Contents

1 Lecture 1

Today, we reviewed:

1. Homology.
2. How to construct cohomology.
3. Made a statement about a sequence splitting.
4. The sequence contains $\ker \delta_n$, which we don't understand.

Understanding this $\ker \delta_n$ will be the topic of the next lecture. For more details, see my handwritten notes.

Recall that **homology** is represented as taking a space X, making a chain complex C_i out of it, and then by a purely algebraic manipulation creating a homology H_i. For a cochain complex, we dualize C_i, then take a cohomology H^i.

$$\begin{tikzcd} \text{Space } X \arrow[r, rightsquigarrow] & \text{chain complex } C_i \arrow[r, rightsquigarrow, "\text{purely algebraic}"] \arrow[d, rightsquigarrow, "\text{dualize}"] & \text{homology} H_i \\ & \text{cochain complex } C_i \arrow[r, rightsquigarrow] & \text{cohomology } H^i \end{tikzcd} \tag{1}$$

Suppose now that we have a chain complex C_i of free Abelian groups:

$$\dots C_{n+1} \xrightarrow{\partial} C_n \to C_{n-1} \to \dots. \tag{2}$$

Take an abelian group G (feel free to think about $G = \mathbb{Z}$) and consider the dual

$$(C_n)^* = \mathrm{Hom}(C_n, G), \text{ an additive group.} \tag{3}$$

We have no natural map from $C_n^* \to C_{n-1}^*$, but we do have one from

$$C_n^* \to C_{n+1}^* \tag{4}$$

$$\partial^* : \mathrm{Hom}(C_n, G) \to \mathrm{Hom}(C_{n+1}, G) \tag{5}$$

$$\varphi \mapsto \varphi \circ \partial \tag{6}$$

giving us

$$\dots \to C_{n-1}^* \to C_n^* \xrightarrow{\delta} C_{n+1}^* \to C_{n+2}^* \to \dots \tag{7}$$

Note that we use the sign convention

$$\delta : C_n^* \to C_{n+1}^* \text{ in } (-1)^{n+1}\partial^* \tag{8}$$

which is ***not*** used in Hatcher & Bredon.

Definition 1.0.1: Cochain Complex

*A **cochain complex** $C^\bullet$ is a collection of Abelian groups C^n with codifferentials δ_n such that $\delta_{n+1}\delta_n = 0$.*

Thus,

$$\mathcal{H}om(C_\bullet, G) = \mathrm{Hom}(C_i, G) \text{ with differential } \delta = (-1)^{n+1}\partial^* \tag{9}$$

is a cochain complex, as defined above. Be cautious of Hom vs. $\mathcal{H}om$.

Definition 1.0.2: Cohomology

*For a cochain complex $C^\bullet$, define **cohomology** as*

$$H^i(C^\bullet) = \frac{\ker \delta_i}{\mathrm{im}\delta_{i-1}}. \tag{10}$$

If $C_\bullet$ is a chain complex, then any choice of abelian group G gives a cochain complex

$$\mathcal{H}om(C_\bullet, G), \tag{11}$$

hence a cohomology and thus gohomology groups, denoted as $H^*(C_\bullet, G)$.

If $C_\bullet$ is the singular cochain complex of a space X, then put

$$H^*(X', G) = H^*(\mathcal{H}om(C, G)). \tag{12}$$

Example 1.0.3: Singuluar Cochain Complex

Let $C_\bullet$ be:

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0. \tag{13}$$

Then we have:

$$H_*(C_\bullet) = \begin{cases} \mathbb{Z} & n = 0, \\ \mathbb{Z}/2\mathbb{Z} & n = 1, \\ 0 & n = 2, \\ \mathbb{Z} & n = 3. \end{cases} \tag{14}$$

If we let $G = \mathbb{Z}$, and dualize, then we obtain:

$$\cdots \longleftarrow 0 \longleftarrow \mathbb{Z} \xleftarrow[0]{} \mathbb{Z} \xleftarrow[\cdot 2]{} \mathbb{Z} \xleftarrow[0]{} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z} \longleftarrow 0. \tag{15}$$

has cohomology:

$$H^*(C_\bullet) = \begin{cases} \mathbb{Z} & n = 0, \\ 0 & n = 1, \\ \mathbb{Z}/2\mathbb{Z} & n = 2, \\ \mathbb{Z} & n = 3. \end{cases} \tag{16}$$

*These are **not** the duals of the homology H_*. Moreover, $G = \mathbb{Z}/2\mathbb{Z}$ has different results.*

Note: Functoriality. If $C_\bullet \to D_\bullet$ is a map of cochain complexes, dualizing gives

$$\mathcal{H}om(D_\bullet, G) \to \mathcal{H}(C_\bullet, G) \tag{17}$$

$$\varphi \mapsto \varphi \circ f \tag{18}$$

which induces a map

$$H^*(\mathcal{H}om(D_\bullet, G)) \to H^*(\mathcal{H}om(C_\bullet, G)). \tag{19}$$

We want to now relate the cohomology of $\mathcal{H}om(C_\bullet, G)$ to the homology of $C_\bullet$.

Lemma 1.0.4: Cohomology and Homology

For any chain complex of free Abelian groups, and any Abelian group G, there is a natural surjection:

$$h : H^n(C_\bullet; G) \to \mathrm{Hom}(H_n(C_\bullet; G)) \tag{20}$$

and the sequence

$$\begin{tikzcd} 0 \arrow[r] & \ker(h) \arrow[r] & H^n(C_\bullet; G) \arrow[r] & \mathrm{Hom}(H_n(C_\bullet; G)) \arrow[l, bend right, dashed] \arrow[r] & 0 \end{tikzcd} \tag{21}$$

splits.

Proof. The proof can be found in my live notes. Please review it in detail. □

2 Lecture 2

Our goal for today is to understand $\ker h$.

Lemma 2.0.1: "Hom is left exact"

Suppose that

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0. \tag{22}$$

is a short exact sequence of Abelian groups and G is a group. Then the dual sequence

$$0 \to \mathrm{Hom}(C; G) \xrightarrow{g^*} \mathrm{Hom}(B; G) \xrightarrow{f^*} \mathrm{Hom}(A; G) \tag{23}$$

is exact. If the original sequence was split, then f^* is surjective.

Proof. See my live notes for the proof. Please review it carefully. □

Remarks.

1. $\mathrm{Hom}(\cdot, G)$ is a contravariant functor, and the property of the lemma is called "left-exactness".
2. $$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0 \tag{24}$$
   is exact, and $G = \mathbb{Z}$. Dualize:
   $$0 \to \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \tag{25}$$
   is *not* surjective. So, Hom in general is not exact.

Now an argument is developed, regarding degree shifts in cochain complexes, short exact sequences are related via the snake lemma, and we make a statement regarding the cokernel, which measures something interesting. See live notes for details.

In fact: $\mathrm{coker}(i^*_{n-1})$ measures the "non-right-exactness" of Hom applied to a specific SES:

$$0 \to B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \to H_{n-1}(C_\bullet) \to 0. \tag{26}$$

Corollary 2.0.2: Cohomology and Homology Homomorphisms

If $H_{n-1}(C_\bullet)$ is free Abelian, then

$$H^n(C_\bullet, G) \cong \mathrm{Hom}(H_n(C_\bullet, G)). \tag{27}$$

Proof. If $H_{n-1}(C_\bullet)$ is free Abelian, then (26) splits. Then apply Hom, get a SES, then the $\mathrm{coker}(i^*_{n-1}) = 0$. □

Definition 2.0.3: Free Resolution

Let H be an Abelian group. A ***free resolution*** *of H is an exact sequence*

$$\cdots \to F_2 \to F_1 \to F_0 \to H \to 0 \tag{28}$$

where all F are free Abelian.

Lemma 2.0.4: Free Resolutions of Length 2

Any Abelian group H has a free resolution (of length 2).

Proof. $f_0 : \bigoplus_{H \setminus e} \mathbb{Z} \xrightarrow{\pi} H$ a homomorphism mapping $1 \in \mathbb{Z}$ to h induced by h. Then

$$0 \to \ker(\pi) \to F_0 \xrightarrow{\pi} H. \tag{29}$$

□

Corollary 2.0.5: Cohomology Independent of Free Resolution

uppose that

$$\cdots \to F_2 \to F_1 \to F_0 \to H \to 0 \tag{30}$$

is a free resolution of H, G is any Abelian group. Consider the cochain complex

$$0 \to \mathrm{Hom}(H, G) \xrightarrow{f_0^*} \mathrm{Hom}(F_0, G) \xrightarrow{f_1^*} \mathrm{Hom}(F_1, G) \to \cdots . \tag{31}$$

The cohomology of this is independent of the free resolution. We call the first cohomology group

$$\ker f_2^*/imf_1^* = \mathrm{Ext}^1(H, G). \tag{32}$$

Ext^1 *is the* ***derived functor*** *of Hom.*

There is also a technical corollary, which I leave in the live notes, since the diagram is complicated.

Corollary 2.0.6: Technical Corollary: Chain Morphisms

See live notes.

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