Mathematical Statistical Physics 2020

Classical part: Intro & 1 The Ising model

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About the course

Quick take-away:

- This course is about systems with very many or even infinitely many degrees of freedom.
- The initial physical motivation is statistical physics and a mathematically rigorous understanding of phase transitions.
- We'll cover systems in thermal equilibrium, classical and quantum.
- On the math side, that means
  - **classical**: probability theory, Gibbs measures, DLR conditions (Dobrushin-Lanford-Ruelle)
  - **quantum**: C^*-algebras, KMS states (Kubo-Martin-Schwinger).

Why care?

Maybe phase transitions and statistical physics bring up unexciting memories of undergraduate thermodynamics.

But, as Robert Helling perhaps explained: we won't do steam engines.

Instead, we want to understand a very fundamental question, namely how to make sense of equilibrium states for infinite systems, where the standard Boltzmann-Gibbs way

$$\frac{1}{Z}\exp(-\beta H)$$

does not work because $Z = \infty$.

And, we'll see how infinite systems enter the mathematics of phase transitions and spontaneous symmetry breaking.

Remark: Many quantum concepts pop up in modern mathematical statistical physics: renormalization, Feynman diagrams, Gaussian free field...

Outline of the classical part

We'll follow the book by Sacha Friedli and Yvan Velenik available at

`https://www.unige.ch/math/folks/velenik/smbook/`

The current plan is to do:

1. Ising model—existence of the thermodynamic limit, Peierls argument: parts of FV Chapter 3.
2. Gibbs measures in infinite volume: parts of FV Chapter 6
3. Absence of continuous symmetry-breaking in dimensions 1 and 2, Mermin-Wagner theorem: parts of FV Chapter 9
4. Reflection positivity, Existence of continuous symmetry-breaking in dimension 3: parts of FV Chapter 10.

Subject to adjustments along the way!

The Ising model:

generalities & motivation

FV Chapter 1.4., Chapter 3

[!!! Notations in FV Chapter 1.4 and Chapter 3 differ a bit. I'll start with notation from Chapter 1.4.]

The model

Spins ± 1 on finite portion of lattice.

Notation: $\Lambda \Subset \mathbb{Z}^d$: finite non-empty subset.

For each lattice site $i \in \Lambda$: a spin $\omega_i \in \{+1, -1\}$ (up $\uparrow$ & down $\downarrow$).

Configuration space:

$$\Omega_\Lambda = \{1, -1\}^\Lambda = \big\{(\omega_i)_{i \in \Lambda} : \omega_i = \pm 1\big\}.$$

Energy at external field $h \in \mathbb{R}$

$$\mathscr{H}_{\Lambda;h}(\omega) = -\sum_{\substack{\{i,j\}: \\ i \sim j}} \omega_i \omega_j - h \sum_{i \in \Lambda} \omega_i.$$

First sum over non-ordered pairs $\{i, j\} \in \Lambda^2$ that are nearest neighbors:

$$i \sim j \;\; :\Leftrightarrow ||i - j|| = 1.$$

$$\mathscr{H}_{\Lambda;h}(\omega) = -\sum_{\substack{i,j\in\Lambda:\\ i\sim j}} \omega_i\omega_j - h\sum_{i\in\Lambda}\omega_i.$$

First sum is minimal if all spins are aligned: $\omega_i = \omega_j$.

Second sum is minimal if spins align on external field: all up $\omega_i \equiv +1$ if $h > 0$, all down $\omega_i \equiv -1$ if $h < 0$.

Partition function at inverse temperature $\beta > 0$

$$\mathbf{Z}_{\Lambda;\beta,h} = \sum_{\omega\in\Omega_\Lambda} \mathrm{e}^{-\beta\mathscr{H}_{\Lambda;h}(\omega)}.$$

Gibbs measure = probability measure on configuration space

$$\mu_{\Lambda;\beta,h}(\omega) = \frac{1}{\mathbf{Z}_{\Lambda;\beta,h}}\,\mathrm{e}^{-\beta\mathscr{H}_{\Lambda;h}(\omega)}.$$

$\beta = 0$: uniform distribution.
$\beta \to \infty$: concentrated on energy minimizer(s).

The question

Is there a phase transition as β and h are varied?

What is a phase transition?

Math: parameter-dependent model. Phase transition = "something" happens as parameters are varied. That's not very precise. . .

Phys: at water boiling point 100°C:

- jump discontinuity of physically relevant quantities: density liquid water $\neq$ density of vapor.
- coexistence of two phases: liquid water and vapor.

Will lead to **two math definitions** of (first-order) phase transitions. Equivalence of definitions not clear at all!

We start with the jump discontinuity aspect.

Pressure, magnetization, thermodynamic limit

Pressure (also called: $-$ free energy per unit volume)

$$\psi_\Lambda(\beta,h) = \frac{1}{\beta\,|\Lambda|}\log \mathbf{Z}_{\Lambda;\beta,h}, \qquad \psi(\beta,h) = \lim_{\Lambda\nearrow\mathbb{Z}^d}\psi_\Lambda(\beta,h).$$

Total magnetization

$$M_\Lambda(\omega) = \sum_{i\in\Lambda}\omega_i.$$

h-derivative $\leftrightarrow$ average magnetization $(\exp(-\beta(-h\sum_i\omega_i)))$

$$\begin{aligned}
\frac{\partial}{\partial h}\psi_\Lambda(\beta,h) &= \frac{1}{|\Lambda|}\frac{1}{\mathbf{Z}_{\Lambda;\beta,h}}\sum_{\omega\in\Omega_\Lambda}\sum_{i\in\Lambda}\omega_i \mathrm{e}^{-\beta\mathscr{H}_{\Lambda;h}(\omega)} \\
&= \frac{1}{|\Lambda|}\sum_{\omega\in\Omega_\Lambda} M_\Lambda(\omega)\mu_{\Lambda;\beta,h}(\omega) \\
&= \frac{1}{|\Lambda|}\langle M_\Lambda\rangle_{\Lambda;\beta,h}.
\end{aligned}$$

$$\frac{\partial}{\partial h}\psi_\Lambda(\beta,h) = \frac{1}{|\Lambda|}\langle M_\Lambda\rangle_{\Lambda;\beta,h} =: m_\Lambda(\beta,h).$$

Can anything interesting happen as β and h are varied?
E.g. discontinuity of the magnetization?

In finite volume, no! Functions $\psi_\Lambda(\beta,h)$ and $m_\Lambda(\beta,h)$ are analytic (= Taylor expansion has non-zero radius of convergence, everywhere).

In order to see jump discontinuities, we have to take the limit $\Lambda \nearrow \mathbb{Z}^d$.

Today, this is standard procedure. But it used to be debated...

A major concern and a much debated issue, in the theoretical physics community at the beginning of the 20th century, was to determine whether phase transitions could be described within the framework of statistical mechanics, still a young theory at that time. [FV p. 37]

In November 1937, during the Van der Waals Centenary Conference, a morning-long debate took place about the following question: does the partition function contain the information necessary to describe a sharp phase transition? As the debate turned out to be inconclusive, Kramers, who was the chairman, put the question to a vote, the result of which was nearly a tie (the "yes" winning by a small margin). [FV Appendix A]

This question was settled using the Ising model. The latter is indeed *the first system of locally interacting units for which it was possible to prove the existence of a phase transition*. This proof was given in the above-mentioned paper by Peierls in 1936, using an argument that would later become a central tool in statistical mechanics.

Its simplicity and the richness of its behavior have turned the Ising model into a preferred laboratory to test new ideas and methods in statistical mechanics. [FV p. 37]

Coming next

Before we can work our way through the historical proof by Peierls, we need to make sure that all objects are well-defined:

1. Be more precise: what exactly do we mean by $\Lambda \nearrow \mathbb{Z}^d$?
2. Check the existence of the thermodynamic limit that was used to define $\psi(\beta, h)$.

This is done at the beginning of FV Chapter 3.