Exercise sheet 1: From uniform law to Gibbs - Laplace Integrals

Exercise 1 - *From uniform measure to Gibbs or from micro-canonical to canonical ensemble.*

Let $n \in \mathbb{N}$ and $\Omega_n := \{0,1\}^n$. For $E \in \mathbb{N}_0$ let $\mathbb{P}_{n,E}$ be the uniform distribution on

$$\Omega_{n,E} := \left\{ \omega = (\omega_1, \dots, \omega_n) \in \Omega_n \;\middle|\; \sum_{j=1}^{n} \omega_j = E \right\}.$$

For $j \in \{1, \dots, n\}$ let X_j be the map

$$\begin{aligned} X_j : \quad \Omega_n &\to \{0,1\} \\ (\omega_1, \dots, \omega_n) &\mapsto \quad \omega_j. \end{aligned}$$

Finally let $u \in (0,1)$ and $(E_n)_{n\in\mathbb{N}}$ a sequence in $\mathbb{N}$ with $\lim_{n\to\infty} \frac{E_n}{n} = u$.

Show that there exists a uniquely defined $\beta \in \mathbb{R}$ such that, for $\varepsilon \in \{0,1\}$

$$\lim_{n\to\infty} \mathbb{P}_{n,E_n}(X_1 = \varepsilon) = \frac{\mathrm{e}^{-\beta\varepsilon}}{Z(\beta)},$$

where $Z(\beta) = \mathrm{e}^{-\beta.0} + \mathrm{e}^{-\beta.1}$.

Exercise 2 - *Generalization to systems with more than one energy value.*

Let $n \in \mathbb{N}$ and $\Omega_n := \mathbb{N}_0^n$ (we have integer energy levels). As before let $E \in \mathbb{N}_0$, $\Omega_{n,E} := \{\omega = (\omega_1, \dots, \omega_n) \in \Omega_n \mid \sum_{j=1}^{n} \omega_j = E\}$ and define

$$Z(\beta) = \sum_{\varepsilon \in \mathbb{N}_0} \mathrm{e}^{-\beta\varepsilon}$$

.

(a) Let $X_1, X_2, \dots$ be i.i.d r.v with law $\mathbb{P}_\beta(X_k = \varepsilon) = \frac{\mathrm{e}^{-\beta\varepsilon}}{Z(\beta)}$. Show that for all $\beta > 0$, $E \in \mathbb{R}$ we have

$$|\Omega_{n,E}| = Z(\beta)^n \mathrm{e}^{\beta E} \mathbb{P}_\beta(X_1 + \dots + X_n = E).$$

(b) Show that

$$\mathbb{E}_\beta[X_1] = -\frac{\mathrm{d}}{\mathrm{d}\beta} \log(Z(\beta)) \quad , \quad \mathrm{Var}_\beta[X_1] = \frac{\mathrm{d}^2}{\mathrm{d}\beta^2} \log(Z(\beta))$$

and that for every $u > 0$ there exists a unique $\beta := \beta(u) > 0$ with $\mathbb{E}_\beta[X_1] = u$.

(c) Let u and $\beta := \beta(u)$ as in (b), $\frac{E_n}{n} \to u$ and $\mathbb{P}_{n,E_n}$ be the uniform distribution on Ω_{n,E_n}. Show that

$$\lim_{n\to\infty} \mathbb{P}_{n,E_n}(X_1 = \varepsilon) = \mathbb{P}_\beta(X_1 = \varepsilon)$$

[*Hint: Use local limit theorem for discrete random variables, i.e. if $X_1, X_2, \dots$ are i.i.d. copies of an integer-valued random variable X of mean μ and variance σ^2. Suppose furthermore that there is no infinite subprogression $a + q\mathbb{Z}$ of $\mathbb{Z}$ with $q > 1$ for which X takes values almost surely in $a + q\mathbb{Z}$. Then one has*

$$\mathbb{P}(S_n = m) = \frac{1}{\sqrt{2\pi n}\sigma} e^{-(m-n\mu)^2/2n\sigma^2} + o(1/n^{1/2})$$

for all $n \geq 1$ and all integers m, where the error term $o(1/n^{1/2})$ is uniform in m.]

Exercise 3 - *Laplace Integrals in $\mathbb{R}^n$.*

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^∞ with compact support B and $\varphi : \mathbb{R}^n \to \mathbb{R}$ a C^∞ function on the interior of B and continuous on B. Let the function φ have a minimum at, and only at, the point $x_0 \in B$ (an interior point) and let the Hessian matrix

$$(H(x_0))_{ij} := \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(x_0)$$

be *positive definite.* For the following so-called Laplace's integral

$$I(\lambda) := \int_{\mathbb{R}^d} f(x) \mathrm{e}^{-\lambda \varphi(x)} \mathrm{d}^n x,$$

show the asymptotics

$$I(\lambda) = f(x_0) \sqrt{\frac{(2\pi/\lambda)^n}{\det(H(x_0))}} \mathrm{e}^{-\lambda\varphi(x_0)}(1 + O(\lambda^{-1})), \quad \lambda \to \infty.$$

[*Hint: Use Taylor's polynomial approximation of f and φ around the point x_0, and try to bound the integral appropriately from both sides.*]

Exercise 4 - *Application of Laplace integral.*

The point of this exercise is to prove the Stirling formula, i.e. that

$$n! \sim \sqrt{2\pi n}(n/e)^n.$$

(a) Use the fact that $n! = \Gamma(n+1)$ to show that

$$n! = n^{n+1} \int_0^\infty e^{-nf(s)} \mathrm{d}s,$$

where $f(s) = s - \log(s)$

(b) Use the last exercise (with an appropriate generalisation to non compact domain) to prove Stirling formula.