Master TMP - Summer 2025
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Exercise sheet 2: Perron Frobenius Theorem - Transfer Matrices

Exercise 1 - *Perron-Frobenius for Positive Matrices.*

Define the following relation for finite-dimensional matrices (and vectors as a special case):

$$(a_{ij}) \succ \mathbf{0}(\text{ resp. } a_{ij} \succeq \mathbf{0}) \iff \forall i, j : a_{ij} > 0(\text{ resp. } a_{ij} \geq 0)$$

and, correspondingly,

$$(a_{ij}) \succ b_{ij}(\text{ resp. } a_{ij} \succeq b_{ij}) \iff \forall i, j : a_{ij} - b_{ij} > 0(\text{ resp. } a_{ij} - b_{ij} \geq 0)$$

Let $n \in \mathbb{N}$ and let $A \in M(n \times n, \mathbb{C})$ be such that $A \succ \mathbf{0}$.

(a) Prove that there exists $\lambda_0 > 0$ and $x_0 \in \mathbb{R}^n$, $x_0 \succ \mathbf{0}$, such that $Ax_0 = \lambda_0 x_0$.
(*Hint: One route you may take is to apply a suitable fixed-point argument. Alternatively, you may define the set* $\Lambda := \{\mu > 0 \mid \exists x \succ \boldsymbol{0} : Ax \succeq \mu x\}$ *and find its supremum.*)

(b) Prove that if $\lambda \neq \lambda_0$ is a (possibly complex) eigenvalue of A, then $|\lambda| < \lambda_0$.

(c) Prove that the eigenvalue λ_0 is simple (i.e., its algebraic multiplicity is 1.)

Exercise 2 - *Transfer Matrices.*

You are given the one dimensional spin chain with the energy function

$$H_\Lambda(\sigma) = \sum_{J \subset \Lambda} \Phi(J) \prod_{x \in J} \sigma_x$$

where the interaction is translationally invariant, i.e., for all $a \in \mathbb{Z}$ and all $J \subset \mathbb{Z}$, we have $\Phi(J) = \Phi(J + a)$ where $J + a := \{j + a \mid j \in J, a \in \mathbb{Z}\}$, and has a finite range: $\Phi(J) = 0$, whenever $\operatorname{diam}(J) > R + 1, R \in \mathbb{N}$ (e.g., for the nearest-neighbour interaction one would have $R = 1$). The partition function is

$$Z_\Lambda(\beta) = \sum_{\sigma \in \{-1,1\}^\Lambda} e^{-\beta H_\Lambda(\sigma)}.$$

(a) Let $B := \{-1, 1\}^R$ and $\Lambda_m := \{1, 2, \ldots, mR\}$. Find functions f_β, $g_\beta : B \to \mathbb{R}_+$ and $K_\beta : B \times B \to \mathbb{R}_+$, such that for all $m \in \mathbb{N}$

$$Z_{\Lambda_m}(\beta) = \sum_{b_1, \ldots, b_m \in B} f_\beta(b_1) K_\beta(b_1, b_2) \cdots K_\beta(b_{m-1}, b_m) g_\beta(b_m)$$

Note that such f_β, g_β and K_β are not unique.

(b) For $R = 1$ express the pressure

$$p(\beta) := \lim_{m\to\infty} \frac{1}{\beta m R} \log(Z_{\Lambda_m}(\beta))$$

in terms of the principal (the largest) eigenvalue of a suitably chosen matrix or linear map. Steps:

- Notice that K_β from the first part of this exercise can be seen as a matrix with positive entries, which can be chosen to be symmetric.
- Show that $Z_{\Lambda_m}(\beta)$ can be represented as some scalar product involving the n^{th} power of K_β.
- Use the results of the first exercise to express the pressure in terms of the principal eigenvalue of K_β.

(c) Let $a, c, d : \mathbb{R} \to \mathbb{R}$ be C^∞ functions and

$$A := \begin{pmatrix} a(t) & c(t) \\ c(t) & d(t) \end{pmatrix}$$

Show that if $A(t_0)$ has two distinct eigenvalues $|\lambda_1(t_0)| > |\lambda_2(t_0)|$, then there is some $\delta > 0$ such that $A(t)$ has two distinct eigenvalues for all $t \in (t_0 - \delta, t_0 + \delta)$, moreover, $\lambda_1(t)$ is a C^∞ function in some neighbourhood of t_0.

(d) Using the results of the second and third parts of the exercise show that $p(\beta)$ is a C^∞ function for $\beta \neq 0$.