Summer term 2025

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Topology II

Sheet 2

Exercise 1. Let A be an abelian group.

(a) Show that $A \otimes -$ is left-adjoint to $\mathrm{Hom}(A, -)$.[1]

(b) Show that $A \otimes -$ is right-exact.[2]

(c) Show that $A \otimes -$ is in general not left-exact. (Abelian groups A for which $A \otimes -$ is left-exact are called *flat.*)

Exercise 2. Let A, B be abelian groups. We define an abelian group $\mathrm{Tor}(A, B)$ as follows: Choose a free resolution $\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} A \to 0$. Then consider the chain complex

$$\cdots \to F_2 \otimes B \xrightarrow{f_2 \otimes id} F_1 \otimes B \xrightarrow{f_1 \otimes id} F_0 \otimes B \to 0$$

and define $\mathrm{Tor}(A, B)$ as the first homology group of this chain complex:

$$\mathrm{Tor}(A, B) = \ker(f_1 \otimes id)/\mathrm{im}(f_2 \otimes id)\,.$$

This group is independent of the choice of free resolution. Prove:

(a) $\mathrm{Tor}(\bigoplus_{i \in I} A_i, B) \cong \bigoplus_{i \in I} \mathrm{Tor}(A_i, B)$

(b) $\mathrm{Tor}(A, B) = 0$ if A is free

(c) $\mathrm{Tor}(\mathbb{Z}/a, B) \cong \ker(B \xrightarrow{a} B)$ for every $a \in \mathbb{Z}_{>0}$.

(d) Compute $\mathrm{Tor}(\mathbb{Z}/a, \mathbb{Z}/b)$ for every $a, b \in \mathbb{Z}_{>0}$.

Exercise 3. Let X be a space and let $C_*(X)$ be its singular chain complex. Let A be an abelian group. We can define a new chain complex by $C_*(X; A) = C_*(X) \otimes A$, or more explicitly

$$\cdots \to C_2(X) \otimes A \xrightarrow{\partial \otimes id} C_1(X) \otimes A \xrightarrow{\partial \otimes id} C_0(X) \otimes A \to 0\,.$$

The homology of $C_*(X; A)$ is called *(singular) homology of X with coefficients in A* and denoted $H_*(X; A)$. It satisfies the usual Eilenberg-Steenrod axioms. Similarly, if X is a CW-complex / Δ-complex we can define cellular / simplicial homology of X with coefficients in A, and it agrees with $H_*(X; A)$

[1] Recall that this means that for all abelian groups B and C there is a bijection $\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(B, \mathrm{Hom}(A, C))$ and this bijection is natural in B and C.

[2] This means that if $0 \to B \to C \to D \to 0$ is a short exact sequence, then $A \otimes B \to A \otimes C \to A \otimes D \to 0$ is still exact.

(a) Compute the homology of the Klein bottle K with coefficients in $\mathbb{Z}/2$ and $\mathbb{Q}$.

(b) Give the Klein bottle the obvious CW-structure with one 0-cell, two 1-cells and one 2-cell. Let $f\colon K \to S^2$ be the map which collapses the 1-skeleton to a point. Compute the induced map

$$f_*\colon H_*(K;A) \to H_*(S^2;A)$$

for $A = \mathbb{Z}$ and $A = \mathbb{Z}/2$.

Exercise 4. (a) Show that tensor product with $\mathbb{Q}$ preserves exact sequences of abelian groups.

(b) Deduce that $H_*(X;\mathbb{Q}) \cong H_*(X) \otimes \mathbb{Q}$.

This sheet will be discussed in the exercise class on 7 May 2025.