

OT

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1 The Economic Problem

1.1 Setting

A **monopolist** designs a menu of products to sell to a heterogeneous population of consumers.

Players:

- **Firm:** Chooses which products to offer and at what prices
- **Consumers:** Each has private willingness-to-pay (WTP) for product attributes
- **Goal:** Maximize profit subject to individual rationality (IR) and incentive compatibility (IC)

Example:

- **Products:** Software bundles with features {basic, analytics, collaboration, API access}
- **Consumers:** Companies with different valuations for each feature
- **Firm must offer a menu** (e.g., “Basic \$10”, “Pro \$25”, “Enterprise \$50”) without knowing each consumer’s exact WTP

1.2 Mathematical Primitives

1.2.1 Consumer Side

Consumer types: Each consumer i is characterized by their WTP vector

$$v_i \in \mathbb{R}^d$$

where d is the number of attributes (features).

Interpretation: $v_i[j]$ = consumer i ’s WTP for attribute j

Population: We model consumers as draws from a distribution

$$\mu \in \mathcal{P}(\mathbb{R}^d)$$

where $\mathcal{P}(\mathbb{R}^d)$ is the space of probability measures on \mathbb{R}^d .

For computation, we observe a finite sample:

$$\{v_1, v_2, \dots, v_N\} \sim \mu$$

1.2.2 Product Side

Bundles: A product is a vector of attributes

$$b \in \{0, 1\}^d$$

where $b[j] = 1$ if feature j is included, 0 otherwise.

Prices: Each bundle has a price

$$p \in \mathbb{R}_+$$

Products in joint space: A product is the PAIR

$$x = (b, p) \in \{0, 1\}^d \times \mathbb{R}_+$$

1.2.3 Menu

A **menu** M is a finite collection of products:

$$M = \{x_1, x_2, \dots, x_L\} \quad (1)$$

$$= \{(b_1, p_1), (b_2, p_2), \dots, (b_L, p_L)\} \quad (2)$$

where L is the menu size.

1.3 Consumer Preferences

Utility: Consumer with type v derives utility from product $x = (b, p)$:

$$u(v, x) = v \cdot b - p$$

This is **quasilinear utility**:

- Positive component: $v \cdot b$ (value from features)
- Negative component: p (cost of purchase)

Outside option: Consumers can choose not to buy, giving utility 0.

Choice: Faced with menu M , consumer v chooses:

$$x^*(v) = \operatorname{argmax}_{x \in M \cup \{\emptyset\}} u(v, x)$$

where \emptyset denotes the outside option with $u(v, \emptyset) = 0$.

1.4 Firm's Problem

Cost function: Producing bundle b costs

$$C(b) = c_1 \cdot |b| + c_2 \cdot |b|^\alpha$$

where $|b|$ = number of features in bundle (typically $\alpha \geq 1$ for convex costs).

Profit: If consumer v chooses product $x = (b, p)$, firm earns

$$\pi(v, x) = \begin{cases} p - C(b) & \text{if } u(v, x) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Total profit: Expected profit over consumer distribution μ :

$$\Pi(M) = \int \pi(v, x^*(v)) d\mu(v)$$

Firm's problem:

$$\max_M \Pi(M) \quad (3)$$

$$\text{subject to: IC: } x^*(v) = \operatorname{argmax}_{x \in M} u(v, x) \quad \forall v \text{ (incentive compatibility)} \quad (4)$$

$$\text{IR: } u(v, x^*(v)) \geq 0 \quad \forall v \text{ (individual rationality)} \quad (5)$$

2 Optimal Transport Formulation

2.1 Why This Is Optimal Transport

The key insight is that the firm's problem is equivalent to **choosing a discrete target measure** in the joint (b, p) space.

Source measure: Consumer types distributed according to μ

Target measure: Menu represented as

$$\nu = \sum_{m=1}^L w_m \cdot \delta_{x_m}$$

where:

- $x_m = (b_m, p_m)$ is the m -th product (an atom in joint space)
- w_m is the mass assigned to product m (market share)
- δ_{x_m} is a Dirac mass at x_m

Transport plan: The assignment of consumers to products is a coupling

$$\gamma \in \Pi(\mu, \nu)$$

where $\Pi(\mu, \nu)$ is the set of joint distributions with marginals μ and ν .

IC constraint in OT language: The transport plan γ must be optimal with respect to the utility cost:

$$\gamma = \operatorname{argmin}_{\pi \in \Pi(\mu, \nu)} \int \int c(v, x) d\pi(v, x)$$

where the **cost function** is:

$$c(v, x) = -(v \cdot b - p) \quad (\text{negative utility})$$

This enforces that consumers choose utility-maximizing products!

2.2 The Monge-Kantorovich Formulation

Primal problem (Monge-Kantorovich):

$$\max_{\nu} \int \max_{x \in \operatorname{supp}(\nu)} \{v \cdot b - p\} d\mu(v) - \int C(b) d\nu(b, p) \quad (6)$$

$$\text{subject to: } \nu = \sum_{m=1}^L w_m \delta_{(b_m, p_m)} \quad (7)$$

$$w_m \geq 0, \sum_m w_m = 1 \quad (8)$$

$$b_m \in \{0, 1\}^d, p_m \in \mathbb{R}_+ \quad (9)$$

Equivalently (since we're maximizing profit):

$$\max_{(b_1, p_1), \dots, (b_L, p_L)} \sum_{m=1}^L w_m(b, p) \cdot [p_m - C(b_m)]$$

where:

$$w_m(b, p) = \mu(\{v : m \in \operatorname{argmax}_j (v \cdot b_j - p_j)\})$$

This is the **semi-discrete optimal transport problem**: continuous source (μ), discrete target (finite support).

2.3 Kantorovich Dual

The **dual problem** provides important structure:

$$\max_{\varphi, \psi} \int \varphi(v) d\mu(v) + \sum_{m=1}^L \psi_m w_m \quad (10)$$

$$\text{subject to: } \varphi(v) + \psi_m \leq v \cdot b_m - p_m \quad \forall v, \forall m \quad (11)$$

Dual variables:

- $\varphi(v)$: Consumer surplus (utility of optimal choice)
- ψ_m : Shadow price of product m

Complementary slackness: At optimum,

$$\varphi(v) + \psi_m = v \cdot b_m - p_m \iff \text{consumer } v \text{ is assigned to product } m$$

Key insight: The dual variables encode:

- $\varphi(v) = \max_m \{v \cdot b_m - p_m\}$: IC constraint (utility-maximizing choice)
- $\varphi(v) \geq 0$: IR constraint (participation)

So the IC/IR constraints from mechanism design are **automatically encoded** in the OT dual structure!

3 Computational Algorithm

3.1 Semi-Discrete OT via Lloyd's Algorithm

The **Lloyd's algorithm** (generalized k -means) solves semi-discrete OT by iteratively:

1. **E-step:** Assign source points to nearest target atom
2. **M-step:** Update target atoms to minimize within-cluster cost

For our problem:

3.1.1 E-step: Consumer Assignment

Given current menu $\{x_1, \dots, x_L\}$, assign each consumer:

$$m^*(v) = \operatorname{argmax}_{m \in \{1, \dots, L\}} u(v, x_m) \quad (12)$$

$$= \operatorname{argmax}_m (v \cdot b_m - p_m) \quad (13)$$

This is the **OT assignment** – minimum cost transport.

Output: Assignment function $m^* : \operatorname{supp}(\mu) \rightarrow \{1, \dots, L\} \cup \{\emptyset\}$

3.1.2 M-step: Atom Update (Voronoi Centroid)

For each product m , update (b_m, p_m) to maximize profit from assigned consumers:

$$(b_m^*, p_m^*) = \operatorname{argmax}_{b,p} \sum_{v:m^*(v)=m} [p - C(b)] \cdot \mathbb{1}\{v \cdot b \geq p\}$$

This is the **Voronoi centroid** in joint (b, p) space.

Key observation: This optimization decomposes!

3.2 Decomposition of the M-Step

Problem: Given assigned consumers $V_m = \{v : m^*(v) = m\}$, find optimal (b, p) .

Decomposition:

Step 1: Fix p , optimize b

For fixed price p , the optimal bundle is:

$$b^*(p) = \operatorname{argmax}_{b \in \{0,1\}^d} |\{v \in V_m : v \cdot b \geq p\}| \cdot (p - C(b))$$

This is a **discrete optimization** over 2^d bundles.

Algorithm: Enumerate all bundles, count demand at price p , compute profit.

Step 2: Fix b , optimize p

For fixed bundle b , the optimal price is found by **threshold pricing**:

$$p^*(b) = \operatorname{argmax}_{p \geq C(b)} |\{v \in V_m : v \cdot b \geq p\}| \cdot (p - C(b))$$

Key insight: Demand is monotone decreasing in p , so optimal p is at a consumer's valuation threshold.

Algorithm:

1. Compute valuations: $T_i = v_i \cdot b$ for $i \in V_m$
2. Sort in descending order: $T_{(1)} \geq T_{(2)} \geq \dots \geq T_{(n)}$
3. For each k , price $p = T_{(k)}$ gives k buyers at margin $T_{(k)} - C(b)$
4. Choose k^* that maximizes $k \cdot (T_{(k)} - C(b))$

Step 3: Coordinate ascent

Alternate between fixing b and fixing p until convergence.

In practice: For small d (e.g., $d = 4$), we enumerate all 2^d bundles and use threshold pricing for each, finding the global optimum directly.

3.3 Complete Algorithm

3.4 Subroutine: Joint (b, p) Optimization

Complexity: $\mathcal{O}(2^d \cdot |V_m| \cdot \log |V_m|)$ per atom update.

For $d = 4$: only 16 bundles to check, very fast.

Algorithm 1 Semi-Discrete OT for Menu Design

Input: Consumer WTP sample $V = \{v_1, \dots, v_N\} \sim \mu$, Menu size L , Max iterations T

Output: Optimal menu $M^* = \{(b_1^*, p_1^*), \dots, (b_L^*, p_L^*)\}$

1. Initialize:

for $m = 1$ to L **do**

 Sample or heuristically choose $(b_m^{(0)}, p_m^{(0)})$

end for

2. Main Loop:

for $t = 1$ to T **do**

// E-step: Assign consumers (OT assignment)

for each consumer i **do**

$m_i = \operatorname{argmax}_m (v_i \cdot b_m^{(t-1)} - p_m^{(t-1)})$

if $\max_m (v_i \cdot b_m^{(t-1)} - p_m^{(t-1)}) < 0$ **then**

$m_i = \emptyset$ *// outside option*

end if

end for

// M-step: Update atoms (Voronoi centroids)

for each product m **do**

$V_m = \{v_i : m_i = m\}$ *// assigned consumers*

if $|V_m| > 0$ **then**

$(b_m^{(t)}, p_m^{(t)}) = \operatorname{argmax}_{b,p} \sum_{v \in V_m} [p - C(b)] \cdot \mathbb{1}\{v \cdot b \geq p\}$

else

$(b_m^{(t)}, p_m^{(t)}) = (b_m^{(t-1)}, p_m^{(t-1)})$

end if

end for

// Check convergence

if $\|\text{atoms}^{(t)} - \text{atoms}^{(t-1)}\| < \varepsilon$ **then**

break

end if

end for

return $M^* = \{(b_m^{(t)}, p_m^{(t)})\}_{m=1}^L$

Algorithm 2 UpdateAtom

Input: Assigned consumers V_m , cost function C

Output: Optimal (b^*, p^*)

best_profit $\leftarrow -\infty$

best_b $\leftarrow \text{None}$

best_p $\leftarrow \text{None}$

for each bundle $b \in \{0, 1\}^d$ **do**

// Compute valuations

$T \leftarrow \{v \cdot b : v \in V_m\}$

// Sort descending

$T_{\text{sorted}} \leftarrow \text{sort}(T, \text{descending}=\text{True})$

// Try each threshold

for $k = 1$ to $|T_{\text{sorted}}|$ **do**

$p \leftarrow T_{\text{sorted}}[k]$

if $p < C(b)$ **then**

break

end if

// Profit: k buyers at margin (p - C(b))

 profit $\leftarrow k \cdot (p - C(b))$

if profit > best_profit **then**

 best_profit \leftarrow profit

 best_b $\leftarrow b$

 best_p $\leftarrow p$

end if

end for

end for

return (best_b, best_p)

Algorithm 3 Multi-Start Lloyd's

Input: V , L , restarts R

```
best_profit  $\leftarrow -\infty$ 
best_menu  $\leftarrow \text{None}$ 

for  $r = 1$  to  $R$  do
  // Random or heuristic initialization
   $M_0 \leftarrow \text{Initialize}(V, L, \text{seed}=r)$ 

  // Run Lloyd's from this init
   $M_r \leftarrow \text{Lloyd}(V, L, \text{init}=M_0)$ 

  // Evaluate
   $\text{profit}_r \leftarrow \text{EvaluateProfit}(M_r, V)$ 

  if  $\text{profit}_r > \text{best\_profit}$  then
    best_profit  $\leftarrow \text{profit}_r$ 
    best_menu  $\leftarrow M_r$ 
  end if
end for

return best_menu
```

3.5 Multi-Start Strategy

Because the problem is **non-convex** (discrete bundles), we use multi-start:

Typical choice: $R = 10\text{--}20$ restarts.

4 Theoretical Properties

4.1 Convergence of Lloyd's Algorithm

Theorem 1 (Lloyd's Convergence). *The Lloyd's algorithm converges to a local optimum in finite iterations.*

Work in Progress!

Note: Global optimum NOT guaranteed (NP-hard problem).

4.2 Optimality Conditions (Kantorovich Dual)

At a local optimum, the **complementary slackness conditions** hold:

$$\varphi(v) + \psi_m = v \cdot b_m - p_m \iff \text{consumer } v \text{ assigned to product } m \quad (14)$$

$$\varphi(v) + \psi_m \geq v \cdot b_m - p_m \iff \text{otherwise} \quad (15)$$

where:

- $\varphi(v) = \max_m \{v \cdot b_m - p_m\}$: consumer surplus

- $\psi_m = -p_m$: shadow price (in standard OT normalization)

Interpretation:

- If consumer v is assigned to product m , they get exactly $\varphi(v)$ utility
- No consumer would strictly prefer any other product
- This encodes both IC (utility maximization) and IR ($\varphi(v) \geq 0$)

5 Summary

5.1 The Mathematical Framework

Problem: Design menu $M = \{(b_1, p_1), \dots, (b_L, p_L)\}$ to maximize profit

OT Formulation:

- **Source:** Consumer distribution $\mu \in \mathcal{P}(\mathbb{R}^d)$
- **Target:** Discrete measure $\nu = \sum_m w_m \delta_{(b_m, p_m)}$
- **Cost:** $c(v, (b, p)) = -(v \cdot b - p) + C(b)$
- **Objective:** $\max_\nu \int \varphi(v) d\mu(v) - \sum_m C(b_m) w_m$

Algorithm: Semi-discrete OT via Lloyd's

- E-step: Assign consumers to utility-maximizing products
- M-step: Update each atom (b_m, p_m) to maximize profit from assigned consumers
- Multi-start: Run from multiple initializations

Theory:

- Dual potentials encode IC/IR
- Lloyd's converges to local optimum
- Complementary slackness at convergence

Extensions:

- Column generation for better bundle search
- Continuous relaxation for smooth optimization
- Dual diagnostics for optimality verification

6 Notation Reference

Sets:

- \mathbb{R}^d : d -dimensional Euclidean space
- \mathbb{R}_+ : Non-negative reals
- $\{0, 1\}^d$: Binary vectors (bundles)
- $\mathcal{P}(X)$: Probability measures on X

Consumer side:

- $v \in \mathbb{R}^d$: WTP vector (type)
- $\mu \in \mathcal{P}(\mathbb{R}^d)$: Type distribution
- $V = \{v_1, \dots, v_N\}$: Sample from μ

Product side:

- $b \in \{0, 1\}^d$: Bundle (features)
- $p \in \mathbb{R}_+$: Price
- $x = (b, p)$: Product atom
- $M = \{x_1, \dots, x_L\}$: Menu

Functions:

- $u(v, x) = v \cdot b - p$: Utility
- $C(b)$: Production cost
- $\pi(v, x)$: Profit from consumer v buying x

OT objects:

- $\nu = \sum_m w_m \delta_{x_m}$: Target measure
- $\gamma \in \Pi(\mu, \nu)$: Transport plan
- $c(v, x)$: Transport cost
- $\varphi(v)$: Consumer surplus (dual potential)
- ψ_m : Shadow price (dual potential)

Algorithm:

- $m^*(v)$: Assignment function
- V_m : Consumers assigned to product m
- T : Max iterations
- R : Number of restarts