1 Lecture 1

Recall that the **tangent bundle** is also a smooth manifold, with the natural footpoint projection given by

$$\pi : TM \to M. \tag{1}$$

A **section** of π is a (smooth) map $X : M \to TM$ such that $\pi \circ X = \mathrm{id}_M$.

Definition 1.0.1: Riemannian Metric & Riemannian Manifold

*A (C^k-)**Riemannian metric** g on M is a family $g = \{g_p\}_{p\in M}$ of inner products (positive definite, symmetric bilinear)*

$$g_p : T_pM \times T_pM \to R \tag{2}$$

such that for any smooth vector fields X, Y on M, the function

$$f : M \to R, \quad p \mapsto g_p(X_p, Y_p) \tag{3}$$

*is smooth (C^k-differentiable). The pair (M, g) is called a **Riemannian manifold** .*

Remarks.

1. Given a C^∞ atlas of M, then a family of inner products $g = \{g_p\}_{p\in M}$ is a (C^k-)Riemannian metric iff for any chart (U, x) in this atlas the function

$$U \to R, \quad p \mapsto g_p\Big(\frac{\partial}{\partial x_j}\Big|_p, \frac{\partial}{\partial x_k}\Big|_p\Big) \tag{4}$$

   is smooth. (To be shown in an exercise class.)

2. Suppose we have an open subset U inside a (C^k-)Riemannian manifold (M, g). Then $(U, g|_U$ is also a (C^k-)Riemannian manifold, where

$$g|_U := \{g_p\}_{p\in U}. \tag{5}$$

3. We sometimes write $\langle \cdot, \cdot \rangle$ instead of g (respectively $\langle \cdot, \cdot \rangle_p$ instead of g_p).

4. Sometimes, also indefinite families of inner products are considered, so-called **semi-Riemannian metrics/manifolds**, e.g. in the theory of relativity where you have Lorentzian metrics, i.e. semi-Riemannian metrics of signature $(1, n)$ like the Lorentzian metric

$$x_0y_0 - \sum_{i=1}^{n} x_iy_i \text{ on } \mathbb{R}^{n+1}. \tag{6}$$

A simple example of a Riemannian manifold: any open subset U of a Euclidean vector space $(V, \langle\cdot,\cdot\rangle)$ is a Riemannian manifold with Riemannian metric $g_p = \langle\cdot,\cdot\rangle_p,\ p \in U$.

Proposition 1.0.2: Riemannian Immersions

Let M a smooth manifold, (N, h) a Riemannian manifold, $\varphi : M \to N$ a smooth (C^{k+1}-)immersion. Then (N, h) induces a Riemannian metric g on M via

$$g_p(v, w) = h_{\varphi(p)}(d\varphi_p v, d\varphi_p w); \quad p \in M;\ v, w \in T_pM. \tag{7}$$

*This Riemannian metric is called the **pullback of h via ϕ, denoted φ^*h.** The immersion $\varphi : (M, g) \to (N, h)$ is called a **Riemannian immersion** .*

Proof. Since $d\varphi_p : T_pM \to T_{\varphi(p)}N$ is linear and injective, $\forall p \in M$, g_p is bilinear and positive definite (as $h_{\varphi(p)}$ is).
Symmetry of $h_{\varphi(p)} \Rightarrow$ symmetry of $g_p \to g_p$ an inner product.
Smoothness. Let's first assume that φ is a (local) diffeomorphism. Then a smooth vector field X on M induces a smooth vector field on N, namely

$$(\varphi^*X) : q \mapsto d\varphi_{\varphi^{-1}(q)} X_{\varphi^{-1}(q)} \tag{8}$$

and we have that

$$\begin{aligned} p \mapsto g_p(X_p, Y_p) &= h_p((\varphi^*X)_{\varphi(p)}, (\varphi^*Y)_{\varphi(p)}) && (9)\\ &= (h((\varphi^*X)_., (\varphi^*Y)_.) \circ \varphi)(p) && (10) \end{aligned}$$

For the general case, one can now apply the local structure for immersions and assume that

$$M = U \subset \mathbb{R}^n \text{ open, } N = V \subset \mathbb{R}^{n+k} \text{ open} \tag{11}$$

and $\varphi = i : U \hookrightarrow V$ the inclusion. Next, extend the vector field X on U to

$$\tilde{X}(p) = (X(p), 0) \text{ on } V. \tag{12}$$

($\to$ details next Wednesday). $\square$

Remarks.

1. By Whitney's immersion theorem, any smooth manifold can be immersed (actually embedded) into some $\mathbb{R}^n$. So it inherits a Riemannian metric from $\mathbb{R}^n$.

2. By Nash's embedding, any ($C^{k\geq 3}$-)Riemannian manifold can be isometrically immersed (embedded) into some $\mathbb{R}^n$.

Note. If g, h are Riemannian metrics on M, then also $(g + h)_p = g_p + h_p$

defines a Riemannian metric on M.

$$f : M \to \mathbb{R} \tag{13}$$

is a positive smooth function, then

$$(f \cdot g)_p = f_p g_p \tag{14}$$

defines a Riemannian metric on M.

Example 1.0.3: Balls and Riemannian Metric.

B_1^n together with the Riemannian metric g, given by

$$g_p(v, w) = \frac{1}{(1 - \langle p, p \rangle)^2} \langle v, w \rangle_p \tag{15}$$

defines a Riemannian manifold isometric to hyperbolic space.

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