Survey and Literature Review on Distributed Observers

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1. Review of Paper: TAC-62(2)-2017
    - Centralized vs distributed observers
    - Problem description

2. Review of Paper: Wang-Morse TAC-63(7)-2018
    - Introduction
    - Observer design equations
    - Solvability analysis of Wang-Morse observers

3. Review of Paper: HTWS TAC-64(1)-2019
    - Introduction
    - Main Theorem
    - Remark

4. Our Project
    - Conjecture

Paper: TAC-62(2)-2017

“Design of Distributed LTI Observers for State Omniscience”

By S. Park and N. C. Martins

Centralized vs distributed observers

Consider LTI plant in state-space form

$$x^+ = Ax, \quad x \in \mathbb{R}^n$$

- Centralized observer:

  $$y = Cx \in \mathbb{R}^r$$

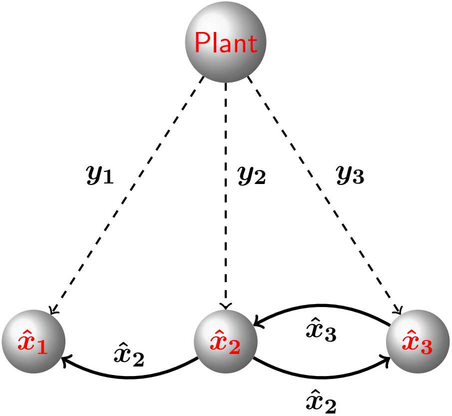
  is available to ensure (C, A) is observable or detectable

- Distributed observer:

  $$y_i = C_i x \in \mathbb{R}^{r_i}, \quad r_i \geq 0, \quad i = 1, 2, \cdots, m$$

  the system can be accessed **separately** at m places, though

  $$(C, A) \text{ with } C = \begin{bmatrix} C_1 \\ \vdots \\ C_m \end{bmatrix} \in \mathbb{R}^r \text{ is observable}$$



Plant with m outputs

$$
\begin{aligned}
x^+ &= Ax \\
y_i &= C_i x, \quad i = 1, \cdots, m
\end{aligned}
$$

Distributed observer:

$$\hat{x}_i^+ = f_i(\hat{x}_i, y_i, \hat{x}_j, j \in \mathcal{N}_i)$$

$\mathcal{N}_i$ "neighbors"

$$i = 1, \cdots, m$$

Centralized (Luenberger) observer:

$$\hat{x}^+ = A\hat{x} + L(y - C\hat{x}), \quad C = \begin{bmatrix} C_1 \\ \vdots \\ C_m \end{bmatrix}$$

NOT realistic! Paper: TAC-62(2)-2017 is a result in this direction

Notation

- The number of observers is a fixed integer

$$m \geq 1$$

- A graph formed by a vertex set $\mathbb{V}$ and an edge set $\mathbb{E}$ is denoted by

$$\mathcal{G} = (\mathbb{V}, \mathbb{E}) \ \text{ with } \ \mathbb{E} \subseteq \mathbb{V} \times \mathbb{V}$$

- $\otimes$ is the Kronecker product
- $\text{diag}(K_1, \cdots, K_p)$ is the block diagonal matrix
-
-
-

Plant and its outputs

Consider LTI plant in state-state form

$$x^+ = Ax, \quad y = Cx, \quad x \in \mathbb{R}^n, \quad y \in \mathbb{R}^r$$

with

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}, \quad C = \begin{bmatrix} C_1 \\ \vdots \\ C_m \end{bmatrix}, \quad y_i = C_i x \in \mathbb{R}^{r_i}, \quad i = 1, \cdots, m$$

The distributed observer by Park-Martins

Structure of the observers

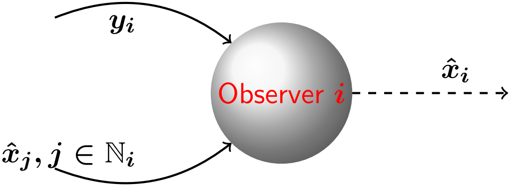
$$\hat{x}_i^+ = A \sum_{j \in \mathbb{N}_i} \mathbf{w}_{ij} \hat{x}_j + \mathbf{K}_i (y_i - C_i \hat{x}_i) + \mathbf{P}_i z_i$$
$$z_i^+ = \mathbf{Q}_i (y_i - C_i \hat{x}_i) + \mathbf{S}_i z_i, \quad i = 1, \cdots, m$$

where the **augmented state**

$$z_i \in \mathbb{R}^{\mu_i}, \quad \sum_{i=1}^{m} \mu_i < m$$

The latter is referred to as the **scalability condition**

- Input and output of the ith observer



$$\hat{x}_i^+ = A \sum_{j \in \mathbb{N}_i} \mathbf{w}_{ij} \hat{x}_j + \mathbf{K}_i (y_i - C_i \hat{x}_i) + \mathbf{P}_i z_i$$

$$z_i^+ = \mathbf{Q}_i (y_i - C_i \hat{x}_i) + \mathbf{S}_i z_i, \quad i = 1, \cdots, m$$

- It is hoped

$$\lim_{k \to \infty} |\hat{x}_i(k) - x(k)| = 0 \text{ and } \sum_{i=1}^{m} \mu_i < m - m_s$$

where $m_s \geq 0$ is the number of **source components**

Paper: Wang-Morse TAC-63(7)-2018

“A Distributed Observer for a Time-Invariant Linear System”

By L. Wang and A. S. Morse

Plant and graph

Consider LTI plant in state-space form

$$\begin{aligned}\dot{x}(t) &= Ax(t) \\ y_i(t) &= C_i x(t), \quad x \in \mathbb{R}^n, \quad y_i \in \mathbb{R}^{s_i}, \quad i \in \mathbf{m} \triangleq \{1, 2, \cdots, m\}\end{aligned}$$

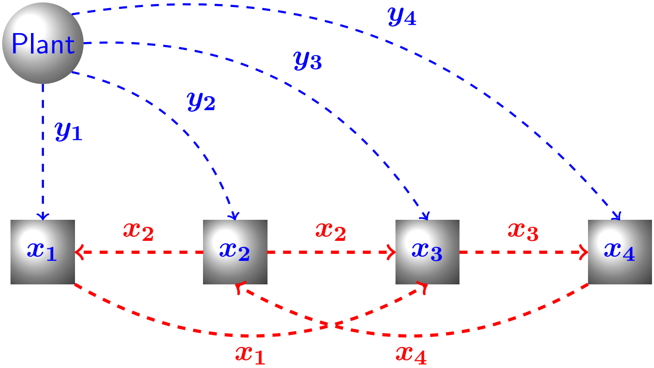
Suppose that

$$(C, A) \text{ with } C = \begin{bmatrix} C_1 \\ \vdots \\ C_m \end{bmatrix} \in \mathbb{R}^r \text{ is observable}$$

For each $i \in \mathbf{m}$,

$\mathcal{N}_i$ is the neighbors (with self-loops) of the ith observer

Illustration



$$m = 4, \quad \mathbf{m} \triangleq \{1, 2, 3, 4\}$$

$$\mathcal{N}_1 = \{1, 2\}, \quad \mathcal{N}_2 = \{2, 4\}, \quad \mathcal{N}_3 = \{1, 2, 3\}, \quad \mathcal{N}_4 = \{3, 4\}$$

Distributed observer by Wang-Morse

Structure of the distributed observer by Wang-Morse

$$\dot{z}_i(t) = \sum_{j \in \mathcal{N}_i} \Big(H_{ij} z_j(t) + K_{ij} y_j(t) \Big)$$
$$x_i(t) = \sum_{j \in \mathcal{N}_i} \Big(M_{ij} z_j(t) + N_{ij} y_j(t) \Big), \quad i \in \mathbf{m}$$

It achieves convergence of the estimation error

$$\lim_{t \to \infty} |\epsilon_i(t)| = 0, \quad \epsilon_i(t) = x_i(t) - x(t), \quad \forall i \in \mathbf{m}$$

In what follows, instead of the above, we focus on a class of distributed observers of the form

$$\dot{z}_i(t) = \sum_{j \in \mathcal{N}_i} H_{ij} z_j(t) + K_i y_i(t), \quad i \in \mathbf{m}$$

with outputs

$$x_i(t) = \sum_{j \in \mathcal{N}_i} M_{ij} z_j(t), \quad i \in \mathbf{m}$$

as the estimates of the trajectory $x(t) \in \mathbb{R}^n$ of the plant

For the distributed observers of the form

$$\dot{z}_i(t) = \sum_{j \in \mathcal{N}_i} H_{ij} z_j(t) + K_i y_i(t), \quad x_i(t) = \sum_{j \in \mathcal{N}_i} M_{ij} z_j(t), \quad i \in \mathbf{m}$$

it should ensure convergence of the estimation error

$$\lim_{t \to \infty} |\epsilon_i(t)| = 0, \quad \epsilon_i(t) = x_i(t) - x(t), \quad \forall i \in \mathbf{m}$$

- **Assumption** (Wang-Morse 2018TAC) The composite system is *jointly observable*, i.e.

$$(C, A) \text{ with } C = \begin{bmatrix} C_1 \\ \vdots \\ C_m \end{bmatrix} \in \mathbb{R}^r \text{ is observable, } C_i \neq 0, \ \forall i \in \mathbf{m}$$

- **Design parameters**

$$i \in \mathbf{m} \ : \ \left\{ (H_{ij}, K_i, M_{ij}) : j \in \mathcal{N}_i \right\}$$

Wang-Morse observer

$$\dot{z}_i(t) = \sum_{j \in \mathcal{N}_i} H_{ij} z_j(t) + K_i y_i(t), \quad x_i(t) = \sum_{j \in \mathcal{N}_i} M_{ij} z_j(t), \quad i \in \mathbf{m}$$

- **Comparison** with Park-Martins observer:
  - First, we outline a construction for systems with **strongly connected neighbor graphs** that enables one to freely adjust the observer's spectrum.
  - Second, the results **apply whether A is singular or not**; the implication of this generalization is that the construction proposed can be used to craft observers for continuous time processes, whereas the construction proposed in [Park-Martins 2017TAC] cannot unless A is nonsingular.

Observer design equations

Consider the plant

$$\Sigma_{plant}: \quad \dot{x} = Ax, \quad y_i = C_i x, \quad i \in \mathbf{m}$$

and its observer

$$\Sigma_{obs}: \quad \dot{z}_i = \sum_{j \in \mathcal{N}_i} H_{ij} z_j + K_i y_i, \quad x_i = \sum_{j \in \mathcal{N}_i} M_{ij} z_j, \quad i \in \mathbf{m}$$

- **Invariance property**[1] There should exist **“immersions”**

  from x system to each z_i system, written by $\theta_i(x)$

  such that

$$\frac{\partial \theta_i(x)}{\partial x} Ax = \sum_{j \in \mathcal{N}_i} H_{ij} \theta_j(x) + K_i C_i x$$

$$x = \sum_{j \in \mathcal{N}_i} M_{ij} \theta_j(x), \quad i \in \mathbf{m}, \quad \forall x \in \mathbb{R}^n$$

[1] Recall the so-called **regulator equations** or **internal model property**

- **Invariance property** There are **“immersions”** $\theta_i(x) \in \mathbb{R}^n$ such that

$$\begin{aligned} \frac{\partial \theta_i(x)}{\partial x} Ax &= \sum_{j \in \mathcal{N}_i} H_{ij}\theta_j(x) + K_i C_i x \\ x &= \sum_{j \in \mathcal{N}_i} M_{ij}\theta_j(x), \quad i \in \mathbf{m}, \quad \forall x \in \mathbb{R}^n \end{aligned}$$

Or equivalently, there are linear mappings $\theta_i(x) = V_i x$ for $i \in \mathbf{m}$ such that

$$V_i A = \sum_{j \in \mathcal{N}_i} H_{ij} V_j + K_i C_i, \quad I_n = \sum_{j \in \mathcal{N}_i} M_{ij} V_j, \quad i \in \mathbf{m}$$

namely **“observer design equations”** in the paper

Solvability analysis of Wang-Morse observers

Consider the observer

$$\Sigma_{obs}: \quad \dot{z}_i = \sum_{j \in \mathcal{N}_i} H_{ij} z_j + K_i y_i, \quad x_i = \sum_{j \in \mathcal{N}_i} M_{ij} z_j, \quad i \in \mathbf{m}$$

If there are linear mappings $\theta_i(x) = V_i x$ for $i \in \mathbf{m}$ satisfying

$$V_i A = \sum_{j \in \mathcal{N}_i} H_{ij} V_j + K_i C_i, \quad I_n = \sum_{j \in \mathcal{N}_i} M_{ij} V_j, \quad i \in \mathbf{m}$$

or re-written as

$$\begin{cases} V_i A = \sum_{j \in \mathcal{N}_i} H_{ij} V_j + K_i C_i \\ \quad 0 = I_n - \sum_{j \in \mathcal{N}_i} M_{ij} V_j, \quad i \in \mathbf{m} \end{cases}$$

Let the observer error state be

$$\epsilon = \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_m \end{bmatrix}, \quad \epsilon_i = z_i - V_i x, \quad i \in \mathbf{m}$$

It gives the error systems

$$\begin{aligned} \widetilde{\Sigma}_{obs}: \quad \dot{\epsilon}_i &= \sum_{j \in \mathcal{N}_i} H_{ij} z_j + K_i y_i - \sum_{j \in \mathcal{N}_i} H_{ij} V_j x - K_i C_i x \\ &= \sum_{j \in \mathcal{N}_i} H_{ij} z_j - \sum_{j \in \mathcal{N}_i} H_{ij} V_j x + K_i C_i x - K_i C_i x \\ &= \sum_{j \in \mathcal{N}_i} H_{ij} \epsilon_j + K_i C_i \epsilon, \quad i \in \mathbf{m} \end{aligned}$$

The error systems

$$\widetilde{\Sigma}_{obs}: \quad \dot{\epsilon}_i = \sum_{j \in \mathcal{N}_i} H_{ij}\epsilon_j + K_i C_i \epsilon, \quad i \in \mathbf{m}$$

can be written in a compact form

$$\widetilde{\Sigma}_{obs}: \quad \dot{\epsilon} = \Xi\epsilon$$

If Ξ is Hurwitz, then $\epsilon = 0$ is exponentially stable and, as $t \to \infty$,

$$z_i(t) - V_i x(t) \to 0 \quad \Rightarrow \quad x(t) - \sum_{j \in \mathcal{N}_i} M_{ij} z_j(t) \to 0$$

Hence, the Wang-Morse observer can be done if, the design parameters

$$i \in \mathbf{m} \;:\; \left\{(H_{ij}, K_i, M_{ij}) : j \in \mathcal{N}_i\right\}$$

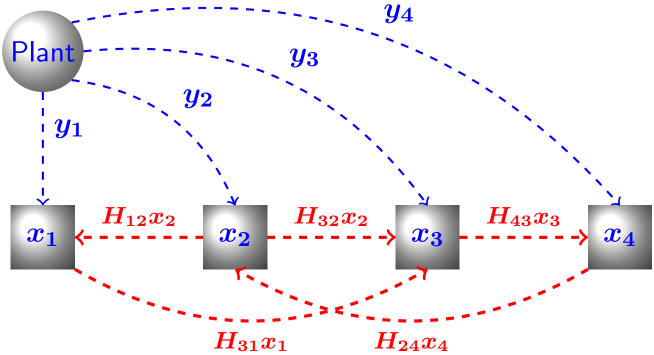
are such that

C1 (immersion condition) there are linear mappings $\theta_i(x) = V_i x$ for $i \in \mathbf{m}$ satisfying

$$\begin{cases} V_i A = \sum_{j \in \mathcal{N}_i} H_{ij} V_j + K_i C_i \\ 0 = I_n - \sum_{j \in \mathcal{N}_i} M_{ij} V_j, \quad i \in \mathbf{m} \end{cases}$$

C2 (stability condition) Ξ is Hurwitz

Question/idea: Regarding the above two key conditions, are they cast to "regulation theory"?



Question (to reduce the communication burden): The design communication gain matrix H_{ij} is restricted, merely relating to output gain C_j for $i \neq j \in \mathcal{N}_i$

Paper: HTWS TAC-64(1)-2019

"A Simple Approach to Distributed Observer Design for Linear Systems"

By W. Han, H. L. Trentelman, Z. Wang, and Y. Shen

Plant and graph

Consider LTI plant in state-space form

$$\begin{aligned}\dot{x}(t) &= Ax(t)\\ y_i(t) &= C_i x(t), \quad x \in \mathbb{R}^n, \quad y_i \in \mathbb{R}^{s_i}, \quad i \in \mathbf{m} \triangleq \{1, 2, \cdots, m\}\end{aligned}$$

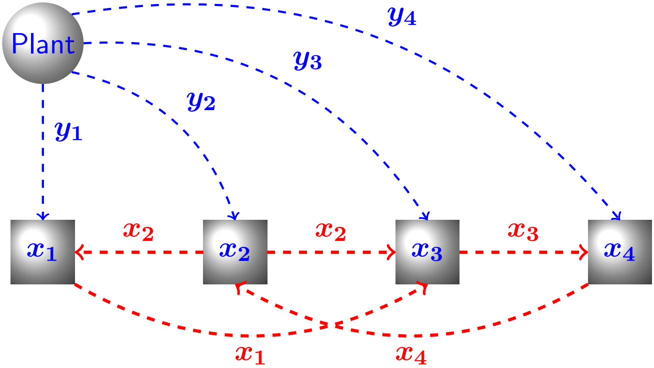
Suppose that

$$(C, A) \;\text{ with }\; C = \begin{bmatrix} C_1 \\ \vdots \\ C_m \end{bmatrix} \in \mathbb{R}^r \;\text{ is observable}$$

For each $i \in \mathbf{m}$,

$\mathcal{N}_i$ is the neighbors (without self-loops) of the ith observer

Illustration



$$m = 4, \quad \mathbf{m} \triangleq \{1, 2, 3, 4\}$$

$$\mathcal{N}_1 = \{2\}, \quad \mathcal{N}_2 = \{4\}, \quad \mathcal{N}_3 = \{1, 2\}, \quad \mathcal{N}_4 = \{3\}$$

Distributed observer by HTWS 2019TAC

Structure of the distributed **(identical)** observer by HTWS 2019TAC

$$\dot{x}_i(t) = Ax_i + L_i(y_i - C_i x_i) + \gamma M_i \sum_{j \in \mathcal{N}_i} a_{ij}\big(x_j(t) - x_i(t)\big), \quad i \in \mathbf{m}$$

It is hoped to achieve convergence of the estimation error

$$\lim_{t \to \infty} |\epsilon_i(t)| = 0, \quad \epsilon_i(t) = x_i(t) - x(t), \quad \forall i \in \mathbf{m}$$

Main Theorem. Assume

- (C, A) is (jointly) observable and
- $\mathcal{G}$ is a strongly connected directed graph.

Then there exists a distributed observer that achieves omniscience asymptotically.

Remark on HTWS 2019TAC

The distributed **(identical)** observer by HTWS 2019TAC

$$\dot{x}_i(t) = Ax_i + L_i(y_i - C_i x_i) + \gamma M_i \sum_{j \in \mathcal{N}_i} a_{ij}\big(x_j(t) - x_i(t)\big), \quad i \in \mathbf{m}$$

Assume

- (C, A) is (jointly) observable and
- $\mathcal{G}$ is a strongly connected directed graph.

Remark.

- The plant matrix A should be the *a prior* or precisely known
- The communication signal at each edge is

$$\gamma a_{ij} M_i x_j(t)$$

possibly burdensome!

Our Project/Plan

Distributed observers based on output interactions: Necessary and sufficient conditions

Hope to achieve distributed estimation of

- less communications
- arbitrary convergence rate
- easy to implement

Plant and graph

Consider LTI plant in state-space form

$$\begin{aligned}\dot{x}(t) &= Ax(t)\\ y_i(t) &= C_i x(t), \quad x \in \mathbb{R}^n, \quad y_i \in \mathbb{R}, \quad i \in \mathbf{m} \triangleq \{1, 2, \cdots, m\}\end{aligned}$$

with **single outputs**. Suppose that

$$(C, A) \;\text{ with }\; C = \begin{bmatrix} C_1 \\ \vdots \\ C_m \end{bmatrix} \in \mathbb{R}^m \;\text{ is observable}$$

For each $i \in \mathbf{m}$,

$\mathcal{N}_i$ is the neighbors **(without self-loops)** of the ith observer

Assumption

- **Assumption**

  A1 A has no eigenvalue with negative real part, and

  A2 $\mathcal{G}$ is a directed graph such that, for each $i \in \mathbf{m}$,

  $$(\bar{C}_i, A) \quad \text{is observable}$$

  (namely, locally jointly observable) where

  $$\bar{C}_i = \begin{bmatrix} C_i \\ C_j, j \in \mathcal{N}_i \end{bmatrix} = \begin{bmatrix} C_i \\ C_{j_1} \\ \vdots \\ C_{j_{l_i}} \end{bmatrix}, \quad l_i = |\mathcal{N}_i|$$

  $$j_1 < \cdots < j_{l_i}, \quad j_k \in \mathcal{N}_i, \quad k = 1, \cdots, l_i$$

- **Remark** The above condition **A2** is stronger than

  (C, A) is (jointly) observable

Structure of the distributed observers

Consider the distributed observer design, taking the form

$$\Sigma_{obs}: \quad \dot{z}_i = M_i z_i + N_i \begin{bmatrix} y_i \\ \hat{y}_j, j \in \mathcal{N}_i \end{bmatrix}$$
$$\hat{x}_i = \Gamma_i z_i, \quad z_i \in \mathbb{R}^n, \quad i \in \mathbf{m}$$

or written by

$$\begin{aligned} \Sigma_{obs}: \quad \dot{z}_i &= M_i z_i + N_i \begin{bmatrix} C_i x \\ C_j \hat{x}_j, j \in \mathcal{N}_i \end{bmatrix} \\ &= M_i z_i + N_i \begin{bmatrix} C_i x \\ C_j x, j \in \mathcal{N}_i \end{bmatrix} + N_i \begin{bmatrix} 0 \\ C_j \hat{x}_j - C_j x, j \in \mathcal{N}_i \end{bmatrix} \\ &= M_i z_i + N_i \bar{C}_i x + N_i \begin{bmatrix} 0 \\ C_j \hat{x}_j - C_j x, j \in \mathcal{N}_i \end{bmatrix} \\ \hat{x}_i &= \Gamma_i z_i, \quad i \in \mathbf{m} \end{aligned}$$

Necessary condition

C1 **(invariance)** For each $i \in \mathbf{m}$, there is a **non-singular matrix** $\Pi_i \in \mathbb{R}^{n \times n}$ such that

$$\Pi_i A = M_i \Pi_i + N_i \bar{C}_i, \quad 0 = I_n - \Gamma_i \Pi_i, \quad \Gamma_i = \Pi_i^{-1}, \quad i \in \mathbf{m}$$

C2 **(convergence or stability)** Moreover, letting

$$\epsilon_i = z_i - \Pi_i x, \quad \epsilon = \begin{bmatrix} \epsilon_1^T & \cdots & \epsilon_m^T \end{bmatrix}^T$$

the error system

$$\begin{aligned} \dot{\epsilon}_i &= M_i \epsilon_i + N_i \begin{bmatrix} 0 \\ C_j \Gamma_j (\epsilon_j + \Pi_j x) - C_j x, j \in \mathcal{N}_i \end{bmatrix} \\ &= M_i \epsilon_i + N_i \begin{bmatrix} 0 \\ C_j \Gamma_j \epsilon_j, j \in \mathcal{N}_i \end{bmatrix}, \quad i \in \mathbf{m} \end{aligned}$$

or written in the compact form

$$\dot{\epsilon} = M \epsilon$$

should be asymptotically stable at $\epsilon = 0$

Conjecture

If

A1 A has no eigenvalue with negative real part, and

A2 $\mathcal{G}$ is a directed graph such that, for each $i \in \mathbf{m}$,

$$(\bar{C}_i, A) \text{ is observable}$$

then, there are design parameters $\{M_i, N_i, \Gamma_i\}$ for $i \in \mathbf{m}$ satisfying the following conditions:

C1 **(invariance)** There are matrices Π_i for $i \in \mathbf{m}$ such that

$$\Pi_i A = M_i \Pi_i + N_i \bar{C}_i, \quad 0 = I_n - \Gamma_i \Pi_i, \quad i \in \mathbf{m}$$

C2 **(convergence or stability)** Moreover, the error system

$$\dot{\epsilon} = M\epsilon$$

is asymptotically stable at $\epsilon = 0$

Not true!

The error system is

$$\dot{\epsilon}_i = M_i \epsilon_i + N_i \begin{bmatrix} 0 \\ C_j \Gamma_j \epsilon_j, j \in \mathcal{N}_i \end{bmatrix}, \quad i \in \mathbf{m}$$

Without loss of generality, suppose the controllable pair (M_i, N_i) is diagonal,

$$M_i = \begin{bmatrix} M_{i1} & 0 \\ 0 & M_{i2} \end{bmatrix}, \quad N_i = \begin{bmatrix} N_{i1} & 0 \\ 0 & N_{i2} \end{bmatrix}$$

with M_i being Hurwitz. Also, let

$$\epsilon_i = \begin{bmatrix} \epsilon_{i1} \\ \epsilon_{i2} \end{bmatrix}, \quad [N_{i2} C_j \Gamma_j \epsilon_j, j \in \mathcal{N}_i] = \begin{bmatrix} \bar{\Gamma}_{i1} & & \\ & \ddots & \\ & & \bar{\Gamma}_{i,2m} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \vdots \\ \epsilon_{m2} \end{bmatrix} =: \bar{\Gamma}_i \epsilon$$

with (because $j \neq i$)

$$\bar{\Gamma}_{i,2i-1} = 0, \quad \bar{\Gamma}_{i,2i} = 0$$

It gives

$$\begin{aligned}\dot{\epsilon}_{i1} &= M_{i1}\epsilon_{i1}\\ \dot{\epsilon}_{i2} &= M_{i2}\epsilon_{i2} + \bar{\Gamma}_i\epsilon, \quad i \in \mathbf{m}\end{aligned}$$

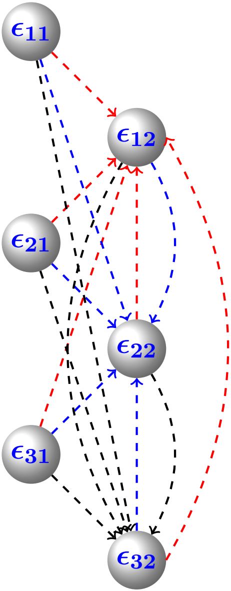
and

$$\dot{\epsilon} = \left[\begin{array}{cc|cc|c|cc} M_{11} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & M_{12} & \bar{\Gamma}_{13} & \bar{\Gamma}_{14} & \cdots & \bar{\Gamma}_{1,2m-1} & \bar{\Gamma}_{1,2m} \\ \hline 0 & 0 & M_{21} & 0 & \cdots & 0 & 0 \\ \bar{\Gamma}_{21} & \bar{\Gamma}_{22} & 0 & M_{12} & \cdots & \bar{\Gamma}_{2,2m-1} & \bar{\Gamma}_{2,2m} \\ \hline \vdots & & \vdots & & \ddots & \vdots & \\ \hline 0 & 0 & 0 & 0 & \cdots & M_{m1} & 0 \\ \bar{\Gamma}_{m1} & \bar{\Gamma}_{m2} & \bar{\Gamma}_{m3} & \bar{\Gamma}_{m4} & \cdots & 0 & M_{m2} \end{array}\right] \epsilon =: \mathcal{M}\epsilon$$

Is $\mathcal{M}$ Hurwitz? Or, any additional condition?

Interconnection structure of

$$\dot{\epsilon} = \mathcal{M}\epsilon$$



Interconnection structure of

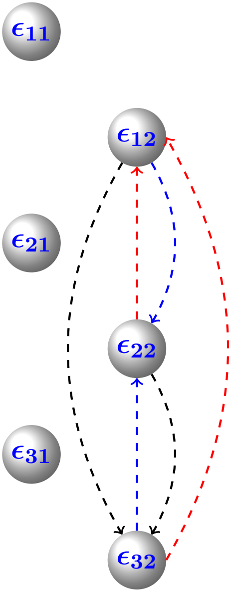
$$\dot{\epsilon} = \mathcal{M}\epsilon$$

Removing ϵ_{11} to ϵ_{m1} subsystems gives

$$\dot{\epsilon}' = \left[\begin{array}{c|c|c|c} M_{12} & \bar{\Gamma}_{14} & \cdots & \bar{\Gamma}_{1,2m} \\ \hline \bar{\Gamma}_{22} & M_{12} & \cdots & \bar{\Gamma}_{2,2m} \\ \hline \vdots & & \ddots & \vdots \\ \hline \bar{\Gamma}_{m2} & \bar{\Gamma}_{m4} & \cdots & M_{m2} \end{array}\right] \epsilon'$$

$$=: \mathcal{M}'\epsilon'$$

Note that if $\mathcal{M}'$ is Hurwitz, then $\mathcal{M}$ is so



Example

- Plant

$$x \in \mathbb{R}^4, \quad A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad A_i = \begin{bmatrix} 0 & 1 \\ -\sigma_i & 0 \end{bmatrix}, \quad \sigma_i > 0, \quad m = 2$$

- **Case-I**

$$C_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}, \quad C_2 = C_1$$

In this case

$$\Sigma_{obs}^{(1)}: \quad \dot{z}_1 = M_1 z_1 + N_1 \begin{bmatrix} y_1 \\ \hat{y}_2 \end{bmatrix}, \quad M_1 \in \mathbb{R}^{4\times 4}, \quad N_1 \in \mathbb{R}^{4\times 2}$$

$$\Sigma_{obs}^{(2)}: \quad \dot{z}_2 = M_2 z_2 + N_2 \begin{bmatrix} y_2 \\ \hat{y}_1 \end{bmatrix}, \quad M_2 \in \mathbb{R}^{4\times 4}, \quad N_2 \in \mathbb{R}^{4\times 2}$$

- Observers

$$\Sigma_{obs}^{(1)}: \quad \dot{z}_1 = M_1 z_1 + N_1 \begin{bmatrix} C_1 x \\ C_2 \hat{x}_2 \end{bmatrix}, \quad M_1 \in \mathbb{R}^{4\times4}, \quad N_1 \in \mathbb{R}^{4\times2}$$

$$\Sigma_{obs}^{(2)}: \quad \dot{z}_2 = M_2 z_2 + N_2 \begin{bmatrix} C_2 x \\ C_1 \hat{x}_1 \end{bmatrix}, \quad M_2 \in \mathbb{R}^{4\times4}, \quad N_2 \in \mathbb{R}^{4\times2}$$

Then, we write the error systems

$$\begin{aligned} \Sigma_{obs}^{(1)}: \quad \dot{z}_1 &= M_1 z_1 + N_1 \bar{C}_1 x + N_1 \begin{bmatrix} 0 \\ C_2 \hat{x}_2 - C_2 x \end{bmatrix} \\ \dot{\epsilon}_1 &= M_1 \epsilon_1 + N_1 \begin{bmatrix} 0 \\ C_2 \epsilon_2 \end{bmatrix} \\ \Sigma_{obs}^{(2)}: \quad \dot{z}_2 &= M_2 z_2 + N_2 \bar{C}_2 x + N_2 \begin{bmatrix} 0 \\ C_1 \hat{x}_1 - C_1 x \end{bmatrix} \\ \dot{\epsilon}_2 &= M_2 \epsilon_2 + N_2 \begin{bmatrix} 0 \\ C_1 \epsilon_1 \end{bmatrix} \end{aligned}$$

- Now, consider the error systems

$$\dot{\epsilon}_1 = M_1\epsilon_1 + N_1 \begin{bmatrix} 0 \\ C_2\epsilon_2 \end{bmatrix}$$
$$\dot{\epsilon}_2 = M_2\epsilon_1 + N_2 \begin{bmatrix} 0 \\ C_1\epsilon_1 \end{bmatrix}$$

- **Caution** (M_i, N_i) for $i = 1, 2$ are not diagonalizable!

Scenario I

Assumption

A1 A has no eigenvalue with negative real part

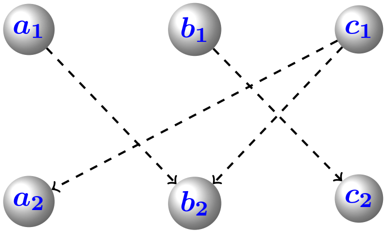
A2 $\mathcal{G}$ is a directed graph such that, for each $i \in \mathbf{m}$,

$$(\bar{C}_i, A) \quad \text{is observable}$$

where

$$\bar{C}_i = \begin{bmatrix} C_i \\ C_j, j \in \mathcal{N}_i \end{bmatrix}$$

(locally jointly observable)



$$\Sigma_{obs}: \quad \dot{z}_i = M_i z_i + N_i \begin{bmatrix} y_i \\ \hat{y}_j, j \in \mathcal{N}_i \end{bmatrix}$$

$$\hat{x}_i = \Gamma_i z_i, \quad z_i \in \mathbb{R}^n, \quad i \in \mathbf{m}$$

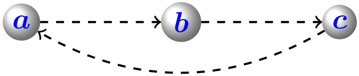
Scenario II

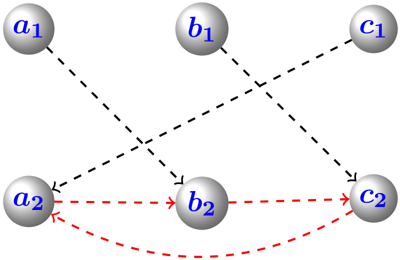
Assumption

A1 A has no eigenvalue with negative real part

A2 (C, A) is jointly observable

The communication graph is





Modified distributed observers

Consider LTI plant in state-space form

$$\dot{x} = Ax, \quad y_i = C_i x, \quad x \in \mathbb{R}^n, \quad y_i \in \mathbb{R}, \quad i \in \mathbf{m}$$

Consider a modified distributed observer

$$\begin{aligned} \Sigma_{obs}: \quad \dot{z}_i &= M_i z_i + N_i \begin{bmatrix} y_i \\ \hat{y}_j, j \in \mathcal{N}_i \end{bmatrix} + \gamma H_i \left[\hat{y}_j - \hat{y}_i, j \in \mathcal{N}_i\right] \\ \hat{x}_i &= \Gamma_i z_i, \quad i \in \mathbf{m} \end{aligned}$$

with

$$\hat{y}_i = C_i \hat{x}_i$$

The error system is

$$\dot{\epsilon}_i = M_i \epsilon_i + N_i \begin{bmatrix} 0 \\ C_j \Gamma_j \epsilon_j, j \in \mathcal{N}_i \end{bmatrix} + \ldots$$