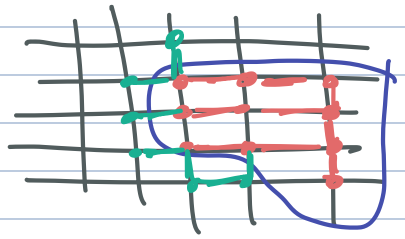
1.2 Thermodynamic Limit of the Pressure.

New notation.



$\Lambda \Subset \mathbb{Z}^d$

$\mathcal{E}_\Lambda := \{ \{i,j\} \subset \Lambda : \underbrace{\|i-j\|=1}_{i \sim j \ ; \ i,j \ \text{n.n.}} \}$

"edge"

$\mathcal{E}^b_\Lambda := \{ \{i,j\} \subset \mathbb{Z}^d : i \sim j, \ i \in \Lambda \text{ or } j \in \Lambda \}$

b boundary

Energy with empty b.c. : $\beta > 0$, $h \in \mathbb{R}$.

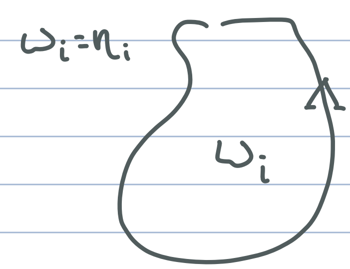
$$\mathcal{H}^\emptyset_{\Lambda;\beta,h} : \Omega_\Lambda \to \mathbb{R}, \quad \mathcal{H}_{\Lambda;\beta,h}(\omega) := -\beta \sum_{\{i,j\} \in \mathcal{E}_\Lambda} \omega_i \omega_j - h \sum_{i \in \Lambda} \omega_i$$

$$Z^\emptyset_{\Lambda;\beta,h} := \sum_{\omega \in \Omega_\Lambda} e^{-\mathcal{H}^\emptyset_{\Lambda;\beta,h}}, \quad \mu^\emptyset_{\Lambda;\beta,h}, \quad \langle \, , \, \rangle^\emptyset_{\Lambda;\beta,h}$$

$$\psi^\emptyset_\Lambda(\beta,h) := \frac{1}{|\Lambda|} \log Z^\emptyset_{\Lambda;\beta,h}.$$

$\Omega := \{+1,-1\}^{\mathbb{Z}^d}$. Formalizing bdy conds: consider

$\eta \in \Omega$ e.g. $\eta_i = +1 \ \forall i \in \mathbb{Z}^d$.



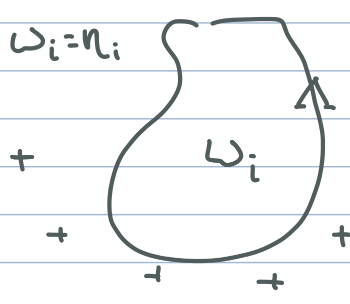
$$\Omega^\eta_\Lambda := \{ \omega \in \Omega : \omega_i = \eta_i \ \forall i \in \mathbb{Z}^d \setminus \Lambda \}$$

$\mathcal{H}_{\Lambda;\beta,h} : \Omega \to \mathbb{R}$

$$\omega \mapsto \mathcal{H}_{\Lambda;\beta,h}(\omega) := -\beta \sum_{\{i,j\} \in \mathcal{E}^b_\Lambda} \omega_i \omega_j - h \sum_{i \in \Lambda} \omega_i$$

Think: $\omega \in \Omega^\eta_\Lambda$; $\mathcal{H}_{\Lambda;\beta,h}(\omega) = \mathcal{H}^\emptyset_{\Lambda;\beta,h}(\omega_\Lambda) - \beta \sum_{\substack{\{i,j\}: i \sim j \\ i \in \Lambda, j \in \Lambda^c}} \omega_i \underset{\omega_j}{\overset{\shortparallel}{\eta_j}}$

(some write $\mathcal{H}_\Lambda(\omega_\Lambda | \eta_j)$)

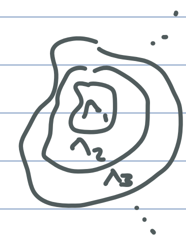


$\omega_\Lambda = (\omega_i)_{i\in\Lambda}$ $\qquad Z^\eta_{\Lambda;\beta,h} := \sum_{\omega\in\Omega^\eta_\Lambda} e^{-\mathcal{H}_{\Lambda;\beta,h}(\omega)}$

$$\mu^\eta_{\Lambda;\beta,h}, \quad \psi^\eta_{\Lambda;\beta,h}(\beta,h)$$

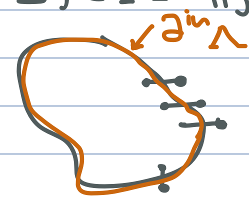
Def: 1.1. $(\Lambda_n)_{n\in\mathbb{N}}$ $\Lambda_n \Subset \mathbb{Z}^d$ converges to $\mathbb{Z}^d$ in the sense of van Hove (or "is a van Hove sequence") if:

(i) $\Lambda_n \uparrow \mathbb{Z}^d$: $\forall n\ \Lambda_n \subset \Lambda_{n+1}$, $\mathbb{Z}^d = \bigcup_{n\in\mathbb{N}} \Lambda_n$



(ii) $\lim_{n\to\infty} \frac{|\partial^{in}\Lambda_n|}{|\Lambda_n|} \longrightarrow 0$

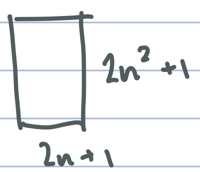
inner boundary $\partial^{in}\Lambda := \{i\in\Lambda : \exists j\in\Lambda^c\ \|j-i\|=1\}$



Example: $\Lambda_n = \{-n,\dots,n\}^d$

$|\partial^{in}\Lambda_n| = O(n^{d-1})$
$|\Lambda_n| = n^d$

$\Lambda_n = \{-n,\dots,n\}\times\{-n^2,\dots,n^2\}$ in $\mathbb{Z}^2$



$|\partial^{in}\Lambda_n| = O(n^2)$ ✓
$|\Lambda_n| = cn^3$

Notation: $\Lambda_n \Uparrow \mathbb{Z}^d$

Theorem. 2.1.

(a) The limit $\psi(\beta,h) = \lim_{n\to\infty} \frac{1}{|\Lambda_n|}\log Z^\#_{\Lambda_n;\beta,h}$

$$\left(= \lim_{n\to\infty} \psi^\#_{\Lambda_n}(\beta,h)\right)$$

exists for all van Hove sequences, for every b.c. $\# = \emptyset$, $\# = \eta\in\Omega$

The value does not depend on the precise choice of van Hove sequence.

(b) " " " " b. c.

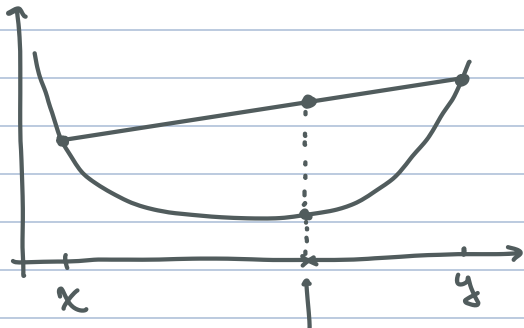
(c) $(0,\infty)\times\mathbb{R}\to\mathbb{R}$, $(\beta,h)\mapsto\psi(\beta,h)$ is convex.

(d) $\forall\beta>0$, $\psi(\beta,h)=\psi(\beta,-h)$

Convex.

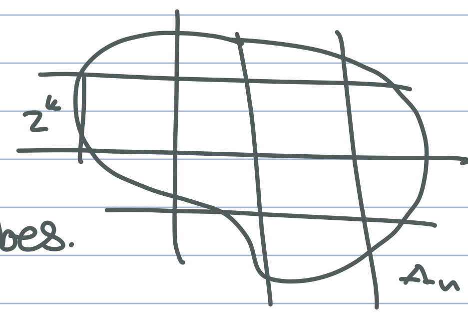
$\forall(\beta_1,h_1)$, $\forall(\beta_2,h_2)$, $\forall t\in[0,1]$

$$\psi((1-t)\beta_1+t\beta_2,\ (1-t)h_1+th_2)\le(1-t)\psi(\beta_1,h_1)+t\psi(\beta_2,h_2)$$

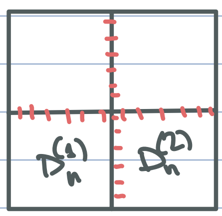


Proof of (a), (b):

① Empty $\# = \emptyset$



$\cancel{\Lambda_n}$ $D_n := [1,\dots,2^n]^d$ dyadic cubes.



$D_{n+1}=\bigcup_{i=1}^{2^d} D_n^{(i)}$ disjoint union

$\uparrow$ shifted copies of D_n

(drop β, h indices) $\mathcal{H}^{\emptyset}_{D_{n+1}}=\sum_{i=1}^{2^d}\mathcal{H}^{\emptyset}_{D_n^{(i)}}+R_n$

$$|R_n(\omega)|\le\beta d\,(2^{n+1})^{d-1}$$

$$\Rightarrow \mathcal{H}^{\emptyset}_{D_{n+1}}\ge\sum_{i=1}^{2^d}\mathcal{H}^{\emptyset}_{D_n^{(i)}}-\beta d\,(2^{n+1})^{d-1}$$

$$\Rightarrow e^{-\mathcal{H}^{\emptyset}_{D_{n+1}}}\le e^{\beta d(2^{n+1})^{d-1}}\prod_{i=1}^{2^d}e^{-\mathcal{H}^{\emptyset}_{D_n^{(i)}}}$$

$$\Rightarrow \quad Z^{\emptyset}_{D_{n+1}} \le e^{\beta d (2^{n+1})^{d-1}} \underbrace{\prod_{i=1}^{2^d} Z^{\emptyset}_{D_n^{(i)}}}_{=\left(Z^{\emptyset}_{D_n}\right)^{2^d}}$$

$$\Rightarrow \quad \psi^{\emptyset}_{D_{n+1}} \le \frac{1}{\underbrace{|D_{n+1}|}_{(2^{n+1})^d}} \beta d \left(2^{n+1}\right)^{d-1} + \frac{1}{|\cancel{D_{n+1}}|} \cancel{2^d} \, |\cancel{D_n}| \, \psi^{\emptyset}_{D_n}$$

$$\psi^{\emptyset}_{D_{n+1}} \le \frac{\beta d}{2^{n+1}} + \psi^{\emptyset}_{D_n}$$

same reasoning: $\quad \ge -\frac{\beta d}{2^{n+1}} + \psi^{\emptyset}_{D_n}$

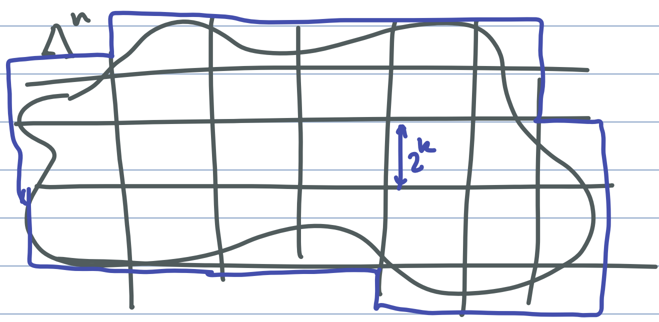
$$\Rightarrow \quad \left|\psi^{\emptyset}_{D_{n+1}} - \psi^{\emptyset}_{D_n}\right| \le \frac{\beta d}{2^{n+1}}$$

(details n book).

$\ldots \longrightarrow \psi_{(\beta,h)} := \lim_{n\to\infty} \psi^{\emptyset}_{D_n}(\beta,h)$ exists $\in \mathbb{R}$.

2 papers, Lieb, Lebowitz. (physics$^{\text{10-20 pg}}$ & math papers$^{\text{100-200 pg}}$)

(2) $\# = \emptyset$, Λ_n van Hove.



$$\Lambda_n \subset \bigcup_j D_k^{(j)} =: [\Lambda_n]$$

Idea: $\quad \log Z^{\emptyset}_{\Lambda_n} \approx \log Z^{\emptyset}_{[\Lambda_n]} \approx \underbrace{\sum_j \underbrace{\log Z^{\emptyset}_{D_k^{(j)}}}_{\approx |D_k|\cdot\psi}}_{\frac{|[\Lambda_n]|}{|\cancel{D_k}|} \cdot |\cancel{D_k}| \cdot \psi}$

Rigorously:

$$|\psi^{\emptyset}_{\Lambda_n} - \psi| \leq |\psi^{\emptyset}_{\Lambda_n} - \psi^{\emptyset}_{[\Lambda_n]}| + |\psi^{\emptyset}_{[\Lambda_n]} - \psi_{D_k}| + |\psi_{D_k} - \psi|$$

defined in ①

ⓒ ⓑ ⓐ

ⓐ Let $\varepsilon > 0$. Part 1 of pf $\Rightarrow \exists k_0 = k_0(\varepsilon, \beta, h)$

$\forall k \geq k_0, \quad |\psi^{\emptyset}_{D_k} - \psi| \leq \varepsilon/3$

ⓑ $$d\ell^{\emptyset}_{[\Lambda_n]} = \sum_j d\ell^{\emptyset}_{D_k^{(j)}} + W_n$$

with

$$W_n \leq \beta d (2^k)^{d-1} \frac{|[\Lambda_n]|}{|D_k|}$$

$\Rightarrow$

$$\frac{|[\Lambda_n]|}{|D_k|} \log Z^{\emptyset}_{D_k} - \frac{\beta d}{2^k}|[\Lambda_n]| \leq \log Z^{\emptyset}_{[\Lambda_n]} \leq \frac{|[\Lambda_n]|}{|D_k|} \log Z^{\emptyset}_{D_k} + \frac{\beta d}{2^k} |[\Lambda_n]|$$

$$\Rightarrow |\psi^{\emptyset}_{[\Lambda_n]} - \psi^{\emptyset}_{D_k}| \leq \frac{\beta d}{2^k}$$ ... To be continued