Def. (M,g), (N,h) Riem mf.

a) A diffeo $\varphi: M \to N$ is called an *isometry* if $\varphi^* h = g$, ie

$$g_p(v,w) = h_{\varphi(p)}(d\varphi_p v, d\varphi_p w)$$

$\forall p \in M$; $v,w \in T_pM$.

If there exists such an isometry, then M and N are called *isometric*.

b) A smooth map $\varphi: M \to N$ is called a *local isometry* if every point $p \in M$ has an open nbd U st $\varphi|_U : U \to \varphi(U)$ is an isometry. (in particular, $\varphi(U)$ open).

c) The set of all isometries of (M,g)

$$\mathrm{Iso}(M,g) = \{\varphi: M \to M \mid \varphi \text{ isometry}\}$$

is a group, the so-called *isometry group*.

Remark.

- If $\varphi: (M,g) \to (N,h)$ is a local isometry, then

$$d\varphi_p : T_pM \to T_pN$$

is a linear isometry.

Example. $\mathbb{R}$ w/ Euclidean inner product g.

$S^1 \subset \mathbb{R}^2 \subset \mathbb{C}^1$ w/ the induced Riem metric.

Claim: $\exp: \mathbb{R} \to S^1$, $t \mapsto e^{it}$ is a local isometry.

$$\left\|(d\exp)_t\left(\tfrac{\partial}{\partial t}\right)\right\| = \left|\tfrac{d}{dt}(e^{it})\right| = |i \cdot e^{it}| = 1 \rightsquigarrow d\exp_t \neq 0$$

$\left|\frac{\partial}{\partial t}\right| = 1$

"preserves norm".

$\Rightarrow$ exp local isometry, but not an isometry: $\mathbb{R} \not\cong S^1$, not even homeo.

Def. Let G a gp. M a smooth mf. A (smooth) *action* of G on M is a homomorphism

$$\Phi: G \to \mathrm{Diff}(M), \quad a \mapsto \Phi(a) = \Phi_a;$$

i.e.,

$$\Phi_{a_1 a_2}(x) = (\Phi_{a_1} \circ \Phi_{a_2})(x) \quad \forall a_1, a_2 \in G,\ x \in M,$$

$$\Phi_e = \mathrm{id}_M, \quad e \in G \text{ neutral elt.}$$

We sometimes denote such an action as $G \curvearrowright M$, and write $\Phi_a(x)$ as ax.

If (M,g) is a Riem mf, and $\Phi(G) \subset \mathrm{Iso}(M,g)$, then we call this action isometric.

Example. $\mathrm{Iso}(M,g) \curvearrowright M$ isometric.

For an action $G \curvearrowright M$ we would like to look at

$$M/G = M/\sim, \text{ where } x \sim y \text{ iff } \exists g \in G : gx = y$$

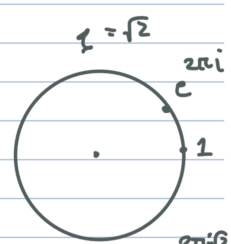
with the quotient topology := the finest topology (ie with the most open sets) s.th. $\pi : M \to M/G$ is continuous, i.e.

$$U \subset M/G \text{ is open} \Leftrightarrow \pi^{-1}(U) \text{ is open in } M.$$

A group action $G \curvearrowright M$ is called free, if $gx = x$ for some $g \in G$, $x \in M$, then $g = e$.

Example.

- $\mathbb{Z} \curvearrowright \mathbb{R}$ $\quad \underset{\mathbb{Z}}{a}\,\underset{\mathbb{R}}{x} := a + x$ is a free action.



- $\mathbb{Z} \curvearrowright S^1 \subset \mathbb{C}$, $\quad \underset{\mathbb{Z}}{a}\,\underset{\mathbb{C}}{z} : e^{2\pi i\sqrt{2}\,a} \cdot z$

If $az : e^{2\pi i q a} \cdot$, $q \in \mathbb{R}$, free iff $q \notin \mathbb{Q}$.

$\mathbb{Z}\cdot 1 \neq \mathbb{Z}\cdot e^{2\pi i\sqrt{2}}$

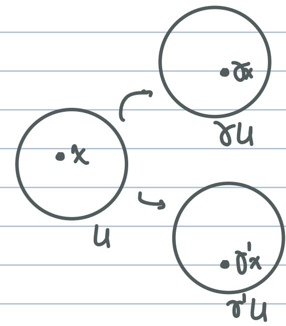
For $q = \sqrt{2}$, $S^1/\mathbb{Z}$ is not Hausdorff.

not separable in the quotient by open sets.

Def: a smooth action $\Gamma \curvearrowright M$ of a group Γ on a smooth mf M is called a covering space action if the following holds:

every point $x \in M$ has an open nbd U st $\gamma U \cap U = \emptyset$ $\forall \gamma \in \Gamma \setminus \{e\}$.

In particular, $\Gamma \curvearrowright M$ is free.

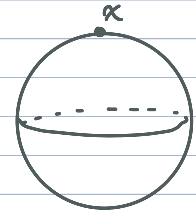


Examples. (for covering space actions).

1) $S^n = \{ x \in \mathbb{R}^{n+1} \mid \|x\| = 1 \}$ Eucl. unit sphere.

Antipodal action: $\underset{\{\pm 1\}}{\mathbb{Z}_2} \curvearrowright S^n$, $(\pm 1) x = \pm x$

with e.g. $U = \{ y \in S^n \mid \langle y, x \rangle > 0 \}$.



2) $\mathbb{Z} \curvearrowright \mathbb{R}$, $ax := a + x$, as for $x \in \mathbb{R}$, we have

$$(x - \tfrac{1}{3}, x + \tfrac{1}{3}) \cap (x + a - \tfrac{1}{3}, x + a + \tfrac{1}{3}) = \emptyset$$

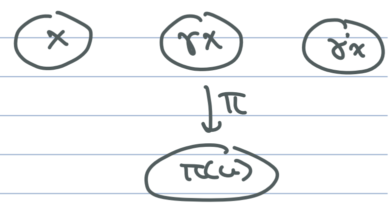
for all $a \in \mathbb{Z} \setminus \{0\}$.

Prop: $\Gamma \curvearrowright M$ smooth covering space action of a group Γ on a mf M. Then M/Γ is a topological manifold with a unique smooth structure w.r.t. which the projection $M \to M/\Gamma$ is a local diffeomorphism.

Proof: π is continuous (by def'n of the quotient topology) and open (ie it maps open sets to open sets): take $U \subset M$ open, $\pi(U)$ open $\Leftrightarrow$ $\pi^{-1}(\pi(U)) = \underbrace{\bigcup_{\gamma \in \Gamma} \underbrace{\gamma U}_{\text{open}}}_{\text{open}}$ open.

For $x \in M$ we choose open nbd U of x s.th. $\underbrace{U \cap \gamma U = \emptyset}_{(*)}$, $\forall \gamma \in \Gamma \setminus \{e\}$. Then

$$\pi|_U : U \to \pi(U)$$



is continuous, open, and bijective

$y, y' \in U$ with $\pi(y) = \pi(y')$

$\Rightarrow \exists \gamma \in \Gamma : \gamma y = y' \overset{(*)}{\Rightarrow} \gamma = e \Rightarrow y = y' \Rightarrow \pi|_U$ is a homeomorphism $\Rightarrow M/\Gamma$ top mfld (π projects a countable base of the topology of M to a . . . of M/Γ.

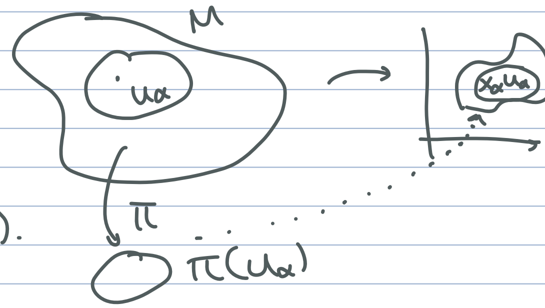
To construct the smooth structure, we choose an atlas $\mathcal{A}_M = \{(U_\alpha, X_\alpha)\}_{\alpha\in I}$ of M s.th.

$$\gamma U_\alpha \cap U_\alpha = \emptyset \quad \forall\, \alpha \in I,\ \gamma\in\Gamma\setminus\{e\}.$$

$\Rightarrow \pi|_{U_\alpha} : U_\alpha \to \pi(U_\alpha)$ is a homeomorphism.

If $\pi: M\to M/\Gamma$ is supposed to be a local diffeo, then the smooth structure on M/Γ has to contain

$$\mathcal{A}_{M/\Gamma} = \{(\pi(U_\alpha), X_\alpha\circ(\pi|_{U_\alpha})^{-1})\}_{\alpha\in I}$$



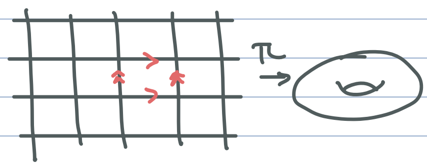
It remains to verify than $\mathcal{A}_{M/\Gamma}$ is a smooth atlas.

($\to$ exercise sesh next week).

Examples.

1) For the antipodal action $\mathbb{Z}_2 \curvearrowright S^n \rightsquigarrow S^n/\mathbb{Z}_2 \cong \mathbb{RP}^n$

2) $\mathbb{Z}^n \curvearrowright \mathbb{R}^n$, $\quad a x := a + x$ is a covering space action,



$$\mathbb{R}^n/\mathbb{Z}^n \cong \underbrace{S^1\times\cdots\times S^1}_{n \text{ times}} = T^n \quad n\text{-torus}.$$

Remark.

- One can show ($\rightsquigarrow$ Topology I WiSe 25/26) that for any smooth manifold M, there exists another simply-connected (smooth) manifold $\tilde{M}$ (the so-called universal cover of M) together with a covering space action $\Gamma \curvearrowright \tilde{M}$ s.t. $\tilde{M}/\Gamma \cong M$.

  In particular, if (M,g) Riem mf then we can pull back g to a Riem metric $\tilde{g}$ on $\tilde{M}$ s.t. $\pi: (\tilde{M},\tilde{g}) \to (M,g)$ is a local isometry.

Conversely, we have

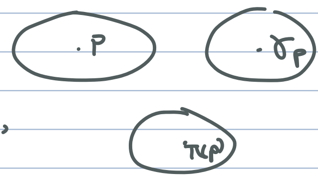
Prop: Let $\Gamma \curvearrowright (M,g)$ be a covering space action by isometries on a Riem manifold (M,g). Then there exists a unique Riem metric $\bar{g}$ on M/Γ s.th. $\pi : (M,g) \to (M/\Gamma, \bar{g})$ is a local isometry.

Proof: For $\pi : M \to M/\Gamma$ to be a local isometry,

the differential of π $\quad d\pi_p : T_pM \to T_{\pi(p)}(M/\Gamma)$ needs to be a linear isometry.

As $d\pi_p$ is an isomorphism, this condition determines (for $p \in M$) an inner product $\bar{g}_{\pi(p)}$ on $T_{\pi(p)}(M/\Gamma)$

Claim: $\bar{g}_{\pi(p)}$ is well-defined.



For $p_1, p_2 \in M$ with $\pi(p_1) = \pi(p_2) = q$,

$\rightsquigarrow \exists \gamma \in \Gamma : \gamma p_1 = p_2$.

Because of $\pi \circ \gamma = \pi$ the following diagram commutes.

$$\begin{tikzcd}
(T_{p_1}M, g_{p_1}) \arrow[rr, "d\gamma_{p_1}", "\text{isometry}"'] \arrow[dr, "d\pi_{p_1}"'] & \circlearrowleft & (T_{p_2}M, g_{p_2}) \arrow[dl, "d\pi_{p_2}"] \\
& T_q(M/\Gamma) &
\end{tikzcd}$$

$d\gamma_{p_1}$ isometry
$\quad \shortparallel$
$\quad d\pi_{p_2} \circ (d\pi_{p_1})^{-1}$. $\quad \square$

Examples:

- S^n unit-sphere. Isometric covering space action on S^n are classified (Wolf, spaces of constant curvature). In this case Γ is always finite.

p, q coprime, $\mathbb{Z}_p \curvearrowright S^3$,

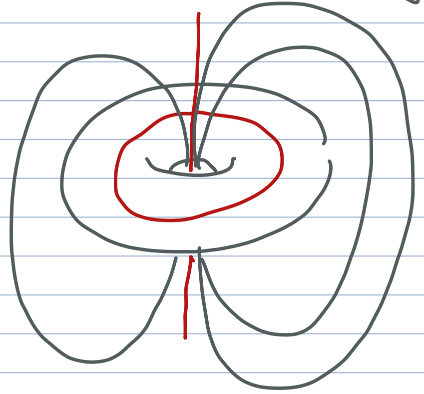
$$e^{\frac{2\pi i}{p}} (z_1, z_2) := (e^{\frac{2\pi i}{p}} z_1, e^{\frac{2\pi q i}{p}} z_2)$$

$S^3 \subset \mathbb{C}^2$

$\rightsquigarrow S^3/\mathbb{Z}_p$ an $L(p,q)$ lens space

$$S^3/\mathbb{Z}_p = S^1 \times D^1 \cup S^1 \times D^1$$

identification on boundary



2) There are many cocompact (compact quotient) isometric covering space actions on hyperbolic space

In particular, any surface of "genus" ≥ 2 arises as such a quotient (and hence admits a "hyperbolic" Riemannian metric).