AB Geometrie & Topologie
Dr. Christian Lange
Panagiotis Papadopoulos

Riemannian geometry

Problem set 2

1. Let $\Gamma \curvearrowright M$ be a smooth covering space action of a group Γ on a smooth manifold M, i.e. every point $p \in M$ has an open neighborhood U such that $\gamma U \cap U = \emptyset$ for all $\gamma \in \Gamma \backslash \{e\}$. Then there exists a smooth atlas $\mathcal{A} = \{(U_\alpha, x_\alpha)\}_{\alpha \in I}$ of M such that $\gamma U_\alpha \cap U_\alpha = \emptyset$ for all $\alpha \in I$ and all $\gamma \in \Gamma \backslash \{e\}$. In the lecture we have seen that M/Γ with the quotient topology is a topological manifold. Show that
$$\bar{\mathcal{A}} = \{(\pi(U_\alpha), x_\alpha \circ (\pi_{|U_\alpha})^{-1})\}_{\alpha \in I}$$
is a smooth atlas on M/Γ which induces the unique smooth structure on M/Γ for which the projection $\pi : M \to M/\Gamma$ is a local diffeomorphism.

2. (i) Show that the action $\mathbb{Z}^n \curvearrowright \mathbb{R}^n$ defined as $ax = 2\pi a + x$ is an isometric covering space action. Let $T^n = \mathbb{R}^n/\mathbb{Z}^n$ be the quotient Riemannian manifold.

   (ii) Let S^1 be the unit circle in $\mathbb{R}^2$ with the induced Riemannian metric. Show that the "flat torus" T^n from (i) is isometric to the Riemannian product $S^1 \times \ldots \times S^1$ of n copies of S^1 (see Problem set 0).

   (iii) Give an isometric embedding of T^n into $\mathbb{R}^{2n}$.

3. Recall that in the Poincaré ball model of hyperbolic space the hyperbolic metric on the unit ball B in $\mathbb{R}^n$ is defined as
$$g_x(v, w) = \frac{4}{(1 - \langle x, x\rangle)^2} \langle v, w\rangle,$$
where $x \in B$, $v, w \in T_xB$ and $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product on $\mathbb{R}^n$. On the upper half space $H := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$ we consider the Riemannian metric defined for $x \in H$ and $v, w \in T_xH$ by
$$h_x(v, w) = \frac{1}{x_n^2} \langle v, w\rangle.$$

   (i) Show that the restriction to H of the inversion with respect to the sphere of radius $\sqrt{2}$ centered at $(0, \ldots, 0, 1) \in H$ defines an isometry $H \to B$.

   (ii) For $n = 2$ we view H as the upper half plane in the complex plane. Show that h is up to multiplication with a positive real number the unique Riemannian metric on H for which the action of $\mathrm{SL}_2(\mathbb{R})$ on H via Möbius transformations
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}$$
is isometric.

   (iii) Show that the action in (ii) is generated by compositions of inversions at circles. What are the analogous statements in higher dimensions?

You can submit your solutions until **Friday** 9.5.2025 at 6 p.m. in Moodle. Solutions will be discussed in the exercise class on Wednesday 14.5.2025.