Wed 12-14. B039.

**Literature.**
- Script.
- Hatcher AT
- Bredon, Geometry & Topology

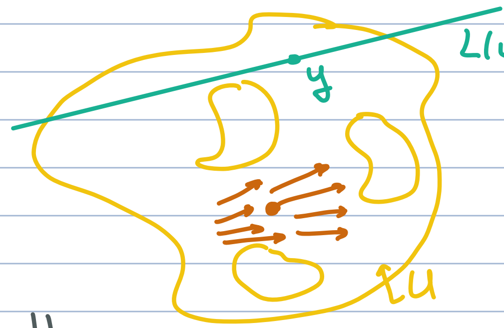
- Older scripts

**Contents.**
- cohomology $H^*(X)$
  - product structures on cohomology
  - Poincaré duality

- higher homotopy groups $\pi_n(X)$

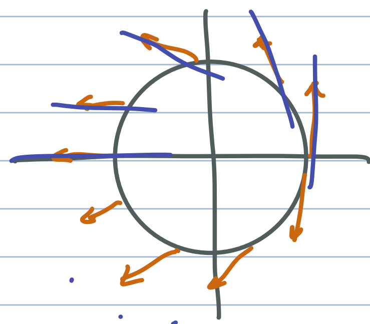
**Why cohomology?** Cohomology is contravariant. If $f: X \to Y$ is continuous, we get induced map $f^*: H^*(Y) \to H^*(X)$

**Toy problem.** $U \subseteq \mathbb{R}^2$ open, and extra structure:
- vfield $V: U \to \mathbb{R}^2$ cts.
- line field $L: U \to \mathbb{RP}^1$ cts




**Question:** given a line field $L$, is there a vfield $V$ s.t. $L(x) = \mathrm{span}(V(x))$ $\forall x \in U$?

Not always!



not valid vectr field,
valid line field.

If this is possible or not $\rightsquigarrow$ lifting problem

$$\begin{tikzcd} & S^1 \arrow[d] \\ U \arrow[r, "L"'] \arrow[ur, dotted, "V\ ?"] & \mathbb{RP}^1 \end{tikzcd}$$

This could be phrased in terms of the fundamental group of $U$ & $\mathbb{RP}^1$.

Can we package this into some kind of <u>obstruction on $U$?</u>

"The problem is the $1 \in \pi_1(\mathbb{RP}^1)$ – if this (or any odd #) is in the image of $L_*$"

We can with cohomology. In $H^1(\mathbb{RP}^1)$, there is a class $\omega$ which measures parity of "how often we turn in $\pi_1$"

Since cohomology is contravariant, we then get

$$L^*\omega \in H^1(U, \mathbb{Z}/2\mathbb{Z})$$

"obstruction to $L = \langle v \rangle$"

If $L^*\omega \neq 0$, $L$ is <u>not</u> defined by the vfield.

This principle is common:

"classifying map"

$$\left(\begin{array}{l}\text{extra structure}\\ \text{on space } X\end{array}\right) \longleftrightarrow \left(\text{map } X \to \mathcal{C}\right)$$

$$f^*\omega \in H^*(X) \leftsquigarrow H^*(\mathcal{C}) \text{ contains}$$

"characteristic classes"

Allows to study/constrain existence of structure on $X$ via study of cohomology.

## Cohomology

Recall: homology:

purely algebraic

$$\text{space } X \rightsquigarrow \text{chain complex } C_i \rightsquigarrow \text{homology } H_i$$

& dualize

$$\text{cochain cplx } C^i \rightsquigarrow \text{coh } H^i.$$

Suppose we have a chain complex $C_i$ of free Abelian groups.

$$\cdots C_{n+1} \xrightarrow{\partial} C_n \to C_{n-1} \to \cdots$$

take a abelian group $G$ (feel free to think about $G = \mathbb{Z}$) and consider the dual $(C_n)^* = \mathrm{Hom}(C_n, G)$. additive group

We have no natural map from $C_n^* \to C_{n-1}^*$, but we do have one from

$$C_n^* \to C_{n+1}^*$$

$$\partial^* : \mathrm{Hom}(C_n, G) \to \mathrm{Hom}(C_{n+1}, G)$$
$$\varphi \longmapsto \varphi \circ \partial$$

$\rightsquigarrow$

$$\cdots \to C_{n-1}^* \to C_n^* \xrightarrow{\delta} C_{n+1}^* \to C_{n+2}^* \to \cdots$$

⚠ Conventions differ on how to define $\delta$. (Hatcher & Bredon do not put this sign)

$\delta : C_n^* \to C_{n+1}^*$ is $(-1)^{n+1}\partial^*$

**Def:** A cochain complex $C^\bullet$ is a collection of Ab groups $C^n$ w/ codifferentials $\delta_n$ s.t. $\delta_{n+1}\delta_n = 0$.

$\rightsquigarrow$ $\mathcal{H}om(C_\bullet, G)$ is a cochain complex, as defined above.

$\mathcal{H}om(C_\bullet, G)^i = \mathrm{Hom}(C_i, G)$ w/ differential $\delta = (-1)^{n+1}\partial^*$

**Def:** for a cochain cx $C^\bullet$, define cohomology as

$$H^i(C^\bullet) = \ker \delta_i \big/ \mathrm{im}\, \delta_{i-1}$$

If $C_\bullet$ is a chain complex, then any choice of Ab group $G$ gives a C.C.

$$\mathcal{H}om(C_\bullet, G),$$

hence a cohomology and thus coh gps. $H^*(C_\bullet; G)$

If $C_\bullet$ is the singular cochain cx of space $X$, put

$$H^*(X; G) = H^*(\mathcal{H}om(C_\bullet, G))$$

Example:

$$C_\bullet \qquad \cdots \to 0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

$$H_*(C_\bullet) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}/2\mathbb{Z} & n=1 \\ 0 & n=2 \\ \mathbb{Z} & n=3 \end{cases}$$

$G = \mathbb{Z}$, dualize:

$$\cdots \leftarrow 0 \leftarrow \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{0} \underset{\mathbb{Z}}{\overset{}{\mathrm{Hom}(\mathbb{Z},\mathbb{Z})}} \leftarrow 0$$

$$\mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}$$

$$H^*(\mathrm{Hom}(C_\bullet, \mathbb{Z})) = \begin{cases} \mathbb{Z} & n=0 \\ 0 & n=1 \\ \mathbb{Z}/2\mathbb{Z} & n=2 \\ \mathbb{Z} & n=3 \end{cases}$$

$\rightsquigarrow$ not the duals of $H_*$

$G = \mathbb{Z}/2\mathbb{Z}$ has different results...

Functoriality. If $C_\bullet \to D_\bullet$ is a map of coch cxs, dualizing gives $\mathrm{Hom}(D_\bullet, G) \to \mathrm{Hom}(C_\bullet, G)$

$$\varphi \mapsto \varphi \circ f$$

induces a map

$$H^*(\mathrm{Hom}(D_\bullet, G)) \to H^*(\mathrm{Hom}(C_\bullet, G)).$$

We want to now relate coh of $\mathrm{Hom}(C_\bullet, G)$ to hom of $C_\bullet$.

**Lemma.** For any ch cx of Free Abelian gps, any Abelian $G$, there is a natural surj.

$$h : H^n(C_\bullet; G) \to \mathrm{Hom}(H_n(C_\bullet); G)$$

and the sequence

$$0 \to \ker(h) \to H^n(C_\bullet; G) \to \mathrm{Hom}(H_n(C_\bullet); G) \to 0$$

splits.

Proof:

- $Z_n = \mathrm{Ker}(\partial_n) \subseteq C_n$, $B_n = \mathrm{im}(\partial_{n+1}) \subseteq C_n$, $H_n(C_\cdot) = Z_n / B_n$.

- Suppose $\chi \in H^n(C_\cdot ; G)$. This is represented by $\varphi \in \mathrm{Hom}(C_n, G)$ and $\delta(\varphi) = 0 \Leftrightarrow \varphi \circ \partial = 0$.

So:

$$\begin{tikzcd} C_n \arrow[r, "\varphi"] \arrow[d] & G \\ C_n/B_n \arrow[ur, "\bar{\varphi}"'] \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}$$

factors since $\varphi|_{B_n} = 0$.

Then $\bar{\varphi}|_{Z_n/B_n} \in \mathrm{Hom}(H_n(C_\cdot), G)$. This is well-defined and additive: (clear) if $\varphi = \delta(\psi)$, then $\varphi = \psi \circ \partial$ hence $\varphi|_{Z_n} \equiv 0 \Rightarrow \bar{\varphi}|_{Z_n/B_n} \equiv 0$.

Naturality easy.

Look at SES:

$$\begin{tikzcd} & & & C_{n-1} \\ 0 \arrow[r] & Z_n \arrow[r] & C_n \arrow[r, "\partial"] & B_{n-1} \arrow[u, phantom, "\cup|" rotate=90] \arrow[l, bend left, "\sigma"] \arrow[r] & 0 \end{tikzcd}$$

Obs: $B_{n-1} \subseteq C_{n-1}$ free Abelian is itself free Abelian!

Hence this sequence splits.

Then get projection

$$p : C_n \to Z_n$$

$$p = \mathrm{id} - \sigma\partial$$

$\Rightarrow$ any $\varphi \in \mathrm{Hom}(Z_n, G)$ extends to $\tilde{\varphi} \in \mathrm{Hom}(C_n, G)$

$(\tilde{\varphi} = \varphi \circ p)$

Denote by $\pi : Z_n \to H_n(C_\cdot)$, build $\longrightarrow$ $\varphi \circ \pi \circ p$

$$\mathrm{Hom}(H_n(C_\cdot), G) \xrightarrow{\Psi} \mathrm{Ker}(\delta_n)$$

defines the section.

Check that this is the desired section.