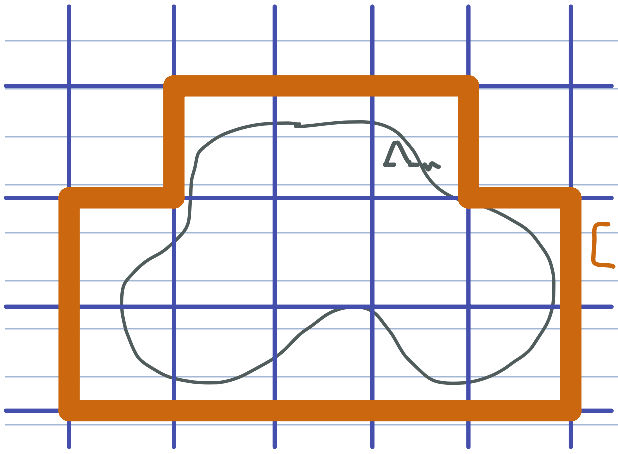


$[\Lambda_n] = \bigcup_j D_k^{(j)}$

$$|\psi^{\emptyset}_{\Lambda_n} - \psi| \le \underbrace{|\psi^{\emptyset}_{\Lambda_n} - \psi^{\emptyset}_{[\Lambda_n]}|}_{c} + \underbrace{|\psi^{\emptyset}_{[\Lambda_n]} - \psi^{\emptyset}_{D_k}|}_{b} + \underbrace{|\psi^{\emptyset}_{D_k} - \psi|}_{a}$$

(a) .. $\exists k_0 \quad n \ge k_0 \Rightarrow (a) \le \frac{\varepsilon}{3}$
$\quad \overset{\shortparallel}{k_0(\varepsilon)}$

(b) $\le \beta d 2^{-k} \rightsquigarrow \exists k_1 : k \ge k_1 \Rightarrow (b) \le \frac{\varepsilon}{3}$

(c) Fix $k \ge \max(k_0, k_1)$, $\Delta_n := [\Lambda_n] \setminus \Lambda_n$

Obs: $|\mathcal{H}^{\emptyset}_{\Lambda_n} - \mathcal{H}^{\emptyset}_{[\Lambda_n]}| \le (2d\beta + |h|)|\Delta_n|$

$$\Rightarrow Z^{\emptyset}_{[\Lambda_n]} = \sum_{\omega \in \Omega_{[\Lambda_n]}} e^{-\beta \mathcal{H}^{\emptyset}_{[\Lambda_n]}(\omega)}$$
$$\le 2^{|\Delta_n|} \underbrace{\left(\sum_{\omega' \in \Omega_{\Lambda_n}} e^{-\mathcal{H}^{\emptyset}_{\Lambda_n}(\omega')} \right)}_{Z^{\emptyset}_{\Lambda_n}} e^{(2d\beta + |h|)|\Delta_n|}$$

similar lower bound

$$\Rightarrow |\log Z^{\emptyset}_{\Lambda_n} - \log Z^{\emptyset}_{[\Lambda_n]}| \le |\Delta_n| \underbrace{(2d\beta + h + \log 2)}_{=:C}$$

Notice:
$$|\Delta_n| \le |\partial^{in} \Lambda_n| \, |D_k|$$

$$\rightsquigarrow |\psi^{\emptyset}_{\Lambda_n} - \psi^{\emptyset}_{[\Lambda_n]}| = \left| \frac{1}{|\Lambda_n|} \log Z^{\emptyset}_{\Lambda_k} - \frac{1}{|[\Lambda_n]|} \log Z^{\emptyset}_{[\Lambda_n]} \right|$$
$$\le \frac{1}{|\Lambda_n|} \underbrace{|\log Z^{\emptyset}_{\Lambda_n} - \log Z^{\emptyset}_{[\Lambda_n]}|}_{(1)\ C \frac{|\Delta_n|}{|\Lambda_n|}} + \underbrace{\left| \left(\frac{1}{|\Lambda_n|} - \frac{1}{|[\Lambda_n]|} \right) \log Z^{\emptyset}_{[\Lambda_n]} \right|}_{(2)\ \left| \frac{|[\Lambda_n]|}{|\Lambda_n|} - 1 \right| \cdot \left| \frac{1}{|[\Lambda_n]|} \log Z^{\emptyset}_{[\Lambda_n]} \right| (3)}$$

① $\frac{|\Delta_n|}{|\Lambda_n|} \to 0$ as $n \to \infty$ at fixed k (van Hove).

② : $\frac{|\Delta_n|}{|\Lambda_n|} \to 0$. ③ is bounded (exercise, bound hamilt. by volume)

$\rightsquigarrow \exists n_0 = n_0(\varepsilon, k) : \forall n \geq n_0$, ⓒ $\leq \frac{\varepsilon}{3}$.

Combine bounds: $\ldots \lim_{n\to\infty} |\psi^{\emptyset}_{\Lambda_n} - \psi| = 0$

③ independence of b.c.

$\eta \in \Omega = \{\pm 1\}^{\mathbb{Z}^d}$.

$\mathcal{H}_{\Lambda_n}(\omega) - \mathcal{H}^{\emptyset}_{\Lambda_n}(\omega') \leq 2d\beta|\partial^{in}\Lambda_n|$

$\omega \in \Omega^{\eta}_{\Lambda_n}$
$\omega_i = \eta_i \ \forall i \in \Lambda_n^c$

$\omega_i' = \omega_i$
$i \in \Lambda_n$

$$\omega_i = \begin{cases} \omega_i' , & i \in \Lambda_n \\ \eta_i , & i \in \Lambda_n^c \end{cases}$$

$\omega' \in \Omega_{\Lambda_n}$

$$e^{-2d\beta|\partial^{in}\Lambda_n|} Z^{\emptyset}_{\Lambda_n} \leq Z_{\Lambda_n} \leq e^{2d\beta|\partial^{in}\Lambda_n|} Z^{\emptyset}_{\Lambda_n}$$

$$\Rightarrow \quad |\psi^{\eta}_{\Lambda_n} - \psi^{\emptyset}_{\Lambda_n}| \leq \frac{1}{|\Lambda_n|} 2d\beta|\partial^{in}\Lambda_n| \xrightarrow[n\to\infty]{} 0 \quad \text{(van Hove)}$$

$\square$ (a), (b) of thm.

Proof of (c) (convexity):

Use Hölder: $p, q \in (1, \infty)$, $\frac{1}{p} + \frac{1}{q} = 1$.

$$\sum_{\omega \in \Omega_\Lambda} |f(\omega) g(\omega)| \leq \Big(\sum_{\omega\in\Omega_\Lambda} |f(\omega)|^p\Big)^{1/p} \Big(\sum_{\omega\in\Omega_\Lambda} |g(\omega)|^q\Big)^{1/q}$$

($p = q = 2$ Cauchy Schwarz)

(β_1, h_1) , $(\beta_2, h_2) \in (0, \infty) \times \mathbb{R}$, $t \in (0,1)$

$\beta := (1-t)\beta_1 + t\beta_2 \,, \qquad h := (1-t)h_1 + t h_2 .$

$p, q \in (1, \infty) \qquad \frac{1}{p} = 1-t \,, \quad \frac{1}{q} = t$

$$\mathcal{H}^{\varnothing}_{\Lambda;\beta,h}(\omega) = (1-t)\,\mathcal{H}^{\varnothing}_{\Lambda;\beta_1,h_1}(\omega) + t\,\mathcal{H}^{\varnothing}_{\Lambda;\beta_2,h_2}(\omega)$$

$\Updownarrow$

$-\beta \sum \omega_i \omega_j - h \sum \omega_i$

$$Z^{\varnothing}_{\Lambda_n}(\beta,h) \le \Big(\sum_{\omega \in \Omega_{\Lambda_n}} \big(e^{-\cancel{(1-t)}\mathcal{H}_{\Lambda_n;\beta_1,h_1}(\omega)} \big)^{\cancel{p}} \Big)^{1/p} \uparrow 1-t$$
$$\times \Big(\sum_{\omega \in \Omega_{\Lambda_n}} \big(e^{-\cancel{t}\mathcal{H}_{\Lambda_n;\beta_2,h_2}(\omega)} \big)^{\cancel{q}} \Big)^{1/q} \uparrow t$$

$$\Rightarrow \log Z^{\varnothing}_{\Lambda_n}(\beta,h) \le (1-t)\log Z^{\varnothing}_{\Lambda_n}(\beta_1,h_1) + t \log Z^{\varnothing}_{\Lambda_n}(\beta_2,h_2)$$

Divide by $|\Lambda_n|$, let $n \to \infty$,

$$\psi(\beta,h) \le (1-t)\psi(\beta_1,h_1) + t\psi(\beta_2,h_2). \qquad \square$$

Proof of Thm 2 1 (d) parity $\quad \psi(\beta,h) = \psi(\beta,-h)$.

$$Z^{\varnothing}_{\Lambda_n}(\beta,-h) = Z^{\varnothing}_{\Lambda_n}(\beta,h) \quad \cdots$$

$$\mathcal{H}^{\varnothing}_{\Lambda_n;\beta,-h}(\omega) = \mathcal{H}^{\varnothing}_{\Lambda_n;\beta,h}(-\omega) \qquad \square$$

Remark.

$$Z^{\eta}_{\Lambda_n}(\beta,-h) = Z^{-\eta}_{\Lambda_n}(\beta,h).$$

1.3 Convexity & Magnetization.

$$M_\Lambda(\omega) = \sum_{i \in \Lambda} \omega_i \,, \qquad \begin{pmatrix} M_\Lambda : \Omega_\Lambda \to \mathbb{R} \\ M_\Lambda : \Omega \to \mathbb{R} \end{pmatrix}$$

$$m^{\#}_{\Lambda}(\beta,h) := \Big\langle \frac{1}{|\Lambda_n|} M_\Lambda \Big\rangle^{\#}_{\Lambda;\beta,h}$$

$$m^{\#}_{\Lambda}(\beta,h) = \frac{\partial}{\partial h}\psi^{\#}_{\Lambda}(\beta,h)$$

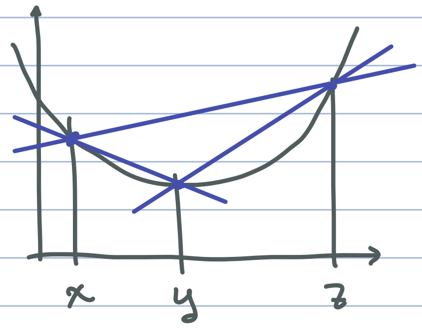
[can we pass to the limit $\Lambda_n \Uparrow \mathbb{Z}^d$?

Some useful properties of convex functions.

$I = (a,b) \subset \mathbb{R}$ open interval.

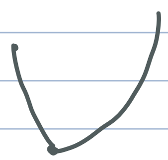
$f : (a,b) \to \mathbb{R}$ convex. Then:

1)



$$\frac{f(y)-f(x)}{y-x} \leq \frac{f(z)-f(x)}{z-x} \leq \frac{f(z)-f(y)}{z-y}$$

2)



$$f'(x_+) := \lim_{y \downarrow x} \frac{f(y)-f(x)}{y-x}$$

$$f'(x_-) := \lim_{y \uparrow x} \frac{f(y)-f(x)}{y-x}$$

exist (in $\mathbb{R}$)

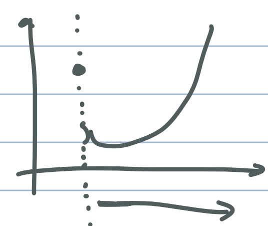
3) $x < y \Rightarrow$

$$f'(x-) \leq f'(x+) \leq f'(y-) \leq f'(y+)$$

4) f is continuous in I ($\leftarrow$ open) ⚠

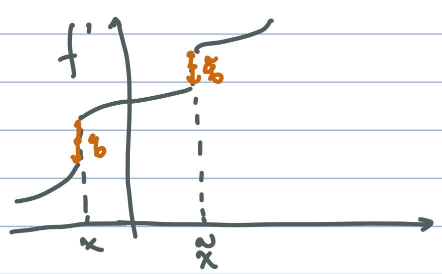
5) $B := \{ x \in (a,b) : f'(x-) < f'(x+) \}$

$= \{ x \in (a,b) : f$ is not differentiable in $x \}$.



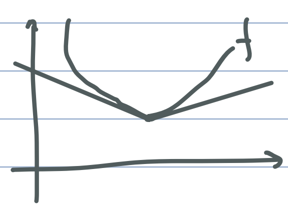
Then B is empty, finite, or countably infinite.

(Proof idea: $x \in B$ $\quad \underbrace{(f'(x-), f'(x+))}_{\text{family of disjoint intervals}}$ pick a $q \in (f'(x-), f'(x+)) \cap \mathbb{Q}$



... injective map $B \to \mathbb{Q}$ ← countable.)

6)



$\forall x, y \in (a,b)$,

$$f(y) \geq f(x) + f'(x+)(y-x)$$
$$f(y) \geq f(x) + f'(x-)(y-x).$$

7) **Lemma 1.3.** $f_n : (a,b) \to \mathbb{R}$, $n \in \mathbb{N}$ convex.

$f : (a,b) \to \mathbb{R}$.

Suppose: $\forall x \in (a,b)$: $\lim_{n\to\infty} f_n(x) = f(x)$.

Then: f is convex, and $\forall x \in (a,b)$:

$$f'(x-) \leq \liminf_{n\to\infty} f_n'(x-) \leq \limsup_{n\to\infty} f_n'(x+) \leq f'(x+)$$

Consequence: · if in addition f_n is differentiable at x, then every accumulation point of $(f_n'(x))_{n\in\mathbb{N}}$ is in $[f'(x-), f'(x+)]$.

· if f_n and f are differentiable at x, then

$$\lim_{n\to\infty} f_n'(x) = f'(x).$$