Contents

1 Lecture 1

Overview of the math part of the lecture. FV = Friedli, Velenik.

1. Ising model: existence of thermodynamic limit of pressure/free energy. We will go over Peierl's argument and phase transitions. *Book references:* FV Chapter 3.

2. Gibbs measures in infinite volume. DLR conditions: Debrushin, Lanford, Ruelle. *Book references:* FV Chapter 6.

3. Mermin-Wagner theorem, absence of continuous symmetry breaking in $d = 1, 2$. *Book references:* FV Chapter 9.

4. If time admits: reflection positivity & existence of symmetry breaking in $d = 3$. *Book references:* FV Chapter 10.

1.1 The Ising Model

1.1.1 The Model

Notation: we will denote by $\Lambda \Subset \mathbb{Z}^d$ that $\Lambda \subset \mathbb{Z}^d$, and that Λ is finite, and non-empty. Often, it will be the case that $\Lambda = \{1, \ldots, L\}^d,\ L \in \mathbb{N}$; i.e., that Λ consists of a square grid (a "lattice").

Configuration space. We denote a configuration space by $\Omega_\Lambda := \{-1, +1\}^\Lambda$. Here,

$$\omega \in \Omega_\Lambda,\ \omega = (\omega_i)_{i \in \Lambda},\ \omega_i \in \{\pm 1\}. \tag{1}$$

More verbosely: Ω_Λ is the set of functions assigning either $+1$ or -1 onto each vertex of the lattice, and each ω is an individual "configuration" which is a collection of $\{\pm 1\}$ assigned to each vertex $i \in \Lambda$.

We also let $h \in \mathbb{R}$ be an external magnetic field. Now, we are in place to define the "**Ising Hamiltonian**", denoted by $\mathscr{H}_{\Lambda;h}$:

$$\begin{aligned} &\mathscr{H}_{\Lambda;h} : \Omega_\Lambda \to \mathbb{R}, \\ &\omega = (\omega_i)_{i\in\Lambda} \mapsto \mathscr{H}_{\Lambda;h}(\omega) = -\sum_{\substack{\{i,j\}\subset\Lambda \\ i,j \text{ n.n.}}} \omega_i\omega_j - h\sum_{i\in\Lambda}\omega_i \end{aligned}$$

where "n.n." means "nearest neighbors in $\mathbb{Z}^d$".

A natural question that arises is: what is (are) the minimizer(s) of $\mathscr{H}$?

$$\begin{aligned} &h > 0 : \omega_i = 1 \ \forall i, \\ &h < 0 : \omega_i = -1 \ \forall i, \\ &h = 0 : 2 \text{ minimizers, all } +1 \text{ or all } -1. \end{aligned}$$

We also introduce now the **partition function** (Zustandssamme):

$$Z_\Lambda = \sum_{\omega\in\Omega_\Lambda} e^{-\beta\mathscr{H}_{\Lambda;h}(\omega)}. \tag{2}$$

where $\beta = \frac{1}{T}$ is the **inverse temperature**.

With this, we may then define the **Gibbs measure**, which is a probability measure on Ω_Λ:

$$\mu_{\lambda;\beta,h}(\{\omega\}) = \frac{1}{Z_\Lambda(\beta,h)} e^{-\beta\mathscr{H}_{\lambda;h}(\omega)} \tag{3}$$

Let's take note of a couple important regimes in different values of the parameter β. First, when $\beta = 0$, we get a uniform distribution: it's completely flat. When $\beta \to \infty$, then the measure concentrates on the minimzer(s).

Now, we can define **pressure/free energy**:

$$\psi_\Lambda(\beta,h) := \frac{1}{\beta|\Lambda|} \log Z_\Lambda(\beta,h). \tag{4}$$

This quantity is dependent upon the system size, inverse temperature, and magnetic field, as we might intuitively expect.

We can also define the **total magnetization**, which is the map:

$$M_\lambda : \Omega_\Lambda \to \mathbb{R}, \tag{5}$$

$$\omega \mapsto M_\Lambda(\omega) = \sum_{i\in\Lambda}\omega_i. \tag{6}$$

Physically, this is just the sum of the spins on each vertex of the lattice Λ.

Observation. We now make an observation regarding the dependence of ψ_Λ on h:

$$\frac{\partial}{\partial h}\psi_\Lambda(\beta,h) = \frac{1}{|\Lambda|}\sum_{\omega\in\Omega_\Lambda} M_\Lambda(\omega)e^{-\beta\mathscr{H}_{\Lambda,h}(\omega)}\frac{1}{Z_\Lambda(\beta,h)} \tag{7}$$

$$= \frac{1}{|\Lambda|}\langle M_\Lambda\rangle_{\Lambda,\beta,h} \tag{8}$$

$$=: m_\Lambda(\beta,h), \tag{9}$$

where $m_\Lambda(\beta,h)$ is the **average magnetization** per unit volume.

2 Lecture 2

2.1 Thermodynamic Limit of the Pressure

We first introduce some new notation. We denote

$$\varepsilon_\Lambda := \{\{i,j\}\subset\Lambda : \underbrace{||i-j||=1}_{i\sim j;\ i,j \text{ n.n.}}\}. \tag{10}$$

We regard ε_Λ as the bulk with interactions across the boundary. Now, for a the bulk which *does* have interactions across the boundary, we denote this as

$$\varepsilon^b_\Lambda := \{\{i,j\}\subset\mathbb{Z}^d : i\sim j,\ i\in\Lambda \text{ or } j\in\Lambda\}. \tag{11}$$

Now, let's consider the energy with *empty boundary*: $\beta>0$, and $h\in\mathbb{R}$. We calculate:

$$\mathscr{H}^\emptyset_{\Lambda,\beta,h} : \Omega_\Lambda\to\mathbb{R}, \quad \mathscr{H}_{\Lambda,\beta,h}(\omega) := -\beta\sum_{\{i,j\}\in\varepsilon_\Lambda}\omega_i\omega_j - h\sum_{i\in\Lambda}\omega_i. \tag{12}$$

Moreover, we may write other thermodynamic quantities with the notion of the "empty boundary condition":

$$Z^\emptyset_{\Lambda,\beta,h} := \sum_{\omega\in\Omega_\Lambda} e^{-\mathscr{H}^\emptyset_{\Lambda,\beta,h}}, \quad \mu^\emptyset_{\Lambda,\beta,h}, \quad \langle\ ,\ \rangle^\emptyset_{\Lambda,\beta,h}, \quad \psi^\emptyset_\Lambda(\beta,h) := \frac{1}{|\Lambda|}\log Z^\emptyset_{\Lambda,\beta,h}. \tag{13}$$

Now, consider an infinite system where we have $\Omega := \{+1,-1\}^{\mathbb{Z}^d}$. We formalize the boundary conditions. Consider $n\in\Omega$, for example $\eta_i = \pm1\ \forall i\in\mathbb{Z}^d$. Then let

$$\Omega^\eta_\Lambda := \{\omega\in\Omega : \omega_i = \eta_i\ \forall i\in\mathbb{Z}^d\setminus\Lambda\}. \tag{14}$$

Then, the energy becomes:

$$\mathscr{H}_{\Lambda,\beta,h} : \Omega\to\mathbb{R} \tag{15}$$

$$\omega\mapsto\mathscr{H}_{\Lambda,\beta,h}(\omega) := -\beta\sum_{\{i,j\}\in\varepsilon^b_\Lambda}\omega_i\omega_j - h\sum_{i\in\Lambda}\omega_i \tag{16}$$

Note that since $\omega \in \Omega^\eta_\Lambda$, then

$$\mathscr{H}_{\Lambda,\beta,h}(\omega) = \mathscr{H}^\emptyset_{\Lambda,\beta,h}(\omega_\Lambda) - \beta \sum_{\substack{\{i,j\}: i\sim j \\ i\in\Lambda, j\in\Lambda^C}} \omega_i \eta_j. \tag{17}$$

(Note that here, $\omega_j = \eta_j$.)

So then with $\omega_\Lambda = (\omega_i)_{i\in\Lambda}$, the partition function is

$$Z^\eta_{\lambda,\beta,h} := \sum_{\omega\in\Omega^\eta_\Lambda} e^{-\mathscr{H}_{\Lambda,\beta,h}(\omega)} \tag{18}$$

$$\mu^\eta_{\Lambda,\beta,h},\ \psi_{\Lambda,\beta,h}(\beta, h). \tag{19}$$

Definition 2.1.1: van Hove Convergence

$(\Lambda_n)_{n\in\mathbb{N}}$, $\Lambda_n \Subset \mathbb{Z}^d$ converges to $\mathbb{Z}^d$ in the sense of ***van Hove*** *(or "is a van Hove sequence") if:*

1. $\Lambda_n \uparrow \mathbb{Z}^d :\ \forall n, \Lambda_n \subset \Lambda_{n+1},\ \mathbb{Z}^d = \cup_{n\in\mathbb{N}} \Lambda_n$.
2. $$\lim_{n\to\infty} \frac{\left|\partial^{\text{in}}\Lambda_{\text{in}}\right|}{|\Lambda_n|} \to 0 \tag{20}$$

   where the inner boundary is

   $$\partial^{\text{in}}\Lambda := \{i \in \Lambda : \exists j \in \Lambda^C \text{ such that } \|j - i\| = 1\} \tag{21}$$

Notation. *We will denote convergence in the sense of van Hove by $\Lambda_n \Uparrow \mathbb{Z}^d$.*

Example 2.1.2: van Hove Sequences

First, let $\Lambda_n = \{-n, \ldots, n\}^d$, then

$$\left|\partial^{\text{in}}\Lambda_n\right| = \mathcal{O}(n^{d-1}), \tag{22}$$

$$|\Lambda_n| = n^d. \tag{23}$$

This is a van Hove sequence. For a second example, now let $\Lambda_n = \{-n, \ldots, n\} \times \{-n^2, \ldots, n^2\}$ in $\mathbb{Z}^2$. Then

$$\left|\partial^{\text{in}}\Lambda_n\right| = \mathcal{O}(n^2), \tag{24}$$

$$|\Lambda_n| = cn^3. \tag{25}$$

Theorem 2.1.3: van Hove sequence properties

1. *The limit* $\psi(\beta, h) = \lim_{n\to\infty} \frac{1}{|\Lambda|} \log \mathbb{Z}^{\#}_{\Lambda_n,\beta,h} = \lim_{n\to\infty} \psi^{\#}_{\Lambda_n}(\beta, h)$ *exists for all van Hove sequences, for every boundary condition* $\# = \emptyset,\ \# = \eta \in \Omega$.

   The value does not depend on the precise choice of van Hove sequence.

2. *The value of the limit* $\psi(\beta, h)$ *does not depend on the precise choice of boundary condition, either.*

3. *The map*

$$(0, \infty) \times \mathbb{R} \to \mathbb{R} \tag{26}$$

$$(\beta, h) \mapsto \psi(\beta, h) \tag{27}$$

   is convex.

4. $\forall \beta > 0,\ \psi(\beta, h) = \psi(\beta, -h)$.

Definition 2.1.4: Convex

For a function to be ***convex*** *means that* $\forall (\beta_1, h_1),\ \forall (\beta_2, h_2),\ \forall t \in [0, 1]$*, then*

$$\psi\Big((1-t)\beta_1 + t\beta_2, (1-t)h_1 + th_2\Big) \leq (1-t)\psi(\beta_1, h_1) + t\psi(\beta_2, h_2). \tag{28}$$

Proof of parts 1, 2 of Theorem (2.1) in Live Notes LN2.

3 Lecture 3

The proof from last lecture was completed today. See live notes, or the textbook.

Remark. Note also that $Z^{\eta}_{\Lambda_n}(\beta, -h) = Z^{-\eta}_{\Lambda_n}(\beta, h)$.

3.1 Convexity & Magnetization

Recall the definitions of the total magnetization and average magnetization. Can we pass to the limit $\Lambda_n \Uparrow \mathbb{Z}^d$?

To do this, we first note a few useful properties of convex functions. Let $I = (a, b) \subset \mathbb{R}$ be an open interval, $f : (a, b) \to \mathbb{R}$ convex. Then:

1. $$\frac{f(y) - f(x)}{y - x} \leq \frac{f(z) - f(x)}{z - x} \leq \frac{f(z) - f(y)}{z - y}. \tag{29}$$

2. The **one-sided limits**

$$f'(x+) := \lim_{y \downarrow x} \frac{f(y) - f(x)}{y - x}, \tag{30}$$

$$f'(x-) := \lim_{y \uparrow x} \frac{f(y) - f(x)}{y - x} \tag{31}$$

exist (in $\mathbb{R}$).

3. $$x < y \implies f'(x-) \leq f'(x+) \leq f'(y-) \leq f'(y+) \tag{32}$$

4. f is continuous in I ($\leftarrow$ open).

5. Let

$$\mathcal{B} := \{x \in (a,b) : f'(x-) < f'(x+)\} \tag{33}$$

$$= \{x \in (a,b) : f \text{ is not differentiable in } x\}. \tag{34}$$

then $\mathcal{B}$ is empty, finite, or countably infinite.

6. For all $x, y \in (a,b)$,

$$f(y) \geq f(x) + f'(x+)(y - x), \tag{35}$$

$$f(y) \geq f(x) + f'(x-)(y - x). \tag{36}$$

Lemma 3.1.1: Convex Functions & Ordering of $\limsup$ and $\liminf$

Let $f_n : (a,b) \to \mathbb{R}$, $n \in \mathbb{N}$ convex, and $f : (a,b) \to \mathbb{R}$. Suppose $\forall x \in (a,b)$ that $\lim_{n\to\infty} f_n(x) = f(x)$. Then:

1. *f is convex.*
2. *$\forall x \in (a,b)$,*

$$f'(x-) \leq \liminf_{n\to\infty} f_n'(x-) \leq \limsup_{n\to\infty} f_n'(x+) \leq f'(x+). \tag{37}$$

Consequences.

1. If in addition f_n is differentiable at x, then every accumulation point of $(f_n'(x))_{n\in\mathbb{N}}$ is in $[f'(x-), f'(x+)]$.
2. If f_n and f are differentiable at x, then

$$\lim_{n\to\infty} f_n'(x) = f'(x). \tag{38}$$

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