant map, so that $F_t(S^{n-1}) \subset A$. Now observe that

$$\{1\} \times D^n \cup [0, 1] \times S^{n-1}$$

is a n–disk homotopic to $\{0\} \times D^n$ in $[0, 1] \times D^n$, fixing the boundary. This shows that f is homotopic (relative to S^{n-1}) to a map with image in $F(\{1\} \times D^n \cup [0, 1] \times S^{n-1}) \subset A$. □

The following is proved like all the previous, similar results:

Lemma 3.10. *Higher homotopy groups are a functor between triples (X, A, p) of spaces, subspaces, and points, and groups.*

Theorem 3.11. *For any triple (X, A, p) there is a long exact sequence*

$$\cdots \to \pi_n(A, p) \to \pi_n(X, p) \to \pi_n(X, A, p) \to \pi_{n-1}(A) \to \cdots \to \pi_1(A) \to \pi_1(X)$$

where

(1) $i : \pi_n(A, p) \to \pi_n(X, p)$ is induced by the inclusion $A \subset X$,
(2) $j : \pi_n(X, p) = \pi_n(X, p, p) \to \pi_n(X, A, p)$ is induced by the inclusion $(X, p, p) \to (X, A, p)$.
(3) $\partial : \pi_n(X, A, p) \to \pi_{n-1}(A)$ is induced by restricting a map $f : (D^n, S^{n-1}, s) \to (X, A, p)$ to S^{n-1}.

Bemerkung 3.12. *One could extend the sequence a bit further, defining relative fundamental groups but we don't need this. Also, one can show (very similarly) a triple sequence.*

Proof. We check exactness at the various places:

(1) $\mathrm{im}(\partial) = \ker(i)$: Suppose we have $[f] \in \pi_n(X, A, p)$, represented by $f : (D^n, S^{n-1}, s_0) \to (X, A, p)$, and denote by $g = f|_{S^{n-1}}$ the restriction (i.e. $\partial[f] = [g]$). Choose a deformation retraction $F : [0, 1] \times D^n \to D^n$ of the ball to s. Then $F(t, s) = f(F(t, s))$ is a homotopy between g and the constant map relative to s. Hence $i[g] = 0$ in $\pi_{n-1}(X)$.

Conversely, suppose that $i[g] = 0$, which means that there is

$$H : [0, 1] \times S^{n-1} \to X$$

a homotopy of g to the constant map with value p, relative to s. Put

$$f(x) = \begin{cases} H(1 - \|x\|, x/\|x\|) & x \neq 0 \\ p & x = 0. \end{cases}$$

This is a map $f : D^n \to X$ which restricts to g on S^{n-1}, showing that $[g] = \partial[f]$.

(2) $\mathrm{im}(i) = \ker(j)$:Suppose that $f : S^n \to X$ is given, representing an element of $\pi_n(X, p)$. Then $j[f]$ is represented by the map

$$g : (D^n, S^{n-1}, p) \to (X, A, p)$$

which is the composition

$$D^n \to D^n/S^{n-1} = S^n \xrightarrow{f} X$$

Thus, if $[f]$ is in the image of i, this map has image in A. By the compression criterion, this is then zero, showing $ji = 0$.

On the other hand, suppose that $j[f] = [g] = 0 \in \pi_1(X, A, p)$. This means, by the compression criterion, that g is homotopic, rel S^{n-1}, to a map $g' : (D^n, S^{n-1}, p) \to (X, A, p)$ with image in A. Since this homotopy is relative to S^{n-1}, it induces a homotopy of f to a map f' with image in A. That then shows that $[f] = [f'] \in \mathrm{im}(i)$.

(3) $\operatorname{im}(j) = \ker(\partial)$: If $f : I^n \to X$ represents a class in $\pi_n(X)$, interpreting it as an element $f \in \pi_n(X, p, p)$ shows directly that the restriction to $I^{n-1} \subset \partial I^n$ is constant.

Conversely, suppose that $f : I^n \to X$ has restriction to I^{n-1} homotopic to the constant map. Then we can "stack" this homotopy onto f to homotope f to a map $I^n \to X$ which is constant equal to p on the boundary (see the lecture blackboards 3-1 for sketches).

□

3.1. **Cellular Approximation.** In order to compute any higher homotopy groups (of spaces that aren't aspherical), we now want to discuss a theorem, which is very interesting in its own right.
This is the so-called *cellular approximation theorem*:

Theorem 3.13. *Suppose that X, Y are CW-complexes, and suppose that $f : X \to Y$ is a continuous map. Then f is homotopic to a map $g : X \to Y$ which is* cellular*, i.e. $g(X^n) \subset Y^n$.*
If f was already cellular on a subcomplex $A \subset X$, then we may choose the homotopy to be trivial on A (in particular, $g|_A = f|_A$.

Before proving this, let's observe:

Corollary 3.14. $\pi_n(S^k) = 0$ *if $k > n$.*

What is the intuition behind the theorem?

(1) For $k < n$, a piece of a k–dimensional disk inside a n–dimensional disk is homotopic (relative to boundary) to a nonsurjective map.
(2) If a map $f : X^n \to Y^{n+1}$ misses a point in any $(n+1)$–cell, then it can be homotoped to have image in Y^n.
(3) CW complexes are nice enough to define homotopies on the cells and then piece them together.

By the way, for $n = 1$, we have proved the first part when we showed $\pi_1(S^n) = 0$.
To prove the first bit, we will "make maps linear" as the statement is clear for such maps. To this, we need some notation:

Definition 3.15. (1) A *convex polyhedron in* $\mathbb{R}^n$ is a compact subspace obtained as the intersection of finitely many halfspaces.
(2) A *polyhedron* is a connected union of finitely many convex polyhedra.
(3) A map $f : X \to \mathbb{R}^k$ from a polyhedron is called *PL (piecewise linear)* if there is a decomposition of X into convex polyhedra $X = P_1 \cup \cdots \cup P_l$, so that the restriction $f|P_i$ is (affinely) linear for each i.

Sometimes, any space obtained from gluing polyhedra is called a polyhedron (which is terrible nomenclature...)

Proposition 3.16. *Suppose*

$$Z = W \cup e^k$$

is a space obtained by attaching a k–cell to W. Suppose that $f : I^n \to Z$ is a continuous map.
Then there are

(1) *a polyhedron $K \subset I^n$,*
(2) *a nonempty open set $U \subset e^k$,*
(3) *and a map $g : I^n \to Z$*

so that

i) *$g(K) \subset e^k$,*
ii) *$g|K$ is PL, under a suitable identification of e^k with $\mathbb{R}^k$,*
iii) *$g^{-1}(U) \subset K$.*

In other words: we can homotope any map f as in the lemma to be piecewise linear on at least a piece which maps to a set with interior. This will be good enough for us later – but there are much better PL approximation results as well...

Proof. This is Lemma 4.10 from Hatcher. □

With this in hand, we could *cell by cell* make a map $f : X^n \to Z \bigcup I \times e^k, k > n$ miss points in the "new cells", and then homotope them to miss all the new cells.
To make this formal, we'll need to talk a little bit about extending homotopies, which is useful anyway.

3.2. **Interlude: The Homotopy Extension Property.** The property we need is the following:

Definition 3.17. Let $A \subset X$ be a subset of a topological space. We say that the pair (X, A) has the *homotopy extension property* if the following holds: if $f : X \to Y$ is a continous map, and $H_0 : [0,1] \times A \to Y$ is a homotopy of the restriction of f to A, then there is a homotopy $H : [0,1] \times X \to Y$ of f extending it.

It is hopefully clear that such a property is very useful.
Here is a useful criterion to check this property:

Lemma 3.18. *(X, A) has the homotopy extension property if and only if $X \times \{0\} \cup A \times [0,1]$ is a retract of $X \times [0,1]$.*

Proof. Assume that the HEP holds. Then, consider the identity map $\mathrm{id} : X \to X$, and the constant homotopy of the restriction $\mathrm{id}|_A$. This is, as a map, simply the identity:

$$H_0 : X \times \{0\} \cup A \times [0,1] \to X \times \{0\} \cup A \times [0,1]$$

But, the HEP implies that it extends to a homotopy of the identity *with the same image*, i.e. there is an extension

$$H : X \times [0,1] \to X \times \{0\} \cup A \times [0,1],$$

which is the desired retraction.
Conversely, suppose that we have a retraction $R : X \times [0,1] \to X \times \{0\} \cup A \times [0,1]$. Then, by precomposition with R, we can extend any partial homotopy and get the HEP. $\square$

Here is a useful example:

Example 3.19. The pair $(D^n, \partial D^n)$ has the HEP. Namely, there is a retraction

$$r : D^n \times [0,1] \to D^n \times \{0\} \cup \partial D^n \times [0,1].$$

This can easiest be seen geometrically, by considering $D^n \times [0,1] \subset \mathbb{R}^{n+1}$ as a cylinder, and pushing along segments starting in a point "above" the cylinder (see blackboards).

The most important example are CW pairs:

Theorem 3.20. *Suppose that X is a CW complex, and $A \subset X$ is a subcomplex. Then (X, A) has the homotopy extension property.*

Proof. Step 1 is to consider a single cell, which is given by the example above. Step 2 is to define a retraction

$$R^n : X^n \times [0,1] \to X^n \times \{0\} \cup (X^{n-1} \cup A) \times [0,1],$$

by performing r on each n–cell of X which is not in A. Let's be a bit more precise here. We know that (since A is a subcomplex)

$$X^n = (X^{n-1} \cup A^n) \cup_f J \times D^n,$$

for J indexing the cells of X^n not in A, and a suitable attaching map f. and thus

$$X^n \times [0,1] = ((X^{n-1} \cup A^n) \cup_f J \times D^n) \times [0,1].$$

Now, last semester, we showed that quotients commute with product–with–compact–space, and thus we also have

$$X^n \times [0,1] = ((X^{n-1} \cup A^n) \times [0,1] \cup_F J \times D^n \times [0,1]).$$

In other words: $X^n \times [0,1]$ is obtained from $(X^{n-1} \cup A^n) \times [0,1]$ by attaching copies of $D^n \times [0,1]$ (along $\partial D^n \times [0,1]$) to $(X^{n-1} \cup A^n) \times [0,1]$. This is still not quite what we want. Namely, we can break the attaching into two steps: first attach $D^n \times \{0\}$ to form $X^n \times \{0\}$, and then do the rest of the attaching. This realises $X^n \times [0,1]$ as a space obtained from $X^n \times \{0\} \cup A^n \times [0,1]$ by attaching cells $D^n \times [0,1]$ along $D^n \times \{0\} \cup \partial D^n \times [0,1]$.
On each such cell, we have the retraction from Step 1, which fit together to form the desired R^n.

Step 3 is to perform all of the R^n after another (this is only tricky if the CW complex is infinite dimensional). To do that, we define a map R as an "infinite concatenation":

$$R = \cdots \circ R^n \circ \cdots \circ R^1, \quad R : X \times [0,1] \to X \times \{0\} \cup A \times [0,1]$$

which is well defined, as any point $p \in X \times [0,1]$ lies in $X^n \times [0,1]$ for some n, and then $R^{n+k}(p) = R^n(p)$. Why is this map continuous? We know that it is continuous on the "skeleta" $X^n \times [0,1]$ (as the R^i are) – so we would be done if we knew that $X \times [0,1]$ has the weak topology with respect to these. This is true, and we will prove it below in the Interlude-of-the-interlude.

□

3.3. **Interlude of the Interlude: Products and CW complexes.** We will show the following:

Lemma 3.21. *Suppose that (X, A) is a CW complex, and suppose that*

$$F : X \times [0,1] \to Y$$

is a map. Then F is continuous if and only if $F : X^n \times [0,1] \to Y$ is continuous.

Bemerkung 3.22. *The actual reason for why this is true is that $X \times [0,1]$ is in fact again a CW complex, with open cells $e \times (0,1)$ for e open cells of X. But, remembering why we wanted to prove this lemma in the first place (checking that gluing and product behave nicely), we see that this may not be so easy to see directly...*

There is the following theorem, which is really good to know, but which we will not prove, but which you may use from now on.

Theorem 3.23. *Suppose that X is a CW complex, and suppose that Y is a locally compact CW complex. Then $X \times Y$ has the structure of a CW complex, whose open cells are of the form $e_X \times e_Y$ for open cells e_X, e_Y of X, Y.*

If you are interested in the details, see e.g. the Appendix of Hatcher. The requirement on Y is really necessary. In general, the product of two CW complexes need not be a CW complex again.
Now, let's show Lemma 3.21. It will actually be an immediate corollary of the following:

Lemma 3.24. *Let (X, A) be a CW complex. Suppose K is Hausdorff and locally compact. Then, a set $O \subset X \times K$ is open (with respect to the quotient topology) if and only if $O \cap (X_n \times K)$ is open.*

Proof. Let

$$\overline{X} = X^{-1} \coprod X^0 \coprod \cdots$$

be the disjoint union of the skeleta of X. The fact that X has the weak topology exactly means that the (obvious) map

$$p : \overline{X} \to X,$$

(which is induced by the inclusions) is a quotient map, i.e.: sets $U \subset X$ are open if and only if

$$p^{-1}(U) = (X^{-1} \cap U) \coprod (X^0 \cap U) \coprod \cdots$$

is open.
Now, recalling a lemma from last semester, since K is locally compact, the induced map

$$p' : \overline{X} \times K \to X \times K$$

is also a quotient map. This means that $O \subset X \times K$ is open if and only if

$$(p')^{-1}(O) = (X^{-1} \times K \cap O) \coprod (X^0 \times K \cap O) \coprod \cdots$$

is open. That shows the lemma. □

3.4. **Finally Proving Cellular Approximation.** We now have all the tools needed to prove cellular approximation.

Proof. Proof of cellular approximation The first part of the proof inductively constructs homotopies to make f cellular on the skeleta of X. The start of the induction works as follows: consider the 0–skeleton X^0 of X. This is a disjoint union of points, and a closed subset of X. Consider $f(X^0)$. This is a collection of points in Y. Choose for each a path joining it to (any) point in Y^0. This can be seen as a homotopy of $f|_{X^0}$. By the HEP of (X, X_0), we can extend this to a homotopy of f which then has $f(X^0) \subset Y^0$.
Now suppose that we have a map

$$f : X \to Y$$

which is already cellular on X^n for any $n \geq 0$. Take any $(n+1)$–cell e. The image $f(e)$ has compact closure, and therefore only intersects finitely many cells of Y (this was shown last semester). If f is not yet cellular on e, then there is a highest–dimensional cell e' of dimenstion $k > n + 1$ of Y which intersects $f(e)$. We can thus write Y as

$$Y^k = Z \cup e'$$

where Z is the space obtained by attaching all k–cells except e' to Y^{k-1}. By our PL-approximation lemma, there map

$$f : e \to Z \cup e'$$

can be homotoped, relative to ∂e, to a map f' so that $f' : U \to V$ is PL for V an open set in e' and $U = f^{-1}(V)$. Now, since e' is a k–cell and e is a $(n+1)$–cell, the image of $f'|_U$ is contained in a finite union of lower-dimensional subspace, and thus in particular not surjective. Hence, f' misses a point in e', and is thus homotopic, rel ∂e to a map f'' which misses the interior of e' completely. In fact, we can do this for all $(n+1)$–cells

simultaneously, and the homotopy extension property of (X, X^{n+1}) implies that we can extend the homotopy to all of X.
This yields homotopies H^k, so that performing

$$H^0 * H^1 * \cdots * H^k$$

makes f cellular on the k–skeleton. We define the homotopy

$$H : [0,1] \times X \to Y$$

by performing H^n on $[1 - 1/2^n, 1 - 1/2^{n+1}]$. This is actually well-defined, as any point in x is in some skeleton X^n, and thus eventually stationary in any subsequent H^{n+i}. Continuity of H then follows by the product CW structure, as H restricted to a skeleton $[0,1] \times X^n$ is a finite concatenation of homotopies. □

3.5. **Whitehead's theorem.** Next, we come to a crucial application of higher homotopy groups (you have to bear with me a little bit longer before we actually compute interesting examples...).

Theorem 3.25. *Suppose that X, Y are connected CW complexes, and let $f : X \to Y$ be a continuous map. If all induced maps*

$$f_* : \pi_n(X) \to \pi_n(Y)$$

are isomorphisms, then f is a a homotopy equivalence. If f is additionally the inclusion of a subcomplex, then it is a deformation retract.

Corollary 3.26. *If X is a connected CW complex, and $\pi_n(X) = 0$ for all n, then X is contractible.*

The proof of the theorem uses the following lemma, which uses the same kind of argument as in the proof of cellular appoximation.

Lemma 3.27. *Suppose (X, A) is a CW pair, and (Y, B) is a pair of topological spaces with $B \neq \emptyset$. Assume that*

$$\pi_n(Y, B, y_0) = 0,$$

for all n so that $X - A$ has cells of dimension n, and all $y_0 \in B$. Then every map

$$f : (X, A) \to (Y, B)$$

is homotopic rel A to a map with image in B.

Proof. As a first step, we homotope f on X^n, assuming that it is already correct on X^{n-1}. As in the proof of cellular approximation, for a cell D^n of $X^n - A$, consider the restricted map $f : D^n \to Y$. The condition on relative homotopy groups exactly allows us to homotope it, rel boundary, to have image in B (observe here, that the basepoint can be chosen arbitrarily, by the assumption).
Then, as a second step, we concatenate the infinitely many homotopies to find the desired element. □

Proof of Whitehead's theorem. We first consider the special cas of the inclusion of a subcomplex. Here, by considering the long exact sequence of the pair, the fact the the inclusion induces isomorphisms on all degrees yields that $\pi_n(Y, X) = 0$ for all n. Then, applying the lemma to the identity map yields the desired homotopy.
For the general case, first homotope f to be cellular. Then, the mapping cylinder

$$M_f = X \times [0,1] \coprod Y/(x,1) \sim f(x)$$

can be equipped with a CW structure, so that $X \times \{0\}$ is a subcomplex.
Recall that M_f deformation retracts to Y, and the map f is the composition of the inclusion of $X \to X \times \{0\}$, followed by this retraction to Y. Thus, the assumption on f gives that the inclusion of X into M_f induces isos on all π_n. The special case now yields that the inclusion of X into M_f is also a homotopy equivalence, and thus f is one as well. □

3.6. **Homotopy groups of spheres II.** Let's try to compute the group $\pi_n(S^n)$. Write

$$S^{n+1} = C_+S^n \cup C_-S^n$$

as the union of the top and bottom cones over the equatorial sphere. Consider the long exact sequence of the pair (C_+S^n, S^n):

$$\cdots \to \pi_{i+1}(C_+S^n) \to \pi_{i+1}(C_+S^n, S^n) \to \pi_i(S^n) \to \pi_i(C_+S^n) \to \cdots$$

Since the cone C_+S^n is contractible, the middle map is an isomorphism. Similarly, consider the long exact sequence for (S^{n+1}, C_-S^n):

$$\cdots \to \pi_{i+1}(C_-S^n) \to \pi_{i+1}(S^{n+1}) \to \pi_{i+1}(S^{n+1}, C_-S^n) \to \pi_i(C_-S^n) \to \cdots$$

which again has an isomorphism in the middle. Together with the inclusion $(C_+S^n, S^n) \to (S^{n+1}, C_-S^n)$ this gives a map

$$\pi_i(S^n) \simeq \pi_{i+1}(C_+S^n, S^n) \to \pi_{i+1}(S^{n+1}, C_-S^n) \simeq \pi_{i+1}(S^{n+1}).$$

This middle map looks like "homology excision". In fact, this map indeed does behave like this:

Theorem 3.28. *Suppose that X is a CW complex, which is the union of two subcomplexes*

$$X = A \cup B$$

so that $C = A \cap B$ is connected. Assume that A is obtained from C by attaching a $m+1$–cells e^{m+1}, and assume that B is obtained from C by attaching a $n+1$ cell e^{n+1}.
Then the map

$$\pi_i(A, C) \to \pi_i(X, B),$$

induced by inclusion, is an isomorphism for $i < m+n$ and a surjection for $i = m+n$.

This is not the most general version of this theorem (we will briefly discuss this later). A corollary of the theorem and the computation above is the following:

Corollary 3.29. *We have*
$$\pi_i(S^n) \simeq \pi_{i+1}(S^{n+1})$$
for $i+1 < 2n$.

In fact, we can be a bit more precise. Namely, the identity map, seen as an element of $\pi_n(S^n)$ maps under the isomorphism above to the identity map, seen as an element of $\pi_{n+1}(S^{n+1})$. Thus, inductively, we see that the identity $S^n \to S^n$ is a generator of $\pi_n(S^n)$ (as we already know that this is the case for $\pi_1(S^1)$).
We thus have

Corollary 3.30. *The degree map*
$$\deg : \pi_n(S^n, s_0) \to \mathbb{Z}$$
is an isomorphism.

Proof. First, we need to show that the degree map is a homomorphism (we know from last semester that it is well-defined). This is easiest to see for the definition of multiplication developed on the problem set. Namely, we have, for $f+g$,
$$S^n \to S^n \vee S^n \to S^n,$$
which in homology gives
$$H_n(S^n) \to H_n(S^n) \oplus H_n(S^n) \to H_n(S^n)$$
where the first map is the diagonal inclusion, and the second map is $f_* + g_*$. By the definition of degree, this shows that $\deg(f+g) = \deg(f) + \deg(g)$. Now, we know from above that the identity map is a generator of $\pi_n(S^n, s_0)$ and we know from last semester that it has degree 1. This shows the corollary. $\square$

Now, we get to the proof of the "excision" theorem.

Proof. Our first goal will be to show surjectivity of
$$\pi_i(A, C) \to \pi_i(X, B).$$
So, we are given a map
$$f : (I^i, \partial I^i, J^i) \to (X, B, x_0)$$
and we aim to homotope it to a map with image in (A, C, x_0).
Before actually achieving this, we will first prepare our map by a homotopy (described below), so that afterwards we have points $p \in e^{m+1}$ (which is the cell in A), and $q \in e^{n+1}$ (which is the cell in B), and a map $\varphi : I^{i-1} \to [0,1)$ so that

(1) $f^{-1}(q)$ is below the graph of φ,

(2) $f^{-1}(p)$ is above the graph of φ,
(3) $\varphi = 0$ on ∂I^{i-1}

Why is this useful? First observe that

$$\pi_i(A,C) = \pi_i(X-q, X-q-p), \quad \pi_i(X,B) = \pi_i(X, X-p)$$

(in both cases, since the right hand side space pair deformation retracts to the left hand pair). We can therefore also think of f as defining an element in $\pi_i(X, X-p)$.

We can now interpret the restrictions of f to the region Q_t above the graph of $t\varphi$ as a homotopy f_t of f. This homotopy (by 3) is relative to J^i. Furthermore, by 2), the boundary is mapped into the complement of p for the whole time, hence $[f] = [f_1]$. By 1) the whole cube is mapped into the complement of q by f_1. Thus, under the identification above, f lies in the image of the map

$$\pi_i(X-q, X-q-p) \to \pi_i(X, X-p).$$

This shows surjectivity of the map we care about.

Before explaining the how to modify f, observe that actually the same kind of proof will show injectivity. Two maps $f_0, f_1 : (I^i, \partial I^i, J^i) \to (A, C, x_0)$ being equal in $\pi_i(X,B)$ means that there is a homotopy

$$H : I^i \times [0,1] \to X$$

(with certain properties). Letting H play the role of f above, we can again "separate $H^{-1}(p), H^{-1}(q)$ by a graph" and then certify that f_0, f_1 are the same element in $\pi_i(A,C)$. Doing this replaces the dimension i by $i+1$.

Now, we will show the claim about f given above. This will be done by using the PL approximation lemma again. Apply that lemma to f to find simplices $\Delta^{m+1} \subset e^{m+1}, \Delta^{n+1} \subset e^{n+1}$, so that

$$f^{-1}(\Delta^{m+1}) = \bigcup P_j^{m+1}, \quad f^{-1}(\Delta^{n+1}) = \bigcup P_j^{n+1}$$

are finite unions of polyhedra P_i^j, and $f|P_j^k$ is (the restriction of a) linear map A_j^k on those.

As a warm-up, suppose that none of the linear maps A_j^{m+1} is surjective. Then, neither is f restricted to $P_1^{m+1} \cup \cdots$, and thus (arguing as in the cellular approximation theorem), we can find a point $p \in e^{m+1}$ which is not in the image of f. This clearly has the property we want, independent of what φ we choose.

Thus, we really only care about the case where some of the A_i^j are surjective. Since the non-surjective ones cannot surject on the image, by choosing the simplices Δ correctly, we may therefore assume that all A_i^j are surjective (or, none of them are, in which case we have already found the point p, q).

Let $\pi : I^i \to I^{i-1}$ be the orthogonal projection. Our goal will be to find points p, q so that $\pi(f^{-1}(p)), \pi(f^{-1}(q))$ are disjoint. If we have that, we can

find an open neighbourhood of U of $\pi(f^{-1}(q))$ disjoint from $\pi(f^{-1}(p))$, and build the desired function φ supported in U.
Now, for some $q \in \Delta^{n+1}$, consider $f^{-1}(q)$. This is a finite union of convex polyhedra, contained in the preimages $(A_j^{n+1})^{-1}(q)$. As

$$A_j^{n+1} : \mathbb{R}^i \to \mathbb{R}^{n+1}$$

is a surjection, these convex polyhedra have dimension $\leq i - n - 1$.
We would like to choose p so that $\pi(f^{-1}(p)) \cap \pi(f^{-1}(q)) = \emptyset$, or, equivalently,

$$f^{-1}(p) \cap \pi^{-1}\pi f^{-1}(q) = \emptyset$$

The set $T = \pi^{-1}\pi f^{-1}(q)$ is the set of all vertical segments $\{z\} \times [0,1]$ which intersect $f^{-1}(q)$. By the dimension bound above, T is a union of convex polyhedra of dimension $\leq i - n$. The image $f(T) \cap e^{m+1}$ is, as linear maps do not increase dimension, also a set of dimension $\leq i - n$. If $m + 1 > i - n$ this means that there is a point $p \in e^{m+1}$ so that $f^{-1}(p) \notin T$. This shows what we want.

Observe that for the surjectivity proof, the bound we needed was $i \leq m + n$. For the injectivity proof, as we apply the argument above to a homotopy (and thus the domain has dimension $i + 1$), the bound we need is $i < m + n$. □

With this in hand, let's compute some more examples.

Lemma 3.31. *Let $n > 1$ be given. We have*

$$\pi_n(S^n \vee \cdots \vee S^n) = \mathbb{Z}^k$$

where k is the number of spheres, and generators are given by the obvious inclusions.

Proof. We use a trick. Namely, consider the CW structure on S^n which has one zero-cell and one n–cell. Then, consider the product

$$P = S^n \times \cdots \times S^n$$

of k spheres. This then has a CW structure whose cells are the products of the cells of the individual S^n (this is a finite product of finite CW complexes, so product structures aren't a problem here!) We observe two important things here:

(1) The n–skeleton of P is exactly $S^n \vee \cdots \vee S^n$.
(2) The CW structure has cells only in dimensions which are a multiple of n.

Because of the second point and cellular approximation we have

$$\pi_i(P) = 0, 0 < i < n,$$

$$\pi_i(P, P^n) = 0, 0 < i < 2n$$

With the long exact sequence, we thus get that the inclusion

$$\pi_n(P^1) \to \pi_n(P)$$

is an isomorphism. But,

$$\pi_n(P) = \prod \pi_n(S^n) = \mathbb{Z}^k$$

by homework. The generation statment follows in the same way. $\square$

Corollary 3.32. *Let $n > 1$ be given. Then*

$$\pi_n(\bigvee_{\alpha \in A} S^n) = \mathbb{Z}[A]$$

even for infinite A.

Proof. This follows since any map $f : S^n \to \bigvee_{\alpha\in A} S^n$ or $H : S^n \times [0,1] \to \bigvee_{\alpha\in A} S^n$ has compact image, hence intersects only finitely many cells, and we can use the lemma. $\square$

As a consequence, we also see that homotopy groups can be really big, even for nice compact spaces:

Lemma 3.33. *$\pi_n(S^1 \vee S^n)$ is not finitely generated for $n \geq 2$.*

Proof. We can compute $\pi_n(S^1 \vee S^n)$ from the universal cover of $S^1 \vee S^n$. This is a real line with a S^n attached at each integer point, hence homotopy equivalent to $\bigvee_{\mathbb{Z}} S^n$. Now this follows from the corollary. $\square$

This computation is in pretty stark contrast to the case of homology, where finite CW complexes always had finitely generated groups in only finitely many dimensions.

3.7. **Homotopy Excision, general case.** Recall that we say that a space X is *k–connected* if it is path-connected and any map $f : S^k \to X$ is nullhomotopic (i.e. $\pi_i(X) = 0, i \leq k$).
We similarly call a pair (X, A) *n–connected* if for any $i \leq n$ and every map

$$(D^i, \partial D^i) \to (X; A)$$

is homotopic, relative to ∂D^i, to a map $D^i \to A$.
With this in place, here is the general excision theorem:

Theorem 3.34. *Suppose that X is a CW complex, and A, B are subcomplexes with*

$$X = A \cup B$$

and so that $C = A \cap B$ is connected. If (A, C) is m–connected and (B, C) is n–connected, then the map induced by inclusion

$$\pi_i(A, C) \to \pi_i(X, B)$$

is an isomorphism for $i < m + n$ and a surjection for $i = n + m$.

We won't prove this (see e.g. Hatcher Theorem 4.23) for a proof, but the main idea is the step we have proved. We record the following consequence, which we will need at some point.

Proposition 3.35. *If (X, A) is a CW pair which is r–connected, and A is s–connected, then the quotient map $X \to X/A$ induces*

$$\pi_i(X, A) \to \pi_i(X/A)$$

which is an isomorphism for $i \leq r + s$ and surjective for $i = r + s + 1$.

3.8. **The Hurewicz theorem.** The Hurewicz theorem states that for such a space, all homology below degree k also vanishes, and the first interesting homotopy group agrees with homology:

Theorem 3.36 (Hurewicz)**.** *Suppose that X is a CW complex which is $(n-1)$–connected for some $n \geq 2$. Then*

$$\widetilde{H}_i(X) = 0, \quad i < n,$$

$$H_n(X) \simeq \pi_n(X)$$

Corollary 3.37. *If X is simply connected, the first nonzero homology and homotopy group occur in the same degree, and are isomorphic.*

As a first step to the proof we need the following.

Lemma 3.38. *If X is a k–connected CW complex, then X is homotopy equivalent to a CW complex Y which has a single zero-cell, and no i–cells for $0 < i \leq k$.*

Proof. You have proved this in the homework (Set 3, Problem 2) – the proof, as in the solution, doesn't attach cells in the dimensions in the range here. □

Proof. By the lemma, we can assume that X has $(n-1)$–skeleton a point, which immediately shows $\widetilde{H}_i(X) = 0$. By cellular approximation, and a lemma from last semester, both $H_n(X), \pi_n(X)$ are already computed by the $(n+1)$–skeleton of X. Thus, we may in fact assume that X has the form

$$X = (\bigvee_\alpha S^n\alpha) \cup_f J \times D^{n+1}$$

Let's compute $\pi_n(X)$, by looking at the long exact sequence of the pair $(X, \bigvee_\alpha S^n\alpha)$:

$$\pi_{n+1}(X, \bigvee_\alpha S^n\alpha) \to \pi_n(\bigvee_\alpha S^n\alpha) \to \pi_n(X) \to \pi_n(X, \bigvee_\alpha S^n\alpha) = 0$$

So, we can think of $\pi_n(X)$ as the cokernel of the boundary map $\pi_{n+1}(X, \bigvee_\alpha S^n\alpha) \to \pi_n(\bigvee_\alpha S^n\alpha)$. Now, the pair $(X, \bigvee_\alpha S^n\alpha)$ is n–connected, and $\bigvee_\alpha S^n\alpha$ is $(n-1)$–connected. By the proposition after the excision theorem, the quotient map gives an isomorphism

$$\pi_{n+1}(X, \bigvee_\alpha S^n\alpha) = \pi_{n+1}(X/\bigvee_\alpha S^n\alpha) = \pi_{n+1}(\bigvee_J S^{n+1})$$

Let's be more precise. The way the isomorphisms in this line work, we see that

$$\pi_{n+1}(X, \bigvee_\alpha S^n\alpha) = \bigoplus_J \mathbb{Z}$$

and the generator 1_j on the right is mapped to the characteristic map $C_j : D^n \to \bigvee_\alpha S^n\alpha$ of the gluing. The boundary of this is exactly the gluing map $f_j : \partial D^n \to \bigvee_\alpha S^n\alpha$. Now, we can identify $\pi_n(\bigvee_\alpha S^n\alpha) = \bigoplus_\alpha \mathbb{Z}$ (*via degree after projection of each component*). This means that the map

$$\bigoplus_J \mathbb{Z} = \pi_{n+1}(X, \bigvee_\alpha S^n\alpha) \to \pi_n(\bigvee_\alpha S^n\alpha) = \bigoplus_\alpha \mathbb{Z}$$

whose cokernel is $\pi_n(X)$, is exactly the boundary map of *cellular* homology of X. This shows that $\pi_n(X) = H_n(X)$. □

3.9. **Fiber Bundles.**

Definition 3.39. We say that $p : E \to B$ has the *homotopy lifting property with respect to* X, if for any homotopy

$$g_t : X \to B$$

and a lift

$$\widetilde{g}_0 : X \to E$$

there is a homotopy $\widetilde{g}_t$ extending it and lifting g_t.

An alternative way of thinking about this is a *lift extension property*. We say p has LEP for (Z, A) if any $Z \to B$, and any partial lift $A \to E$, admits an extension to Z. Then HLP is LEP for $(X \times [0,1], X \times \{0\})$.
Here are two fairly obvious examples:

Example 3.40.
(1) Covering spaces have this for all spaces X (we've showed that last semester!) and we used it to great effect.
(2) Products also have this for all X: consider

$$p : B \times F \to B$$

and homotopy $H : X \times I \to B$ of a map $f : X \to B$. Suppose we have an initial lift $\widetilde{f} : X \to B \times F$. Being a lift means that it has the form

$$\widetilde{f}(x) = (f(x), g(x))$$

for some $g : X \to F$. Then simply put

$$\widetilde{H}(x,t) = (H(x,t), g(x)).$$

Nonobvious examples will follow later.
We can (as always) also define a relative version:

Definition 3.41. We say that $p : E \to B$ has the *homotopy lifting property for a pair* (X, A) if it has the lift extension property for $(X \times [0,1], X \times \{0\} \cup A \times [0,1])$. Intuitively, we can lift homotopies of maps, extending an initial lift and a partial homotopy.

Lemma 3.42. *A map p has the HLP for all D^k if and only if it has the HLP for all CW pairs (X, A).*

Proof. First observe that

$$(D^k \times [0,1], D^k \times \{0\}), \quad (D^k \times [0,1], D^k \times \{0\} \cup \partial D^k \times [0,1])$$

are homeomorphic pairs. As a consequence, the HLP for D^k is equivalent to the one for $(D^k, \partial D^k)$ (by interpreting it as lift extension properties). Now, we argue exactly as in the proof of the homotopy extension property: we define the desired lift skeleton-by-skeleton, cell-by-cell, where it reduces to the HLP of $(D^k, \partial D^k)$ by precomposing with characteristic maps of the cells. ☐

Definition 3.43. A *fibration* is a map $p : E \to B$ which has the HLP for all X. A *Serre fibration* is a map p which has the HLP for disks.

The reason we are interested in this notion is that there is a long exact sequence in homotopy groups for (Serre) fibrations.

Theorem 3.44. *Suppose $p : E \to B$ is a Serre fibration. Let $b_0 \in B, x_0 \in F = p^{-1}(b_0)$ be any basepoints.*
The map

$$p_* : \pi_n(E, F, x_0) \to \pi_n(B, b_0)$$

is an isomorphism for all $n \geq 1$.

Proof.
We want to show that p_* is surjective. Suppose we are given

$$f : ([0,1]^n, \partial[0,1]^n) \to (B, b_0),$$

and we think of f as a homotopy of $f|_{[0,1]^{n-1}}$. The constant map (with value x_0) is a lift of $f|_{[0,1]^{n-1}}$. The constant map (with value x_0) is also a lift of the (constant) partial homotopy of the restriction to the boundary of $[0,1]^{n-1}$. Hence, the HLP implies that we can lift f.
Similarly, suppose that we have

$$g_0, g_1 : ([0,1]^n, \partial[0,1]^n, J^{n-1}) \to (E, F, x_0)$$

and there is a homotopy

$$H_t : ([0,1]^n, \partial[0,1]^n) \to (B, b_0)$$

between $f_0 = pg_0, f_1 = p_g 1$.
Define

$$G : [0,1]^n \times \{0\} \cup [0,1]^n \times \{1\} \cup J^n \times [0,1] \to E$$

by gluing g_0, g_1, and the constant map with value x_0. This is a partial lift of H, and by thinking of the second-to-last coordinate as time instead, the HLP shows that it can be extended to a lift. This shows p_* is injective.
☐

Now, suppose that p is a Serre fibration. We consider the long exact sequence for the pair (E, F) to get

$$\cdots \to \pi_n(F, X_0) \to \pi_n(E, x_0) \to \pi_n(E, F, x_0) \to \pi_{n-1}(F, x_0) \to \cdots$$

Using the theorem, we can replace each $\pi_n(E, F, x_0)$ by $\pi_n(B, b_0)$, to get a long exact sequence

$$\cdots \to \pi_n(F, X_0) \to \pi_n(E, x_0) \to \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \cdots$$

What are the maps here?

- $\pi_n(F, X_0) \to \pi_n(E, x_0)$ is induced by inclusion,
- $\pi_n(E, x_0) \to \pi_n(B, b_0)$ is induced by p (as it is induced by $(E, \emptyset) \to (E, F)$ from the original LES, composed with p_*).
- $\pi_n(B, b_0) \to \pi_{n-1}(F, x_0)$ is the interesting one. It is the composition

  $$\pi_n(B, b_0) \to \pi_n(E, F, x_0) \to \pi_{n-1}(F, x_0)$$

  where the first is p_*^{-1} and the second is restriction to the boundary. In other words, it is obtained by lifting a map $f : ([0,1]^n, \partial[0,1]^n) \to B$ to E, then restricting to $[0,1]^{n-1}$.

We record this

Corollary 3.45. *If B is path-connected, there is a long exact sequence*

$$\cdots \to \pi_n(F, X_0) \to \pi_n(E, x_0) \to \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \cdots \to \pi_0(E, x_0)$$

Now we want to try and obtain interesting examples of fibrations. These will be given by a class of maps that are interesting anyway.

Definition 3.46. A map $p : E \to B$ is a *fiber bundle with fiber F* if every point $b \in B$ has a neighbourhood U with the following property: there is a homeomorphism $h : p^{-1}(U) \to U \times F$ so that $p = p_1 h$ where p_1 is projection on the first coordinate.

Intuitively, each preimage $p^{-1}(b)$ is homeomorphic to F, in such a way that the map p behaves locally like a projection.
We call F the *fiber*, B the *base space*, E the *total space* of the fiber bunle.
h is called a *local trivialisation.*
Let's see some examples to see that these are important maps:

(1) Products.
(2) Covering spaces are exactly fiber bundles with discrete fibers.
(3) Vector bundles (if you know what these are) are fiber bundles with fiber $\mathbb{R}^k$ (and extra compatibility!)
(4) Projective spaces $\mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n$
(5) The *Möbius band*

$$M = [0,1] \times [-1,1] / \sim$$

induced by $(0, x) \sim (1, -x)$. This is locally a product, but not globally!

(6) A bit more generally, if F is any space, and $\phi : F \to F$ is a homeomorphism, build
$$M_\phi = F \times [0,1]/\sim$$
where $(0, x) \sim (1, f(x))$. This is called a *mapping torus*.

(7) A very powerful construction generalising the last one: suppose that F is a topological space, and B is any path-connected topological space with a universal cover $\widetilde{B}$, and
$$\rho : \pi_1(B) \to \mathrm{Homeo}(F),$$
define an action $\hat{\rho}$ of $\pi_1(B)$ on $\widetilde{B} \times F$ diagonally. Then the quotient
$$B \times_\phi F = \widetilde{B} \times F/\hat{\rho}.$$
(we used a construction like this for covering spaces last semester!)

(8) Lie group quotients

(9) $\mathrm{Homeo}(S) \to S$ via $f \mapsto f(p)$.

Theorem 3.47. *A fiber bundle $p : E \to B$ is a Serre fibration.*

Proof. We need to show that we have the HLP for cubes. So, given
$$G : I^n \times I \to B$$
a homotopy, and
$$\widetilde{g}_0 : I \to F$$
an initial lift.
Choose an open cover U_i of B and local trivialisations $h_i : p^{-1}(U_i) \to U_i \times F$. By compactness of the cube $I^n \times [0,1]$, we can decompose into subcubes $C_j \times [t_r, r_{r+1}]$ so that each of them is mapped by G into one of the U_i.
Fix one pair j, r. We know that
$$G(C_j \times [t_r, t_{r+1}]) \subset U_i$$
for some i. Thus, by applying the fact that a product has the HLP, the homotopy can be lifted, extending a lift at time t_r. By induction, this shows that the homotopy can be extended over all t. □

This concludes our discussion of homotopy theory. There is of course much more...