Mathematical Statistical Physics 2020

Classical part 1 The Ising model.

1.2 Existence of the thermodynamic limit

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Friedli-Velenik Sections 3.1 and 3.2.1, 3.2.2

Overview

- Some additional vocabulary and notation.
- Van Hove sequence—how do we let $\Lambda \nearrow \mathbb{Z}^d$?
- Existence of the thermodynamic limit.
  Bulk free energy does not depend:
  - shape of the container (limit along growing cubes or balls?)
  - boundary conditions.

Notation: some minor changes

Previously:

$$\mathscr{H}_{\Lambda;h}(\omega) = -\sum_{\substack{i,j\in\Lambda:\\ i\sim j}} \omega_i\omega_j - h\sum_{i\in\Lambda}\omega_i,$$

Boltzmann weight $\exp(-\beta\mathscr{H}_{\Lambda;h}(\omega))$.

From now on, instead:

$$\mathscr{H}^{\varnothing}_{\Lambda;\beta,h}(\omega) = -\beta \sum_{\{i,j\}\in\mathscr{E}_\Lambda} \sigma_i(\omega)\sigma_j(\omega) - h\sum_{i\in\Lambda}\sigma_i(\omega),$$

Boltzmann weight $\exp(-\mathscr{H}_{\Lambda;\beta,h}(\omega))$. New notation $\sigma_i(\omega) = \omega_i$,

$$\mathscr{E}_\Lambda = \big\{\{i,j\} : i,j\in\Lambda,\ i\sim j\big\}$$

set of nearest neighbor edges within Λ. Superscript $\varnothing$ = empty/ free boundary conditions = no interactions with the outside Λ^{c}.

Boundary conditions

Configuration space for infinite lattice $\mathbb{Z}^d$:

$$\Omega = \{+1, -1\}^{\mathbb{Z}^d} = \left\{\omega = (\omega_i)_{i\in\mathbb{Z}^d} : \omega_i = \pm 1\right\}.$$

Fix a configuration $\eta \in \Omega$, e.g. $\eta_i \equiv +1$. Look at configurations ω that can be anything they want inside Λ but with frozen degrees of freedom $\omega_i = \eta_i$ outside Λ

$$\Omega_\Lambda^\eta = \left\{\omega \in \Omega : \ \omega_i = \eta_i \text{ for all } i \in \mathbb{Z}^d \setminus \Lambda\right\}.$$

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Include nearest neighbor edges that cross the border of Λ

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Energy with boundary condition η

$$\mathscr{H}_{\Lambda;\beta,h}^\eta(\omega) = -\beta \sum_{\{i,j\}\in\mathscr{E}_\Lambda^{\mathrm{b}}} \sigma_i(\omega)\sigma_j(\omega) - h \sum_{i\in\Lambda} \sigma_i(\omega)$$

includes some contributions $\sigma_i(\omega)\eta_j(\omega)$.

Gibbs measure with b.c.

$$\mathscr{H}^{\eta}_{\Lambda;\beta,h}(\omega) = -\beta \sum_{\{i,j\}\in\mathscr{E}^{\mathrm{b}}_{\Lambda}} \sigma_i(\omega)\sigma_j(\omega) - h\sum_{i\in\Lambda}\sigma_i(\omega)$$

Partition function

$$\mathbf{Z}^{\eta}_{\Lambda;\beta,h} = \sum_{\omega\in\Omega^{\eta}_{\Lambda}} \exp\Big(-\mathscr{H}^{\eta}_{\Lambda;\beta,h}(\omega)\Big)$$

Gibbs measure

$$\mu^{\eta}_{\Lambda;\beta,h}(\omega) = \frac{1}{\mathbf{Z}^{\eta}_{\Lambda;\beta,h}} \exp\Big(-\mathscr{H}^{\eta}_{\Lambda;\beta,h}(\omega)\Big), \quad \omega\in\Omega^{\eta}_{\Lambda}.$$

Another b.c.: periodic boundary conditions $\rightarrow$ book.

Notation covering both free boundary conditions and b.c. η:
Superscript $\# = \varnothing,\ \eta$.

Van Hove sequences

Wanted:
a notion of convergence $\Lambda \nearrow \mathbb{Z}^d$ that makes boundaries $\partial\Lambda$ irrelevant.

Why?

Heuristics:
Each volume $\Lambda \Subset \mathbb{Z}^d$ is assigned a free energy $\Psi_\Lambda(\beta, h) = \log \mathbf{Z}_\Lambda$. We would like to say

$$\Psi_\Lambda(\beta, h) = |\Lambda|\, \psi(\beta, h) + \mathrm{const}_{\beta,h}\, |\partial\Lambda|$$

i.e. free energy $=$ volume $\times$ free energy density $+$ a boundary correction.

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The bulk contribution proportional to the volume should be the dominant term—we want

$$|\partial\Lambda| \text{ small compared to } |\Lambda|.$$

Given: sequence $(\Lambda_n)_{n\in\mathbb{N}}$ of domains $\Lambda_n \Subset \mathbb{Z}^d$.

A weak notion of convergence: $\Lambda_n \uparrow \mathbb{Z}^d$ if

1. $\Lambda_n \subset \Lambda_{n+1}$ for all n.
2. $\bigcup_{n\in\mathbb{N}} \Lambda_n = \mathbb{Z}^d$.

We want something stronger.

Definition (Λ_n) converges to $\mathbb{Z}^d$ in the sense of van Hove if $\Lambda_n \uparrow \mathbb{Z}^d$ and

$$\boxed{\lim_{n\to\infty} \frac{|\partial^{\mathrm{in}}\Lambda_n|}{|\Lambda_n|} = 0}$$

where $\partial^{\mathrm{in}}\Lambda = \{i \in \Lambda : \ \exists j \notin \Lambda, \ j \sim i\}$. Notation:

$$\Lambda_n \Uparrow \mathbb{Z}^d.$$

Call $(\Lambda_n)_{n\in\mathbb{N}}$ a van Hove sequence.

Existence of the thermodynamic limit

Theorem (FV Thm 3.6)

(a) *The limit*

$$\psi(\beta, h) = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \log \mathbf{Z}^{\#}_{\Lambda_n;\beta,h}$$

exists for every van Hove sequence $\Lambda_n \Uparrow \mathbb{Z}^d$, and its value does not depend on the precise choice of van Hove sequence.

(b) *The limit does not depend on the choice of boundary condition.*

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(c) *(The function $\mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $(\beta, h) \mapsto \psi(\beta, h)$ is convex.)*

(d) *(The function $h \mapsto \psi(\beta, h)$ is even: $\psi(\beta, -h) = \psi(\beta, h)$.)*

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In particular: it doesn't matter whether you take limits along growing sequences cubes, balls, parallelipipeds. . .

The thermodynamic potential $\psi(\beta, h)$ does not remember the shape of the container.

(Side remark: clashes with elasticity theory.)

Proof strategy for (a), (b)

Proof of existence of the limit for free b.c. in two steps:

1. Prove the existence of the limit along sequences of cubes with sidelength 2^n
$$D_n = \{1, \ldots, 2^n\}^d.$$
Note: we do not have $D_n \uparrow \mathbb{Z}^d$ but it is true that $|\partial^{\mathrm{in}} D_n|/|D_n| \to 0$.

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Then, move over to general b.c.

3. Show that when you change the boundary condition, the energy and the logarithm $\log Z_\Lambda^\#$ change by a term of order $|\partial^{\mathrm{in}}\Lambda|$.

Coming next

Convexity.

What kind of information do we get for free out of the existence of the thermodynamic limit?

Phase transition.