GL Applied Data Science Program

Network Analysis

August 25, 2021

Overview

Overview of this week / module:

- Data collection and visualization for exploratory data analysis
- Network analysis
- Unsupervised learning - clustering

Overview of this lecture:

- Examples of networks and representing networks
- Summary statistics of a network
- Centrality measures - finding important nodes in a network

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Network research:

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  - e.g. Euler's celebrated 1735 solution of the Königsberg bridge problem
- In recent years network research witnessed a big change:
  - From study of a single graph on 10-100 nodes to the statistical properties of large networks on millions of nodes
  - Characterize the structure of networks
  - Identify important nodes / edges in a network
  - Identify missing links in a network

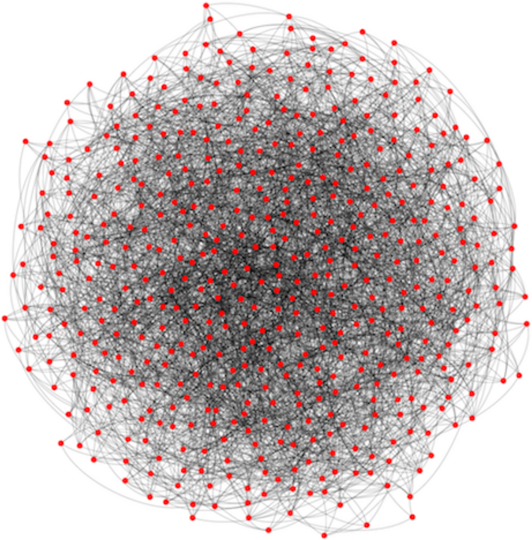
Examples of networks

Network	Vertex	Edge
World Wide Web	web page	hyperlink
Internet	computer	network protocol interaction
power grid	generating station / substation	transmission line
friendship network	person	friendship
gene regulatory network	gene	regulatory effect
neural network	neuron	synapse
food web	species	who-eats-who
phylogenetic tree	species	evolution
Netflix	person / movie	rating

Different kinds of networks

- simple network: undirected network with at most one edge between any pair of vertices and no self-loops
  - e.g. Internet, power grid, telephone network
- multigraph: self-loops and multiple links between vertices possible
  - e.g. neural network, road network
- directed network: $(i, j) \in E$ does not imply $(j, i) \in E$
  - e.g. World Wide Web, food web, citation network
- weighted network: with edge weights or vertex attributes
- tree: graph with no cycles
  - e.g. phylogenetic tree
- acyclic network: graph with no directed cycles
  - e.g. food web, citation network
- bipartite network: edges between but not within classes
  - e.g. recommender systems, Netflix
- hypergraph: generalized 'edges' for interaction between > 2 nodes
  - e.g. protein-protein interaction network

Large networks look like hairballs



Representation of a network

Two common representations of a network $G = (V, E)$:

- **adjacency list**
  - undirected graph $1 - 2 - 3$: $E = \{\{1,2\},\{2,3\}\}$
  - directed graph $1 \to 2 \leftarrow 3$: $E = \{(1,2),(3,2)\}$

- **adjacency matrix** of size $n \times n$ (where $n = |V|$) with

$$A_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

  - For weighted graph, A_{ij} can be non-binary

How does the adjacency matrix of an undirected graph look like? How to count the number of friends or suggest new friends in a social network?

Representation of a network

Quantitative measures of networks

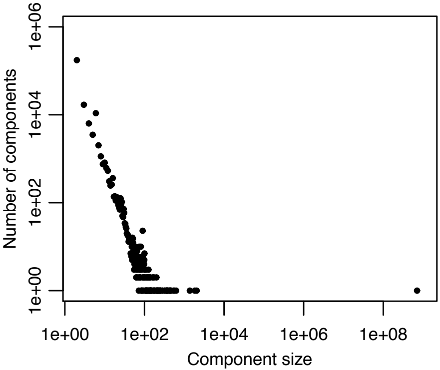
Some quantitative measures of networks to describe structural patterns of a network and to compare networks:

- connected components
- degree distribution
- diameter and average path length
- homophily or assortative mixing

Connected Components

Connected component: set of nodes that are reachable from one another

- Many networks consist of one large component and many small ones



Component size distribution in the 2011 Facebook network on a log-log scale. Most vertices (99.91%) are in the largest component.

Degree distribution

- Degree of node i: k_i
- Average degree: $\frac{1}{n}\sum_i k_i = \frac{\sum_{i,j} A_{ij}}{n} = \frac{2m}{n}$, where $|V| = n$, $|E| = m$
- More information captured by degree distribution
  - histogram of fraction of nodes with degree k.

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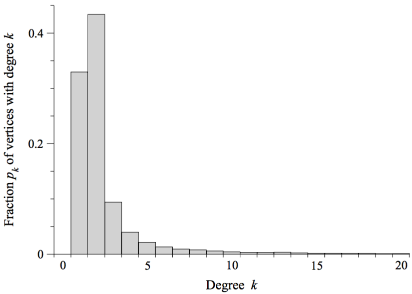
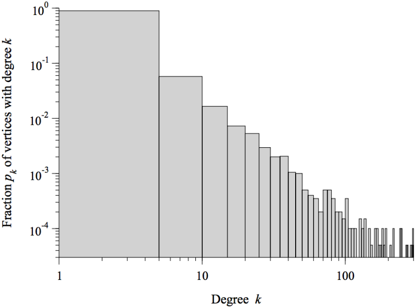
  - histogram of fraction of nodes with degree k.
- Special type of degree distribution: power-law distribution:

$$\log p_k = -\alpha \log k + c \quad \text{for some } \alpha, c > 0$$

  - tail of distribution is fat, i.e., there are many nodes with high degrees
  - appears linear on a log-log plot
  - appear in wide variety of settings including WWW, Internet

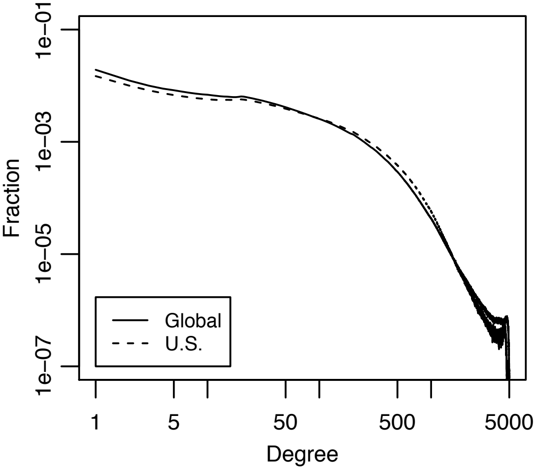
Degree distribution of the Internet





Figures from Chapter 8 in “Networks: An Introduction” by M.E.J. Newman (2010)

Degree distribution of Facebook network



From "The Anatomy of the Facebook Social Graph" by Ugander et al. (2011)

Diameter and average distance

- Let d_{ij} denote the length of the geodesic path (or shortest path) between node i and j
- The diameter of a network is the largest distance between any two nodes in the network:

$$\text{diameter} = \max_{i,j \in V} d_{ij}$$

- The average path length is the average distance between any two nodes in the network:

$$\text{average path length} = \frac{1}{\binom{n}{2}} \sum_{i \leq j} d_{ij}$$

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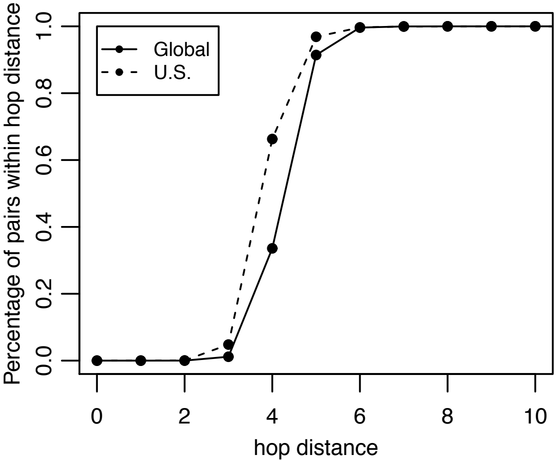
- If network is not connected, one often computes the diameter and the average path length in the largest component.
- Algorithms for finding shortest paths: breadth-first search for unweighted graph, Dijkstra's algorithm for weighted graphs

Small-world and 6 degrees of separation

- Concept of 6 degrees of separation was made famous by sociologist Stanley Milgram and his study "The Small World Problem" (1967)
- In his experiment participants from a particular town were asked to get a letter to a particular person in a different town by passing it from acquaintance to acquaintance.
- 18 out of 96 letters made it in an average of 5.9 steps
- Any reasons why we should take the conclusion of 6 degrees of separation with a grain of salt?

Diameter of Facebook (2011)



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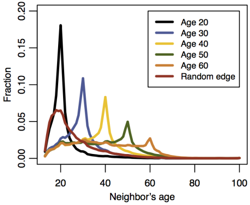
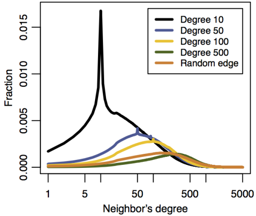
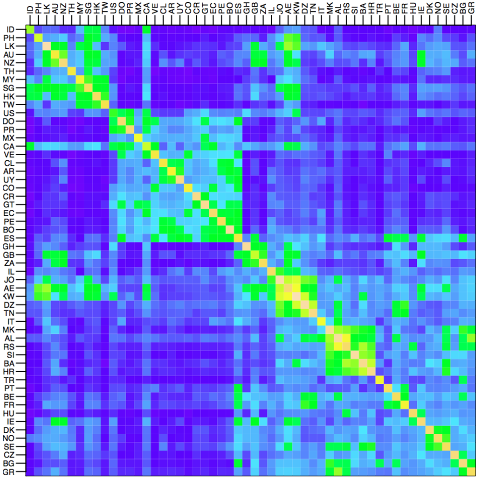
Homophily (or **assortative mixing**): tendency of people to associate with others that are similar







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Characteristics of different networks

	Network	Type	n	m	c	S	ℓ	α	C
Social	Film actors	Undirected	449 913	25 516 482	113.43	0.980	3.48	2.3	0.20
	Company directors	Undirected	7 673	55 392	14.44	0.876	4.60	–	0.59
	Math coauthorship	Undirected	253 339	496 489	3.92	0.822	7.57	–	0.15
	Physics coauthorship	Undirected	52 909	245 300	9.27	0.838	6.19	–	0.45
	Biology coauthorship	Undirected	1 520 251	11 803 064	15.53	0.918	4.92	–	0.088
	Telephone call graph	Undirected	47 000 000	80 000 000	3.16			2.1	
	Email messages	Directed	59 812	86 300	1.44	0.952	4.95	1.5/2.0	
	Email address books	Directed	16 881	57 029	3.38	0.590	5.22	–	0.17
	Student dating	Undirected	573	477	1.66	0.503	16.01	–	0.005
	Sexual contacts	Undirected	2 810					3.2	
Information	WWW `nd.edu`	Directed	269 504	1 497 135	5.55	1.000	11.27	2.1/2.4	0.11
	WWW AltaVista	Directed	203 549 046	1 466 000 000	7.20	0.914	16.18	2.1/2.7	
	Citation network	Directed	783 339	6 716 198	8.57			3.0/–	
	Roget's Thesaurus	Directed	1 022	5 103	4.99	0.977	4.87	–	0.13
	Word co-occurrence	Undirected	460 902	16 100 000	66.96	1.000		2.7	
Technological	Internet	Undirected	10 697	31 992	5.98	1.000	3.31	2.5	0.035
	Power grid	Undirected	4 941	6 594	2.67	1.000	18.99	–	0.10
	Train routes	Undirected	587	19 603	66.79	1.000	2.16	–	
	Software packages	Directed	1 439	1 723	1.20	0.998	2.42	1.6/1.4	0.070
	Software classes	Directed	1 376	2 213	1.61	1.000	5.40	–	0.033
	Electronic circuits	Undirected	24 097	53 248	4.34	1.000	11.05	3.0	0.010
	Peer-to-peer network	Undirected	880	1 296	1.47	0.805	4.28	2.1	0.012
Biological	Metabolic network	Undirected	765	3 686	9.64	0.996	2.56	2.2	0.090
	Protein interactions	Undirected	2 115	2 240	2.12	0.689	6.80	2.4	0.072
	Marine food web	Directed	134	598	4.46	1.000	2.05	–	0.16
	Freshwater food web	Directed	92	997	10.84	1.000	1.90	–	0.20
	Neural network	Directed	307	2 359	7.68	0.967	3.97	–	0.18

$n = |\text{nodes}|$, $m = |\text{edges}|$, c: mean degree, S: prop. largest component, ℓ: mean geodesic, α: exp. power-law degree distribution, C: clustering coeff.

Find important nodes in a network

- **Centrality measure**: A measure that captures importance of a node's position in the network
- There are many different centrality measures

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- **Centrality measure**: A measure that captures importance of a node's position in the network
- There are many different centrality measures
  - degree centrality (indegree / outdegree)
  - "propagated" degree centrality (score that is proportional to the sum of the score of all neighbors)
  - closeness centrality
  - betweenness centrality

Degree centrality

- For undirected graphs the degree k_i of node i is the number of edges connected to i, i.e. $k_i = \sum_j A_{ij}$
- For directed graphs the indegree of node i is $k_i^{\mathrm{in}} = \sum_j A_{ji}$ and the outdegree is $k_i^{\mathrm{out}} = \sum_j A_{ij}$

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- Simple, but intuitive: individuals with more connections have more influence and more access to information.
- Does not capture "cascade of effects": importance better captured by having connections to important nodes

Eigenvector centrality

- gives each node a score that is proportional to the sum of the scores of all its neighbors

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- need to know scores of all neighbors, which we don't know
- start with equal centrality: $x_i^{(0)} = 1$ for all nodes $i = 1, \ldots, n$
- update each centrality by the centrality of the neighbors:

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- iterate this process: $x^{(k)} = A^k x^{(0)}$
- if there exists $m > 0$ such that $A^m > 0$, then one can show that

$$x^{(k)} \stackrel{k \to \infty}{\longrightarrow} \alpha \lambda_{\max}^k v,$$

where $\lambda_{\max}$ is the largest eigenvalue and $v \geq 0$ the corresponding eigenvector; α depends on choice of $x^{(0)}$ (Perron-Frobenius theorem)

Eigenvector centrality

Interpretation: $v_i = \frac{1}{\lambda_{\max}} \sum_{j=1}^{n} A_{ij} v_j$

- node is important if it has important neighbors
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What happens when G is directed?

- right eigenvector: $v_i = \frac{1}{\lambda_{\max}} \sum_{j=1}^{n} A_{ij} v_j$
  - importance comes from nodes i points to
  - Example: determining malfunctioning genes
- left eigenvector: $w_i = \frac{1}{\lambda_{\max}} \sum_{j=1}^{n} w_j A_{ji}$
  - importance comes from nodes pointing to i
  - Example: ranking websites
  - Is the foundation for Google's PageRank algorithm

Other centrality measures

- **Closeness centrality:** Tracks how close a node is to any other node:

$$C_i = \left(\frac{1}{n-1} \sum_{j \neq i} d_{ij} \right)^{-1},$$

where d_{ij} is the distance between nodes i and j

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- **Betweenness centrality:** Measures the extent to which a node lies on paths between other nodes:

$$B_i = \frac{1}{n^2} \sum_{s,t} \frac{n_{st}^i}{g_{st}},$$

where n_{st}^i is number of shortest paths between s and t that pass through i, and g_{st} is total number of shortest paths between s and t

Which centrality measure to use

Choice of centrality measure depends on application!

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In a friendship network:

- high degree centrality: most popular person
- high eigenvector centrality: most popular person that is friends with popular people
- high closeness centrality: person that could best inform the group
- high betweenness centrality: person whose removal could best break the network apart

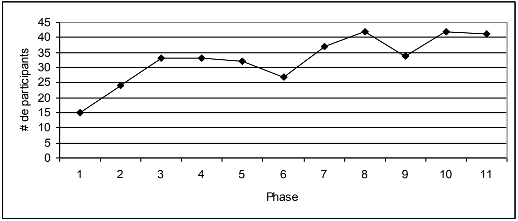
Case study: CAVIAR (criminal network in Montreal)

- Data based on 11 wiretap warrants from 1994-1996 $\rightarrow$ 11 periods
- Mandate of CAVIAR project: Seize drugs, arrests only in period 11
- 11 seizures total with monetary losses for traffickers of $32 mio
  - phase 4: 1 seizure $ 2.5mio, 300kg of marijuana
  - phase 6: 3 seizures $ 1.3mio, 2 x 15kg of marijuana, 1 x 2 kg of cocaine
  - phase 7: 1 seizure $ 3.5mio, 401kg of marijuana
  - phase 8: 1 seizure $ 0.4mio, 9kg of cocaine
  - phase 9: 2 seizures $ 4.3mio, 2kg of cocaine + 1 x 500kg marijuana
  - phase 10: 1 seizure $ 18.7mio, 2200kg of marijuana
  - phase 11: 2 seizures $ 1.3mio, 12kg of cocaine + 11kg of cocaine

Unique opportunity to study changes in the structure of a criminal network in upheaval by police forces

Case study: CAVIAR (criminal network in Montreal)

- network consists of 110 (numbered) players: 1-82 are traffickers, 83-110 are non-traffickers (financial investors, accountants, owners of various importation businesses, etc.)
- initially, investigation targeted Daniel Serero, alleged mastermind of drug network in downtown Montreal
- initially marijuana was imported to Canada from Morocco
- after first seizure in phase 4, traffickers reoriented to cocaine import from Colombia, transiting through the United States



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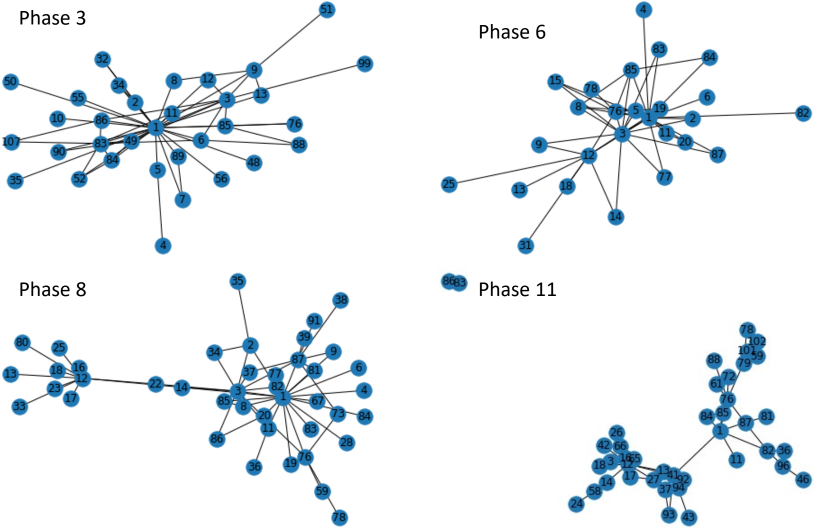
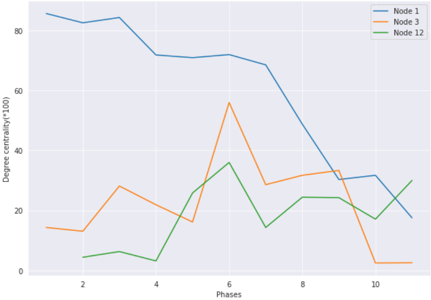
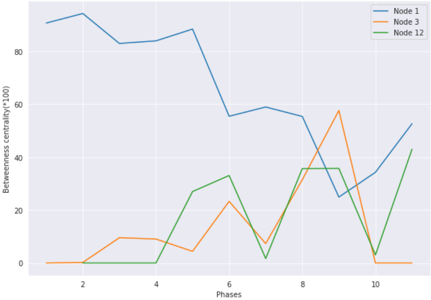


Case study: CAVIAR (criminal network in Montreal)

Role of the different actors:

- Daniel Serero (node 1): mastermind of the network
- Pierre Perlini (node 3): principal lieutenant of Serero (executes his instructions)
- Ernesto Morales (node 12): principal organizer of the cocaine import, intermediary between the Colombians and the Serero organization





Additional thoughts: Criminal networks

- Given a social network and k criminal suspects, how to determine other suspects?
- Same question is extremely important in biology: given certain genes that are known to cause a certain disease, determine other candidate genes (e.g. based on protein-protein interaction network for determining autism genes: http://dx.doi.org/10.1101/057828)
- How do we identify nodes that are "between" a given set of seed nodes?

Steiner trees

Determine a small subnetwork that contains the given suspects / genes and connects these nodes

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$\Rightarrow$ use collection of approximate Steiner trees for further analysis: autism interactome / criminal interactome

For genomics applications, see: http://fraenkel-nsf.csbi.mit.edu/steinernet/tutorial.html

$\Rightarrow$ compute nodes with high betweenness centrality in interactome to obtain candidate genes / suspects

References

- Chapters 1 - 10 (but mostly chapters 6 - 8) in

  M. E. J. Newman. *Networks: An Introduction*. 2010.

- For an analysis of the Facebook network:

  J. Ugander, B. Karrer, L. Backstrom and C. Marlow. *The Anatomy of the Facebook Social Graph*. 2011.

- For more information on the CAVIAR network:

  C. Morselli. Inside Criminal Networks (Springer, New York). Chapter 6: Law-enforcement disruption of a drug-importation network. 2009.